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► **To cite this version:**

Françoise Pene. An asymptotic estimate of the variance of the self-intersections of a planar periodic Lorentz process. 2013. hal-00799504

HAL Id: hal-00799504

<https://hal.science/hal-00799504>

Preprint submitted on 12 Mar 2013

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AN ASYMPTOTIC ESTIMATE OF THE VARIANCE OF THE SELF-INTERSECTIONS OF A PLANAR PERIODIC LORENTZ PROCESS

FRANÇOISE PÈNE

ABSTRACT. We consider a \mathbb{Z}^2 -periodic planar Lorentz process with strictly convex obstacles and finite horizon. This process describes the displacement of a particle moving in the plane with unit speed and with elastic reflection on the obstacles. We call number of self-intersections of this Lorentz process the number V_n of couples of integers (k, ℓ) smaller than n such that the particle hits a same obstacle both at the k th and at the ℓ th collision times. The aim of this article is to prove that the variance of V_n is equivalent to n^2 (such a result has recently been proved for simple planar random walks in [6]).

1. INTRODUCTION

We consider a finite number of convex open sets $O_1, \dots, O_I \subset \mathbb{R}^2$ with boundary C^3 -smooth and with non null curvature. We repeat these sets \mathbb{Z}^2 -periodically by defining $U_{i,\ell} = O_i + \ell$ for every $(i, \ell) \in \{1, \dots, I\} \times \mathbb{Z}^2$. We suppose that the closures of the $U_{i,\ell}$ are pairwise disjoint. We assume that **the horizon is finite**, which means that every line meets the boundary of at least one obstacle (i.e. there is no infinite free flight). We consider a particle moving in the domain $Q := \mathbb{R}^2 \setminus \bigcup_{i=1}^I \bigcup_{\ell \in \mathbb{Z}^2} U_{i,\ell}$ with unit speed and with respect to the Descartes reflection law at its reflection times (reflected angle=incident angle). We assume that the particle starts from $[0, 1]^2 \cap Q$ with uniform distribution in position and in speed. The **Lorentz process** describes the evolution of the particle in Q . Because of the \mathbb{Z}^2 -periodicity, it is strongly related to the Sinai billiard, the ergodic properties of which have been studied namely by Sinai in [9] (for its ergodicity), Bunimovich and Sinai [2, 3], Bunimovich, Chernov and Sinai [4, 5] (for central limit theorems), Young [11] (for exponential rate of decorrelation). The similarity of behaviour of the Lorentz process with a simple planar random walk has been investigated by many authors ([10, 7],...). The number of auto-intersections up to time n of a random walk $(\tilde{S}_n)_n$ is $\tilde{V}_n := \sum_{k,\ell=1}^n \mathbf{1}_{\tilde{S}_k = \tilde{S}_\ell}$. This quantity is linked with random walks in random sceneries [1, 6]. Recently, in [6], Deligiannidis and Utev proved that $Var(\tilde{V}_n) \sim \tilde{c}n^2$ with an explicit \tilde{c} . This improved the estimation in $O(n^2 \log n)$ by Bolthausen [1]. For the Lorentz process, we define (\mathcal{I}_k, S_k) in $\{1, \dots, I\} \times \mathbb{Z}^2$ for the index of the obstacle hit at the k -th reflection time ((I_0, S_0) being the index of the obstacle at the reflection time just before time 0). Recall that $(k^{-1/2}S_k)_{k \geq 1}$ admits an asymptotic positive variance matrix Σ^2 . We call **number of self-intersections** of the Lorentz process up to the n -th reflection time the quantity $V_n := \sum_{k,\ell=1}^n \mathbf{1}_{S_k = S_\ell, \mathcal{I}_k = \mathcal{I}_\ell}$. In [8], we proved that $\mathbb{E}[V_n] \sim c_0 n \log n$ with $c_0 := \frac{\sum_{i=1}^I (|\partial O_i|^2)}{(\sum_{i=1}^I |\partial O_i|)^2 \pi \sqrt{\det \Sigma^2}}$, where $|\partial O_i|$ stands for the length of ∂O_i . In [8], $Var(V_n) = O(n^2 \log n)$ was enough for our study of the planar Lorentz

Date: March 12, 2013.

