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Christian Licht[a,b], Somsak Orankitjaroen[b]

[a] *LMGC, UMR-CNRS 5508, Université Montpellier II, Case courier 048, Place Eugène Bataillon, 34095 Montpellier Cedex 5, France*
[b] *Department of Mathematics, Faculty of Science, Mahidol University, Bangkok 10400, Thailand; Centre of Excellence in Mathematics, CHE, Bangkok 10400, Thailand*

Abstract

We extend the study [1] devoted to the dynamic response of a structure made of two linearly elastic bodies connected by a thin soft adhesive layer made of a Kelvin Voigt type nonlinear viscoelastic material to the case of a Generalized Standard Material with a positive definite quadratic density of free energy. A concise formulation in terms of an evolution equation in a Hilbert space of possible states with finite energy makes it possible to identify the asymptotic behavior, when some geometrical and mechanical parameters tend to their natural limits, like the response of the two bodies connected by a mechanical constraint. Its law has the same structure as that of the adhesive but with coefficients accounting for the relative behavior of the parameters.

Résumé

On étend au cas d'un matériau standard généralisé l'étude [1] consacrée à la réponse dynamique d'un assemblage de deux corps linéairement élastiques liés par une couche adhésive mince et molle constituée d'un matériau viscoélastique non linéaire de type Kelvin-Voigt. Une formulation concise en termes d'équation d'évolution dans un espace de Hilbert d'états possibles d'énergie mecánique finie permet d'identifier le comportement asymptotique, lorsque des paramètres géométriques et mécaniques tendent vers leurs limites naturelles, comme celui de la réponse de l'assemblage des deux corps par une liaison mécanique. Sa loi a la même structure que celle de l'adhésif mais avec des coefficients rendant compte du comportement relatif des paramètres.

Keywords: Bonding problem, Generalized Standard Material, Dynamics, Maximal Monotone Operators

1. Introduction

For both theoretical and practical reasons, it is important to study the behavior of thin adhesively bonded joints not only in static or quasistatic cases but also in dynamic ones. Here we extend a previous study [1] devoted to a Kelvin Voigt type nonlinear viscoelastic material to a general inelastic one. More precisely, the adhesive is assumed to be made of a generalized standard material [2, 3] with a positive definite quadratic density of free energy. The key point is a concise formulation of the problem of determining the dynamic response of a structure made of two linearly elastic bodies perfectly connected by a thin soft layer in terms of an evolution equation in a Hilbert space of possible states with finite energy. Hence, it is possible to adapt the strategy of [1] in order to first obtain existence and uniqueness results and then to study the asymptotic behavior when some geometrical and mechanical data, now regarded as parameters, tend to their natural limits. The limit behavior, which prompted our proposal of a simplified but accurate enough model for the initial physical situation, corresponds to the dynamic response to the initial load of two linearly elastic bodies linked by a mechanical constraint along the surface the adhesive layer shrinks to. Its constitutive equations keep the memory of the adhesive joint in the sense that they have the same generalized standard structure but with various coefficients depending on the relative behaviors of the parameters. In the following, we focus on specifying the necessary adjustments to [1].

Email addresses: `clicht@univ-montp2.fr` (Christian Licht), `somsak.ora@mahidol.ac.th` (Somsak Orankitjaroen)

2. Setting the problem

We study the dynamic response of a structure consisting of two adherents connected by a thin adhesive layer which is subjected to a given load. The reference configuration of the structure is a bounded connected open set of $\mathbb{R}^3$ with a Lipschitz-continuous boundary $\partial\Omega$. Let $\{ e^1, e^2, e^3 \}$ be an orthonormal basis of $\mathbb{R}^3$ assimilated to the Euclidean physical space and, for all $\xi = (\xi_1, \xi_2, \xi_3)$ in $\mathbb{R}^3$, $\widehat{\xi}$ stands for (ξ_1, ξ_2). The intersection S of Ω with $\{ x_3 = 0 \}$ is assumed to have a positive two-dimensional Hausdorff measure $\mathcal{H}^2(S)$, and it is also assumed that there exists $\varepsilon_0 > 0$ such that $B_{\varepsilon_0} := \{ x \in \Omega;\ |x_3| < \varepsilon_0 \}$ is equal to $S \times (-\varepsilon_0, \varepsilon_0)$. Let $\varepsilon < \varepsilon_0$, then the adhesive occupies the layer B_ε while each of the two adherents occupies $\Omega_\varepsilon^\pm := \{ x \in \Omega;\ \pm x_3 > \varepsilon \}$, and let $\Omega_\varepsilon = \Omega_\varepsilon^+ \cup \Omega_\varepsilon^-$. Adherents and adhesive are assumed to be perfectly stuck together along $S_\varepsilon = S_\varepsilon^+ \cup S_\varepsilon^-$, $S_\varepsilon^\pm := \{ x \in \Omega; x_3 = \pm\varepsilon \}$. The structure is clamped on a part Γ_0 of $\partial\Omega$, with $\mathcal{H}^2(\Gamma_0) > 0$, and is subjected to body forces in Ω and surface forces on $\Gamma_1 = \partial\Omega \setminus \Gamma_0$ having densities f and g, respectively, during the time interval $[0, T]$; let $\Gamma_0^\pm = \Gamma_0 \cap \{ \pm x_3 > 0 \}$. The adherents are modeled as linearly elastic materials with a strain energy density W, such that

$$\begin{cases} W(x, e) = \dfrac{1}{2} a(x) e \cdot e \quad \text{a.e. } x \in \Omega,\ \forall e \in S^3 \\ a \in L^\infty\big(\Omega; \mathrm{Lin}(S^3)\big); \quad \exists c_m, c_M > 0 \ \text{ s.t. } c_m |e|^2 \le a(x) e \cdot e \le c_M |e|^2 \qquad \forall e \in S^3, \end{cases} \tag{2.1}$$

where S^3 is the space of (3×3) symmetric matrices with the usual inner product and norm denoted $\cdot$ and $|\ |$ (as for $\mathbb{R}^3$), and $\mathrm{Lin}(S^3)$ denotes the space of linear mappings from S^3 into S^3. The adhesive is assumed to be made of a homogeneous generalized standard material [2, 3]. In addition to strain e, there exists another inner or hidden state variable α which takes values in a finite dimensional space Ξ (whose inner product and norm are also denoted $\cdot$ and $|\ |$ and the same for $\Theta := S^3 \times \Xi$), and the densities of free energy and dissipation potential read as $(W_{\lambda\mu}, b\mathcal{D})$ where

$$\begin{cases} -\lambda, \mu, b \text{ are strictly positive real numbers,} \\ -W_{\lambda\mu}(e, \alpha) = \lambda W_1(e_{\mathrm{sph}}, j\alpha) + \mu W_2(e, \alpha) \quad \forall (e, \alpha) \in \Theta, \\ -e_{\mathrm{sph}} \text{ is the spherical part of } e, \\ -j \text{ is a not necessarily injective element of } \mathrm{Lin}(\Xi), \\ -W_1, W_2 \text{ are quadratic forms satisfying :} \\ \qquad \exists c_m^i, c_M^i > 0; \quad c_m^i |\theta|^2 \le W_i(\theta) \le c_M^i |\theta|^2 \quad \forall (i, \theta) \in \{ 1, 2 \} \times \Theta, \\ -\mathcal{D} \text{ is a convex function on } \Theta \text{ satisfying:} \\ \qquad \exists p \in [1, 2], \exists c'_m, c'_M > 0 \text{ s.t. } c'_m |\dot\theta|^p \le \mathcal{D}(\dot\theta) \le c'_M |\dot\theta|^p \quad \forall \dot\theta \in \Theta, \end{cases} \tag{2.2}$$

with the upper dot $\dot{}$ denoting the time derivative.

