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# ON NONLINEAR SCHRÖDINGER TYPE EQUATIONS WITH NONLINEAR DAMPING

PAOLO ANTONELLI, RÉMI CARLES, AND CHRISTOF SPARBER

ABSTRACT. We consider equations of nonlinear Schrödinger type augmented by nonlinear damping terms. We show that nonlinear damping prevents finite time blow-up in several situations, which we describe. We also prove that the presence of a quadratic confinement in at least one spatial direction is sufficient to ensure that the solution of our model decays to zero for large time. In the case without external potential we prove that the solution may not go to zero for large time due to (non-trivial) scattering.

## 1. INTRODUCTION

We consider, for  $a > 0$  and  $\lambda \in \mathbf{R}$ , the following class of damped nonlinear Schrödinger type equations (NLS):

$$(1.1) \quad \begin{cases} i\partial_t u + \frac{1}{2}\Delta u = V(x)u + \lambda|u|^{2\sigma_1}u - ia|u|^{2\sigma_2}u, & t \geq 0, x \in \mathbf{R}^d, \\ u|_{t=0} = u_0 \in \Sigma, \end{cases}$$

where  $\Sigma$  denotes the energy space associated to the harmonic oscillator, i.e.

$$\Sigma = \{f \in H^1(\mathbf{R}^d), x \mapsto |x|f(x) \in L^2(\mathbf{R}^d)\},$$

equipped with the following norm:

$$\|u\|_{\Sigma} := \|u\|_{L^2} + \|\nabla u\|_{L^2} + \|xu\|_{L^2}.$$

In the following, we allow  $u_0$  to be arbitrarily large within  $\Sigma$ , i.e. we shall not be concerned with solutions corresponding to small initial data. We make the following standard assumption on the nonlinearities:

$$0 < \sigma_1, \sigma_2 < \frac{2}{(d-2)_+},$$

where  $(d-2)_+$  denotes the positive part. Thus, if  $d \leq 2$ , we impose no size restriction on  $\sigma_1, \sigma_2 > 0$ . For  $d \geq 3$  the above assumption ensures that both nonlinearities are  $H^1$ -subcritical. The external potential  $V$  is supposed to be harmonic (or zero),

$$V(x) = \frac{1}{2} \sum_{j=1}^d \omega_j^2 x_j^2, \quad \omega_j \geq 0.$$

As we shall indicate below, all of our results can be generalized to the case of potentials  $V(x) \geq 0$ , growing at most quadratically at infinity. In the case  $a = 0$ , it is well known that (1.1) is a Hamiltonian equation (its mass and energy are

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conserved); see e.g. [10, 19]. In fact, the Hamiltonian counterpart of our model, i.e. (1.1) with  $a \in i\mathbf{R}$  (yielding a nonlinear Schrödinger equation with combined power-law nonlinearities) has been studied in [20]. In the present case  $a > 0$ , the last term in (1.1) is dissipative, which is the reason why we consider non-negative times only. Indeed, the dissipative nature of (1.1) can easily be seen from the fact that the local conservation law for the particle density  $\rho = |u|^2$  is augmented as follows:

$$(1.2) \quad \partial_t \rho + \operatorname{div} J = -2a\rho^{\sigma_2+1},$$

where, as usual  $J = \operatorname{Im}(\bar{u}\nabla u)$  denotes the current density. For  $a > 0$  the right hand side describes a nonlinear damping mechanism for the density.

Equations of the form (1.1) arise as phenomenological models in different areas of Physics. For example, in nonlinear optics, equation (1.1) with  $V = 0$  models the propagation of a laser pulse within an optical fiber ( $d = 1$ ) under the influence of additional multi-photon absorption processes, see, e.g., [5, 12]. Another application arises from quantum mechanics, where NLS type models arise in the description of Bose-Einstein condensates in harmonic traps (which are experimentally required to produce these condensates). In this context the nonlinear damping is a model for the reduction of the condensate wave function through higher order particle-interactions, cf. [1, 4].

From a mathematical point of view, NLS type equation with nonlinear damping terms have been studied in [13] as a possibility to continue the solution of NLS beyond the point of finite-time blow-up (see also [18] for an earlier study in this direction). In [3], global existence for the particular case  $\sigma_1 = 1$  and  $1 < \sigma_2 \leq 2$  in dimensions  $d \leq 3$  has been studied. Notice that in  $d = 3$ , this allows to take into account an  $H^1$ -critical damping term (i.e. a quintic nonlinearity with  $\sigma_2 = 2$ ). In [11] the particular case of a mass critical nonlinearity  $\sigma_1 = 2/d$  and  $V = 0$  has been studied. In there, global in-time existence of solutions is established if  $\sigma_2 \geq 2/d$  and it is claimed that finite time blow-up in the log-log regime occurs if  $\sigma_2 < 2/d$ . A more complete understanding of the possibility of finite time blow-up remains an open problem, however (numerical simulations can be found in [18, 13]).

In the following, we shall develop a more systematic study of NLS type equations with nonlinear damping, generalizing the results mentioned above in several aspects:

- We extend the results of global well-posedness to the case of general (energy-subcritical) nonlinearities.
- We prove that in the case without external potential, the solution is asymptotically close to the solution of the free equation for  $t \rightarrow +\infty$ , i.e. we establish scattering for positive times.
- We show that in the case where a quadratic external confinement is present in at least one spatial dimension, the  $L^2$  norm of solution vanishes asymptotically with a certain (not necessarily sharp) rate. (This result corrects a mistake in the proof of Corollary 2.5 in [3]).
- We compare the results above to the ones which can be obtained for  $x \in M$ , a compact manifold without boundaries.

**Theorem 1.1.** *Let  $d \geq 1$ ,  $a > 0$ ,  $\omega_1, \dots, \omega_d \geq 0$ , and  $u_0 \in \Sigma$ . Then, the Cauchy problem (1.1) has a unique solution  $u \in C(\mathbf{R}_+; \Sigma)$  in either one of the following cases:*

- (1)  $\lambda \geq 0$  (defocusing nonlinearity) and  $0 < \sigma_1, \sigma_2 < 2/(d-2)_+$ ;
- (2)  $\lambda < 0$  (focusing nonlinearity) with, either
  - (a)  $0 < \sigma_1 < 2/d$  and  $0 < \sigma_2 < 2/(d-2)_+$ , or
  - (b)  $2/d \leq \sigma_1 < \sigma_2 < 2/(d-2)_+$ , or
  - (c)  $\sigma_1 = \sigma_2 = 2/d$ , or
  - (d)  $2/d < \sigma_1 = \sigma_2 =: \sigma < 2/(d-2)_+$  and  $a \geq \min(\sigma, \sqrt{\sigma}) |\lambda|$ .

If in addition there exists at least one  $j$  such that  $\omega_j > 0$ , then  $u$  vanishes asymptotically as  $t \rightarrow +\infty$ :

$$\lim_{t \rightarrow +\infty} \|u(t)\|_{L^2(\mathbf{R}^d)} = 0.$$

If, however,  $\omega_j = 0$  for all  $j$ ,  $u$  may behave like the free evolution of a non-trivial asymptotic state  $u_+ \in L^2(\mathbf{R}^d)$ :  $\|u(t) - e^{i\frac{t}{2}\Delta} u_+\|_{H^1(\mathbf{R}^d)} \rightarrow 0$  as  $t \rightarrow +\infty$ .

The paper is organized as follows: In the next section we establish the local (in-time) well-posedness of our model in the energy space. In Section 3, we extend this to global in-time well-posedness using a modified energy functional. The long time behavior and the possible extinction of solutions is studied in Section 4. Finally, we briefly discuss the case of compact manifolds in the appendix.

## 2. BASIC PROPERTIES OF THE CAUCHY PROBLEM

In this section we shall show that (1.1) is locally well-posed for any  $u_0 \in \Sigma$  and we also establish a blow-up alternative.

**2.1. Local well-posedness.** We denote by  $U(t) = e^{-itH}$ , the Schrödinger group generated by  $H = -\frac{1}{2}\Delta + V$ . We first recall the standard Strichartz estimates (see e.g. [10]).

**Definition 2.1.** A pair  $(q, r)$  is admissible if  $2 \leq r < \frac{2d}{d-2}$  ( $2 \leq r \leq \infty$  if  $d = 1$ ,  $2 \leq r < \infty$  if  $d = 2$ ) and

$$\frac{2}{q} = \delta(q) := d \left( \frac{1}{2} - \frac{1}{r} \right).$$

**Proposition 2.2** (Strichartz estimates). *Let  $T > 0$ . There exists  $\eta > 0$  such that the following holds:*

- (1) For any admissible pair  $(q, r)$ , there exists  $C_q$  such that

$$\|U(\cdot)\varphi\|_{L^q([0, \eta]; L^r)} \leq C_q \|\varphi\|_{L^2}, \quad \forall \varphi \in L^2(\mathbf{R}^d).$$

- (2) For  $s \in \mathbf{R}$ , denote

$$D_s(F)(t, x) = \int_s^t U(t - \tau) F(\tau, x) d\tau.$$

For all admissible pairs  $(q_1, r_1)$  and  $(q_2, r_2)$ , there exists  $C = C_{q_1, q_2}$  independent of  $s \in \mathbf{R}$  such that

$$(2.1) \quad \|D_s(F)\|_{L^{q_1}([s, s+\delta]; L^{r_1})} \leq C \|F\|_{L^{q_2}'([s, s+\delta]; L^{r_2}')},$$

for all  $F \in L^{p_2}'(I; L^{q_2}')$  and  $0 \leq \delta \leq \eta$ .

- (3) In the case without potential,  $V = 0$ , the above results remain true with  $\eta = \infty$ .

