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INVARIANT HILBERT SCHEMES AND DESINGULARIZATIONS OF SYMPLECTIC REDUCTIONS FOR CLASSICAL GROUPS

RONAN TERPEREAU

ABSTRACT. Let $G \subset GL(V)$ be a reductive algebraic subgroup acting on the symplectic vector space $W = V^{\oplus n'} \oplus V^{*\oplus n'}$, and $\mu : W \rightarrow Lie(G)^*$ the corresponding moment map. In this article, we use the theory of invariant Hilbert schemes to construct canonical desingularizations of the irreducible components of the symplectic reduction $\mu^{-1}(0)//G$ for classes of examples where $G = GL(V)$, $O(V)$, or $Sp(V)$. Such components are isomorphic to closures of nilpotent orbits in simple Lie algebras and, for these classes of examples, we determine all the cases where the Hilbert-Chow morphism is a symplectic desingularization.

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1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

We work over the field of complex numbers \mathbb{C} . Let V and V' be two vector spaces of dimensions n and n' respectively, and let

$$(1) \quad W := \text{Hom}(V', V) \times \text{Hom}(V, V'),$$

which is naturally equipped with an action of $GL(V') \times GL(V)$:

$$\forall (g', g) \in GL(V') \times GL(V), \forall (u_1, u_2) \in W, (g', g) \cdot (u_1, u_2) := (g \circ u_1 \circ g'^{-1}, g' \circ u_2 \circ g^{-1}).$$

The $GL(V)$ -module W is *symplectic*, that is, equipped with a $GL(V)$ -invariant non-degenerate skew-symmetric bilinear form Ω defined by:

$$(2) \quad \forall (u_1, u_2), (u'_1, u'_2) \in W, \Omega((u_1, u_2), (u'_1, u'_2)) := \text{tr}(u'_1 \circ u_2) - \text{tr}(u_1 \circ u'_2),$$

where $\text{tr}(\cdot)$ denotes the trace. The symplectic $GL(V)$ -module W has a $GL(V)$ -equivariant *moment map* defined by:

$$(3) \quad \mu_{GL(V)} : \begin{array}{ccc} W & \rightarrow & \mathfrak{gl}(V)^* \\ (u_1, u_2) & \mapsto & (h \mapsto \text{tr}(u_2 \circ h \circ u_1)) \end{array}$$

where $\mathfrak{gl}(V)$ denotes the Lie algebra of $GL(V)$. If $G \subset GL(V)$ is any algebraic subgroup, and \mathfrak{g} is the Lie algebra of G , then the moment map $\mu_G : W \rightarrow \mathfrak{g}^*$ for the action of G on W is obtained by composing $\mu_{GL(V)}$ with the linear projection

$\mathfrak{gl}(V)^* \rightarrow \mathfrak{g}^*$. To simplify the notation, we will denote μ instead of μ_G . The map μ is G -equivariant, and thus the set-theoretic fiber $\mu^{-1}(0)$ is a G -stable closed subset of W . From now on, we suppose that G is reductive, and we denote $W//G$ the *symplectic reduction* of W by G , that is, the categorical quotient $\mu^{-1}(0)//G := \text{Spec}(\mathbb{C}[\mu^{-1}(0)]^G)$. Let us mention that if G' is an algebraic subgroup of the G -equivariant automorphism group $\text{Aut}^G(\mu^{-1}(0))$, then G' acts on $W//G$ and the quotient morphism $\mu^{-1}(0) \rightarrow W//G$ is G' -equivariant.

The aim of this article is to construct canonical desingularizations of the irreducible components of $W//G$ for classes of examples where $G = GL(V)$, $O(V)$, or $Sp(V)$. The symplectic reduction $W//G$ for $G = Sp(V)$ or $O(V)$ was considered by Becker ([Bec09]), and the case $G = GL(V)$ will be studied in Section 3.1. The symplectic reduction $W//G$ is

- (1) the closure of a nilpotent orbit in the Lie algebra of $G' = GL(V')$ resp. $Sp(V' \oplus V'^*)$, resp. $SO(V' \oplus V'^*)$, if $G = GL(V)$ resp. $O(V)$, resp. $Sp(V)$ with $(n' > n$ or n' odd);
- (2) the union of two (isomorphic) closures of nilpotent orbits in the Lie algebra of $G' = SO(V' \oplus V'^*)$ if $G = Sp(V)$ with $(n' \leq n$ and n' even).

The geometry of nilpotent orbits has been extensively studied ([KP79, KP81, KP82, Fu03a, Fu03b, Fu06b, Nam06]). In particular, the normalizations of such closures are *symplectic varieties* (as defined by Beauville in [Bea00]) whose symplectic desingularizations are the so-called *Springer desingularizations*, obtained by collapsing the cotangent bundle over some flag varieties (see Section 2 for details).

Before stating our results, we need to recall briefly the definition of the *invariant Hilbert scheme*, constructed by Alexeev and Brion (see [AB05, Bri] for more details), in the particular case we are interested. Let $G \subset GL(V)$ be a reductive algebraic subgroup, $\text{Irr}(G)$ the set of isomorphism classes of irreducible representations of G , and $h : \text{Irr}(G) \rightarrow \mathbb{N}$ a *Hilbert function*. If $X \subset W$ a G -stable closed subscheme, then the invariant Hilbert scheme $\text{Hilb}_h^G(X)$ is the moduli space that parametrizes the G -stable closed subschemes Z of X such that

$$\mathbb{C}[Z] \cong \bigoplus_{M \in \text{Irr}(G)} M^{\oplus h(M)}$$

as a G -module. We now suppose that $X//G$ is a variety. If $h = h_0$ is the Hilbert function of the general fibers of the quotient morphism $\nu : X \rightarrow X//G$ (that is, the fibers over a nonempty open subset of $X//G$), then there exists a projective morphism

$$\gamma : \text{Hilb}_{h_0}^G(X) \rightarrow X//G,$$

called the *Hilbert-Chow morphism*, that sends a closed subscheme $Z \subset X$ to the point $Z//G \subset X//G$. If $G' \subset \text{Aut}^G(X)$ is any algebraic subgroup, then it is known that G' acts on $X//G$ and on $\text{Hilb}_{h_0}^G(X)$ such that ν and γ are G' -equivariant. The Hilbert-Chow morphism induces an isomorphism over the flat locus $U \subset X//G$ of ν . The *main component* of $\text{Hilb}_{h_0}^G(X)$ is the irreducible component defined by

$$\text{Hilb}_{h_0}^G(X)^{\text{main}} := \overline{\gamma^{-1}(U)}.$$

Then the restriction $\gamma : \text{Hilb}_{h_0}^G(X)^{\text{main}} \rightarrow X//G$ is a projective birational morphism, and thus γ is a candidate for a canonical desingularization of $X//G$. It is an open problem to determine whether this restriction is always a desingularization or not.

In [Terb, Tera], we studied the invariant Hilbert scheme for classical groups acting on classical representations. We obtained classes of examples where the Hilbert-Chow morphism is a desingularization of the categorical quotient, and further examples where it is not. In this paper, we use the results of [Tera] to prove the following:

Theorem A. [Sections 3.3 and 4.3] *With the above notation, let $G = GL(V)$, $O(V)$ or $Sp(V)$, and X denote $\mu^{-1}(0)$ in Case (1) resp. the preimage of one of the two irreducible components of $W//G$ by the quotient morphism $\mu^{-1}(0) \rightarrow W//G$ in Case (2). Then the Hilbert-Chow morphism $\gamma : \text{Hilb}_{h_0}^G(X)^{\text{main}} \rightarrow X//G$ is a Springer desingularization (and the unique one) if and only if*

- $G = GL(V)$, $n \geq n' - 1$, and n' is even; or
- $G = O(V)$, and $n \geq 2n' - 1$; or
- $G = Sp(V)$, n and n' are even, and $n \geq 2n' - 2$.

Moreover, γ is still a desingularization (that strictly dominates the Springer desingularizations when they exist) if

- $G = GL(V)$ and $((n = 1$ and $n' \geq 3)$ or $(n = 2$ and $n' \geq 4))$; or
- $G = O(V)$ and $(n = 1 < n'$ or $n = 2 \leq n')$; or
- $G = Sp(V)$ and $(n = 2 < n'$ or $n = 4 \leq n')$.