Let $\rho > 0$ and $\overline{\rho}_M > \overline{\rho}_m > 0$. If $\overline{\rho}$ is a measurable function such that $\overline{\rho}_m \le \overline{\rho}(x) \le \overline{\rho}_M$ a.e. x in Ω, the density γ of the structure is $\overline{\rho}$ in Ω_ε and ρ in B_ε. Thus, the problem (P_s) of determining the dynamic evolution of the assembly involves a quintuplet $s := (\varepsilon, \lambda, \mu, b, \rho)$ of data, and hereafter all the fields involved will be indexed by s. In the following, t denotes the times, $e(u)$ is the linearized strain tensor associated with the field of displacment u, and $\partial J(v)$ denotes the subdifferential at v of any lower semi-continuous convex function J, while $DJ(v)$ stands for the differential at v of any Fréchet differentiable function J. Hence, the constitutive equations of the adhesive read as:

$$(\sigma_s, 0) \in DW_{\lambda\mu}(e(u_s), \alpha_s) + b\partial\mathcal{D}\left(e\left(\frac{\partial u_s}{\partial t}\right), \frac{\partial \alpha_s}{\partial t} \right), \tag{2.3}$$

where σ_s denotes the field of stresses, so if $U_s^o = (u_s^0, \alpha_s^0, v_s^0)$ is the initial state, a formulation of (P_s) could be

$$(P_s) \quad \begin{cases} \text{Find } u_s \text{ sufficiently smooth in } \Omega \times [0,T] \text{ such that } u_s = 0 \text{ on } \Gamma_0 \times (0,T], \\ \left(u_s(\cdot,0), \alpha_s(\cdot,0), \dfrac{\partial u_s}{\partial t}(\cdot,0)\right) = U_s^o \text{ and there exists } \zeta \text{ in } \partial\mathcal{D}\left(e\left(\dfrac{\partial u_s}{\partial t}\right), \dfrac{\partial \alpha_s}{\partial t}\right) \text{ satisfying:} \\ \displaystyle\int_\Omega \gamma \frac{\partial^2 u_s}{\partial t^2} \cdot v\, dx + \int_{\Omega_\varepsilon} ae(u_s) \cdot e(v)\, dx + \int_{B_\varepsilon} \big(DW_{\lambda\mu}\big(e(u_s), \alpha_s\big) + b\zeta\big) \cdot (e(v), \alpha)\, dx = \\ \qquad\qquad\qquad\qquad = \displaystyle\int_\Omega f \cdot v\, dx + \int_{\Gamma_1} g \cdot v\, d\mathcal{H}^2 \\ \text{for all } (v,\alpha) \text{ sufficiently smooth in } \Omega \text{ and } v \text{ vanishing on } \Gamma_0. \end{cases}$$

3. Existence and uniqueness

Assuming

$$(f,g) \in BV\big(0,T; L^2(\Omega;\mathbb{R}^3)\big) \times BV^{(2)}\big(0,T; L^2(\Gamma_1;\mathbb{R}^3)\big), \tag{H1}$$

where for any Banach space X, $BV(0,T;X)$ is the subspace of $L^1(0,T;X)$ consisting of all elements whose time derivative in the sense of distributions is a bounded X-valued measure on $(0,T)$, and $BV^{(2)}(0,T;X)$ is the subspace of $BV(0,T;X)$ made of all elements whose time derivative in the sense of distributions belongs to $BV(0,T;X)$, we seek $z_s := (u_s, \alpha_s)$ in the form

$$z_s = z_s^e + z_s^r, \tag{3.1}$$

where z_s^e is the unique solution to

$$z_s^e(t) \in Z_s; \quad \varphi_s(z_s^e(t), z) = L(t)(v) \quad \forall z \in Z_s,\ \forall t \in [0,T], \tag{3.2}$$

with

$$\begin{gathered} Z_s = H^1_{\Gamma_0}(\Omega;\mathbb{R}^3) \times L^2(B_\varepsilon;\Xi), \\ H^1_{\Gamma_0}(\Omega;\mathbb{R}^3) := \{\, v \in H^1(\Omega;\mathbb{R}^3); v = 0 \text{ on } \Gamma_0 \text{ in the sense of traces} \,\}, \end{gathered} \tag{3.3}$$

$$\begin{gathered} \varphi_s(z,z') := \int_{\Omega_\varepsilon} ae(v) \cdot e(v')\, dx + \int_{B_\varepsilon} DW_{\lambda\mu}\big(e(v),\alpha\big) \cdot (e(v'),\alpha')\, dx \quad \forall z = (v,\alpha), z' = (v',\alpha') \in Z_s, \\ \Phi_s(z) := \varphi_s(z,z), \end{gathered} \tag{3.4}$$

$$L(t)(v) := \int_{\Gamma_1} g(x,t) \cdot v(x)\, d\mathcal{H}^2 \quad \forall v \in H^1_{\Gamma_0}(\Omega;\mathbb{R}^3),\ \forall t \in [0,T]. \tag{3.5}$$

As $g \mapsto z_s^e$ is linear continuous from $L^2(\Gamma_1;\mathbb{R}^3)$ into Z_s, we have

$$z_s^e \in BV^{(2)}\big(0,T;Z_s\big). \tag{3.6}$$

The remaining part z_s^r will therefore be involved in an evolution equation governed by a maximal monotone operator A_s defined in a Hilbert space H_s of possible states with finite total mechanical energy. Given the kinetic forms k_s, K_s

$$k_s(v,v') := \int_\Omega \gamma(x) v(x) \cdot v'(x)\, dx, \quad K_s(v) := k_s(v,v), \quad \forall v,v' \in L^2(\Omega;\mathbb{R}^3), \tag{3.7}$$

H_s reads as

$$H_s = Z_s \times L^2(\Omega;\mathbb{R}^3) \tag{3.8}$$

where, for all $U^i = (z^i, v^i)$, $i = 1,2$, the inner product and norm are

$$(U^1,U^2)_s := \varphi_s(z^1,z^2) + k_s(v^1,v^2), \quad |U^i|_s^2 := (U^i,U^i)_s, \tag{3.9}$$

while A_s is defined by

$$\begin{cases} D(A_s) = \left\{ U = (z,v) \in H_s; \begin{cases} \text{i)} v \in H^1_{\Gamma_0}(\Omega;\mathbb{R}^3) \\ \text{ii)} \exists (w,\beta,\zeta) \in L^2(\Omega;\mathbb{R}^3) \times L^2(B_\varepsilon;\Xi) \times \partial\mathcal{D}(e(v),\beta) \text{ with} \end{cases} \right. \\ \qquad\qquad k_s(w,v') + \varphi_s(z,z') + b\int_{B_\varepsilon} \zeta \cdot (e(v'),\alpha')\,dx = 0 \quad \forall z' = (v',\alpha') \in Z_s \Big\} \\ A_s U = (-v,0,0) + \{\, (0,-\beta,-w);\ \text{ ii) of the definition of } D(A_s) \text{ is satisfied} \,\}. \end{cases} \tag{3.10}$$