**Proposition 2.3** (Local existence). *Let  $\lambda, a \in \mathbf{R}$ ,  $\omega_1, \dots, \omega_d \geq 0$ , and  $\sigma_j > 0$  with  $\sigma_j < 2/(d-2)$  if  $d \geq 3$ . For all  $u_0 \in \Sigma$ , there exists  $T$  and a unique solution  $u$  of (1.1), such that*

$$u, \nabla u, xu \in C([0, T]; L^2(\mathbf{R}^d)) \cap \bigcap_{j=1,2} L^{\frac{4\sigma_j+4}{d\sigma_j}}([0, T]; L^{2\sigma_j+2}(\mathbf{R}^d)).$$

*Proof.* We present the main steps of the classical argument, which can be found for instance in [10, 14] in the case  $V = 0$  (see also [8] in the presence of a potential). Duhamel's formulation for (1.1) reads

$$(2.2) \quad u(t) = U(t)u_0 - i\lambda \int_0^t U(t-\tau) (|u|^{2\sigma_1} u)(\tau) d\tau - a \int_0^t U(t-\tau) (|u|^{2\sigma_2} u)(\tau) d\tau.$$

Denote the right hand side by  $\Phi(u)(t)$ . Proposition 2.3 follows from a fixed point argument in a ball of the space

$$X_T = \left\{ u \in C([0, T]; \Sigma) ; u, xu, \nabla u \in \bigcap_{j=1,2} L_T^{\frac{4\sigma_j+4}{d\sigma_j}} L^{2\sigma_j+2} \right\},$$

where  $L_T^q L^r$  stands for  $L^q([0, T]; L^r(\mathbf{R}^d))$ . For  $j = 1, 2$ , introduce the Lebesgue exponents

$$(2.3) \quad r_j = 2\sigma_j + 2 \quad ; \quad q_j = \frac{4\sigma_j + 4}{d\sigma_j} \quad ; \quad \theta_j = \frac{2\sigma_j(2\sigma_j + 2)}{2 - (d-2)\sigma_j}.$$

Then  $(q_j, r_j)$  is admissible, and

$$\frac{1}{r'_j} = \frac{2\sigma_j}{r_j} + \frac{1}{r_j} \quad ; \quad \frac{1}{q'_j} = \frac{2\sigma_j}{\theta_j} + \frac{1}{q_j}.$$

Proposition 2.2 and Hölder inequality yield, for  $j = 1, 2$

$$\begin{aligned} \|\Phi(u)\|_{L_T^{q_j} L^{r_j} \cap L_T^\infty L^2} &\leq C \|u_0\|_{L^2} + C \sum_{\ell=1,2} \| |u|^{2\sigma_\ell} u \|_{L_T^{q'_\ell} L^{r'_\ell}} \\ &\leq C \|u_0\|_{L^2} + C \sum_{\ell=1,2} \|u\|_{L_T^{\theta_\ell} L^{r_\ell}}^{2\sigma_\ell} \|u\|_{L_T^{q_\ell} L^{r_\ell}}, \end{aligned}$$

where  $C$  is independent of  $T \leq \eta$ . We note that if  $\sigma_\ell \leq 2/d$  for  $\ell = 1, 2$ , then  $\theta_\ell \leq q_\ell$ , and

$$(2.4) \quad \|\Phi(u)\|_{L_T^{q_j} L^{r_j} \cap L_T^\infty L^2} \leq C \|u_0\|_{L^2} + C \sum_{\ell=1,2} T^{2\sigma_\ell(1/\theta_\ell - 1/q_\ell)} \|u\|_{L_T^{q_\ell} L^{r_\ell}}^{2\sigma_\ell+1}.$$

If  $\sigma_\ell > 2/d$  for  $\ell = 1$  or  $2$ , using Sobolev embedding,

$$\|\Phi(u)\|_{L_T^{q_j} L^{r_j} \cap L_T^\infty L^2} \leq C \|u_0\|_{L^2} + C \sum_{\ell=1,2} T^{2\sigma_\ell/\theta_\ell} \|u\|_{L_T^\infty H^1}^{2\sigma_\ell} \|u\|_{L_T^{q_\ell} L^{r_\ell}}.$$

We have

$$\begin{aligned} \nabla \Phi(u)(t) &= U(t) \nabla u_0 - i\lambda \int_0^t U(t-\tau) \nabla (|u|^{2\sigma_1} u)(\tau) d\tau \\ &\quad - a \int_0^t U(t-\tau) \nabla (|u|^{2\sigma_2} u)(\tau) d\tau - i \int_0^t U(t-\tau) (\Phi(u)(\tau) \nabla V(\tau)) d\tau. \end{aligned}$$

We estimate the second term of the right hand side as above, and, for all admissible pairs  $(q, r)$ , the new term is estimated by

$$\begin{aligned} \left\| \int_0^t U(t-\tau) (\Phi(u)(\tau) \nabla V(\tau)) d\tau \right\|_{L_T^q L^r} &\leq C \|\Phi(u) \nabla V\|_{L_T^1 L^2} \\ &\leq CT (\|\Phi(u)\|_{L_T^\infty L^2} + \|x\Phi(u)\|_{L_T^\infty L^2}), \end{aligned}$$

where we have written an estimate which is valid in the more general case where  $V$  is at most quadratic ( $\partial^\alpha V \in L^\infty(\mathbf{R}^d)$  for  $|\alpha| \geq 2$ ). Similarly, to estimate  $x\Phi(u)$ , a new term appears, which is controlled by

$$CT \|\nabla \Phi(u)\|_{L_T^\infty L^2}.$$

Choosing  $T$  sufficiently small, one can then prove that  $\Phi$  maps a suitable ball in  $X_T$  into itself. Contraction for the norm  $\|\cdot\|_{L^{q_1}(I_T; L^{r_1})}$  is proved similarly, and one concludes by remarking that  $X_T$  equipped with this norm is complete.  $\square$

*Remark 2.4.* The above result can be extended to the case where the first assumption is replaced with  $u \in L^\infty(\mathbf{R}_+; \mathcal{F}(H^s))$  for some  $s > 0$ , up to changing the application of Hölder inequality in the proof.

*Remark 2.5* (Energy-critical damping). When  $d \geq 3$ , the case  $\sigma_2 = 2/(d-2)$  could be considered, like in [3] for the case  $d = 3$ . This requires a different presentation in the proofs, which is the reason why this case is not studied here.

*Remark 2.6* (More general potentials). As suggested in the course of the proof, Proposition 2.3 remains valid if we assume more generally that  $V(x)$  is smooth, and at most quadratic, i.e.  $\partial^\alpha V \in L^\infty(\mathbf{R}^d)$  for all  $|\alpha| \geq 2$ .

**2.2. Basic *a priori* estimates and blow-up alternative.** In order to extend the obtained local in-time solution to arbitrary time intervals, we shall derive several *a priori* estimates. In a first step, we recall (1.2) to infer the following.

**Lemma 2.7** (Mass dissipation). *The local in time solution  $u(t) \in \Sigma$  satisfies*

$$(2.5) \quad \frac{d}{dt} \|u(t)\|_{L^2}^2 + 2a \|u(t)\|_{L^{2\sigma_2+2}}^{2\sigma_2+2} = 0, \quad \forall t \in [0, T].$$

*As a consequence, we have that  $u \in L^\infty([0, T] \times L^2(\mathbf{R}^d)) \cap L^{2\sigma_2+2}([0, T] \times \mathbf{R}^d)$ .*

*Proof.* We multiply (1.1) by  $\bar{u}$  and integrate with respect to  $x \in \mathbf{R}^d$ . Taking the real part, yields (2.5), which consequently implies

$$\frac{d}{dt} \|u(t)\|_{L^2}^2 \leq -2a \|u(t)\|_{L^{2\sigma_2+2}}^{2\sigma_2+2} \leq 0,$$

and thus  $\|u(t)\|_{L^2} \leq \|u_0\|_{L^2}$ ,  $\forall t \in [0, T]$ . In addition, we can integrate (2.5) with respect to  $t$  to infer

$$2a \int_0^T \|u(t)\|_{L^{2\sigma_2+2}}^{2\sigma_2+2} dt = \|u_0\|_{L^2}^2 - \|u(T)\|_{L^2}^2 \leq \|u_0\|_{L^2}^2,$$

and thus  $u \in L^{2\sigma_2+2}([0, T] \times \mathbf{R}^d)$ .  $\square$

*Remark 2.8* (Non-existence of steady states). An immediate consequence of (2.5) is the non-existence of non-trivial steady states. In the Hamiltonian case ( $a = 0$ ), they are found by inserting the ansatz  $u(t, x) = \psi(x)e^{i\mu t}$  with  $\mu \in \mathbf{R}$  into (1.1) and study the resulting elliptic equation for  $\psi$ . In our case, (2.5) together with the fact that for stationary states  $|u(t, x)|^2 = |\psi(x)|^2$ , immediately implies that  $\psi = 0$ .

Proposition 2.3 and Lemma 2.7 allow us to infer the following blow-up alternative.

**Corollary 2.9** (Blow-up alternative). *Let  $\lambda \in \mathbf{R}$ ,  $a, \omega_1, \dots, \omega_d \geq 0$ ,  $\sigma_j > 0$  with  $\sigma_j < 2/(d-2)$  if  $d \geq 3$ , and  $u_0 \in \Sigma$ . Either the solution to (1.1) exists for all  $t \geq 0$ , i.e.*

$$(2.6) \quad u, \nabla u, xu \in C(\mathbf{R}_+; L^2(\mathbf{R}^d)) \cap \bigcap_{j=1,2} L_{\text{loc}}^{\frac{4\sigma_j+4}{d\sigma_j}}(\mathbf{R}_+; L^{2\sigma_j+2}(\mathbf{R}^d)),$$

or there exists  $T > 0$ , such that

$$\|\nabla u(t)\|_{L^2} \xrightarrow[t \nearrow T]{} +\infty.$$

In the case  $\sigma_1 = \sigma_2 = \frac{2}{d}$ , or in the fully mass-subcritical case  $\sigma_1, \sigma_2 < 2/d$ , the solution is global, that is, (2.6) is satisfied.