2010 Mathematics Subject Classification. 60F99, 37D50.

Key words and phrases. Sinai billiard, Lorentz process, self intersection

Françoise Pène is supported by the french ANR projects GEODE (ANR-10-JCJC-0108) and PERTURBATIONS (ANR-10-BLAN-0106).

process in random scenery. Our proof of the following result uses decorrelation and precised local limit theorems established in [8]. It provides an alternative strategy to the one of [6].

Theorem 1. $\text{Var}(V_n) \sim cn^2$ with

$$c := c_0^2 \left(1 + 2J - \frac{\pi^2}{6} \right) \text{ and } J := \int_{[0,1]^3} \frac{(1 - (u + v + w)) \mathbf{1}_{\{u+v+w \leq 1\}} du dv dw}{uv + uw + vw}.$$

2. PROOF OF THEOREM 1

Observe that the distribution of $(S_k - S_0, \mathcal{I}_k)_k$ under \mathbb{P} and under $\bar{\nu}$ considered in [8] are the same (by \mathbb{Z}^2 -periodicity and by construction of $\bar{\nu}$). We write $E_{k,\ell} := \{S_k = S_\ell, \mathcal{I}_k = \mathcal{I}_\ell\}$. According to [8], we have

$$\mathbb{P}(E_{k,\ell}) = \mathbb{P}(E_{0,|\ell-k|}) = c_1 |\ell - k|^{-1} + O(|\ell - k|^{-2}), \text{ with } c_1 := \frac{\sum_i^I (\mathbb{P}(\mathcal{I}_0 = i))^2}{2\pi \sqrt{\det \Sigma^2}} = \frac{c_0}{2}. \quad (1)$$

Observe that we have $V_n = n + 2 \sum_{1 \leq k < \ell \leq n} \mathbf{1}_{S_k = S_\ell, \mathcal{I}_k = \mathcal{I}_\ell}$ and so

$$\text{Var}(V_n) = 4 \sum_{1 \leq k_1 < \ell_1 \leq n} \sum_{1 \leq k_2 < \ell_2 \leq n} D_{k_1, \ell_1, k_2, \ell_2} = 8A_1 + 8A_2 + 8A_3 + 4A_4,$$

with $D_{k_1, \ell_1, k_2, \ell_2} := \mathbb{P}(E_{k_1, \ell_1} \cap E_{k_2, \ell_2}) - \mathbb{P}(E_{k_1, \ell_1})\mathbb{P}(E_{k_2, \ell_2})$ and

$$\begin{aligned} A_1 &:= \sum_{1 \leq k_1 < \ell_1 \leq k_2 < \ell_2 \leq n} D_{k_1, \ell_1, k_2, \ell_2}, \quad A_2 := \sum_{1 \leq k_1 \leq k_2 < \ell_1 \leq \ell_2 \leq n} D_{k_1, \ell_1, k_2, \ell_2}, \\ A_3 &:= \sum_{1 \leq k_1 < k_2 < \ell_2 < \ell_1 \leq n} D_{k_1, \ell_1, k_2, \ell_2}, \quad A_4 := \sum_{1 \leq k < \ell \leq n} [\mathbb{P}(E_{k,\ell}) - (\mathbb{P}(E_{k,\ell}))^2]. \end{aligned}$$

- Control of A_1 .

Due to [8], if $k_1 < \ell_1 \leq k_2 < \ell_2$, then $|D_{k_1, \ell_1, k_2, \ell_2}| \leq C_1 \tau_1^{k_2 - \ell_1} / ((\ell_1 - k_1)(\ell_2 - k_2))$ for some $C_1 > 0$ and some $\tau_1 \in (0, 1)$. Hence $A_1 = O(n \log^2 n) = o(n^2)$.

- Control of A_4 .

Due to (1) or [10], $A_4 \leq C_2 \sum_{1 \leq k < \ell \leq n} (\ell - k)^{-1} = O(n \log n) = o(n^2)$.

- Control of A_2 .