Proposition 3.1. *Operator A_s is maximal monotone and, for all $\psi = (\psi^1,\psi^2,\psi^3)$ in H_s,*

$$\begin{cases} \overline{U}_s = (\overline{u}_s,\overline{\alpha}_s,\overline{v}_s) \text{ s.t.} \\ \overline{U}_s + A_s\overline{U}_s \ni \psi \end{cases} \quad \Leftrightarrow \quad \begin{cases} Z_s \ni \overline{\xi}_s = (\overline{v}_s,\overline{\alpha}_s); \quad J_s(\overline{\xi}_s) \le J_s(z) \quad \forall z \in Z_s, \\ J_s(z) := \frac{1}{2}K_s(v) - k_s(\psi^3,v) + \frac{1}{2}\Phi_s(z) + \varphi_s((\psi^1,0),z) + \\ \qquad\qquad b\int_{B_\varepsilon} \mathcal{D}\big(e(v),\alpha-\psi^2\big)\,dx \quad \forall z = (v,\alpha) \in Z_s \\ \overline{u}_s = \overline{v}_s + \psi^1. \end{cases} \tag{3.11}$$

Proof. Let $U^i = (z^i,v^i)$ in $D(A_s)$ and $V^i = -(v^i,\beta^i,w^i)$ in A_sU^i, $i = 1,2$, the definition of $D(A_s)$ implies that there exist ζ^i in $\partial\mathcal{D}\big(e(v^i),\beta^i\big)$ such that

$$k_s(w^1-w^2,v^1-v^2) + \varphi_s(z^1-z^2,((v^1-v^2,\beta^1-\beta^2)) + b\int_{B_\varepsilon}(\zeta^1-\zeta^2)\cdot(e(v^1-v^2),\beta^1-\beta^2)\,dx = 0,$$

Hence,

$$(V^1-V^2,U^1-U^2)_s = -\varphi_s((v^1-v^2,\beta^1-\beta^2),z^1-z^2) - k_s(w^1-w^2,v^1-v^2) = b\int_{B_\varepsilon}(\zeta^1-\zeta^2)\cdot(e(v^1-v^2),\beta^1-\beta^2)\,dx$$

and the monotonicity of A_s stems from that of $\partial\mathcal{D}$.

If $\overline{U}_s + A_s\overline{U}_s \ni \psi$, the very definition of A_s means that $\overline{u}_s - \overline{v}_s = \psi^1$ and that there exists ζ_s in $\partial\mathcal{D}\big(e(\overline{v}_s),\overline{\alpha}_s-\psi^2\big)$ such that

$$k_s(\overline{v}_s-\psi^3,v) + \varphi_s(\overline{v}_s+\psi^1,\overline{\alpha}_s),(v,\alpha)) + b\int_{B_\varepsilon}\zeta_s\cdot(e(v),\alpha)\,dx = 0 \quad \forall (v,\alpha) \in Z_s$$

that is to say, $(\overline{v}_s,\overline{\alpha}_s)$ is the unique minimizer in Z_s of the strictly convex, continuous and coercive function J_s. Conversely, if $\overline{u}_s := \overline{v}_s + \psi^1$, then clearly $\overline{U}_s := (\overline{u}_s,\overline{\alpha}_s,\overline{v}_s)$ belongs to $D(A_s)$ and $\overline{U}_s + A_s\overline{U}_s \ni \psi$. □

Then, taking (H1), (3.1), (3.2), (3.6), (3.10) into account, it is straightforward that (P_s) is formally equivalent to

$$\frac{dU_s^r}{dt} + A_sU_s^r \ni F_s := \Big(-\frac{dz_s^e}{dt}, f/\gamma\Big), \quad U_s^r(0) = U_s^o - \big(z_s^e(0),0\big) \tag{3.12}$$

and consequently [4] we have the following result

Theorem 3.1. *If (f,g) satisfies* (H1) *and $U_s^o \in \big(z_s^e(0),0\big) + D(A_s)$, then* (3.12) *has a unique solution such that U_s^r belongs to $W^{1,\infty}(0,T;H_s)$ and* (3.12) *is satisfied almost everywhere in* $[0,T]$. *Hence, there exists a unique $z_s = (u_s,\alpha_s)$ in $W^{1,\infty}\big(0,T;Z_s)$ with u_s in $W^{2,\infty}(0,T;L^2(\Omega;\mathbb{R}^3)\big)$ which satisfy*

$$\begin{cases} \exists\zeta_s \in \partial\mathcal{D}\Big(e\left(\frac{du_s}{dt}\right),\frac{d\alpha_s}{dt}\Big) \text{ such that} \\ \displaystyle\int_\Omega \gamma\frac{d^2u_s}{dt^2}v\,dx + \int_{\Omega_\varepsilon} ae(u_s)\cdot e(v)\,dx + \int_{B_\varepsilon}\big(DW_{\lambda\mu}\big(e(u_s),\alpha_s\big) + b\zeta_s\big)\cdot(e(v),\alpha)\,dx \\ \displaystyle\qquad = \int_\Omega f\cdot v\,dx + \int_{\Gamma_1} g\cdot v\,d\mathcal{H}^2, \quad \forall (v,\alpha) \in Z_s, \ a.e.\ t \in (0,T] \\ \Big(z_s(0),\frac{du_s}{dt}(0)\Big) = U_s^o. \end{cases} \tag{3.13}$$

We set

$$U_s^e = \left(z_s^e, 0\right), \quad U_s = U_s^r + U_s^e. \tag{3.14}$$

4. Asymptotic behavior

Now we regard the quintuplet s of geometrical and mechanical *data* as a quintuplet of *parameters* taking values in a countable subset of $(0, \varepsilon_0) \times (0, +\infty)^4$ with a single cluster point $\bar{s}$ and study the asymptotic behavior of U_s in order to obtain a simplified but accurate enough model for the initial physical situation. The following assumptions account for the magnitudes of thickness, stiffness and density:

$$\begin{cases} \text{i)} & \bar{s} \in \{0\} \times [0, +\infty)^2 \times [0, +\infty] \times [0, +\infty) \\ \text{ii)} & \exists(\bar{\lambda}, \bar{\mu}) \in [0, +\infty]^2 \text{ s.t. } (\lambda/2\varepsilon, \mu/2\varepsilon) \to (\bar{\lambda}, \bar{\mu}) \\ \text{iii)} & \lim\limits_{s \to \bar{s}} b\varepsilon = 0, \exists \bar{b} \in [0, +\infty] \text{ s.t. } \bar{b} = \lim\limits_{s \to \bar{s}} b/(2\varepsilon)^{p-1} \\ \text{iv)} & \bar{\mu} \in (0, +\infty] \text{ if } \min\{\mathcal{H}^2(\Gamma_0^{\pm})\} = 0 \\ \text{v)} & \overline{\lim\limits_{s \to \bar{s}}}\, \varepsilon^2/\mu < +\infty \\ \text{vi)} & \exists r \in [0, 1) \text{ s.t. } \overline{\lim\limits_{s \to \bar{s}}}\, \varepsilon^r/\rho < +\infty. \end{cases} \tag{H2}$$