*Proof.* Let  $M > 0$ . Lemma 2.7 shows that  $t \mapsto \|u(t)\|_{L^2}^2$  is non-increasing function. Thus, the only obstruction to well-posedness on  $[0, M]$  is the existence of a time  $0 < T < M$  such that

$$\|xu(t)\|_{L^2} + \|\nabla u(t)\|_{L^2} \xrightarrow[t \nearrow T]{} +\infty.$$

As long as  $u \in C([0, t]; \Sigma)$ , we have

$$\begin{aligned} \frac{d}{dt} \int_{\mathbf{R}^d} x_j^2 |u(t, x)|^2 dx &= 2 \operatorname{Re} \int x_j^2 \bar{u}(t, x) \partial_t u(t, x) dx = 2 \operatorname{Im} \int x_j^2 \bar{u}(t, x) i \partial_t u(t, x) dx \\ &= - \operatorname{Im} \int x_j^2 \bar{u}(t, x) \Delta u(t, x) - a \int x_j^2 |u(t, x)|^{2\sigma_2+2} dx \\ &= 2 \operatorname{Im} \int x_j \bar{u}(t, x) \partial_j u(t, x) - a \int x_j^2 |u(t, x)|^{2\sigma_2+2} dx \\ &\leq 2 \|xu(t)\|_{L^2} \|\nabla u(t)\|_{L^2}, \end{aligned}$$

where we have used Cauchy–Schwarz inequality and the assumption  $a \geq 0$ . Suppose  $u \in L^\infty([0, T]; H^1)$ . Then the above estimate and Gronwall’s lemma show that  $xu \in L^\infty([0, T]; L^2)$ , hence a contradiction. Hence the first part of the corollary. The second part follows from the standard criterion for  $L^2$ -critical problems (see e.g. [10]): if the solution does not satisfy (2.6), then there exists  $T > 0$  such that

$$\int_0^T \|u(t)\|_{L^{2+4/d}}^{2+4/d} dt = \infty.$$

This is also a direct consequence of the proof of Proposition 2.3 (if  $\sigma_1 = \sigma_2 = 2/d$ , then  $\theta_j = q_j$ ). Lemma 2.7 rules out this possibility; this point has already been noticed in [11].

Finally, if  $\sigma_1, \sigma_2 < 2/d$ , the standard argument given initially in [21], following again from Proposition 2.3, shows that (2.6) follows if  $u \in L^\infty(\mathbf{R}_+; L^2(\mathbf{R}^d))$ , which in turn is a direct consequence of Lemma 2.7.  $\square$

## 3. GLOBAL WELL-POSEDNESS

The *a-priori* bounds obtained in Section 2.2 are not sufficient to infer global well-posedness for  $\sigma_j \geq 2/d$  (unless  $\sigma_1 = \sigma_2 = 2/d$ ). In order to obtain further *a-priori* estimates, one possible approach would be to follow [11], where the author studies the time-evolution of  $\|\nabla u(t)\|_{L^2}$  and shows that for  $\sigma_2 > \sigma_1 = 2/d$  one can obtain a bound of the form

$$\|\nabla u(t)\|_{L^2} \leq \|\nabla u_0\|_{L^2} e^{Ct},$$

for some  $C > 0$  depending on the involved parameters  $a, \lambda, \sigma_2$ . Indeed, an analogous result could also be obtained in our, more general, situation. However, we shall rather follow the approach of [3] based on a modified energy functional  $E(t)$  which will allow us to infer (under certain conditions) uniform in-time bounds on different quantities involving  $u(t)$ .

**3.1. Bounds on a modified energy functional.** In the following, we denote

$$(3.1) \quad \begin{aligned} E(t) = & \frac{1}{2} \|\nabla u(t)\|_{L^2}^2 + \int_{\mathbf{R}^d} V(x) |u(t, x)|^2 dx \\ & + \frac{\lambda}{\sigma_1 + 1} \int_{\mathbf{R}^d} |u(t, x)|^{2\sigma_1 + 2} dx + \kappa \int_{\mathbf{R}^d} |u(t, x)|^{2\sigma_2 + 2} dx, \end{aligned}$$

for some  $\kappa > 0$  (to be made precise below). Clearly,  $E$  is well defined on  $[0, T]$  for any  $u \in C([0, T]; \Sigma)$ , by Sobolev embedding. Even though this energy functional is not conserved, we shall prove that it is uniformly bounded in time, provided that some assumptions on the involved parameters hold true.

**Proposition 3.1** (Energy bound). *Let  $0 < \kappa < \frac{a}{\sigma_2^2 + \sigma_2}$  and assume that*

- (1) *Either  $\lambda \geq 0$ ,*
- (2) *Or  $\lambda < 0$  and  $\sigma_2 > \sigma_1$ .*

*Then there exists a  $C = C(\|u_0\|_{L^2}) \geq 0$  such that:*

$$E(t) \leq E(0) + C(\|u_0\|_{L^2}), \quad \forall t \in [0, T],$$

*where  $T > 0$  is the existence time obtained in Proposition 2.3.*

*Proof.* We first assume that  $u(t)$  is sufficiently regular and decaying so that all of the following formal manipulations can be carried out. Once the final result is established, a standard density argument allows to conclude that it also holds for  $u \in C([0, T]; \Sigma)$ .

We compute the time-derivative of the energy functional (3.1), using equation (1.1), which yields:

$$(3.2) \quad \begin{aligned} \frac{d}{dt} E(t) = & a \int_{\mathbf{R}^d} |u|^{2\sigma_2} \operatorname{Re}(u \Delta \bar{u}) dx - \kappa(\sigma_2 + 1) \int_{\mathbf{R}^d} |u|^{2\sigma_2} \operatorname{Im}(\bar{u} \Delta u) dx \\ & - 2a \int_{\mathbf{R}^d} V(x) |u|^{2\sigma_2 + 2} dx - 2a\lambda \int_{\mathbf{R}^d} |u|^{2\sigma_2 + 2\sigma_1 + 2} dx \\ & - 2a\kappa(\sigma_2 + 1) \int_{\mathbf{R}^d} |u|^{4\sigma_2 + 2} dx. \end{aligned}$$

Consider the first term on the right hand side; since  $\Delta |u|^2 = 2 \operatorname{Re}(\bar{u} \Delta u) + 2 |\nabla u|^2$ , notice it can be rewritten, using integration by parts,

$$\int_{\mathbf{R}^d} |u|^{2\sigma_2} \operatorname{Re}(\bar{u} \Delta u) dx = - \int_{\mathbf{R}^d} |u|^{2\sigma_2} |\nabla u|^2 dx - 2\sigma_2 \int_{\mathbf{R}^d} |u|^{2\sigma_2} |\nabla |u||^2 dx.$$



Now, we rewrite the second term on the right hand side:

$$\int_{\mathbf{R}^d} |u|^{2\sigma_2} \operatorname{Im}(\bar{u}\Delta u) dx = \int_{\mathbf{R}^d} |u|^{2\sigma_2} \operatorname{div}(\operatorname{Im}(\bar{u}\nabla u)) dx = - \int_{\mathbf{R}^d} \nabla |u|^{2\sigma_2} \cdot \operatorname{Im}(\bar{u}\nabla u) dx.$$

Here we use the polar factorisation introduced in [2] (see also [9]), to show the above integral equals

$$(3.3) \quad -2\sigma_2 \int_{\mathbf{R}^d} |u|^{2\sigma_2} \operatorname{Re}(\bar{\phi}\nabla u) \cdot \operatorname{Im}(\bar{\phi}\nabla u) dx,$$

where  $\phi$  is the polar factor related to  $u$ ,

$$\phi(t, x) := \begin{cases} |u(t, x)|^{-1} u(t, x) & \text{if } u(t, x) \neq 0, \\ 0 & \text{if } u(t, x) = 0. \end{cases}$$

This indeed can first be proved by replacing  $\phi$  with

$$\phi^\varepsilon(t, x) = \frac{u(t, x)}{\sqrt{|u(t, x)|^2 + \varepsilon^2}},$$

and then passing to the limit  $\varepsilon \rightarrow 0$  in  $H^1$ , as in [2], [9]. Let us rewrite (3.3) as

$$\begin{aligned} \sigma_2 \int_{\mathbf{R}^d} |u|^{2\sigma_2} |\operatorname{Re}(\bar{\phi}\nabla u) - \operatorname{Im}(\bar{\phi}\nabla u)|^2 - |u|^{2\sigma_2} \left( |\operatorname{Re}(\bar{\phi}\nabla u)|^2 + |\operatorname{Im}(\bar{\phi}\nabla u)|^2 \right) dx \\ = \sigma_2 \int_{\mathbf{R}^d} |u|^{2\sigma_2} |\operatorname{Re}(\bar{\phi}\nabla u) - \operatorname{Im}(\bar{\phi}\nabla u)|^2 dx - \sigma_2 \int_{\mathbf{R}^d} |u|^{2\sigma_2} |\nabla u|^2 dx. \end{aligned}$$