Let us mention that the case $G = Sp(V)$ with $n = 2$ and $n' = 3$ was already handled by Becker in [Bec11]. In this case, $W//G$ is a closure of a nilpotent orbit that admits two Springer desingularizations, and Becker showed that $\gamma : \text{Hilb}_{h_0}^G(\mu^{-1}(0)) \rightarrow W//G$ is a desingularization that dominates them both. To obtain this result, Becker first used the existence of natural morphisms from the invariant Hilbert scheme to Grassmannians to identify $\text{Hilb}_{h_0}^G(\mu^{-1}(0))^{\text{main}}$ with the total space of a homogeneous line bundle over a Grassmannian, and then she showed that $\text{Hilb}_{h_0}^G(\mu^{-1}(0)) = \text{Hilb}_{h_0}^G(\mu^{-1}(0))^{\text{main}}$ by computing the tangent space of $\text{Hilb}_{h_0}^G(\mu^{-1}(0))$ at every point of the main component. In the present article, our most important theoretical result is a *reduction principle* that generalizes the first step of Becker's method.

Reduction Principle. [Propositions 3.8 and 4.8] *We take the notation of Theorem A, and we make the extra assumption that $n' \geq 2n$ or $(n' < 2n$ and n' is even) when $G = GL(V)$, and $n' \geq n$ or $(n' < n$ and n', n are even) when $G = Sp(V)$.*

Then the main component $\text{Hilb}_{h_0}^G(X)^{\text{main}}$ is the total space of a homogeneous fiber bundle over a flag variety whose fiber is isomorphic to $\text{Hilb}_{h_{W'}}^G(W')^{\text{main}}$, where $W' = V^{\oplus N} \oplus V^{\oplus N}$ with $N := h_0(V^*)$, and $h_{W'}$ is the Hilbert function of the general fibers of the quotient morphism $W' \rightarrow W'//G$.*

A similar reduction principle (but in a different setting) was already obtained in [Tera]. Once we have performed the reduction step, we use [Tera, §1, Theorem] to identify $\text{Hilb}_{h_{W'}}^G(W')^{\text{main}}$. The last step to prove Theorem A is then to compare the Hilbert-Chow morphism $\gamma : \text{Hilb}_{h_0}^G(X)^{\text{main}} \rightarrow X//G$ with the Springer desingularizations of $X//G$. We thus have a general method to determine $\text{Hilb}_{h_0}^G(X)^{\text{main}}$ that will certainly be helpful to treat further examples when $G = GL(V)$, $O(V)$ or $Sp(V)$. In addition, the author thinks that the same method can be used when $G = SL(V)$. However, when $G = SO(V)$, we do not know how to obtain a reduction principle as above.

Let now $G \subset GL(V)$ be any reductive algebraic subgroup, $X \subset \mu^{-1}(0)$ a maximal G -stable closed subset such that $X//G$ is an irreducible component of $W///G$, and h_0 the Hilbert function of the general fibers of the quotient morphism $X \rightarrow X//G$. Then it is generally a difficult problem to determine whether $\text{Hilb}_{h_0}^G(X)$ is irreducible, that is, equals its main component. In this direction, we obtain

Proposition B. [Propositions 3.18 and 4.10] *With the notation of Theorem A, if $G = GL(V)$ and $n' \geq 2n$, then the invariant Hilbert scheme $\text{Hilb}_{h_0}^G(X)$ has at least two irreducible components (and exactly two when $n = 1$). On the other hand, if $G = O(V)$ or $Sp(V)$, and $n' \geq n = 2$, then $\text{Hilb}_{h_0}^G(X)$ is irreducible.*

In Section 2, we recall some basic facts about symplectic varieties and closures of nilpotent orbits in simple Lie algebras. The case of $GL(V)$ is treated in Section 3, and the case of $Sp(V)$ is treated in Section 4. The case of $O(V)$ is quite similar to the case of $GL(V)$, and details can be found in the thesis [Terb, §3.4] from which this article is extracted.

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2. GENERALITIES ON SYMPLECTIC VARIETIES AND CLOSURES OF NILPOTENT ORBITS

2.1. Symplectic varieties and symplectic desingularizations. Let us first recall the definitions of symplectic variety and symplectic desingularization (see [Bea00] or the survey [Fu06a] for more details). Let Y be a normal variety whose regular locus Y_{reg} admits a *symplectic form* Ω (that is, Ω is a holomorphic 2-form which is closed and non-degenerate at every point of Y_{reg}) such that, for any desingularization $f : \tilde{Y} \rightarrow Y$, the 2-form $f^*(\Omega)$ extends to a 2-form on the whole \tilde{Y} , then we say that Y is a *symplectic variety*. Moreover, if $f : \tilde{Y} \rightarrow Y$ is a desingularization such that $f^*(\Omega)$ extends to a symplectic form on \tilde{Y} , then we say that f is a *symplectic desingularization* of Y . It must be emphasized that symplectic varieties do not always admit symplectic desingularizations, and when they do, there may be several of them.

Let $W = \text{Hom}(V', V) \times \text{Hom}(V, V')$, and $G \subset GL(V)$ a reductive algebraic subgroup. We consider the symplectic reduction $W///G$ as in the introduction. The following conjecture motivates the study (and the name!) of $W///G$:

Conjecture 2.1 (Kaledin, Lehn, Sorger). With the above notation, the irreducible components Y_1, \dots, Y_r of $W///G$ are symplectic varieties. Moreover, if every Y_i admits a symplectic desingularization, then the quotient $\text{Hom}(V', V)//G$ is smooth.

When G is a finite group, Conjecture 2.1 was proved by Kaledin and Verbitsky, but the general case remains open. Let us mention that Becker showed in [Bec09] that the converse of the second part of Conjecture 2.1 holds for $G = Sp(V)$ with $n = 2$. In our setting, that is when $G = GL(V)$, $O(V)$ or $Sp(V)$, one easily checks that Conjecture 2.1 holds (see [Terb, §A.2] for details).

2.2. Closures of nilpotent orbits. We now recall some basic facts concerning the closures of nilpotent orbits (see [CM93, Fu03a] for more details). Let \mathfrak{g}' be a semi-simple Lie algebra, and G' the adjoint group of \mathfrak{g}' . The following well-known result is due to Kostant, Kirillov, Souriau, and Panyushev:

Theorem 2.2. *The normalization of the closure of an adjoint orbit in a semi-simple Lie algebra is a symplectic variety.*

When \mathfrak{g}' is a simple Lie algebra of classical type, one can associate partitions to nilpotent orbits. In particular:

- If $\mathfrak{g}' = \mathfrak{sl}_{n'}$, then there exists a one-to-one correspondence between the partitions $\mathbf{d} = (d_1 \geq \dots \geq d_k)$ of n' and the nilpotent orbits $\mathcal{O}_{[d_1, \dots, d_k]}$ of \mathfrak{g}' ([CM93, §3.1]).
- If $\mathfrak{g}' = \mathfrak{sp}_{2n'}$, then there exists a one-to-one correspondence between the partitions $\mathbf{d} = (d_1 \geq \dots \geq d_k)$ of $2n'$, in which every odd d_i occurs with even multiplicity, and the nilpotent orbits $\mathcal{O}_{[d_1, \dots, d_k]}$ of \mathfrak{g}' ([CM93, §5.1]).
- If $\mathfrak{g}' = \mathfrak{so}_{2n'}$, then there exists a correspondence (which is not bijective anymore) between some partitions of $2n'$ and the nilpotent orbits of \mathfrak{g}' ([CM93, §5.1]). Let $\mathbf{d} = (d_1 \geq \dots \geq d_k)$ be a partition of $2n'$ such that each even d_i occurs with even multiplicity. If at least one of the d_i is odd, then one associates to \mathbf{d} a unique nilpotent orbit $\mathcal{O}_{[d_1, \dots, d_k]}$; else, one associates to \mathbf{d} two distinct nilpotent orbits $\mathcal{O}_{[d_1, \dots, d_k]}^I$ and $\mathcal{O}_{[d_1, \dots, d_k]}^{II}$.

If $\mathfrak{g}' = \mathfrak{sl}_{n'}$, then $\overline{\mathcal{O}_{\mathbf{d}}}$ is always normal ([KP79]). For the other classical types, the geometry of $\overline{\mathcal{O}_{\mathbf{d}}}$ was studied in [KP82]; in particular, if $\mathfrak{g}' = \mathfrak{sp}_{2n'}$ and $d_1 + d_2 \leq 4$ resp. if $\mathfrak{g}' = \mathfrak{so}_{2n'}$ and $d_1 \leq 2$, then $\overline{\mathcal{O}_{\mathbf{d}}}$ is normal. From now on, we only consider partitions \mathbf{d} such that each $d_i \leq 2$. By Theorem 2.2, the variety $\overline{\mathcal{O}_{\mathbf{d}}}$ is symplectic, and we are going to describe its symplectic desingularizations (see [Fu03a, Fu06b, FN04] for details).