According to [8], we have

$$A_2 = \sum_{1 \leq k_1 < k_2 < \ell_1 < \ell_2 \leq n} c_1^2 \left[\left(\sum_x \frac{e^{-\langle (\Sigma^2)^{-1} x, x \rangle \left(\frac{1}{k_2 - k_1} + \frac{1}{\ell_1 - k_2} + \frac{1}{\ell_2 - \ell_1} \right)}}{2\pi \sqrt{\det \Sigma^2} (k_2 - k_1)(\ell_1 - k_2)(\ell_2 - \ell_1)} \right) - \frac{1}{(\ell_1 - k_1)(\ell_2 - k_2)} \right] + o(n^2), \quad (2)$$

where $\sum_x = \sum_{x \in \mathbb{Z}^2 : |x| \leq \|S_1\|_\infty \min(k_2 - k_1, \ell_1 - k_2, \ell_2 - \ell_1)}$ and $\langle \cdot, \cdot \rangle$ is the usual scalar product in \mathbb{R}^2 .

$$\begin{aligned} - \text{First } A_{2,0} &:= \sum_{1 \leq k_1 < k_2 < \ell_1 < \ell_2 \leq n} \sum_x \frac{e^{-\langle (\Sigma^2)^{-1} x, x \rangle \left(\frac{1}{k_2 - k_1} + \frac{1}{\ell_1 - k_2} + \frac{1}{\ell_2 - \ell_1} \right)}}{2\pi \sqrt{\det \Sigma^2} (k_2 - k_1)(\ell_1 - k_2)(\ell_2 - \ell_1)} \\ &= \sum_{(k_1, m_0, m_1, m_2) \in E_n} \sum_{|x| \leq \|S_1\|_\infty \min(m_0, m_1, m_2)} \frac{e^{-\langle (\Sigma^2)^{-1} x, x \rangle \left(\frac{1}{m_0} + \frac{1}{m_1} + \frac{1}{m_2} \right)}}{2\pi \sqrt{\det \Sigma^2} m_0 m_1 m_2}, \end{aligned}$$

with $E_n := \{(k_1, m_0, m_1, m_2) \in \mathbb{Z}_+ : k_1 + m_0 + m_1 + m_2 \leq n\}$. Observe that, using a comparison series-integral, we obtain

$$\sup_{\|S_1\|_\infty \leq a \leq 3\|S_1\|_\infty} \left| \sum_{x \in \mathbb{Z}^2 : |x| \leq am} e^{-\frac{\langle (\Sigma^2)^{-1}x, x \rangle}{2m}} - 2\pi m \sqrt{\det \Sigma^2} \right| = O(\sqrt{m}). \quad (3)$$

$$\text{So } A_{2,0} = \sum_{(k_1, m_0, m_1, m_2) \in E_n} \frac{1 + O(\min(m_0, m_1, m_2)^{-1/2})}{m_0 m_1 + m_0 m_2 + m_1 m_2} \sim n^2 J.$$

$$- \text{Second } \sum_{1 \leq k_1 < k_2 < \ell_1 < \ell_2 \leq n} \frac{1}{(\ell_1 - k_1)(\ell_2 - k_2)} = A_{2,1} + 2A_{2,2},$$

$$\text{with } A_{2,1} := \sum_{k=1}^n \sum_{\max(1, 2k-n) \leq m \leq k} \frac{n - (2k - m) + 1}{k^2} \leq \sum_{k=1}^n \sum_{m=0}^k \frac{n}{k^2} = O(n \log n) = o(n^2),$$

$$\text{and } A_{2,2} := \sum_{1 \leq k < \ell \leq n} \sum_{\max(0, k+\ell-n) \leq m \leq k} \frac{n - (k + \ell - m) + 1}{k\ell}$$