The last equality follows from the identity

$$|\nabla u|^2 = |\operatorname{Re}(\bar{\phi}\nabla u)|^2 + |\operatorname{Im}(\bar{\phi}\nabla u)|^2, \quad \text{a.e. in } \mathbf{R}^d,$$

see formulas (30) in [2] and (5.15) in [9]. Hence, by resuming the second term on the right hand side of (3.2) is equal to

$$-\kappa(\sigma_2^2 + \sigma_2) \int_{\mathbf{R}^d} |u|^{2\sigma_2} |\operatorname{Re}(\bar{\phi}\nabla u) - \operatorname{Im}(\bar{\phi}\nabla u)|^2 dx + \kappa(\sigma_2^2 + \sigma_2) \int_{\mathbf{R}^d} |u|^{2\sigma_2} |\nabla u|^2 dx.$$

In summary this yields:

$$\begin{aligned} \frac{d}{dt} E(t) = & - (a - \kappa(\sigma_2^2 + \sigma_2)) \int_{\mathbf{R}^d} |u|^{2\sigma_2} |\nabla u|^2 dx - 2a\sigma_2 \int_{\mathbf{R}^d} |u|^{2\sigma_2} |\nabla |u||^2 \\ & - \kappa(\sigma_2^2 + \sigma_2) \int_{\mathbf{R}^d} |u|^{2\sigma_2} |\operatorname{Re}(\bar{\phi}\nabla u) - \operatorname{Im}(\bar{\phi}\nabla u)|^2 dx - 2a \int_{\mathbf{R}^d} V(x) |u|^{2\sigma_2+2} dx \\ & - 2a\lambda \int_{\mathbf{R}^d} |u|^{2\sigma_2+2\sigma_1+2} dx - 2a\kappa(\sigma_2 + 1) \int_{\mathbf{R}^d} |u|^{4\sigma_2+2} dx. \end{aligned}$$

Under the assumption  $0 < \kappa < \frac{a}{\sigma_2^2 + \sigma_2}$ , and if  $\lambda \geq 0$  (defocusing case), all the terms on the right hand side are non-positive, and we infer that  $E$  is a non-increasing function:  $E(t) \leq E(0)$ ,  $\forall t \in [0, T]$ .

If  $\lambda < 0$  (focusing case), however, the term involving the  $L^{2\sigma_1+2\sigma_2+2}$  norm is positive. But since  $\sigma_2 > \sigma_1$  by assumption, we can interpolate this term using:

$$\|u\|_{L^{2\sigma_1+2\sigma_2+2}} \leq \|u\|_{L^{4\sigma_2+2}}^\theta \|u\|_{L^{2\sigma_2+2}}^{1-\theta},$$

with

$$\frac{1}{\sigma_1 + \sigma_2 + 1} = \frac{\theta}{2\sigma_2 + 1} + \frac{1-\theta}{\sigma_2 + 1} : \quad \theta = \frac{\sigma_1(2\sigma_2 + 1)}{\sigma_2(\sigma_1 + \sigma_2 + 1)} \in (0, 1).$$

Denoting  $\gamma = \sigma_1/\sigma_2$ , this implies that

$$\|u\|_{L^{2\sigma_1+2\sigma_2+2}}^{2\sigma_1+2\sigma_2+2} \leq \|u\|_{L^{4\sigma_2+2}}^{\gamma(4\sigma_2+2)} \|u\|_{L^{2\sigma_2+2}}^{(1-\gamma)(2\sigma_2+2)} \leq \varepsilon \|u\|_{L^{4\sigma_2+2}}^{4\sigma_2+2} + \frac{1}{\varepsilon^{\gamma/(1-\gamma)}} \|u\|_{L^{2\sigma_2+2}}^{2\sigma_2+2},$$

where we have used Young inequality. In summary this yields

$$\frac{d}{dt} E(t) \leq -2a(\kappa(\sigma_2 + 1) - |\lambda|\varepsilon) \int_{\mathbf{R}^d} |u|^{4\sigma_2+2} dx + \frac{2a|\lambda|}{\varepsilon^{\gamma/(1-\gamma)}} \int_{\mathbf{R}^d} |u|^{2\sigma_2+2} dx.$$

For  $0 < \varepsilon \ll 1$ , the coefficient in front of the first term is negative, which allows to conclude, after an integration with respect to time, that:

$$\forall t \in [0, T] : E(t) \leq E(0) + C \int_0^T \int_{\mathbf{R}^d} |u|^{2\sigma_2+2} dx dt < E(0) + C(\|u_0\|_{L^2}),$$

since we have  $u \in L^{2\sigma_2+2}([0, T] \times \mathbf{R}^d)$  from Lemma 2.7.  $\square$

We are now in the position to prove global well-posedness of (1.1) under various conditions on the parameters. For the sake of a simpler presentation, we shall treat the case  $\sigma_1 = \sigma_2$  separately, see Section 3.3 below.

**3.2. The case  $\sigma_1 \neq \sigma_2$ .** In the following we shall consider two different powers and impose the following assumption.

**Assumption 3.2** (Nonlinearity). *Let  $\sigma_1 \neq \sigma_2$  and, in addition:*

(1) *Defocusing case. If  $\lambda \geq 0$ , we assume*

$$0 < \sigma_1, \sigma_2 < \frac{2}{d-2} \quad (0 < \sigma_1, \sigma_2 \text{ if } d \leq 2).$$

(2) *Focusing case. If  $\lambda < 0$ , we assume*

- *Either,  $0 < \sigma_1 < \frac{2}{d}$  and  $0 < \sigma_2 < \frac{2}{(d-2)_+}$ ,*
- *Or  $\frac{2}{d} \leq \sigma_1 < \sigma_2 < \frac{2}{(d-2)_+}$ .*

*Remark 3.3.* Assumption 3.2 can be understood as follows: If without damping ( $a = 0$ ), the solution of (1.1) is global in time, then the strength of the nonlinear damping plays no role. On the other hand, if finite time blow-up may occur in the Hamiltonian case (i.e.,  $\lambda < 0$  and  $\sigma_1 \geq 2/d$ ), then the damping is assumed to be stronger than the attractive interaction, in terms of the power  $\sigma_2$ .

**Theorem 3.4** (Global existence I). *Let  $d \geq 1$ ,  $a > 0$ , and  $\omega_1, \dots, \omega_d \geq 0$ . Under Assumption 3.2, for any  $u_0 \in \Sigma$ , (1.1) has a unique solution*

$$u \in C(\mathbf{R}_+; \Sigma) \cap L^\infty(\mathbf{R}_+; H^1) \cap L^{2\sigma_2+2}(\mathbf{R}_+ \times \mathbf{R}^d) \cap \bigcap_{j=1,2} L_{\text{loc}}^{\frac{4\sigma_j+4}{d\sigma_j}}(\mathbf{R}_+; L^{2\sigma_j+2}(\mathbf{R}^d)).$$

*Moreover, if  $\omega_j > 0$  for all  $j \in \{1, \dots, d\}$ , then we also have  $u \in L^\infty(\mathbf{R}_+; \Sigma)$ .*

*Proof.* In the defocusing situation  $\lambda > 0$ , Proposition 3.1 immediately implies a uniform (in-time) bound on  $\|\nabla u(t)\|_{L^2}^2 \lesssim E(0)$ , since  $E(t)$  is the sum of four non-negative terms. In view of the blow-up alternative, stated in Corollary 2.9, we thus infer global well-posedness.

For the focusing situation  $\lambda < 0$ , we note that the mass-subcritical case  $\sigma_1, \sigma_2 < 2/d$ , has already been dealt with in Corollary 2.9 (including  $\sigma_1 = \sigma_2 = 2/d$ ). The

moment  $\sigma_1 \geq 2/d$ , we require  $\sigma_2 > \sigma_1$  which allows us to interpolate the potential energy of the attractive nonlinearity similarly to the calculation above, namely:

$$\|u(t)\|_{L^{2\sigma_1+2}}^{2\sigma_1+2} \leq \|u(t)\|_{L^2}^{2\beta} \|u(t)\|_{L^{2\sigma_2+2}}^{(1-\beta)(2\sigma_2+2)} \leq \varepsilon^{(\beta-1)/\beta} \|u(t)\|_{L^2}^2 + \varepsilon \|u(t)\|_{L^{2\sigma_2+2}}^{2\sigma_2+2},$$

for any  $\varepsilon > 0$  and an appropriately chosen  $0 < \beta < 1$ . Now fix  $\varepsilon = 2\kappa(\sigma_1 + 1)/|\lambda|$  to obtain

$$\|\nabla u(t)\|_{L^2}^2 \leq 2E(t) + C\|u(t)\|_{L^2}^2,$$

for some  $C = C(\lambda, \kappa, \sigma_j) > 0$ . In view of Corollary 2.9 and Proposition 3.1 this yields  $\|\nabla u(t)\|_{L^2}^2 \lesssim E(0) + \|u_0\|_{L^2}^2$  and we are done.  $\square$

**3.3. The case  $\sigma_1 = \sigma_2$ .** In the case where  $\sigma_1 = \sigma_2 \equiv \sigma$ , (1.1) simplifies to

$$(3.4) \quad i\partial_t u + \frac{1}{2}\Delta u = V(x)u + (\lambda - ia)|u|^{2\sigma}u, \quad u|_{t=0} = u_0 \in \Sigma.$$

Formally, this model can be considered as the diffusionless limit of the (generalized) complex Ginzburg-Landau equation. Concerning global existence of (3.4), we shall only be interested in the  $L^2$ -supercritical situation ( $\sigma > 2/d$ ), in view of the last assertion in Corollary 2.9.