Let $f: Z \rightarrow \overline{\mathcal{O}_{\mathbf{d}}}$ be a symplectic desingularization. Then, by [Fu03a, Proposition 3.1], the group G' acts naturally on Z in such a way that f is G' -equivariant. One says that f is a *Springer desingularization* if there exists a parabolic subgroup $P \subset G'$ and a G' -equivariant isomorphism between Z and the total space of the cotangent bundle over G'/P , denoted $\mathcal{T}^*(G'/P)$. Then, under this isomorphism, the map f becomes

$$\mathcal{T}^*(G'/P) \cong G' \times^P \mathfrak{u} \longrightarrow \mathfrak{g}', \quad (g', x) \longmapsto Ad(g').x,$$

where \mathfrak{u} is the nilradical of the Lie algebra of P , and $G' \times^P \mathfrak{u} := (G' \times \mathfrak{u}) / \equiv$ with $(g', u) \equiv (g' \circ p^{-1}, Ad(p).u)$.

Theorem 2.3. ([Fu03a, Theorem 3.3]) *With the above notation, if $f: Z \rightarrow \overline{\mathcal{O}_{\mathbf{d}}}$ is a symplectic desingularization, then f is a Springer desingularization.*

When \mathfrak{g}' is of classical type, the Springer desingularizations of $\overline{\mathcal{O}_{\mathbf{d}}}$ are known (up to isomorphism). In particular:

- Let $\mathfrak{g}' = \mathfrak{sl}_{n'}$ and $\mathbf{d} = [2^N, 1^{n'-2N}]$. We denote $\text{Gr}(p, \mathbb{C}^{n'})$ the Grassmannian of p -dimensional subspaces of $\mathbb{C}^{n'}$, and \mathcal{T}_1^* resp. \mathcal{T}_2^* , the cotangent bundle over $\text{Gr}(N, \mathbb{C}^{n'})$ resp. over $\text{Gr}(n' - N, \mathbb{C}^{n'})$. By [Fu06b, §2], if $N < \frac{n'}{2}$, then \mathcal{T}_1^* and \mathcal{T}_2^* are the two Springer desingularizations of $\overline{\mathcal{O}_{\mathbf{d}}}$; else, $\mathcal{T}_1^* = \mathcal{T}_2^*$ is the unique Springer desingularization of $\overline{\mathcal{O}_{\mathbf{d}}}$.

- Let $\mathfrak{g}' = \mathfrak{sp}_{2n'}$ and $\mathbf{d} = [2^N, 1^{2(n'-N)}]$. Then $\overline{\mathcal{O}_{\mathbf{d}}}$ admits a Springer desingularization if and only if $N = n'$ ([Fu03a, Proposition 3.19]). We denote $\mathrm{IGr}(p, \mathbb{C}^{2n'})$ the Grassmannian of isotropic p -dimensional subspaces of $\mathbb{C}^{2n'}$, and \mathcal{T}^* the cotangent bundle over $\mathrm{IGr}(n', \mathbb{C}^{2n'})$. By [FN04, Proposition 3.5], if $N = n'$, then \mathcal{T}^* is the unique Springer desingularization of $\overline{\mathcal{O}_{\mathbf{d}}}$.
- Let $\mathfrak{g}' = \mathfrak{so}_{2n'}$ and $\mathbf{d} = [2^N, 1^{2(n'-N)}]$ (with N even). We recall that if $d_1 = \dots = d_k = 2$, then one associates to \mathbf{d} two distinct nilpotent orbits $\mathcal{O}_{\mathbf{d}}^I$ and $\mathcal{O}_{\mathbf{d}}^{II}$. By [Fu03a, Proposition 3.20], the variety $\overline{\mathcal{O}_{\mathbf{d}}}$ admits a Springer desingularization if and only if $N \in \{n' - 1, n'\}$. We denote $\mathrm{OGr}(p, \mathbb{C}^{2n'})$ the Grassmannian of isotropic p -dimensional subspaces of $\mathbb{C}^{2n'}$. The Grassmannian $\mathrm{OGr}(p, \mathbb{C}^{2n'})$ is irreducible except if $p = n'$, in which case $\mathrm{OGr}(n', \mathbb{C}^{2n'}) = \mathrm{OG}^I \cup \mathrm{OG}^{II}$ is the union of two irreducible components (exchanged by the natural action of the orthogonal group $O(\mathbb{C}^{2n'})$). We denote \mathcal{T}_I^* resp. \mathcal{T}_{II}^* , the cotangent bundle over OG^I , resp. over OG^{II} . If $N = n' - 1$, then \mathcal{T}_I^* and \mathcal{T}_{II}^* are the two Springer desingularizations of $\overline{\mathcal{O}_{[2^{n'-1}, 1^2]}}$ by [Fu06b, §2]. If $N = n'$, then \mathcal{T}_I^* resp. \mathcal{T}_{II}^* , is the unique Springer desingularization of $\overline{\mathcal{O}_{[2^{n'}]}}^I$ resp. of $\overline{\mathcal{O}_{[2^{n'}]}}^{II}$, by [FN04, Proposition 3.5].

Remark 2.4. We will see in Sections 3 and 4 that the irreducible components of the symplectic reduction $W///G$, when $G = GL(V)$ resp. $G = Sp(V)$, are closures of nilpotent orbits $\mathcal{O}_{\mathbf{d}}$ in the Lie algebra of $G' = GL(V')$ resp. of $G' = SO(V' \oplus V'^*)$, where every $d_i \leq 2$. If $G = O(V)$, then it is shown in [Terb, §3.4] that $W///G$ is the closure of the nilpotent orbit $\mathcal{O}_{[2^N, 1^{2(n'-N)}]}$ in the Lie algebra of $G' = Sp(V' \oplus V'^*)$, where $N := \min(n', n)$.

3. CASE OF GL_n

In this section, we take $G = GL(V)$ and $G' = GL(V')$, both acting on $W = \mathrm{Hom}(V', V) \times \mathrm{Hom}(V, V')$ as in the introduction. We denote \mathfrak{g} resp. \mathfrak{g}' , the Lie algebra of G resp. of G' , and $N := \min(E(\frac{n'}{2}), n)$, where $n := \dim(V)$, $n' := \dim(V')$, and $E(\cdot)$ is the lower integer part.

3.1. The quotient morphism. The two main results of this section are Proposition 3.3, which describes the symplectic reduction $W///G := \mu^{-1}(0)//G$, and Corollary 3.6, which gives the Hilbert function h_0 of the general fibers of the quotient morphism $\nu: \mu^{-1}(0) \rightarrow W///G$.

By (3), the zero fiber of the moment map $\mu: W \rightarrow \mathfrak{g}^*$ is

$$\mu^{-1}(0) = \{(u_1, u_2) \in W \mid u_1 \circ u_2 = 0\},$$

which is a $G' \times G$ -stable closed subset of W . Let us determine the irreducible components of $\mu^{-1}(0)$ as well as their dimensions. Let $m \in \{0, \dots, n'\}$; we define the closed subset

$$(4) \quad X_m := \left\{ (u_1, u_2) \in W \mid \begin{array}{l} \mathrm{Im}(u_2) \subset \mathrm{Ker}(u_1); \\ \mathrm{rk}(u_2) \leq \min(n, m); \\ \dim(\mathrm{Ker}(u_1)) \geq \max(n' - n, m). \end{array} \right\} \subset \mu^{-1}(0),$$

and we consider the diagram

$$\begin{array}{ccc}
 Z_m := \{(u_1, u_2, L) \in W \times \text{Gr}(m, V') \mid \text{Im}(u_2) \subset L \subset \text{Ker}(u_1)\} & & \\
 p_1 \downarrow & \searrow p_2 & \\
 X_m & & \text{Gr}(m, V')
 \end{array}$$

where the p_i are the natural projections. We fix $L_0 \in \text{Gr}(m, V')$; the second projection equips Z_m with a structure of homogeneous vector bundle over $\text{Gr}(m, V')$ whose fiber over L_0 is isomorphic to $F_m := \text{Hom}(V'/L_0, V) \times \text{Hom}(V, L_0)$. Hence, Z_m is a smooth variety of dimension $m(n' - m) + n'n$.

Proposition 3.1. *The irreducible components of $\mu^{-1}(0)$ are*

$$\begin{cases} X_0, \dots, X_{n'} & \text{if } n' \leq n; \\ X_{n'-n}, \dots, X_n & \text{if } n < n' < 2n; \\ X_n & \text{if } n' \geq 2n; \end{cases}$$

where X_m is defined by (4).