4.1. The limit behavior

From our previous study [1], it is easy to guess what the limit behavior may be and thus introduce the following concepts. We will consider five cases indexed by I : I = 1 if $(\bar{\lambda}, \bar{\mu}) \in [0, +\infty) \times \{0\}$; I = 2 if $(\bar{\lambda}, \bar{\mu}) \in \{+\infty, 0\}$; I = 3 if $(\bar{\lambda}, \bar{\mu}) \in [0, \infty) \times (0, +\infty)$; I = 4 if $(\bar{\lambda}, \bar{\mu}) \in \{+\infty\} \times (0, +\infty)$; I = 5 if $\bar{\mu} = +\infty$. As any element in $H^1_{\Gamma_0}(\Omega \setminus S; \mathbb{R}^3)$ has restrictions $u^\pm$ to $\Omega^\pm = \Omega \cap \{\pm x_3 > 0\}$ in $H^1(\Omega^\pm; \mathbb{R}^3)$, we denote the difference between the traces on S of u^+ and u^- by $[u]$ which belongs to $L^2(S; \mathbb{R}^3)$ and represents the relative displacement of bodies occupying $\Omega^\pm$. Let

$$\begin{aligned} &{}^1\mathrm{H}^d := H^1_{\Gamma_0}(\Omega \setminus S; \mathbb{R}^3), \quad {}^2\mathrm{H}^d := \{u \in {}^1\mathrm{H};\ [u]_3 = 0\}, \quad {}^3\mathrm{H}^d := {}^1\mathrm{H}^d, \quad {}^4\mathrm{H}^d := {}^2\mathrm{H}^d, \quad {}^5\mathrm{H}^d := H^1_{\Gamma_0}(\Omega; \mathbb{R}^3), \\ &{}^3\mathrm{X} := L^2(S; \Xi), \quad {}^4\mathrm{X} := \{\alpha \in {}^3\mathrm{X};\ j\alpha = 0\}, \\ &{}^{\mathrm{I}}\mathrm{Z} := {}^{\mathrm{I}}\mathrm{H}^d \text{ if } \mathrm{I} \in \{1, 2, 5\}, \quad {}^{\mathrm{I}}\mathrm{Z} := {}^{\mathrm{I}}\mathrm{H}^d \times {}^{\mathrm{I}}\mathrm{X} \text{ if } \mathrm{I} \in \{3, 4\}, \end{aligned} \tag{4.1}$$

such that ${}^{\mathrm{I}}\mathrm{Z}$ will be the space of the state variables describing the limit behavior; this suggests that when I = 1, 2, 5 an additional state variable to the displacement is not necessary! It is convenient to introduce some operators:

$$\begin{aligned} &\mathrm{I} = 3, 4 \quad z = (u, \alpha) \in {}^{\mathrm{I}}\mathrm{Z} \mapsto (z^d = u, z^h = \alpha, \Pi z = ([u] \otimes_S e^3, \alpha)) \in {}^{\mathrm{I}}\mathrm{Z} \times L^2(S; \Theta) \\ &\mathrm{I} = 1, 2, 5 \quad z (= u) \in {}^{\mathrm{I}}\mathrm{Z} \mapsto (z^d = u, \Pi z = [u] \otimes_S e^3) \in {}^{\mathrm{I}}\mathrm{Z} \times L^2(S; S^3) \end{aligned} \tag{4.2}$$

where $(\xi \otimes_S \zeta)_{ij} = \frac{1}{2}/(\xi_i \zeta_j + \xi_j \zeta_i) \quad \forall (\xi, \zeta) \in \mathbb{R}^3$.

The following forms define an inner product and a norm on ${}^{\mathrm{I}}\mathrm{Z}$:

$${}^{\mathrm{I}}\varphi(z, z') := \int_{\Omega \setminus S} a e(z^d) \cdot e(z'^d)\, dx + \int_S D\overline{W}_{\bar{\lambda}\bar{\mu}}(\Pi z) \cdot \Pi z'\, d\hat{x}, \quad {}^{\mathrm{I}}\Phi(z) := {}^{\mathrm{I}}\varphi(z, z) \tag{4.3}$$

where:

$$\begin{cases} \mathrm{I} = 1 & \overline{W}_{\bar{\lambda}\bar{\mu}}(e) = \bar{\lambda} W_1^{\perp}(e_{\mathrm{sph}}), \quad W_1^{\perp}(e) = \inf\{W_1(e, \alpha); \alpha \in \Xi\} \\ \mathrm{I} = 2, 5 & \overline{W}_{\bar{\lambda}\bar{\mu}} = 0 \\ \mathrm{I} = 3 & \overline{W}_{\bar{\lambda}\bar{\mu}}(\Pi z) = \bar{\lambda} W_1([z^d]_3 e^3 \otimes_S e^3, j z^h) + \bar{\mu} W_2(\Pi z) \\ \mathrm{I} = 4 & \overline{W}_{\bar{\lambda}\bar{\mu}}(\Pi z) = \bar{\mu} W_2(\Pi z), \end{cases}$$

so the Hilbert space of possible “limit”states with finite energy should be

$$ {}^{\mathrm{I}}\mathrm{H} = {}^{\mathrm{I}}\mathrm{Z} \times L^2(\Omega;\mathbb{R}^3) \tag{4.4} $$

where, for all $U^i = (z^i, v^i)$ in ${}^{\mathrm{I}}\mathrm{H}$, the inner product and norm are:

$$ ((U^1, U^2))_{\mathrm{I}} := {}^{\mathrm{I}}\varphi(z^1, z^2) + k(v^1, v^2) \quad \|U\|^2_{\mathrm{I}} := ((U, U))_{\mathrm{I}}. \tag{4.5} $$

with

$$ k(v, v') := \int_\Omega \overline{\rho} v \cdot v' \, dx \quad \forall (v, v') \in L^2(\Omega;\mathbb{R}^3). \tag{4.6} $$

The limit global potential of dissipation will be the functional $\int_S \overline{\mathcal{D}}(\dot z)\, dx$, with

$$ \begin{aligned} &\mathrm{I} = 3,4 \qquad \overline{\mathcal{D}}(\dot z) = \begin{cases} \overline{b}\mathcal{D}^{\infty,p}(\Pi \dot z) & \text{if } \overline{b} \in [0,+\infty) \\ I_{\{0\}}(\Pi \dot z) & \text{if } \overline{b} = \infty, \end{cases} \\ &\mathrm{I} = 1,2,5 \quad \overline{\mathcal{D}}(\dot z) = \begin{cases} \overline{b}(\mathcal{D}^{\infty,p})^{\perp}(\Pi \dot z) & \text{if } \overline{b} \in [0,+\infty) \\ I_{\{0\}}(\Pi \dot z) & \text{if } \overline{b} = \infty, \end{cases} \end{aligned} \tag{4.7} $$

where $(\mathcal{D}^{\infty,p})^{\perp}(\dot e) = \mathrm{Inf}\{\mathcal{D}^{\infty,p}(\dot e, \dot\alpha),\ \dot\alpha \in \Xi\}$, I_C is the indicator function of any convex set C and $\mathcal{D}^{\infty,p}(e) = \overline{\lim}_{t\to\infty} \mathcal{D}(te)/t^p$, with

$$ \exists c_\delta > 0 \text{ and } \delta \in (0,p);\ |\mathcal{D}(\theta) - \mathcal{D}^{\infty,p}(\theta)| \le c_\delta(1 + |\theta|^\delta) \quad \forall \theta \in \Theta. \tag{H3} $$