**Theorem 3.5** (Global existence II). *Let  $d \geq 1$ ,  $a > 0$ ,  $\sigma > 2/d$ , and  $\omega_1, \dots, \omega_d \geq 0$ . Assume that*

- (1) *Either  $\lambda \geq 0$ ,*
- (2) *Or  $\lambda < 0$  and  $a \geq \min(\sigma, \sqrt{\sigma})|\lambda|$ .*

*Then, for all  $u_0 \in \Sigma$ , (3.4) has a unique solution*

$$u \in C(\mathbf{R}_+; \Sigma) \cap L^\infty(\mathbf{R}_+; H^1) \cap L^{2\sigma+2}(\mathbf{R}_+ \times \mathbf{R}^d) \cap L_{\text{loc}}^{\frac{4\sigma+4}{d\sigma}}(\mathbf{R}_+; L^{2\sigma+2}(\mathbf{R}^d)).$$

*Moreover, if  $\omega_j > 0$  for all  $j \in \{1, \dots, d\}$ , then we also have  $u \in L^\infty(\mathbf{R}_+; \Sigma)$ .*

*Proof.* The defocusing case  $\lambda \geq 0$ , can be treated as in the proof of Theorem 3.4 above. It therefore remains to consider the case  $\lambda < 0$  and  $\sigma > 2/d$ . Note that we cannot invoke Proposition 3.1, which requires  $\sigma_1 < \sigma_2$ . Instead (following an idea in [3]), we consider the linear energy functional:

$$E_{\text{lin}}(t) = \frac{1}{2} \|\nabla u(t)\|_{L^2}^2 + \int_{\mathbf{R}^d} V(x)|u(t, x)|^2 dx.$$

Obviously, it suffices to prove that  $E_{\text{lin}}(t)$  is uniformly bounded in time. Differentiating  $E_{\text{lin}}(t)$  and using Equation (3.4) yields

$$(3.5) \quad \begin{aligned} \frac{d}{dt} E_{\text{lin}}(t) &= a \int_{\mathbf{R}^d} |u|^{2\sigma} \operatorname{Re}(\bar{u}\Delta u) dx + |\lambda| \int_{\mathbf{R}^d} |u|^{2\sigma} \operatorname{Im}(u\Delta \bar{u}) dx \\ &\quad - 2a \int_{\mathbf{R}^d} V(x)|u|^{2\sigma+2} dx. \end{aligned}$$

Using the same arguments as in the proof of Proposition 3.1, we infer

$$\begin{aligned} \frac{d}{dt} E_{\text{lin}}(t) &\leq -(a - |\lambda|\sigma) \int_{\mathbf{R}^d} |u|^{2\sigma} |\nabla u|^2 dx - 2a\sigma \int_{\mathbf{R}^d} |u|^{2\sigma} |\nabla |u||^2 \\ &\quad - |\lambda|\sigma \int_{\mathbf{R}^d} |u|^{2\sigma} |\operatorname{Re}(\bar{\phi}\nabla u) - \operatorname{Im}(\bar{\phi}\nabla u)|^2 dx - 2a \int_{\mathbf{R}^d} V(x)|u|^{2\sigma+2} dx. \end{aligned}$$

Thus, if  $a \geq \sigma|\lambda|$ , we obtain  $E_{\text{lin}}(t) \leq E_{\text{lin}}(0) < \infty$ , for all  $t \in [0, T]$ . On the other hand, from (3.5) we have

$$\begin{aligned} \frac{d}{dt} E_{\text{lin}}(t) &= -a\sigma \int_{\mathbf{R}^d} |u|^{2\sigma} |\nabla|u||^2 dx - a \int_{\mathbf{R}^d} |u|^{2\sigma} |\nabla u|^2 dx \\ &\quad + 2\sigma|\lambda| \int_{\mathbf{R}^d} |u|^{2\sigma} \nabla|u| \cdot \text{Im}(\bar{\phi} \nabla u) dx - 2a \int_{\mathbf{R}^d} V(x) |u|^{2\sigma+2} dx. \end{aligned}$$

By using Cauchy-Schwarz and then Young inequality, we see that the third term in the right hand side is bounded by

$$a\sigma \int |u|^{2\sigma} |\nabla|u||^2 dx + \frac{|\lambda|^2 \sigma}{a} \int |u|^{2\sigma} |\nabla u|^2 dx,$$

hence if  $|\lambda|\sqrt{\sigma} \leq a$ , then again we have  $E_{\text{lin}}(t) \leq E_{\text{lin}}(0) < \infty$ , for all  $t \in [0, T]$ . This establishes global well-posedness of (3.4), in the second case.  $\square$

*Remark 3.6.* The second case of Theorem 3.5 shows that if the damping is of the same strength (in terms of its power) as an attractive interaction nonlinearity, the relative size of the respective coefficients starts to play a role. Note that the larger the dimension  $d \geq 1$ , the smaller  $a$  can be chosen to ensure global existence. We remark that the numerical experiments presented in [13] always consider  $a \ll |\lambda|$  and thus they do not show the aforementioned possibility for global existence.

At this point, we have proved global existence in all the cases listed in Theorem 1.1. We now turn to the large time behavior, in cases where the solution is defined for all  $t \geq 0$ .

#### 4. LARGE TIME BEHAVIOR

In view of the dissipation equation (1.2), the damping is expected to have a significant influence on the long time behavior of solutions to (1.1). Indeed, if we consider, for a moment, the case of  $x$ -independent solutions, then (1.2) simplifies to the following ordinary differential equation

$$\partial_t \rho = -2a\rho^{\sigma_2+1}, \quad \rho|_{t=0} = \rho_0 := |u_0|^2,$$

the solution of which is given by

$$(4.1) \quad \rho(t) = \frac{\rho_0}{(1 + 2at\rho_0^{\sigma_2})^{1/\sigma_2}}.$$

Thus, one might expect the solution to vanish like  $\|u(t)\|_{L^2}^2 = \mathcal{O}(t^{-1/\sigma_2})$ , as  $t \rightarrow +\infty$ . We shall see below, however, that this rather naive argument does not yield the correct long time behavior of  $u$  in the case of where  $V = 0$ . The idea is that the dispersion due to the Laplacian may prevent the damping from taking the wave function  $u$  to zero. On the other hand, in the presence of a confining potential (i.e.,  $\omega_j > 0$  for *some*  $j$ ), there is no (large time) dispersion, and the ODE mechanism remains, at least qualitatively.

**4.1. Asymptotic extinction with full confinement.** In this subsection, we consider the case of a fully confining potential, in the sense that we suppose  $\omega_j > 0$  for *all*  $j$ . We start with a simple estimate, which can be viewed as a dual version of Nash inequality.

**Lemma 4.1** (Localization). *Let  $d \geq 1$ . For all  $p \geq 2$ , there exists  $C$  such that for all  $f \in \mathcal{S}(\mathbf{R}^d)$ ,*

$$\|f\|_{L^2(\mathbf{R}^d)} \leq C \|f\|_{L^p(\mathbf{R}^d)}^\theta \|xf\|_{L^2(\mathbf{R}^d)}^{1-\theta}, \quad \theta = \frac{1}{1+d(1/2-1/p)} \in (0, 1].$$

*Proof.* For  $R > 0$ , write

$$\begin{aligned} \|f\|_{L^2(\mathbf{R}^d)} &\leq \|f\|_{L^2(|x|<R)} + \|f\|_{L^2(|x|>R)} \\ &\leq C_d R^{d(1/2-1/p)} \|f\|_{L^p(\mathbf{R}^d)} + \frac{1}{R} \|xf\|_{L^2(\mathbf{R}^d)}, \end{aligned}$$

where we have applied Hölder's inequality for the first term, and simply multiplied and divided by  $|x|$  for the second term. Both terms on the right hand side have the same order of magnitude if  $R^{1+d(1/2-1/p)} = \|xf\|_{L^2}/\|f\|_{L^p}$ . Using this value of  $R$  yields the result.  $\square$

Recall that in the case with confining potential ( $\omega_j > 0$  for all  $j \in \{1, \dots, d\}$ ), the solution satisfies  $u \in L^\infty(\mathbf{R}_+; \Sigma)$ . This can be used to obtain an estimate for the time-decay of the solution.

**Proposition 4.2** (Asymptotic extinction I). *Let  $d \geq 1$ ,  $a > 0$ ,  $\omega_1, \dots, \omega_d > 0$ , and  $u_0 \in \Sigma$ . In either of the cases mentioned in Theorem 1.1, the solution to (1.1) satisfies  $u \in L^\infty(\mathbf{R}_+; \Sigma)$  and there exists  $C > 0$  such that*

$$\|u(t)\|_{L^2}^2 \leq C t^{-\frac{2}{(d+2)\sigma_2}}, \quad \forall t \geq 1.$$

*Proof.* In view of Lemma 4.1 with  $p = 2\sigma_2 + 2$ , we have, since  $u \in L^\infty(\mathbf{R}_+; \Sigma)$  by assumption,

$$\|u(t)\|_{L^2} \leq C \|u(t)\|_{L^{2\sigma_2+2}}^\theta, \quad \theta = \frac{2\sigma_2 + 2}{(d+2)\sigma_2 + 2}.$$

Along with (2.5), this yields

$$\frac{d}{dt} \|u(t)\|_{L^2}^2 + 2aC \|u(t)\|_{L^2}^{\frac{2\sigma_2+2}{\theta}} \leq 0.$$

Therefore,  $y(t) = \|u(t)\|_{L^2}^2$  satisfies a differential inequality of the form

$$\dot{y}(t) + C y(t)^p \leq 0, \quad p = 1 + \frac{d+2}{2}\sigma_2.$$

The comparison with the ordinary differential equation yields  $y(t) = \mathcal{O}(t^{-1/(p-1)})$  for  $t \geq 1$ , since  $p > 1$ , hence the result.  $\square$

We shall see in Section 4.2 that the presence of at least some confinement is essential for the time-decay of the  $L^2$  norm.