Proof. We have

$$\mu^{-1}(0) = \{(u_1, u_2) \in W \mid \text{Im}(u_2) \subset \text{Ker}(u_1)\} = \bigcup_{i=0}^{n'} X_i.$$

Furthermore, for every $m \in \{0, \dots, n'\}$, the morphism p_1 is surjective and Z_m is irreducible, hence X_m is irreducible.

If $n' \geq 2n$, then

$$\begin{cases} X_0 \subset \dots \subset X_n; \\ X_n = \dots = X_{n'-n}; \\ X_{n'-n} \supset \dots \supset X_{n'}; \end{cases}$$

and thus $\mu^{-1}(0) = X_n$.

If $n' < 2n$, then

$$\begin{cases} X_0 \subset \dots \subset X_{\max(0, n'-n)}; \\ X_{\min(n', n)} \supset \dots \supset X_{n'}; \end{cases}$$

and one easily checks that there is no other inclusion relation between the X_m . \square

Corollary 3.2. *The dimension of $\mu^{-1}(0)$ is*

$$\dim(\mu^{-1}(0)) = \begin{cases} nn' + \frac{1}{4}n'^2 & \text{if } n' < 2n \text{ and } n' \text{ is even;} \\ nn' + \frac{1}{4}(n'^2 - 1) & \text{if } n' < 2n \text{ and } n' \text{ is odd;} \\ 2nn' - n^2 & \text{if } n' \geq 2n. \end{cases}$$

Proof. By Proposition 3.1, it suffices to compute the dimension of X_m for some m . If $m \leq n$ or $m \geq n' - n$, one may check that the map $p_1 : Z_m \rightarrow X_m$ is birational, and thus $P(m) := \dim(X_m) = \dim(Z_m) = m(n' - m) + n'n$. It remains simply to study the variations of the polynomial P to obtain the result. \square

Let $l \in \{0, \dots, N\}$. We fix a basis \mathcal{B} of V resp. \mathcal{B}' of V' , and we introduce some notation that we will use in the proofs of Proposition 3.3 and Lemma 3.5:

$$(5) \quad \bullet \quad (u_1^l, u_2^l) := \left(\begin{bmatrix} 0_{l, n'-l} & I_l \\ 0_{n-l, n'-l} & 0_{n-l, l} \end{bmatrix}, \begin{bmatrix} I_l & 0_{l, n-l} \\ 0_{n'-l, l} & 0_{n'-l, n-l} \end{bmatrix} \right) \in W;$$

$$(6) \quad \bullet \quad f_l := \begin{bmatrix} 0_{l, n'-l} & I_l \\ 0_{n'-l, n'-l} & 0_{n'-l, l} \end{bmatrix} \in \mathfrak{g}'.$$

Proposition 3.3. *Let $\mathcal{O}_{\mathbf{d}} \subset \mathfrak{g}' \cong \mathfrak{gl}_{n'}$ denote the nilpotent orbit associated to a partition \mathbf{d} of n' . Then the symplectic reduction of W by G is $W///G = \overline{\mathcal{O}_{[2N, 1^{n'-2N}]}}$.*

Proof. If $f \in W///G$, then there exists $(u_1, u_2) \in \mu^{-1}(0)$ such that $f = u_2 \circ u_1$, and thus $f \circ f = (u_2 \circ u_1) \circ (u_2 \circ u_1) = u_2 \circ (u_1 \circ u_2) \circ u_1 = 0$, whence the inclusion " \subset ". Now, let $f \in \overline{\mathcal{O}_{[2N, 1^{n'-2N}]}}$. Up to conjugation by an element of G' , we can suppose that $f = f_l$ for some $l \leq N$, where f_l is defined by (6). But then $u_2^l \circ u_1^l = f_l$ and $u_1^l \circ u_2^l = 0$, where u_1^l and u_2^l are defined by (5), and thus $f \in W///G$. \square

Corollary 3.4. *The symplectic reduction $W///G \subset \mathfrak{g}'$ is irreducible and decomposes into $N+1$ orbits for the adjoint action of G' :*

$$U_i := \mathcal{O}_{[2^i, 1^{n'-2i}]}, \quad \text{for } i = 0, \dots, N.$$

The closures of the nilpotent orbits U_i are nested in the following way:

$$\{0\} = \overline{U_0} \subset \dots \subset \overline{U_N} = W///G.$$

Hence, $W///G$ is a symplectic variety (see Section 2), of dimension $2N(n' - N)$ ([CM93, Corollary 6.1.4]), and whose singular locus is $\overline{U_{N-1}}$ ([KP81, §3.2]).

By Corollary 3.2, the dimension of the general fibers of the quotient morphism ν is

$$(7) \quad \begin{cases} nn' - \frac{1}{4}n'^2 & \text{if } n' < 2n \text{ and } n' \text{ is even;} \\ nn' - \frac{1}{4}(n'^2 - 1) & \text{if } n' < 2n \text{ and } n' \text{ is odd;} \\ n^2 & \text{if } n' \geq 2n. \end{cases}$$

If $n' < 2n$, then $N = E(\frac{n'}{2})$, and we denote

$$(8) \quad H := \left\{ \begin{bmatrix} M & 0_{n-N, N} \\ 0_{N, n-N} & I_N \end{bmatrix}, M \in GL_{n-N} \right\} \cong GL_{n-N},$$

which is a reductive algebraic subgroup of $G \cong GL_n$.

Proposition 3.5. *The general fibers of the quotient morphism $\nu : \mu^{-1}(0) \rightarrow W///G$ are isomorphic to*

$$\begin{cases} G & \text{if } n' \geq 2n; \\ G/H & \text{if } n' < 2n \text{ and } n' \text{ is even;} \end{cases}$$

where $H \subset G$ is the subgroup defined by (8).

Proof. We first suppose that $n' < 2n$ and n' is even (that is, $N = \frac{n'}{2}$). With the notation (5), and by a result of Luna (see [SB00, §1.6.2.5, Theorem 10]), we have the equivalence

$$G.(u_1^N, u_2^N) \text{ is closed in } \mu^{-1}(0) \Leftrightarrow C_G(H).(u_1^N, u_2^N) \text{ is closed in } \mu^{-1}(0).$$

We have $C_G(H) = \left\{ \begin{bmatrix} M & 0 \\ 0 & \lambda I_{n-N} \end{bmatrix}, M \in GL_N, \lambda \in \mathbb{G}_m \right\}$, where \mathbb{G}_m denotes the multiplicative group. Hence

$$C_G(H) \cdot (u_1^N, u_2^N) = \left\{ \left(\begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} M^{-1} & 0 \\ 0 & 0 \end{bmatrix} \right), M \in GL_N \right\} \subset \mu^{-1}(0)$$

is a closed subset, and thus $G \cdot (u_1^N, u_2^N)$ is the unique closed orbit contained in the fiber $\nu^{-1}(f_N)$, where f_N is defined by (6). One may check that $\text{Stab}_G((u_1^N, u_2^N)) = H$. Furthermore, $\dim(G/H) = N(2n - N)$, which is also the dimension of the general fibers of ν by (7), and thus $\nu^{-1}(f_N) \cong G/H$.

We now suppose that $n' \geq 2n$ (that is, $N = n$). One may check that $\text{Stab}_G((u_1^n, u_2^n)) = Id$, and thus the fiber $\nu^{-1}(f_n)$ contains a unique closed orbit isomorphic to G . But $\dim(G) = n^2$ is the dimension of the general fibers of ν by (7), hence $\nu^{-1}(f_n) \cong G$. \square

Corollary 3.6. *The Hilbert function h_0 of the general fibers of the quotient morphism $\nu: \mu^{-1}(0) \rightarrow W//G$ is given by:*

$$\forall M \in \text{Irr}(G), h_0(M) = \begin{cases} \dim(M) & \text{if } n' \geq 2n; \\ \dim(M^H) & \text{if } n' < 2n \text{ and } n' \text{ is even;} \end{cases}$$

where $H \subset G$ is the subgroup defined by (8).

If $n' < 2n$ and n' is odd, then the situation is more complicated (except the case $n' = 1$ which is trivial) because the general fibers of the quotient morphism ν are reducible. From now on, we will only consider the cases where $n' \geq 2n$ or ($n' < 2n$ and n' is even).

3.2. The reduction principle for the main component. In this section we prove the reduction principle when $G = GL(V)$ (Proposition 3.8).