$$\begin{aligned} &= \sum_{\ell=1}^{\lfloor n/2 \rfloor} \sum_{k=1}^{\ell-1} \sum_{m=0}^k \cdots + \sum_{\ell=\lfloor n/2 \rfloor + 1}^n \sum_{k=1}^{n-\ell} \sum_{m=0}^k \cdots + \sum_{\ell=\lfloor n/2 \rfloor + 1}^n \sum_{k=n-\ell+1}^{\ell-1} \sum_{m=k+\ell-n}^k \cdots \\ &= o(n^2) + \sum_{\ell=1}^{\lfloor n/2 \rfloor} \sum_{k=1}^{\ell-1} \frac{2(n-\ell) - k}{2\ell} + \sum_{\ell=\lfloor n/2 \rfloor + 1}^n \sum_{k=1}^{n-\ell} \frac{2(n-\ell) - k}{2\ell} + \sum_{\ell=\lfloor n/2 \rfloor + 1}^n \sum_{k=n-\ell+1}^{\ell-1} \frac{(n-\ell)^2}{2k\ell} \\ &= o(n^2) + \sum_{\ell=1}^{\lfloor n/2 \rfloor} \frac{4n - 5\ell}{4} + \sum_{\ell=\lfloor n/2 \rfloor + 1}^n \frac{3(n-\ell)^2}{4\ell} + \sum_{\ell=\lfloor n/2 \rfloor + 1}^n \frac{(n-\ell)^2}{2\ell} \log \left(\frac{\ell}{n-\ell} \right) \\ &\sim n^2 \left(-\frac{1}{8} + \frac{3}{4} \log 2 + \frac{I}{2} \right), \end{aligned}$$

with

$$\begin{aligned} I &:= \int_{1/2}^1 \frac{(1-u)^2}{u} \log \left(\frac{u}{1-u} \right) du \\ &= \left[Li_2(u) + \frac{1}{2} (u + \log u (u^2 + \log u - 4u) + \log(1-u)(-u^2 + 4u - 3)) \right]_{1/2}^1, \end{aligned}$$

with $Li_2(z) := \sum_{k \geq 1} \frac{z^k}{k^2}$. So $I = Li_2(1) - Li_2(1/2) + \frac{1}{4} - \frac{\log^2 2}{2} - \frac{3}{2} \log 2 = \frac{\pi^2}{6} - (\frac{\pi^2}{12} - \frac{\log^2 2}{2}) + \frac{1}{4} - \frac{\log^2 2}{2} - \frac{3}{2} \log 2 = \frac{\pi^2}{12} + \frac{1}{4} - \frac{3}{2} \log 2$. Hence we have $A_{2,1} + 2A_{2,2} \sim \frac{\pi^2}{12} n^2$.

• **Control of A_3 .**

Notice that $\sum_{1 \leq k_1 < k_2 < \ell_2 < \ell_1 \leq n} \mathbb{P}(E_{k_1, \ell_1} \cap E_{k_2, \ell_2})$ and $\sum_{1 \leq k_1 < k_2 < \ell_2 < \ell_1 \leq n} \mathbb{P}(E_{k_1, \ell_1}) \mathbb{P}(E_{k_2, \ell_2})$ are in $n^2 \log n$. But we will see that their difference is in n^2 . According to [8] and to (1), we have:

$$A_3 = c_1 \sum_{1 \leq k_1 \leq k_2 < \ell_2 < \ell_1 \leq n} \left[\left(\sum_x \frac{e^{-\frac{\langle (\Sigma^2)^{-1}x, x \rangle}{2} \left(\frac{1}{k_2 - k_1} + \frac{1}{\ell_1 - \ell_2} \right)}}{2\pi \sqrt{\det \Sigma^2} (k_2 - k_1)(\ell_1 - \ell_2)} \right) - \frac{1}{(\ell_1 - k_1)} \right] \mathbb{P}(E_{k_2, \ell_2}) + o(n^2) \quad (4)$$

with the same notations as for (2). Using again (3) and (1), we obtain

$$\begin{aligned}
A_3 &= o(n^2) + c_1^2 \sum_{1 \leq k_1 < k_2 < \ell_2 \leq \ell_1 \leq n} \frac{1}{\ell_2 - k_2} \left[\frac{1}{(\ell_1 - k_1) - (\ell_2 - k_2)} - \frac{1}{(\ell_1 - k_1)} \right] \\
&= o(n^2) + c_1^2 \sum_{1 \leq k_1 < k_2 < \ell_2 \leq \ell_1 \leq n} \frac{1}{(\ell_1 - k_1)[(\ell_1 - k_1) - (\ell_2 - k_2)]} \\
&\sim c_1^2 n^2 \int_{[0,1]^4} \frac{\mathbf{1}_{\{t+u+v+w < 1\}} dt du dv dw}{(u+w)(u+v+w)} = c_1^2 n^2 \int_{0 \leq u \leq r \leq s \leq 1} \frac{(1-s) du dr ds}{rs} = \frac{c_1^2}{2} n^2.
\end{aligned}$$

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