**4.2. Non-vanishing solutions.** In the following, we assume that there is no confining potential, i.e.  $V = 0$  in (1.1). Then the solution must not be expected to vanish as  $t \rightarrow +\infty$ , as shown by the following result.

**Proposition 4.3** (Scattering without potential). *Let  $d \geq 1$ ,  $\lambda, a \in \mathbf{R}$  and  $2/d \leq \sigma_1, \sigma_2 \leq 2/(d-2)$ . Let  $R > 0$ . There exists  $T$  depending only on  $d, |\lambda|, |a|$  and  $R$  such that for all  $u_+ \in H^1(\mathbf{R}^d)$  with  $\|u_+\|_{H^1} \leq R$ , there exists a solution  $u \in C([T, \infty); H^1)$  to*

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda|u|^{2\sigma_1} u - ia|u|^{2\sigma_2} u,$$

such that

$$\lim_{t \rightarrow +\infty} \|u(t) - e^{i\frac{t}{2}\Delta} u_+\|_{H^1(\mathbf{R}^d)} = 0.$$

*Sketch of the proof.* This result follows from [15] (see also [14, Proposition 3.1] for a simplification), and stems from a fixed point argument applied to Duhamel's formulation of the above problem:

$$u(t) = U(t)u_+ + i\lambda \int_t^\infty U(t-s) (|u|^{2\sigma_1} u)(s) ds + a \int_t^\infty U(t-s) (|u|^{2\sigma_2} u)(s) ds.$$

Denote by  $\Phi(u)(t)$  the right hand side, and  $u_{\text{free}}(t) = e^{i\frac{t}{2}\Delta} u_+$ . Let  $r_j = 2\sigma_j + 2$  and  $q_j > 2$  be such that  $(q_j, 2\sigma_j + 2)$  is admissible,  $j = 1, 2$ , like in (2.3). With the notation  $L_T^\beta Y = L^\beta([T, \infty); Y)$ , we introduce:

$$X_T := \left\{ u \in C([T, \infty); H^1) ; \|u\|_{L_T^{q_j} W^{1, 2\sigma_j+2}} \leq 2C_{q_j} \|u_+\|_{H^1}, \right. \\ \left. \|u\|_{L_T^\infty H^1} \leq 2\|u_+\|_{H^1}, \quad \|u\|_{L_T^{q_j} L^{2\sigma_j+2}} \leq 2\|u_{\text{free}}\|_{L_T^{q_j} L^{2\sigma_j+2}}, j = 1, 2 \right\},$$

where  $C_{q_j}$  is given by Proposition 2.2. Resume the numerology of the proof of Proposition 2.3: we have

$$\frac{1}{r'_j} = \frac{1}{r_j} + \frac{2\sigma_j}{r_j}, \quad \frac{1}{q'_j} = \frac{1}{q_j} + \frac{2\sigma}{\theta_j},$$

where  $q_j \leq \theta_j < \infty$  since  $2/d \leq \sigma_j < 2/(d-2)$ . For  $u \in X_T$ , Strichartz estimates and Hölder inequality yield:

$$\|\Phi(u)\|_{L_T^{q_j} W^{1, 2\sigma_j+2}} \leq C_{q_j} \|u_+\|_{H^1} + C \sum_{\ell=1,2} \left( \| |u|^{2\sigma_\ell} u \|_{L_T^{q'_\ell} L^{r'_\ell}} + \| |u|^{2\sigma_\ell} \nabla u \|_{L_T^{q'_\ell} L^{r'_\ell}} \right) \\ \leq C_{q_j} \|u_+\|_{H^1} + C \sum_{\ell=1,2} \|u\|_{L_T^{\theta_\ell} L^{r_\ell}}^{2\sigma_\ell} \left( \|u\|_{L_T^{q_\ell} L^{r_\ell}} + \|\nabla u\|_{L_T^{q_\ell} L^{r_\ell}} \right) \\ \leq C_{q_j} \|u_+\|_{H^1} + C \sum_{\ell=1,2} \|u\|_{L_T^{q_\ell} L^{r_\ell}}^{2\sigma_\ell \eta_\ell} \|u\|_{L_T^\infty L^{r_\ell}}^{2\sigma_\ell(1-\eta_\ell)} \|u\|_{L_T^{q_\ell} W^{1, r_\ell}},$$

for  $\eta_\ell = q_\ell/\theta_\ell \in (0, 1]$ . Sobolev embedding and the definition of  $X_T$  then imply:

$$\|\Phi(u)\|_{L_T^{q_j} W^{1, 2\sigma_j+2}} \leq C_{q_j} \|u_+\|_{H^1} + C \sum_{\ell=1,2} \|u_{\text{free}}\|_{L_T^{q_\ell} L^{r_\ell}}^{2\sigma_\ell \eta_\ell} \|u\|_{L_T^\infty H^1}^{2\sigma_\ell(1-\eta_\ell)} \|u\|_{L_T^\ell W^{1, 2\sigma_\ell+2}}.$$

We have similarly

$$\|\Phi(u)\|_{L_T^\infty H^1} \leq \|u_+\|_{H^1} + C \sum_{\ell=1,2} \|u_{\text{free}}\|_{L_T^{q_\ell} L^{r_\ell}}^{2\sigma_\ell \eta_\ell} \|u\|_{L_T^\infty H^1}^{2\sigma_\ell(1-\eta_\ell)} \|u\|_{L_T^{q_\ell} W^{1, 2\sigma_\ell+2}}, \\ \|\Phi(u)\|_{L_T^{q_j} L^{2\sigma_j+2}} \leq \|u_{\text{free}}\|_{L_T^{q_j} L^{2\sigma_j+2}} + C \sum_{\ell=1,2} \|u_{\text{free}}\|_{L_T^{q_\ell} L^{r_\ell}}^{2\sigma_\ell \eta_\ell} \|u\|_{L_T^\infty H^1}^{2\sigma_\ell(1-\eta_\ell)} \|u\|_{L_T^{q_\ell} W^{1, 2\sigma_\ell+2}}.$$

From Strichartz estimates,  $u_{\text{free}} \in L^{q_j}(\mathbf{R}; L^{r_j})$ , so

$$\|u_{\text{free}}\|_{L_T^{q_j} L^{2\sigma_j+2}} \rightarrow 0 \quad \text{as } T \rightarrow +\infty.$$

Since  $\eta_\ell > 0$ , we infer that  $\Phi$  sends  $X_T$  to itself, for  $T$  sufficiently large.

We have also, for  $u_2, u_1 \in X_T$ :

$$\begin{aligned} \|\Phi(u_2) - \Phi(u_1)\|_{L_T^{q_j} L^{r_j}} &\lesssim \sum_{\ell=1,2} \max_{k=1,2} \|u_k\|_{L_T^{\theta_\ell} L^{r_\ell}}^{2\sigma_\ell} \|u_2 - u_1\|_{L_T^{q_\ell} L^{r_\ell}} \\ &\lesssim \sum_{\ell=1,2} \|u_{\text{free}}\|_{L_T^{q_\ell} L^{r_\ell}}^{2\sigma_\ell \eta_\ell} \|u_+\|_{H^1}^{2\sigma_\ell(1-\eta_\ell)} \|u_2 - u_1\|_{L_T^{q_\ell} L^{r_\ell}}. \end{aligned}$$

Up to choosing  $T$  larger,  $\Phi$  is a contraction on  $X_T$ , equipped with the  $L_T^{q_1} L^{r_1} \cap L_T^{q_2} L^{r_2}$ -norm. Since this space is complete (see e.g. [10, Section 4.4]), the proposition follows from the standard fixed point argument.  $\square$

Proposition 4.3 rules out the asymptotic extinction of the solution, since, by invoking the triangle inequality, we directly infer

$$\begin{aligned} \lim_{t \rightarrow +\infty} \|u(t)\|_{L^2} &\geq \lim_{t \rightarrow +\infty} \left| \|e^{i\frac{t}{2}\Delta} u_+\|_{L^2} - \|u(t) - e^{i\frac{t}{2}\Delta} u_+\|_{L^2} \right| \\ &= \lim_{t \rightarrow +\infty} \|e^{i\frac{t}{2}\Delta} u_+\|_{L^2} = \|u_+\|_{L^2}, \end{aligned}$$

due to mass conservation. The solution of the damped equation therefore does not decay to zero, due to the possibility of radiation escaping to infinity. The latter is no longer true if we consider (1.1) on a compact manifold, instead of  $\mathbf{R}^d$  (see the appendix), or in the presence of a partially confining potential on  $\mathbf{R}^d$ , as we show in the next subsection.