The subset $\mu^{-1}(0) \subset W$ being $G' \times G$ -stable, it follows from [Bri, Lemma 3.3] that the invariant Hilbert scheme

$$\mathcal{H} := \text{Hilb}_{h_0}^G(\mu^{-1}(0))$$

is a G' -stable closed subscheme of $\text{Hilb}_{h_0}^G(W)$. The scheme $\text{Hilb}_{h_0}^G(W)$ was studied in [Tera]; let us recall

Proposition 3.7. ([Tera, §4.4]) *Let h_0 be the Hilbert function given by Corollary 3.6, and $G' = GL(V')$ acting naturally on $\text{Gr}(n' - h_0(V), V'^*) \times \text{Gr}(n' - h_0(V^*), V')$. Then there exists a G' -equivariant morphism*

$$\rho: \text{Hilb}_{h_0}^G(W) \rightarrow \text{Gr}(n' - h_0(V), V'^*) \times \text{Gr}(n' - h_0(V^*), V')$$

given on closed points by $Z \mapsto (\text{Ker}(f_Z^1), \text{Ker}(f_Z^2))$, where $f_Z^1: V'^* \cong \text{Mor}^G(W, V) \rightarrow \text{Mor}^G(Z, V)$ and $f_Z^2: V' \cong \text{Mor}^G(W, V^*) \rightarrow \text{Mor}^G(Z, V^*)$ are the restriction maps.

By Corollary 3.6, we have $h_0(V) = h_0(V^*) = N$. We identify $\text{Gr}(n' - N, V'^*)$ with $\text{Gr}(N, V')$, and we denote

$$A_i := \{(L_1, L_2) \in \text{Gr}(N, V') \times \text{Gr}(n' - N, V') \mid \dim(L_1 \cap L_2) = N - i\}, \text{ for } i = 0, \dots, N.$$

The A_i are the $N + 1$ orbits for the action of G' on $\text{Gr}(N, V') \times \text{Gr}(n' - N, V')$, and

$$A_0 = \overline{A_0} \subset \overline{A_1} \subset \dots \subset \overline{A_N} = \text{Gr}(N, V') \times \text{Gr}(n' - N, V').$$

In particular, A_N is the unique open orbit and

$$(9) \quad A_0 = \mathcal{F}_{N, n'-N} := \{(L_1, L_2) \in \text{Gr}(N, V') \times \text{Gr}(n' - N, V') \mid L_1 \subset L_2\},$$

which is a partial flag variety, is the unique closed orbit. Let

- $a_0 := (L_1, L_2) \in A_0$, and P the parabolic subgroup of G' stabilizing a_0 ;
- $W' := \{(u_1, u_2) \in W \mid L_2 \subset \text{Ker}(u_1) \text{ and } \text{Im}(u_2) \subset L_1\}$, which is a $P \times G'$ -module contained in $\mu^{-1}(0)$;
- $\mathcal{H}' := \text{Hilb}_{h_0}^G(W')$, and $\mathcal{H}'^{\text{main}}$ its main component.

Let us note that, if $n' \geq 2n$ or ($n' < 2n$ and n' is even), then h_0 coincides with the Hilbert function of the general fibers of the quotient morphism $W' \rightarrow W'//G$ by [Tera, Proposition 4.13]; in particular, $\mathcal{H}'^{\text{main}}$ is well-defined. We are going to show

Proposition 3.8. *If $n' \geq 2n$ or ($n' < 2n$ and n' is even), and with the above notation, we have a G' -equivariant isomorphism*

$$\mathcal{H}^{\text{main}} \cong G' \times^P \mathcal{H}'^{\text{main}}.$$

First of all, we need

Lemma 3.9. *If $n' \geq 2n$ or ($n' < 2n$ and n' is even), then the morphism ρ of Proposition 3.7 sends $\mathcal{H}^{\text{main}}$ onto A_0 , the G' -variety defined by (9).*

Proof. As the quotient morphism $\nu : \mu^{-1}(0) \rightarrow W//G$ is flat over the open orbit U_N , the restriction of the Hilbert-Chow morphism γ to $\gamma^{-1}(U_N)$ is an isomorphism. We fix $f_N \in U_N$, and we denote $Q := \text{Stab}_{G'}(f_N)$, and Z_N the unique point of \mathcal{H} such that $\gamma(Z_N) = f_N$. As γ is G' -equivariant, Z_N is Q -stable. In addition, ρ is also G' -equivariant, hence $\rho(Z_N)$ is a fixed point for the action of Q . But one may check that $\text{Gr}(N, V') \times \text{Gr}(n' - N, V')$ has a unique fixed point for Q , which is contained in A_0 . Then, as A_0 is G' -stable, we have $\rho(Z) \in A_0$, for every $Z \in \gamma^{-1}(U_N)$. Hence, $\rho^{-1}(A_0)$ is a closed subscheme of $\text{Hilb}_{h_0}^G(W)$ containing $\gamma^{-1}(U_N)$, and the result follows. \square

The restriction $\rho|_{\mathcal{H}^{\text{main}}} : \mathcal{H}^{\text{main}} \rightarrow A_0$ is G' -equivariant, hence $\mathcal{H}^{\text{main}}$ is the total space of a G' -homogeneous fiber bundle over A_0 . Let F be the scheme-theoretic fiber of $\rho|_{\mathcal{H}^{\text{main}}}$ over a_0 . The action of P on $\mathcal{H}^{\text{main}}$, induced by the action of G' , stabilizes F , and we have a G' -equivariant isomorphism

$$(10) \quad \mathcal{H}^{\text{main}} \cong G' \times^P F.$$

Hence, to show Proposition 3.8, we have to determine F as a P -scheme. We start by considering F' , the scheme-theoretic fiber of the restriction $\rho|_{\mathcal{H}} : \mathcal{H} \rightarrow \text{Gr}(N, V') \times \text{Gr}(n' - N, V')$ over a_0 , as a P -scheme. The proof of the next lemma is analogous to the proof of [Tera, Lemma 3.7].

Lemma 3.10. *With the above notation, we have a P -equivariant isomorphism*

$$F' \cong \mathcal{H}',$$

where P acts on \mathcal{H}' via its action on W' .

As $\mathcal{H}^{\text{main}}$ is a variety of dimension $2N(n' - N)$, we deduce from (10) that F is a variety of dimension N^2 . By Lemma 3.10, the fiber F is isomorphic to a subvariety of $\mathcal{H}'^{\text{main}}$, but $\dim(\mathcal{H}'^{\text{main}}) = N^2$, and thus we have a P -equivariant isomorphism

$$(11) \quad F \cong \mathcal{H}'^{\text{main}},$$

and Proposition 3.8 follows.

Remark 3.11. The scheme \mathcal{H}' is P -stable and identifies with a closed subscheme of \mathcal{H} , hence we have an inclusion of G' -schemes $G' \times^P \mathcal{H}' \subset \mathcal{H}$.

3.3. Proof of Theorem A for $GL(V)$. First, we recall the following result:

Theorem 3.12. [Tera, §1, Theorem] *Let $G = GL(V)$, $W = \text{Hom}(V', V) \times \text{Hom}(V, V')$, and h_W the Hilbert function of the general fibers of the quotient morphism $W \rightarrow W//G$. We denote $n := \dim(V)$, $n' := \dim(V')$, and Y_0 the blow-up of $W//G = \text{End}(V')^{\leq n} := \{f \in \text{End}(V') \mid \text{rk}(f) \leq n\}$ at 0. In the following cases, the invariant Hilbert scheme $\mathcal{H}' := \text{Hilb}_{h_W}^G(W)$ is a smooth variety and the Hilbert-Chow morphism is the succession of blow-up described as follows:*

- if $n \geq 2n' - 1$, then $\mathcal{H}' \cong W//G = \text{End}(V')$;
- if $n' > n = 1$ or $n' = n = 2$, then $\mathcal{H}' \cong Y_0$;
- if $n' > n = 2$, then \mathcal{H}' is isomorphic to the blow-up of Y_0 along the strict transform of $\text{End}(V')^{\leq 1}$.

Let us now consider the following diagram

$$(12) \quad \begin{array}{ccc} & \mathcal{F}_{N, n'-N} & \\ p_1 \swarrow & & \searrow p_2 \\ \text{Gr}(N, V') & & \text{Gr}(n' - N, V') \end{array}$$

where $\mathcal{F}_{N, n'-N}$ is defined by (9), p_1 and p_2 being the natural projections. We denote \underline{V}' the constant vector bundle over $\mathcal{F}_{N, n'-N}$ with fiber V' , and T_1 resp. T_2 , the pull-back of the tautological bundle over $\text{Gr}(N, V')$ by p_1 , resp. over $\text{Gr}(n' - N, V')$ by p_2 . In particular, if $N = \frac{n'}{2}$, then $\mathcal{F}_{N, n'-N} = \text{Gr}(N, V')$ and $T := T_1 = T_2$ is the tautological bundle over $\text{Gr}(N, V')$.