Finally, the evolution operator ${}^{\mathrm{I}}A$ can be defined by:

$$ \begin{cases} D({}^{\mathrm{I}}A) = \left\{ U = (z,v) \in {}^{\mathrm{I}}\mathrm{H}; \begin{cases} \text{i)} & v \in {}^{\mathrm{I}}\mathrm{H}^d \text{ and } [v] = 0 \text{ if } \overline{b} = \infty, \\ \text{ii)} & \mathrm{I} = 1,2,5 \quad \exists (w,\zeta) \in L^2(\Omega;\mathbb{R}^3) \times \partial\overline{\mathcal{D}}(v) \text{ s.t.} \\ & \mathrm{I} = 3,4 \quad \exists (w,\beta,\zeta) \in L^2(\Omega;\mathbb{R}^3) \times L^2(S;\Xi) \times \partial\overline{\mathcal{D}}(v,\beta) \text{ s.t.} \end{cases} \right. \\ \qquad\qquad k(w, z'^d) + {}^{\mathrm{I}}\varphi(z,z') + \int_S \zeta \cdot \Pi(z')\, d\hat x = 0 \quad \forall z' \in {}^{\mathrm{I}}\mathrm{Z}. \Big\} \\ {}^{\mathrm{I}}AU = \begin{cases} (-v,0,0) + \{(0,-\beta,-w);\ \text{ii) of the definition of } D({}^{\mathrm{I}}A) \text{ is satisfied }\} \text{ if } \mathrm{I} = 3,4, \\ (-v,0) + \{(0,-w);\ \text{ii) of the definition of } D({}^{\mathrm{I}}A) \text{ is satisfied }\} \text{ if } \mathrm{I} = 1,2,5. \end{cases} \end{cases} \tag{4.8} $$

When arguing as in the case of A_s, it can easily be checked that ${}^{\mathrm{I}}A$ is maximal monotone and, especially, that for all $\psi = (\psi^1, \psi^2)$ in ${}^{\mathrm{I}}\mathrm{H}$:

$$ \begin{cases} {}^{\mathrm{I}}\overline{U} = ({}^{\mathrm{I}}\overline{z}, {}^{\mathrm{I}}\overline{v}) \text{ s.t.} \\ {}^{\mathrm{I}}\overline{U} + {}^{\mathrm{I}}A{}^{\mathrm{I}}\overline{U} \ni \psi \end{cases} \Leftrightarrow \begin{cases} {}^{\mathrm{I}}\mathrm{Z} \ni {}^{\mathrm{I}}\overline{\xi} = \begin{cases} ({}^{\mathrm{I}}\overline{v}, {}^{\mathrm{I}}\overline{z}^h) & I = 3,4 \\ {}^{\mathrm{I}}\overline{v} & I = 1,2,5 \end{cases} \text{ s.t. } {}^{\mathrm{I}}J({}^{\mathrm{I}}\overline{\xi}) \le {}^{\mathrm{I}}J(z) \quad \forall z \in {}^{\mathrm{I}}\mathrm{Z} \\ {}^{\mathrm{I}}J(z) := \frac{1}{2}K(z^d) - k(\psi^2, z^d) + \frac{1}{2}{}^{\mathrm{I}}\Phi(z) + {}^{\mathrm{I}}\varphi(\check\psi^1, z) + \int_S \overline{\mathcal{D}}(z - \tilde\psi^1)\, d\hat x \\ \qquad \check\psi^1 = (\psi^{1d}, 0),\ \tilde\psi^1 = (0, \psi^{1h}) \text{ if } \mathrm{I} = 3,4, \quad \check\psi^1 = \psi^1,\ \tilde\psi^1 = 0 \text{ if } \mathrm{I} = 1,2,5 \\ {}^{\mathrm{I}}\overline{u} (= {}^{\mathrm{I}}\overline{z}^d) = {}^{\mathrm{I}}\overline{v} + \psi^{1d} \end{cases} \tag{4.9} $$

Consequently, the same statement as that of theorem 3.1 is valid for the following equation, which will be shown to describe the asymptotic behavior of z_s:

$$ \frac{d{}^{\mathrm{I}}U^r}{dt} + {}^{\mathrm{I}}A{}^{\mathrm{I}}U^r \ni {}^{\mathrm{I}}F := \Big(-\frac{d{}^{\mathrm{I}}z^e}{dt}, f/\overline{\rho}\Big), \quad {}^{\mathrm{I}}U^r(0) = {}^{\mathrm{I}}U^{ro} \tag{4.10} $$

with

$$ {}^{\mathrm{I}}z^e \in BV^{(2)}(0,T;{}^{\mathrm{I}}\mathrm{Z}); {}^{\mathrm{I}}\varphi({}^{\mathrm{I}}z^e(t), z) = L(t)(z^d) \quad \forall z \in {}^{\mathrm{I}}\mathrm{Z}, \forall t \in [0,T]. \tag{4.11} $$

We set

$$ {}^{\mathrm{I}}U^e = \left({}^{\mathrm{I}}z^e, 0\right), \quad {}^{\mathrm{I}}U = {}^{\mathrm{I}}U^e + {}^{\mathrm{I}}U^r. \tag{4.12}$$

4.2. Convergence

As in [1], to prove the convergence of z_s toward ${}^{\mathrm{I}}z = {}^{\mathrm{I}}z^e + {}^{\mathrm{I}}z^r$, we will use the framework of a nonlinear version of Trotter's theory of convergence of semigroups acting on *variable spaces* ([5], [6] and consider the Appendix of [7]) because z_s^r and ${}^{\mathrm{I}}z^r$ *do not inhabit the same space*.

First, to introduce a linear operator ${}^{\mathrm{I}}P_s$ from ${}^{\mathrm{I}}\mathrm{H}$ into H_s in order to compare the elements of ${}^{\mathrm{I}}\mathrm{H}$ and H_s it suffices, if need be, to add to that of [1] an obviously suitable operator to deal with the additional state variable; let R_ε be the smoothing operator, which is also linear continuous from ${}^{\mathrm{I}}\mathrm{H}^d$ into $H^1_{\Gamma_0}(\Omega;\mathbb{R}^3)$ defined by

$$ R_\varepsilon u(x) = \begin{cases} u^s(x) + \mathrm{Min}\{|x_3|/\varepsilon, 1\}u^a(x) & \forall x \text{ in } B_\varepsilon \\ u(x) & \forall x \text{ in } \Omega_\varepsilon \end{cases} \tag{4.13}$$

where $u^s(x) = \frac{1}{2}\left(u(\hat{x}, x_3) + u(\hat{x}, -x_3)\right)$, $u^a(x) = \frac{1}{2}\left(u(\hat{x}, x_3) - u(\hat{x}, -x_3)\right)$. Then operator ${}^{\mathrm{I}}P_s$ defined by

$$ \begin{cases} \mathrm{I} = 1 & U = (u, v) \in {}^{1}\mathrm{H} & \mapsto {}^{1}P_s U = (R_\varepsilon u, 0, v) \in H_s \\ \mathrm{I} = 2 & U = (u, v) \in {}^{2}\mathrm{H} & \mapsto {}^{2}P_s U = (R_\varepsilon(\hat{u}, 0) + (0, u_3), 0, v) \in H_s \\ \mathrm{I} = 3 & U = (u, \alpha, v) \in {}^{3}\mathrm{H} & \mapsto {}^{3}P_s U = (R_\varepsilon u, \alpha/2\varepsilon, v) \in H_s \\ \mathrm{I} = 4 & U = (u, \alpha, v) \in {}^{4}\mathrm{H} & \mapsto {}^{4}P_s U = (R_\varepsilon(\hat{u}, 0) + (0, u_3), \alpha/2\varepsilon, v) \in H_s \\ \mathrm{I} = 5 & U = (u, \alpha, v) \in {}^{5}\mathrm{H} & \mapsto {}^{5}P_s U = (u, \alpha/2\varepsilon, v) \in H_s, \end{cases} \tag{4.14}$$

where $\alpha/2\varepsilon$ is also considered to be a function of $x \in B_\varepsilon : \alpha(x)/2\varepsilon = \alpha(\hat{x})/2\varepsilon$, has the fundamental properties :