**4.3. Asymptotic extinction with partial confinement.** Suppose  $d \geq 2$ , and decompose

$$x = (x', x'') \in \mathbf{R}^k \times \mathbf{R}^{d-k}, \quad 1 \leq k \leq d-1,$$

with  $\omega_j > 0$  if  $1 \leq j \leq k$  and  $\omega_j = 0$  if  $j \geq k+1$ . Consider the linear equation

$$(4.2) \quad i\partial_t u_{\text{lin}} = H u_{\text{lin}}, \quad H = -\frac{1}{2}\Delta + \frac{|x'|^2}{2}, \quad u_{\text{lin}}|_{t=0} = u_0.$$

In this case, a generalized Mehler formula is available (see, e.g., [17]), yielding the linear solution for  $t > 0$ , in the following form

$$u_{\text{lin}}(t, x) = \frac{1}{(2i\pi \sin t)^{\frac{k}{2}}} \frac{1}{(2i\pi t)^{\frac{d-k}{2}}} \int_{\mathbf{R}^d} e^{\frac{i}{\sin t} \sum_{j=1}^k \left( \frac{x_j^2 + y_j^2}{2} \cos t - x_j y_j \right)} e^{i \frac{|x'' - y''|^2}{2t}} u_0(y) dy,$$

where  $x = (x_1, x')$  and the square roots are defined in such a way that Maslov index is taken into account. In the case where  $u_0$  is a tensor product,  $u_0(x) = f(x')g(x'')$ , we have

$$u_{\text{lin}}(t, x) = \left( e^{-it \left( -\frac{1}{2}\Delta_{\mathbf{R}^k} + \frac{|x'|^2}{2} \right)} f(x') \right) \otimes \left( e^{i\frac{t}{2}\Delta_{\mathbf{R}^{d-k}}} g(x'') \right).$$

Even though the evolution of  $f$  leads to no global dispersion (the solution is periodic in time), the standard dispersive properties of the free Schrödinger group allow to write that, for  $2 < p < \infty$ , it holds:

$$\begin{aligned} \|u_{\text{lin}}(t)\|_{L^p(\mathbf{R}^d)} &= \left\| e^{-it \left( -\frac{1}{2}\Delta_{\mathbf{R}^k} + \frac{|x'|^2}{2} \right)} f \right\|_{L^p(\mathbf{R}^k)} \left\| e^{i\frac{t}{2}\Delta_{\mathbf{R}^{d-k}}} g \right\|_{L^p(\mathbf{R}^{d-k})} \\ &= \mathcal{O} \left( t^{-(d-k)(1/2-1/p)} \right) \quad \text{as } t \rightarrow +\infty. \end{aligned}$$

For a general  $u_0 \in \Sigma$ , it is possible to recover such a decay rate by invoking suitable vector fields. In view of the relations (see e.g. [8])

$$\begin{aligned} A(t) &= e^{-itH} (i\nabla_{x'}, x'') e^{itH} f = (- (\sin t) x' f + i (\cos t) \nabla_{x'} f, (x'' + it\nabla_{x''}) f) \\ &= \left( i (\cos t) e^{-i\frac{|x'|^2}{2} \tan t} \nabla_{x'} \left( e^{i\frac{|x'|^2}{2} \tan t} f \right), ite^{i\frac{|x''|^2}{2t}} \nabla_{x''} \left( e^{-i\frac{|x''|^2}{2t}} f \right) \right), \\ B(t) &= e^{-itH} x e^{itH} f = ((\cos t) x' f + i (\sin t) \nabla_{x'} f, (x'' + it\nabla_{x''}) f) \\ &= \left( i (\sin t) e^{i\frac{|x'|^2}{2} \cot t} \nabla_{x'} \left( e^{-i\frac{|x'|^2}{2} \cot t} f \right), ite^{i\frac{|x''|^2}{2t}} \nabla_{x''} \left( e^{-i\frac{|x''|^2}{2t}} f \right) \right), \end{aligned}$$

the first equality shows that  $A$  and  $B$  commute with the linear group  $e^{-itH}$ , so  $\|A(t)u_{\text{lin}}\|_{L^2}$  and  $\|B(t)u_{\text{lin}}\|_{L^2}$  are conserved quantities, and the last equality, combined with Gagliardo-Nirenberg inequality, yields

$$\|u_{\text{lin}}(t)\|_{L^p(\mathbf{R}^d)} \leq \frac{C}{t^{(d-k)(1/2-1/p)}} \|u_0\|_{\Sigma}, \quad 2 \leq p < \frac{2}{(d-2)_+}.$$

In view of the previous subsection, the effect of nonlinear damping terms, in combination with partial confinement, does not seem obvious, because without damping, there may be scattering for the nonlinear analogue of (4.2) (provided that the nonlinearity is short range in  $\mathbf{R}^{d-k}$ ).

The present situation can also be compared to the framework of [16] and [22]. There, cubic or quintic Schrödinger equation is considered on a product space of the form  $\mathbf{R}^{d-k} \times M^k$ , where  $M^k$  is a  $k$ -dimensional compact manifold, and scattering is established at least for small data. Let us describe more precisely the results in [16]: The quintic defocusing equation is considered on  $\mathbf{R} \times \mathbf{T}^2$ , a situation which may be compared to the case of a partial harmonic confinement in the last two spatial directions:

$$i\partial_t u + \Delta_{\mathbf{R} \times \mathbf{T}^2} u = |u|^4 u \quad ; \quad u|_{t=0} = u_0.$$

Z. Hani and B. Pausader prove that even though global dispersion is not possible in the last two components, dispersion associated to the first component is enough to ensure scattering: assuming that  $\|u_0\|_{H^1(\mathbf{R} \times \mathbf{T}^2)}$  is sufficiently small (a condition which is conjectured to be unnecessary in [16]), then there exists  $v_+ \in H^1(\mathbf{R} \times \mathbf{T}^2)$  such that

$$\|u(t) - e^{it\Delta_{\mathbf{R} \times \mathbf{T}^2}} v_+\|_{H^1(\mathbf{R} \times \mathbf{T}^2)} \xrightarrow{t \rightarrow +\infty} 0.$$

In [22], a similar result is proved for cubic (focusing or defocusing) Schrödinger equation on  $\mathbf{R}^n \times M^k$ ,  $n \geq 2$ , in a more cumbersome function space. In view of Proposition 4.3, one might believe that adding a nonlinear damping is not enough to drive the solution to zero (in  $L^2$ ) for large time.

We shall see that nonlinear damping always leads to asymptotic extinction in the case of a *partial* confinement, provided that the solution to (1.1) is defined for all  $t \geq 0$ . The key to prove this assertion is an anisotropic counterpart of Lemma 4.1:

**Lemma 4.4** (Anisotropic localization). *Let  $d \geq 2$  and  $1 \leq k \leq d-1$ . For all  $\sigma_2 > 0$ , there exists  $C$  such that for all  $f \in \mathcal{S}(\mathbf{R}^d)$ ,*

$$\|f\|_{L^2(\mathbf{R}^d)}^2 \leq C \|f\|_{L^{2\sigma_2+2}(\mathbf{R}^d)}^{(2\sigma_2+2)/((k+1)\sigma_2+1)} \|x' f\|_{L^2(\mathbf{R}^d)}^{(2k\sigma_2)/((k+1)\sigma_2+1)},$$

where  $x = (x', x'') \in \mathbf{R}^k \times \mathbf{R}^{d-k}$ .



*Proof.* Let

$$\rho(x') = \int_{\mathbf{R}^{d-k}} |f(x', x'')|^2 dx'',$$

and write

$$\|f\|_{L^2(\mathbf{R}^d)}^2 = \int_{\mathbf{R}^k} \rho(x') dx' = \int_{|x'| < R} \rho(x') dx' + \int_{|x'| > R} \rho(x') dx'.$$

The second integral can be estimated exactly like in the proof of Lemma 4.1, i.e.

$$\int_{|x'| > R} \rho(x') dx' \leq R^{-2} \int_{|x'| > R} |x'|^2 \rho(x') dx' = R^{-2} \|x' f\|_{L^2(\mathbf{R}^d)}^2.$$

For the first integral we estimate (this step is analogous to Lemma 4.1):

$$\int_{|x'| < R} \rho(x') dx' \leq C_k R^{k\sigma_2/(\sigma_2+1)} \left( \int_{|x'| < R} |\rho(x')|^{\sigma_2+1} dx' \right)^{1/(\sigma_2+1)}.$$

In view of Minkowski's inequality, we have

$$\left( \int_{|x'| < R} |\rho(x')|^{\sigma_2+1} dx' \right)^{1/(\sigma_2+1)} \leq \int_{|x'| < R} \left( \int_{\mathbf{R}^{d-k}} |f(x', x'')|^{2\sigma_2+2} dx'' \right)^{1/(\sigma_2+1)} dx'.$$

Using (the same) Hölder inequality, we infer

$$\int_{|x'| < R} \left( \int_{\mathbf{R}^{d-k}} |f(x', x'')|^{2\sigma_2+2} dx'' \right)^{1/(\sigma_2+1)} dx' \leq C_k R^{k\sigma_2/(\sigma_2+1)} \|f\|_{L^{2\sigma_2+2}(\mathbf{R}^d)}^2,$$

and for all  $R > 0$ ,

$$\|f\|_{L^2(\mathbf{R}^d)}^2 \leq C_k R^{2k\sigma_2/(\sigma_2+1)} \|f\|_{L^{2\sigma_2+2}(\mathbf{R}^d)}^2 + R^{-2} \|x' f\|_{L^2(\mathbf{R}^d)}^2.$$

Optimizing in  $R$  yields the result.  $\square$

In view of Section 3 and of the proof of Proposition 4.2, we readily infer:

**Corollary 4.5** (Asymptotic extinction II). *Let  $d \geq 2$ ,  $1 \leq k \leq d-1$ ,  $a > 0$ ,  $\omega_1, \dots, \omega_k > 0$ ,  $\omega_j = 0$  for  $j \geq k+1$ , and  $u_0 \in \Sigma$ . In either of the cases mentioned in Theorem 1.1, the solution to (1.1) satisfies  $u \in L^\infty(\mathbf{R}_+; H^1(\mathbf{R}^d))$ ,  $x'u \in L^\infty(\mathbf{R}_+; L^2(\mathbf{R}^d))$ , and there exists  $C > 0$  such that*

$$\|u(t)\|_{L^2}^2 \leq Ct^{-\frac{2}{(k+1)\sigma_2}}, \quad \forall t \geq 1.$$

*Remark 4.6* (Comparison with the fully confining case). The decay rate is always weaker than in Proposition 4.2. Technically, this is due to the fact that in the proof of Lemma 4.4, we have used Minkowski's inequality and an extra Hölder inequality, so we have lost more information than in the proof of Lemma 4.1. Note however that none of the rates in Proposition 4.2 or Corollary 4.5 is claimed to be sharp.