We deduce from Proposition 3.8 and Theorem 3.12 that we have the following G' -equivariant isomorphisms

$$(13) \quad \mathcal{H}^{\text{main}} \cong \begin{cases} \text{Hom}(\underline{V}'/T, T) & \text{if } n \geq n' - 1 \text{ and } n' \text{ is even;} \\ \text{Hom}(\underline{V}'/T_2, T_1) & \text{if } n = 1 \text{ and } n' \geq 3; \\ \text{Bl}_0(\text{Hom}(\underline{V}'/T_2, T_1)) & \text{if } n = 2 \text{ and } n' \geq 4; \end{cases}$$

where $\text{Bl}_0(\cdot)$ denotes the blow-up along the zero section. In all these cases, $\mathcal{H}^{\text{main}}$ is smooth, and thus the Hilbert-Chow morphism $\gamma : \mathcal{H}^{\text{main}} \rightarrow W//G$ is a canonical desingularization.

On the other hand, we saw in Section 2 that the Springer desingularizations of $W//G$ are the cotangent bundles $\mathcal{T}_1^* := \mathcal{T}^* \text{Gr}(N, V')$ and $\mathcal{T}_2^* := \mathcal{T}^* \text{Gr}(n' - N, V') \cong \mathcal{T}^* \text{Gr}(N, V'^*)$. We then distinguish between two cases:

- (1) If $N < \frac{n'}{2}$, then let us show by contradiction that $\gamma : \mathcal{H}^{\text{main}} \rightarrow W//G$ cannot be a Springer desingularization. First, we consider the isomorphism of $G' \times G$ -modules $W \cong W^*$. Denoting $\mathcal{H}^* := \text{Hilb}_{h_0}^G(\mu^{*-1}(0))$, where μ^* is the moment map for the natural action of G on W^* , we have an isomorphism of G' -varieties $\mathcal{H}^{\text{main}} \cong \mathcal{H}^{*\text{main}}$. Now if we suppose that (say) $\mathcal{H}^{\text{main}} \cong \mathcal{T}_1^*$, then we get that $\mathcal{H}^{*\text{main}} \cong \mathcal{T}_2^*$, and thus $\mathcal{T}_1^* \cong \mathcal{T}_2^*$ as a G' -variety, which is absurd.

However, one easily checks that if $n \in \{1, 2\}$ and $n' \geq 2n + 1$, then $\gamma : \mathcal{H}^{\text{main}} \rightarrow W//G$ dominates the two Springer desingularizations \mathcal{T}_1^* and \mathcal{T}_2^* (see [Terb, §A.2.2] for details).

- (2) If $N = \frac{n'}{2}$, then $\mathcal{T}^* := \mathcal{T}_1^* = \mathcal{T}_2^*$ is the unique Springer desingularization of $W///G$. Let us show that $\gamma : \mathcal{H}^{\text{main}} \rightarrow W///G$ is the Springer desingularization if and only if $n \geq n' - 1$. The implication " \Leftarrow " is given by (13) since $\mathcal{T}^* \cong \text{Hom}(V'/T, T)$. The other implication is given by:

Lemma 3.13. *If $N = \frac{n'}{2}$ and the Hilbert-Chow morphism $\gamma : \mathcal{H}^{\text{main}} \rightarrow W///G$ is the Springer desingularization, then $n \geq n' - 1$.*

Proof. We suppose that $\gamma : \mathcal{H}^{\text{main}} \rightarrow W///G$ is the Springer desingularization, that is, $\mathcal{H}^{\text{main}} \cong \mathcal{T}^*$ as a G' -variety. We fix $L \in \text{Gr}(N, V')$, and we define $P \subset G'$, W' , and $\mathcal{H}'^{\text{main}}$ as in Section 3.2. We have $\mathcal{T}^* \cong G' \times^P \text{Hom}(V'/L, L)$, and it follows from (10) and (11) that $\mathcal{H}^{\text{main}} \cong G' \times^P \mathcal{H}'^{\text{main}}$. Hence, $\mathcal{H}'^{\text{main}} \cong \text{Hom}(V'/L, L)$ as a P -variety. We denote $\gamma' : \mathcal{H}'^{\text{main}} \rightarrow W'///G$ the restriction of the Hilbert-Chow morphism. As γ' is projective and birational, and $W'///G = \text{Hom}(V'/L, L)$ is smooth, Zariski's Main Theorem implies that γ' is an isomorphism. It follows that the quotient morphism $\nu' : W' \rightarrow W'///G$ is flat, and thus $n \geq 2N - 1$ by [Tera, Corollary 4.12]. \square

In addition, if $n' = 4$ and $n = 2$, then by (13) we have $\mathcal{H}^{\text{main}} \cong \text{Bl}_0(\mathcal{T}^*)$, and thus γ dominates the unique Springer desingularization of $W///G$.

3.4. Reducibility of the invariant Hilbert scheme. The aim of this section is to show Proposition B for $GL(V)$, that is, to show

Proposition 3.14. *If $n' \geq 2n$, then the invariant Hilbert scheme \mathcal{H} is reducible.*

We suppose that $n' \geq 2n$, then $N = n$. We fix

$$(14) \quad a_n = (L'_1, L'_2) \in A_n$$

a point of the open G' -orbit of $\text{Gr}(n, V') \times \text{Gr}(n' - n, V')$, and we consider

$$\begin{aligned} W'' &:= \{(u_1, u_2) \in W \mid L'_2 \subset \text{Ker}(u_1) \text{ and } \text{Im}(u_2) \subset L'_1\} \\ &\cong \text{Hom}(V'/L'_2, V) \times \text{Hom}(V, L'_1), \end{aligned}$$

which is a G -submodule of W . As $V' = L'_1 \oplus L'_2$, we have a natural identification $W'' \cong \text{Hom}(L'_1, V) \times \text{Hom}(V, L'_1)$ as a G -module. We saw at the beginning of Section 1 that the G -module W'' is symplectic and has a G -equivariant moment map that we denote $\mu'' : W'' \rightarrow \mathfrak{g}^*$. The proof of the next lemma is analogous to the proof of [Tera, Lemma 3.7].

Lemma 3.15. *We suppose that $n' \geq 2n$, and let $\rho : \mathcal{H} \rightarrow \text{Gr}(n, V') \times \text{Gr}(n' - n, V')$ be the morphism of Proposition 3.7. The scheme-theoretic fiber F'' of ρ over the point a_n , defined by (14), is isomorphic to the invariant Hilbert scheme $\text{Hilb}_{h_0}^G(\mu''^{-1}(0))$, where h_0 is the Hilbert function defined by $h_0(M) = \dim(M)$, for every $M \in \text{Irr}(G)$, and $\mu'' : W'' \rightarrow \mathfrak{g}^*$ is the moment map defined above.*

Remark 3.16. The Hilbert function h_0 of Lemma 3.15 does not generally coincide with the Hilbert function of the general fibers of the quotient morphism $\mu''^{-1}(0) \rightarrow \mu''^{-1}(0)//G$.

By Lemma 3.9, the morphism $\rho : \text{Hilb}_{h_0}^G(W) \rightarrow \text{Gr}(n, V') \times \text{Gr}(n' - n, V')$ of Proposition 3.7 sends $\mathcal{H}^{\text{main}}$ onto A_0 . Hence, to show Proposition 3.14, it is enough, by Lemma 3.15, to show that $\text{Hilb}_{h_0}^G(\mu''^{-1}(0))$ is non-empty.

We denote $V'' := L'_1$, and we equip $W'' \cong \text{Hom}(V'', V) \times \text{Hom}(V, V'')$ with the natural action of $G'' := GL(V'')$. Then we have

$$(15) \quad \begin{aligned} \mathbb{C}[W'']_2 \cong & (S^2(V'') \otimes S^2(V^*)) \oplus (S^2(V''^*) \otimes S^2(V)) \\ & \oplus (\Lambda^2(V'') \otimes \Lambda^2(V^*)) \oplus (\Lambda^2(V''^*) \otimes \Lambda^2(V)) \\ & \oplus ((sl(V'') \oplus M_0) \otimes (sl(V) \oplus V_0)) \text{ as a } G'' \times G\text{-module,} \end{aligned}$$

where M_0 resp. V_0 , is the trivial G'' -module resp. the trivial G -module, and $sl(V'') := \{f \in \text{End}(V'') \mid \text{tr}(f) = 0\}$.