Proposition 4.1.

i) *There exists a strictly positive constant C_I such that $|{}^{\mathrm{I}}P_s U|_s \le C_I \|U\|_I$, $\forall U \in {}^{\mathrm{I}}\mathrm{H}$,*
ii) *When s tends to $\bar{s}$, $|{}^{\mathrm{I}}P_s U|_s$ converges toward $\|U\|_I$ for all U in ${}^{\mathrm{I}}\mathrm{H}$.*

Next we state that:

$$ (U_s) \text{ in } H_s \textit{ converges in the sense of Trotter toward } U \textit{ in } {}^{\mathrm{I}}\mathrm{H} \textit{ if } \lim_{s\to\bar{s}} |{}^{\mathrm{I}}P_s U - U_s|_s = 0. \tag{4.15}$$

Even if this is the right notion from a mechanical point of view, it could be of interest to consider this convergence with respect to some classical conventional notions:

Proposition 4.2. *For all $U = (z, v)$ in ${}^{\mathrm{I}}\mathrm{H}$, if $U_s = (u_s, \alpha_s, v_s)$ in H_s converges in the sense of Trotter toward U, then :*

i) $1_{\Omega_\varepsilon} e(u_s)$ *converges strongly in $L^2(\Omega \setminus S\,; S^3)$ toward $e(z^d)$ and, for all positive η, the sequence (u_s) converges strongly in $H^1_{\Gamma_0}(\Omega_\eta;\mathbb{R}^3)$ toward $u := z^d$;*
ii) *the traces on $S^\pm_\varepsilon$ of u_s regarded as elements of $L^2(S\,;\mathbb{R}^3)$ converge strongly in $L^2(S\,;\mathbb{R}^3)$ toward the traces on S of $u^\pm$;*
iii) $\int_{-\varepsilon}^{\varepsilon} e(u_s)(\cdot, x_3)\,dx_3$ *converges strongly in $L^2(S\,;\mathbb{R}^3)$ toward $[u] \otimes_S e^3$ if $\overline{\mu} \in (0, +\infty]$;*
iv) (u_s) *is bounded in $L^2(\Omega;\mathbb{R}^3)$ and converges strongly in $L^2(\Omega;\mathbb{R}^3)$ toward u, when $\overline{\lim_{s\to\bar{s}}}\,\varepsilon^2/\mu = 0$, in $L^q(\Omega;\mathbb{R}^3)$, $\forall q < 2$, when $\overline{\lim_{s\to\bar{s}}}\,\varepsilon^2/\mu \in (0, +\infty)$;*
v) $\int_{-\varepsilon}^{\varepsilon} \alpha_s(\cdot, x_3)\,dx_3$ *converges strongly toward α in $L^2(S\,;\Xi)$ if $\overline{\mu} \in (0, +\infty]$;*
vi) $1_{\Omega_\varepsilon} v_s$ *converges strongly in $L^2(\Omega;\mathbb{R}^3)$ toward v and v_s converges strongly in $L^{2/(1+r)}(\Omega;\mathbb{R}^3)$ toward u.*

Of course, 1_{Ω_ε} denotes the characteristic function of Ω_ε, and the new points iii) and v) are a simple consequence of the Cauchy-Schwarz inequality.

Lastly, we conclude by using a suitable nonlinear version (see [6, 7]) of Trotter's theory of convergence of semigroups, where it suffices to make an additional assumption (H5) about the initial states and to establish the following "static" result :

Proposition 4.3. *We have*

i) $\forall\psi \in {}^{\mathrm{I}}\mathrm{H}$, $\quad \lim_{s\to\bar{s}} |{}^{\mathrm{I}}P_s(I + {}^{\mathrm{I}}A)^{-1}\psi - (I + A_s)^{-1}\, {}^{\mathrm{I}}P_s\psi|_s = 0,$

ii) $\lim\limits_{s\to\bar{s}} |{}^{\mathrm{I}}P_s U^e(t) - U_s^e(t)|_s = 0$ *uniformly on* $[0, T]$,

iii) $\lim\limits_{s\to\bar{s}} \displaystyle\int_0^T |{}^{\mathrm{I}}P_s F(t) - F_s(t)|_s \, dt = 0.$

As ${}^{\mathrm{I}}P_s F$ reads as $(\times, 0, \times)$, the proof of ii)-iii) given in [1] which necessitates the additional assumption

$$\begin{aligned}
&\text{i) } f \in BV\big(0, T; L^{2/(1-r)}(B_{\varepsilon_0}; \mathbb{R}^3)\big) \text{ where } r \text{ was defined in (H2)-vi).}\\
&\text{ii) } \operatorname{supp}(g) \cap \overline{B}_{\varepsilon_0} = \varnothing \quad \forall t \in [0, T] \text{ and}\\
&\quad \text{if } \operatorname{Min}\{\mathcal{H}^2(\Gamma_0^\pm)\} = 0, \text{ say } \mathcal{H}^2(\Gamma_0^-) = 0, \text{ then } \operatorname{supp} g \cap (\partial\Omega_{\varepsilon_0}^-) = \varnothing
\end{aligned} \tag{H4}$$

is still valuable. As regards point i), we use the same strategy as in [1] which, taking due account of (3.11) and (4.9), establishes the variational convergence toward ${}^{\mathrm{I}}J$ of $\widetilde{J}_s$ defined by

$$\widetilde{J}_s(z) = \frac{1}{2}K_s(v) - k_s(\psi^2, v) + \frac{1}{2}\Phi_s(z) + \varphi_s(({}^{\mathrm{I}}P^1\psi)^1, z) + b\int_{B_\varepsilon} \mathcal{D}\big(e(v), \alpha - \tilde{\psi}^1\big)\, dx$$