We will see in the appendix that in the geometry considered in [16] and [22], the introduction of a nonlinear damping term also leads to asymptotic vanishing of the solution, in the same spirit as the above result.

## APPENDIX A. DAMPED NLS ON A COMPACT MANIFOLD

In the following we shall consider  $x \in M$ , a  $d$ -dimensional compact Riemannian manifold without boundary. A particular example is  $M = \mathbf{T}^d \equiv (\mathbf{R}/2\pi\mathbf{Z})^d$ , the  $d$ -dimensional torus. Since  $M$  is compact by assumption, we do not gain anything from the inclusion of a confining potential  $V$ . We thus set  $\omega_j = 0$ , for all  $j = 1, \dots, d$  and consider the equation

$$(A.1) \quad \begin{cases} i\partial_t u + \frac{1}{2}\Delta u = \lambda|u|^{2\sigma_1}u - ia|u|^{2\sigma_2}u, & t \geq 0, x \in M, \\ u|_{t=0} = u_0 \in H^1(M), \end{cases}$$

where  $\Delta$  denotes the Laplace–Beltrami operator on  $M$ .

The Hamiltonian analogue of (A.2) has been studied in [7]. Obviously, we cannot expect the dispersive nature of the free Schrödinger group  $U(t)$  to hold on a compact manifold (a simple counterexample is given by the eigenfunctions of the Laplace–Beltrami operator on  $M$ ). It turns out that this is true even locally in-time, see Remark 2.6 in [7]. The possibility of obtaining Strichartz type estimates on  $M$  is therefore severely restricted and any proof of a possible bound on  $\|U(t)f\|_{L^q(I;L^r(M))}$  requires totally different techniques from those needed in the case  $M = \mathbf{R}^d$ . In view of this, we shall restrict ourselves to the situation  $d \leq 3$ , only, in which case the following local well-posedness result can be directly deduced from [7] (the case  $d = 1$  is not treated in [7], as it is straightforward using the Sobolev embedding  $H^1(M) \hookrightarrow L^\infty(M)$ , that is, without Strichartz estimates):

**Lemma A.1** (Local well-posedness on compact manifolds). *Let  $\lambda, a \in \mathbf{R}$ , and assume that it holds*

- (1)  $\sigma_1, \sigma_2 > 0$ , if  $d \leq 2$ , and
- (2)  $0 < \sigma_1, \sigma_2 \leq 1$ , if  $d = 3$ .

*Then, for any  $u_0 \in H^1(M)$ , there exists a time  $T > 0$  depending only on  $\|u_0\|_{H^1(M)}$  and a unique solution  $u \in C([0, T]; H^1(M))$  of (A.2).*

*Remark A.2.* In the particular case  $M = \mathbf{T}^d$  several additional results are available. For example, in  $d = 1$  local well-posedness in  $L^2$  holds for nonlinearities which are smaller than quintic [6]. In higher dimensions, though, the situation seems to be more involved (at least if one seeks for strong solutions in the energy space).

The proofs given in the case of  $\mathbf{R}^d$  readily yield the following result:

**Lemma A.3** (A-priori estimates). *Let  $d \leq 3$ ,  $\lambda, a \in \mathbf{R}$ ,  $\sigma_1, \sigma_2 > 0$ , with  $\sigma_1, \sigma_2 \leq 1$  if  $d = 3$ , and  $u_0 \in H^1(M)$ . On any time intervals  $I \ni 0$  such that  $u \in C(I; H^1(M))$ , we have:*

- (1) *Mass dissipation:*

$$\frac{d}{dt} \|u(t)\|_{L^2}^2 + 2a \|u(t)\|_{L^{2\sigma_2+2}}^{2\sigma_2+2} = 0, \quad \forall t \in I.$$

- (2) *Control of the modified energy: for  $0 < \kappa < \frac{a}{\sigma_2^2 + \sigma_2}$ , let*

$$E(t) = \frac{1}{2} \|\nabla u(t)\|_{L^2}^2 + \frac{\lambda}{\sigma_1 + 1} \int_M |u(t, x)|^{2\sigma_1+2} dx + \kappa \int_M |u(t, x)|^{2\sigma_2+2} dx.$$

*If either  $\lambda \geq 0$  or  $\lambda < 0$  and  $\sigma_2 > \sigma_1$ , there exists a  $C = C(\|u_0\|_{L^2}) \geq 0$  such that:*

$$E(t) \leq E(0) + C(\|u_0\|_{L^2}), \quad \forall t \in I.$$

**Proposition A.4** (Decay rate on compact manifolds). *Let  $d \leq 3$ ,  $\lambda, a \in \mathbf{R}$ ,  $\sigma_1, \sigma_2 > 0$ , with  $\sigma_1, \sigma_2 \leq 1$  if  $d = 3$ , and  $u_0 \in H^1(M)$ . If either  $\lambda \geq 0$  or  $\lambda < 0$  and  $\sigma_2 > \sigma_1$ , (A.2) possesses a unique global solution (in the future),  $u \in C(\mathbf{R}_+; H^1(M))$ . In addition, we have*

$$\|u(t)\|_{L^2}^2 \leq \frac{1}{(2at)^{1/\sigma_2} |M|}, \quad \forall t > 0,$$

and thus  $u$  vanishes asymptotically as  $t \rightarrow +\infty$ .

*Proof.* Global existence for positive time stems directly from Lemma A.1 and Lemma A.3. Since  $M$  is compact, Hölder inequality yields

$$\|u\|_{L^2(M)} \leq |M|^{\sigma_2/(2\sigma_2+2)} \|u\|_{L^{2\sigma_2+2}(M)}.$$

We infer from the equation of mass dissipation, to get

$$\frac{d}{dt} \|u(t)\|_{L^2}^2 + 2a|M|^{\sigma_2} \|u(t)\|_{L^{2\sigma_2+2}}^{2\sigma_2+2} \leq 0.$$

The ODE mechanism sketched in Section 4 yields

$$\|u(t)\|_{L^2}^2 \leq \frac{\|u_0\|_{L^2}^2}{(1 + 2a|M|^{\sigma_2} t \|u_0\|_{L^{2\sigma_2+2}}^{2\sigma_2+2})^{1/\sigma_2}},$$

hence the result.  $\square$

The rate established above is the one indicated by the naive ODE argument (4.1). Obviously, the decay on  $M$  is faster than in the case of partial, or full confinement, but as underlined above, none of these rates is claimed to be sharp.

*Remark A.5.* The case of compact manifolds corresponds to the one found in numerical simulations (for example,  $M = \mathbf{T}^d$  in the case of spectral methods), so the rate of Proposition A.4 should be (at least) the one observed numerically.

To conclude this paper, we examine the case shortly presented in Section 4.3, namely the case where the spatial domain is of the form  $M^k \times \mathbf{R}^{d-k}$ , with  $k \geq 1$ .

**Proposition A.6.** *Let  $k, n \geq 1$ , and  $M$  a compact manifold of dimension  $k$ . Let  $a > 0$ ,  $\lambda \in \mathbf{R}$ ,  $\sigma_1, \sigma_2 > 0$   $u_0 \in \Sigma$ , and suppose that the following Cauchy problem*

$$(A.2) \quad \begin{cases} i\partial_t u + \frac{1}{2} \Delta_{M \times \mathbf{R}^n} u = \lambda |u|^{2\sigma_1} u - ia |u|^{2\sigma_2} u, & t \geq 0, x \in M \times \mathbf{R}^n, \\ u|_{t=0} = u_0, \end{cases}$$

has a solution  $u \in C(\mathbf{R}_+; L^2)$ , satisfying

$$\frac{d}{dt} \|u(t)\|_{L^2}^2 + 2a \|u(t)\|_{L^{2\sigma_2+2}}^{2\sigma_2+2} = 0, \quad \forall t \geq 0.$$

Then  $u$  vanishes asymptotically as  $t \rightarrow +\infty$ :

$$\|u(t)\|_{L^2}^2 \leq \frac{1}{(2at)^{1/\sigma_2} |M|^2}, \quad \forall t > 0.$$

*Proof.* It suffices to notice that Lemma 4.4 is readily adapted to the present framework. Writing  $x = (x', x'') \in M \times \mathbf{R}^n$ , and let

$$\rho(x') = \int_{\mathbf{R}^n} |f(x', x'')|^2 dx''.$$

Hölder inequality yields

$$\int_M \rho(x') dx' \leq |M|^{\sigma_2/(\sigma_2+1)} \left( \int_M |\rho(x')|^{\sigma_2+1} dx' \right)^{1/(\sigma_2+1)}.$$

In view of Minkowski's inequality, we have

$$\left( \int_M |\rho(x')|^{\sigma_2+1} dx' \right)^{1/(\sigma_2+1)} \leq \int_M \left( \int_{\mathbf{R}^n} |f(x', x'')|^{2\sigma_2+2} dx'' \right)^{1/(\sigma_2+1)} dx'.$$

Using (the same) Hölder inequality, we infer

$$\int_M \left( \int_{\mathbf{R}^n} |f(x', x'')|^{2\sigma_2+2} dx'' \right)^{1/(\sigma_2+1)} dx' \leq |M|^{\sigma_2/(\sigma_2+1)} \|f\|_{L^{2\sigma_2+2}(M \times \mathbf{R}^n)}^2,$$

hence

$$\|f\|_{L^2(M \times \mathbf{R}^n)}^2 \leq |M|^{2\sigma_2/(\sigma_2+1)} \|f\|_{L^{2\sigma_2+2}(M \times \mathbf{R}^n)}^2.$$

We can then resume the proof of Proposition A.4, with  $|M|$  replaced by  $|M|^2$ .  $\square$

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