We denote I_0 the ideal of $\mathbb{C}[W'']$ generated by $(sl(V'') \otimes V_0) \oplus (M_0 \otimes V_0) \oplus (M_0 \otimes sl(V)) \subset \mathbb{C}[W'']_2$. The ideal I_0 is homogeneous, $G'' \times G$ -stable, and contains the ideal generated by the homogeneous G'' -invariants of positive degree of $\mathbb{C}[W'']$. In particular, I_0 identifies with an ideal of $\mathbb{C}[\mu''^{-1}(0)]$.

Proposition 3.17. *Let $I_0 \subset \mathbb{C}[W'']$ be the ideal defined above, then I_0 is a point of the invariant Hilbert scheme $\text{Hilb}_{h_0}^G(\mu''^{-1}(0))$ defined in Lemma 3.15.*

Proof. We have to check that the ideal I_0 has the Hilbert function h_0 , that is,

$$\mathbb{C}[W'']/I_0 \cong \bigoplus_{M \in \text{Irr}(G)} M^{\oplus \dim(M)}$$

as a G -module. To do that, we are going to adapt the method used by Kraft and Schwarz to prove [KS, Theorem 9.1]. The result [loc. cit.] was used in [Terb, §2.1.3 and §3.3.2].

We denote $R := V''^* \otimes V$, which is an irreducible $G'' \times G$ -submodule of $W''^* \cong R \oplus R^*$. Then R and R^* are orthogonal modulo I_0 , which means that the image of the $G'' \times G$ -submodule $R \otimes R^* \subset \mathbb{C}[W'']_2$ in $\mathbb{C}[W'']/I_0$ is isomorphic to the highest weight component of $R \otimes R^*$ (that is, $sl(V'') \otimes sl(V)$). Then, by [Bri85, Lemme 4.1], any irreducible $G'' \times G$ -submodule of $\mathbb{C}[R]$ is orthogonal to any irreducible $G'' \times G$ -submodule of $\mathbb{C}[R^*]$, and thus the natural morphism

$$\phi: \mathbb{C}[R]^{U'' \times U} \otimes \mathbb{C}[R^*]^{U'' \times U} \rightarrow (\mathbb{C}[W'']/I_0)^{U'' \times U}$$

is surjective, where U'' resp. U , denotes the unipotent radical of a Borel subgroup $B'' \subset G''$ resp. $B \subset G$. Furthermore, if $T'' \subset B''$ resp. $T \subset B$, is a maximal torus, then ϕ is $T'' \times T$ -equivariant.

Now by [Pro07, §13.5.1] we have the following isomorphisms of $T'' \times T$ -algebras $\mathbb{C}[R]^{U'' \times U} \cong \mathbb{C}[x_1, \dots, x_n]$, where $x_i \in \Lambda^i V'' \otimes \Lambda^i V^*$ is a highest weight vector, and $\mathbb{C}[R^*]^{U'' \times U} \cong \mathbb{C}[y_1, \dots, y_n]$, where $y_j \in \Lambda^j V''^* \otimes \Lambda^j V$ is a highest weight vector. Hence, we have an exact sequence

$$0 \rightarrow K_0 \rightarrow \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n] \rightarrow (\mathbb{C}[W'']/I_0)^{U'' \times U} \rightarrow 0,$$

where K_0 is the kernel of ϕ . One may check that the ideal K_0 is generated by the products $x_r y_s$ with $r + s > n$ (see [KS, §9, Proof of Theorem 9.1(1)]).

We denote $\Lambda = \langle \epsilon_1, \dots, \epsilon_n \rangle$ the weight lattice of the linear group GL_n with its natural basis, and $\Lambda_+ \subset \Lambda$ the subset of dominant weights, that is, weights of the form $r_1 \epsilon_1 + \dots + r_n \epsilon_n$, with $r_1 \geq \dots \geq r_n$. If $\lambda \in \Lambda_+$, then we denote $S^\lambda(\mathbb{C}^n)$ the irreducible GL_n -module of highest weight λ . We fix $\lambda = k_1 \epsilon_1 + \dots + k_t \epsilon_t - k_{t+1} \epsilon_{t+1} - \dots - k_n \epsilon_n \in \Lambda_+$, where each k_i is a nonnegative integer. One easily checks that the weight of the monomial

$$x_{n-t}^{k_{t+1}} x_{n-t-1}^{k_{t+2}-k_{t+1}} x_{n-t-2}^{k_{t+3}-k_{t+2}} \dots x_1^{k_n-k_{n-1}} y_t^{k_t} y_{t-1}^{k_{t-1}-k_t} y_{t-2}^{k_{t-2}-k_{t-1}} \dots y_1^{k_1-k_2}$$

for the action of $T'' \times T$ is (λ^*, λ) , where λ^* denotes the highest weight of the GL_n -module $S^\lambda(\mathbb{C}^{n^*})$, and that λ uniquely determines this monomial. We get that the isotypic component of the G -module $S^\lambda(V)$ in $\mathbb{C}[W'']/I_0$ is the $G'' \times G$ -module $S^\lambda(V''^*) \otimes S^\lambda(V)$. As $\dim(V) = \dim(V'') = n$, we have $\dim(S^\lambda(V)) = \dim(S^\lambda(V''^*))$ for every $\lambda \in \Lambda^+$. In other words, each irreducible G -module M occurs in $\mathbb{C}[W'']/I_0$ with multiplicity $\dim(M)$. \square

By Proposition 3.17, the scheme $\text{Hilb}_{h_0}^G(\mu'^{-1}(0))$ is non-empty, and thus \mathcal{H} has an irreducible component, different from $\mathcal{H}^{\text{main}}$, of dimension greater or equal to $\dim(A_n) = 2n(n' - n)$, which implies Proposition 3.14.

3.5. Study of the case $n = 1$. We saw in Section 3.3 that $\mathcal{H}^{\text{main}}$ is a smooth variety, and in Section 3.4 that \mathcal{H} is always reducible. In this section, we determine the irreducible components of \mathcal{H} when $n = 1$.

We suppose that $n' \geq 2$ (the case $n' = 1$ being trivial). Then $G = \mathbb{G}_m$ is the multiplicative group, $W // G = \overline{\mathcal{O}_{[2,1^{n'-2}]}} \subset \mathfrak{g}'$, and the morphism of Proposition 3.7 is $\rho : \text{Hilb}_{h_0}^G(W) \rightarrow \mathbb{P}(V') \times \mathbb{P}(V'^*)$. The Segre embedding gives a G' -equivariant isomorphism $\mathbb{P}(V') \times \mathbb{P}(V'^*) \cong \mathbb{P}(\mathfrak{g}'^{\leq 1})$, where $\mathfrak{g}'^{\leq 1} := \{f \in \mathfrak{g}' \mid \text{rk}(f) \leq 1\}$, and thus we can consider $\rho' : \text{Hilb}_{h_0}^G(W) \rightarrow \mathbb{P}(\mathfrak{g}'^{\leq 1})$, the morphism induced by ρ .

Proposition 3.18. *We equip \mathcal{H} with its reduced structure. If $n' > n = 1$ and with the above notation, we have a G' -equivariant isomorphism*

$$\mathcal{H} \cong \left\{ (f, L) \in \overline{\mathcal{O}_{[2,1^{n'-2}]}} \times \mathbb{P}(\mathfrak{g}'^{\leq 1}) \mid f \in L \right\}.$$

In particular, \mathcal{H} is the union of two smooth irreducible components of dimension $2n' - 2$ defined by:

- $C_1 := \left\{ (f, L) \in \overline{\mathcal{O}_{[2,1^{n'-2}]}} \times \mathbb{P}(\overline{\mathcal{O}_{[2,1^{n'-2}]}}) \mid f \in L \right\} = \mathcal{H}^{\text{main}}$, and the Hilbert-Chow morphism $\gamma : \mathcal{H}^{\text{main}} \rightarrow \overline{\mathcal{O}_{[2,1^{n'-2}]}}$ is the blow-up of $\overline{\mathcal{O}_{[2,1^{n'-2}]}}$ at 0;
- $C_2 := \left\{ (0, L) \in \overline{\mathcal{O}_{[2,1^{n'-2}]}} \times \mathbb{P}(\mathfrak{g}'^{\leq 1}) \right\} \cong \mathbb{P}(\mathfrak{g}'^{\leq 1})$, and the Hilbert-Chow morphism is the zero map.