$\forall z = (v, \alpha) \in Z_s$, $\psi = (\psi^1, \psi^2)$, $\tilde{\psi}^1 = \psi^{1h}$ if $\mathrm{I} = 3, 4$, $\tilde{\psi}^1 = 0$ if $\mathrm{I} = 1, 2, 5$. Indeed, when $\overline{\mu} \in (0, +\infty]$ or $\overline{b} \in (0, +\infty]$, a simple use of Hölder and Jensen inequalities gives the following additional (with respect to Lemma 4.2 of [1]) compactness property for a sequence $(w_s) = ((v_s, \alpha_s))$ such that $\widetilde{J}_s(w_s)$ is uniformly bounded: $(\int_{-\varepsilon}^{\varepsilon} e(v_s)(\cdot, x_3)\, dx_3, \int_{-\varepsilon}^{\varepsilon} \alpha_s(\cdot, x_3)\, dx_3)$ *converges (up to a subsequence) weakly in* $L^q(S; \Theta)$ *toward* $([v] \otimes_S e^3, \alpha)$ *with* $q = 2$ if $\overline{\mu} \in (0, +\infty]$ or $q = p$ if $\overline{b} \in (0, +\infty]$. This dramatically simplifies the proof of the optimal lower bound for $\widetilde{J}_s(w_s)$ by a simple use of the Jensen inequality and a standard lower semi-continuity argument for convex integral functionals in $L^q(S; \Theta)$ and is the source of terms like $W_1^\perp$ and $(\mathcal{D}^{\infty,p})^\perp$ when $\overline{\mu} = 0!$ ... As $W_1(e, 0)$ or $\mathcal{D}^{\infty,p}(\dot{e}, 0)$ generally differs from $W_1^\perp(e)$ or $(\mathcal{D}^{\infty,p})^\perp(\dot{e})$, it is worthwhile to note that in the cases $\mathrm{I} = 1, 2, 5$, where the additional state variable disappears, it cannot be replaced by 0 in order to uniformize formulation of the spaces, functionals and equations! Thus, we deduce the convergence uniformly on $[0, T]$ in the sense of Trotter of the solution of (3.12) toward that of (4.10) with ${}^{\mathrm{I}}U^{ro} = {}^{\mathrm{I}}U^o - {}^{\mathrm{I}}U^e(0)$ and the additional conditions of convergence and compatibility between the initial state and loading:

$$\exists^{\mathrm{I}}U^o \in {}^{\mathrm{I}}U^e(0) + D({}^{\mathrm{I}}A); \quad U_s^o \in U_s^e(0) + D(A_s) \text{ and } \lim_{s\to\bar{s}} |{}^{\mathrm{I}}P_s{}^{\mathrm{I}}U^o - U_s^o|_s = 0. \tag{H5}$$

This can be rephrased in a more explicit way with respect to (P_s):

Theorem 4.1. *The solution to*

$$\frac{dU_s}{dt} + A_s(U_s - U_s^e) \ni (0, f/\gamma), \quad U_s(0) = U_s^o \tag{4.16}$$

converges toward the solution to

$$\frac{d^{\mathrm{I}}U}{dt} + {}^{\mathrm{I}}A({}^{\mathrm{I}}U - {}^{\mathrm{I}}U^e) \ni (0, f/\bar{\rho}), \quad {}^{\mathrm{I}}U(0) = {}^{\mathrm{I}}U^o \tag{4.17}$$

in the sense $\lim\limits_{s\to\bar{s}} |{}^{\mathrm{I}}P_s{}^{\mathrm{I}}U(t) - U_s(t)|_s = 0$, $\lim\limits_{s\to\bar{s}} |U_s(t)|_s = \|{}^{\mathrm{I}}U(t)\|_{\mathrm{I}}$ *uniformly on* $[0, T]$.

5. Concluding result

A more explicit way of writing (4.17) is:

- if $\overline{b} < +\infty$, $\exists \zeta \in \partial\overline{\mathcal{D}}\big(\frac{d^{\mathrm{I}}z}{dt}\big)$ such that

$$\int_\Omega \overline{\rho}\frac{d^2{}^{\mathrm{I}}u}{dt^2} z^d\,dx + \int_{\Omega\setminus S} ae({}^{\mathrm{I}}u)\cdot e(z^d)\,dx + \int_S \big(D\overline{W}_{\overline{\lambda}\overline{\mu}}(\Pi^{\mathrm{I}}z) + \zeta\big)\cdot \Pi z\,d\hat{x}$$
$$= \int_\Omega f\cdot z^d\,dx + \int_{\Gamma_1} g\cdot z^d\,d\mathcal{H}^2 \quad \forall z \in {}^{\mathrm{I}}\mathrm{Z}.$$

- if $\overline{b} = \infty$, $\big[\frac{d^{\mathrm{I}}u}{dt}\big] = 0$ and

$$\int_\Omega \overline{\rho}\frac{d^2{}^{\mathrm{I}}u}{dt^2} z^d\,dx + \int_{\Omega\setminus S} ae({}^{\mathrm{I}}u)\cdot e(z^d)\,dx + \int_S D\overline{W}_{\overline{\lambda}\overline{\mu}}\big([\Pi^{\mathrm{I}}z\big)\cdot \Pi z\,d\hat{x}$$
$$= \int_\Omega f\cdot z^d\,dx + \int_{\Gamma_1} g\cdot z^d\,d\mathcal{H}^2 \quad \forall z \in {}^{\mathrm{I}}\mathrm{Z}\cap\{\,[z^d]=0\,\}.$$

$\big({}^{\mathrm{I}}z(0), \frac{d^{\mathrm{I}}u}{dt}(0)\big) = {}^{\mathrm{I}}U^o$ with ${}^{\mathrm{I}}z = ({}^{\mathrm{I}}u, \alpha)$ if $\mathrm{I} = 3, 4$ ${}^{\mathrm{I}}z = {}^{\mathrm{I}}u$ if $\mathrm{I} = 1, 2, 5$.

Hence, the limit behavior concerns the dynamic response to the initial loads (f, g) of the assembly of two linearly elastic adherents occupying $\Omega^\pm$ as reference configurations and linked along S by a dissipative mechanical constraint, which can be written:

$$({}^{\mathrm{I}}\sigma e^3, 0) \in D\overline{W}_{\overline{\lambda}\overline{\mu}}\big([\Pi^{\mathrm{I}}z\big) + \partial\overline{\mathcal{D}}\Big(\frac{d^{\mathrm{I}}z}{dt}\Big)$$

where ${}^{\mathrm{I}}\sigma e^3$ is the stress vector along S. This constitutive equation is of the same algebraic form as that of the adhesive layer. It enters the formalism of Generalized Standard Materials [2, 3]. The contact state is described by the relative displacement $[{}^{\mathrm{I}}z^d]$ and possibly an additional state variable ${}^{\mathrm{I}}z^h$ when $\overline{\mu} \in (0, \infty)$ while the constitutive equation is derived from a free energy like $W_{\overline{\lambda}\overline{\mu}}(\Pi^{\mathrm{I}}z)$ and a potential of dissipation $\overline{\mathcal{D}}\big(\frac{d^{\mathrm{I}}z}{dt}\big)$. It may degenerate when the values of one of the coefficients $\overline{\lambda}, \overline{\mu}, \overline{b}$ are in $\{\,0, +\infty\,\}$.

- $\overline{b} = 0$

		$\overline{\mu}$: 0, $z = u$	$\overline{\mu}$: finite and positive, $z = (u, \alpha)$	$\overline{\mu}$: $+\infty$, $z = u$
$\overline{\lambda}$	0	$\sigma e^3 = 0$	$(\sigma e^3, 0) = \overline{\mu} DW_2(\Pi z)$	$[u] = 0$
$\overline{\lambda}$	finite and positive	$\sigma_T = 0$ $\sigma_N = (\overline{\lambda} DW_1^{\perp}([u]_N e^3 \otimes_S e^3)$	$(\sigma e^3, 0) = DW_{\overline{\lambda}\overline{\mu}}(\Pi^{\mathrm{I}}z)$	$[u] = 0$
$\overline{\lambda}$	$+\infty$	$\sigma_T = 0$ $[u]_N = 0$	$(\sigma e^3, 0) = \overline{\mu} DW_2(\Pi z)$ $[u]_N = 0, j\alpha = 0$	$[u] = 0$

with $[u]_N = [u]_3$, $\sigma_N = \sigma e^3 \cdot e^3$, $\sigma_T = \sigma e^3 - \sigma_N e^3$. They are elastic constraints.

- $\overline{b} \in (0, +\infty)$: we have to add some element ζ in $\partial\overline{\mathcal{D}}\big(\frac{d^{\mathrm{I}}z}{dt}\big)$ in the previous left upper 2×2 block. The other boxes on the right are not changed whereas we have to add ζ_T to the bottom left boxes. Thus, as seen in [6], the case $(\overline{\lambda}, \overline{\mu}) = (+\infty, 0)$ corresponds to generalized Norton Hoff ($1 < p \le 2$) or Tresca ($p = 1$) tangential friction with bilateral contact.
- $\overline{b} = +\infty$: the relative displacement along S is always equal to its initial value (which is zero if $\overline{\mu} = \infty$ and has a vanishing normal component if $\overline{\lambda} = \infty$), regardless of the values of $\overline{\lambda}, \overline{\mu}$, the relative motion along S is frozen!
  In practice, the geometrical and mechanical data obviously "do not tend to some limits", so our proposal of a simplified but accurate enough model for the behavior of the real structure is that obtained in the case $\mathrm{I} = 3$ by replacing $\overline{\lambda}, \overline{\mu}, \overline{b}$ by the real values $\lambda/2\varepsilon$, $\mu/2\varepsilon$ and $b/(2\varepsilon)^{p-1}$!

A major defect in this modeling is that it supplies a mechanical constraint which permits the unrealistic interpenetration of the two bodies. This is due to the framework of small deformations used, which is rather questionable because

λ, μ are assumed to take rather low values. That is why, in order to avoid the highly delicate framework of large deformations, especially in the dynamic case, we introduced not only one "stiffness parameter" μ but also a couple (λ, μ) of "stiffness coefficients" which, when $\overline{\lambda} = \infty$, supplies realistic bilateral contact conditions. When $\overline{\mu} = 0$, the limit constraint corresponds to a Norton-or Tresca-like friction and, when $\overline{\mu} \in (0, \infty)$, the limit constraint involves a new type (but entering the formalism of Generalized Standard Materials) of law of friction involving an additional state variable.

For the sake of simplification and to condense the presentation, we considered a special type $W_{\lambda\mu}$ of free energy for the adhesive involving *only one couple* of "stiffness coefficients" (like in a linearized Hooke law) with the defect that the additional state variable disappears when $\overline{\mu} = 0$. However, by systematically considering the fact that if $\int_{-\varepsilon}^{\varepsilon} q_s(\cdot, x_3)\, dx_3$ has a weak limit $\overline{q}$ in $L^p(S\,; X)$, $X \in \{ S^3, \Xi \}$, when (ε, b) tends to zero, then $\varliminf_{(\varepsilon,b)\to 0} b \int_{B_\varepsilon} \mathcal{W}(q_{\varepsilon b}(x))\, dx \geq \lim_{(\varepsilon,b)\to 0} \Big(\frac{b}{\varepsilon^{q-1}} \Big) \int_S \mathcal{W}(\overline{q}(\hat{x}))\, d\hat{x}$ for all q-positively homogeneous convex functions $\mathcal{W}$ and the trick of introducing $\mathcal{W}^\perp$, it is possible to treat many more generalized standard materials. Here are two important examples where α is the inelastic strain.

Example 5.1 (Poynting-Thomson like material)**.**
Free energy: $W(e, \alpha) = W_{\lambda_1,\mu_1}(e) + W_{\lambda_2\mu_2}(e - \alpha)$, $W_{\lambda\mu}(e) = \lambda |\operatorname{tr} e|^2 + \mu |e|^2$, *dissipation potential:* $\mathcal{D}(\dot{e}, \dot{\alpha}) = \mathcal{D}(\dot{\alpha})$. *If* $\overline{\mu}_i$, $\overline{\lambda}_i$, $\overline{b}_i$ *are the expected limits of* $\mu_i/2\varepsilon$, $\lambda_i/2\varepsilon$, $b_i/(2\varepsilon)^{p-1}$. *The state variables for the limit contact law are* $[u]$, $\overline{\alpha} \in S^3$ *(only* $[u]$ *if* $\overline{\mu}_1 = \overline{\mu}_2 = 0$*) while the free energy and dissipation potential are*

$$W_{\overline{\lambda}_1\overline{\mu}_1}([u] \otimes_S e^3) + W_{\overline{\lambda}_2\overline{\mu}_2}([u] \otimes_S e^3 - \overline{\alpha}), \quad \overline{b}\mathcal{D}^{\infty,p}(\dot{\alpha})$$

so that the case $\overline{\lambda}_1 = \overline{\lambda}_2 = \infty$, $\overline{\mu}_1 = 0$, $\overline{\mu}_2 \in (0, \infty)$ *supplies a kind of "Maxwell friction" with bilateral contact :*

$$[u_N] = 0, \quad \sigma_T \in \overline{b}\partial\mathcal{D}(\dot{\overline{\alpha}}) = 2\overline{\mu}_2([u]_T/2 - (\alpha e^3)_T), \alpha_{\text{sph}} = 0.$$

Example 5.2 (Elastoplasticity with strain hardening)**.**
Free energy: $W_{\lambda_1\mu_1}(e - \alpha) + W_{\lambda_2\mu_2}(\alpha)$, *dissipation potential:* $b|\dot{\alpha}|$. *The state variables of the limit constraint are* $[u]$, $\overline{\alpha} \in S^3$ *(only* $[u]$ *if* $\overline{\mu}_1 = 0$.*) If* $\overline{\mu}_1 = 0$ *the bodies are free to separate (*$\sigma e^3 = 0$*), in the other cases the surface energy and dissipation potential read as:*

$$W_{\overline{\lambda}_1\overline{\mu}_1}([u] \otimes_S e^3 - \overline{\alpha}) + W_{\overline{\lambda}_2\overline{\mu}_2}(\overline{\alpha}), \quad \overline{b}|\dot{\alpha}|$$

and the case $\overline{\lambda}_1 = \overline{\lambda}_2 = \infty$, $\overline{\mu}_1 \in (0, \infty)$, $\overline{\mu}_2 = 0$ *corresponds to a kind of "elasto-perfecto plastic friction" with bilateral contact:*

$$[u]_N = 0 \quad \sigma_T \in b\partial|\,|(\dot{\overline{\alpha}}) = 2\overline{\mu}_1([u]_T/2 - (\alpha e^3)_T), \alpha_{\text{sph}} = 0$$

Eventually, we obtained a result of existence and unicity for the dynamic response of two elastic bodies in bilateral contact with a tangential friction law given through a very general dissipation potential and quadratic convex energy because this problem can be formulated as (4.17) . . .

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