Proof. By [Tera, §1, Theorem], we have a G' -equivariant isomorphism

$$\gamma \times \rho' : \text{Hilb}_{h_0}^G(W) \rightarrow \left\{ (f, L) \in \mathfrak{g}'^{\leq 1} \times \mathbb{P}(\mathfrak{g}'^{\leq 1}) \mid f \in L \right\}.$$

Since $\mathcal{H} \hookrightarrow \text{Hilb}_{h_0}^G(W)$, we have a G' -equivariant closed embedding

$$\gamma \times \rho' : \mathcal{H} \hookrightarrow \mathcal{Y} := \left\{ (f, L) \in \overline{\mathcal{O}_{[2,1^{n'-2}]}} \times \mathbb{P}(\mathfrak{g}'^{\leq 1}) \mid f \in L \right\}.$$

One may check that \mathcal{Y} is the union of the two closed subset C_1 and C_2 , both of dimension $2n' - 2$. The morphism $\gamma \times \rho'$ sends $\mathcal{H}^{\text{main}}$ into C_1 ; the varieties $\mathcal{H}^{\text{main}}$ and C_1 have the same dimension, hence $\gamma \times \rho' : \mathcal{H}^{\text{main}} \rightarrow C_1$ is an isomorphism. On the other hand, we saw in Section 3.4 that \mathcal{H} admits another irreducible component, denoted \mathcal{H}_2 , of dimension at least $2n' - 2$, which is the dimension of C_2 , and thus $\gamma \times \rho'$ is an isomorphism between \mathcal{H}_2 and C_2 . \square

Remark 3.19. One may check that the component C_2 of Proposition 3.18 consists of the homogeneous ideals of $\mathbb{C}[\mu^{-1}(0)]$.

When $n' \geq 2n \geq 4$, irreducible components of dimension greater than $\dim(\mathcal{H}^{\text{main}})$ may appear. For instance, if $n = 2$ and $n' \geq 4$, then one may check that the irreducible component consisting of the homogeneous ideals of $\mathbb{C}[\mu^{-1}(0)]$ is of dimension $4n' - 5$, whereas the main component $\mathcal{H}^{\text{main}}$ is of dimension $4n' - 8$. In addition, we showed in Section 3.4 that \mathcal{H} has at least two components, but \mathcal{H} may have more components.

4. CASE OF Sp_n

In this section, we denote $E := V' \oplus V'^*$ on which we fix a non-degenerate quadratic form q , and we take $G = Sp(V)$ and $G' = SO(E)$. We identify $W = \text{Hom}(V', V) \times \text{Hom}(V, V') \cong \text{Hom}(E, V)$, equipped with the naturel action of $G' \times G$. We denote \mathfrak{g} resp. \mathfrak{g}' , the Lie algebra of G resp. of G' , and we recall that we denote $n := \dim(V)$ (which is even), and $n' := \dim(V')$.

4.1. The quotient morphism. The main results of this section are Proposition 4.3, which describes the irreducible components of the symplectic reduction $W///G := \mu^{-1}(0)//G$, and Corollary 4.6, which gives the Hilbert function of the general fibers of the quotient morphism for each irreducible component of $W///G$. Contrary to the case of $GL(V)$ studied in Section 3, we will see that $W///G$ is reducible when $n' \leq n$ and n' is even.

As G resp. G' , preserves a non-degenerate bilinear form on V resp. on E , we have a canonical isomorphism $V \cong V^*$ resp. $E \cong E^*$. If $w \in \text{Hom}(E, V)$, we denote ${}^t w \in \text{Hom}(V^*, E^*) \cong \text{Hom}(V, E)$ the transpose of w . Then, by [Bec09, Proposition 3.1], the zero fiber of the moment map $\mu: W \rightarrow \mathfrak{g}^*$ is

$$\mu^{-1}(0) = \{w \in W \mid w \circ {}^t w = 0\},$$

which is a $G' \times G$ -stable closed subset of W .

Remark 4.1. One may check that the biggest subgroup of $GL(E)$ that stabilizes $\mu^{-1}(0)$ in W is the orthogonal group $O(E)$. However, we prefer to consider the action of $G' = SO(E)$ for practical reasons.

The proof of the next proposition is analogous to those of Proposition 3.1 and Corollary 3.2.

Proposition 4.2. *The zero fiber of the moment map $\mu: W \rightarrow \mathfrak{g}^*$ is*

- *a variety of dimension $2n'n - \frac{1}{2}n(n+1)$ if $n' > n$;*
- *the union of two varieties of dimension $n'n + \frac{1}{2}n'(n'-1)$ if $n' \leq n$.*

The following result was proved by Becker ([Bec09, Proposition 3.6]):

Proposition 4.3. *Let \mathbf{d} be a partition of $2n'$, and $\mathcal{O}_{\mathbf{d}}$ resp. $\mathcal{O}_{\mathbf{d}}^I$ and $\mathcal{O}_{\mathbf{d}}^{II}$, denote the nilpotent orbit(s) of $\mathfrak{g}' \cong \mathfrak{so}_{2n'}$ associated to \mathbf{d} . Then the symplectic reduction of W by G is*

$$W///G = \begin{cases} \overline{\mathcal{O}_{[2^n, 1^{2(n'-n)]}}} & \text{if } n' > n; \\ \overline{\mathcal{O}_{[2^{n'-1}, 1^2]}} & \text{if } n' < n \text{ and } n' \text{ is odd}; \\ \overline{\mathcal{O}_{[2^{n'}]}^I \cup \mathcal{O}_{[2^{n'}]}^{II}} & \text{if } n' \leq n \text{ and } n' \text{ is even.} \end{cases}$$

Corollary 4.4. *The orbits for the adjoint action of G' on $W///G$ are*

- $U_i := \mathcal{O}_{[2^i, 1^{2(n'-i)}]}$, for $i = 0, 2, \dots, n$, if $n' > n$;
- $U_i := \mathcal{O}_{[2^i, 1^{2(n'-i)}]}$, for $i = 0, 2, \dots, n' - 1$, if $n' < n$ and n' is odd;

- $U_i := \mathcal{O}_{[2^i, 1^{2(n'-i)}]}$, for $i = 0, 2, \dots, n' - 2$, and $U_{n'}^I := \mathcal{O}_{[2^{n'}]}^I$, $U_{n'}^{II} := \mathcal{O}_{[2^{n'}]}^{II}$, if $n' \leq n$ and n' is even.

The closures of the nilpotent orbits U_i are nested in the following way:

$$\begin{cases} \{0\} = \overline{U_0} \subset \overline{U_2} \subset \dots \subset \overline{U_n} & \text{if } n' > n; \\ \{0\} = \overline{U_0} \subset \overline{U_2} \subset \dots \subset \overline{U_{n'-1}} & \text{if } n' < n \text{ and } n' \text{ is odd}; \\ \{0\} = \overline{U_0} \subset \overline{U_2} \subset \dots \subset \overline{U_{n'-2}} = \overline{U_{n'}^I} \cap \overline{U_{n'}^{II}} & \text{if } n' \leq n \text{ and } n' \text{ is even.} \end{cases}$$

If ($n' > n$ or n' is odd) resp. if ($n' \leq n$ and n' is even), then the symplectic reduction $W///G$ is the closure of a nilpotent orbit resp. the union of two closures of nilpotent orbits, and thus the irreducible components of $W///G$ are symplectic varieties (see Section 2). If $n' > n$, then $W///G$ is of dimension $2n'n - n(n+1)$, and its singular locus is $\overline{U_{n-2}}$. On the other hand, if $n' \leq n$, then each irreducible component of $W///G$ is of dimension $n'(n'-1)$, and the singular locus of $W///G$ is $\overline{U_{n'-2}}$ resp. $\overline{U_{n'-3}}$, when n' is even resp. when n' is odd. The dimension of the irreducible components of $W///G$ is given by [CM93, Corollary 6.1.4], and the singular locus of $W///G$ is given by [KP82, Theorem 2].

We are now interested by the Hilbert function of the general fibers of the quotient morphism for each irreducible component of $W///G$. We will distinguish between the following cases:

- If $n' > n$, then $W///G$ is irreducible, and we denote h_0 the Hilbert function of the general fibers of the quotient morphism $\nu : \mu^{-1}(0) \rightarrow W///G$. By Proposition 4.2, the dimension of these fibers is $\frac{1}{2}n(n+1)$.
- If $n' \leq n$ and n' is even, then by Proposition 4.2, the zero fiber $\mu^{-1}(0)$ is the union of two irreducible components that we denote X_I and X_{II} . Let $\nu_I : X_I \rightarrow Y_I$ and $\nu_{II} : X_{II} \rightarrow Y_{II}$ be the quotient morphisms. Up to the exchange of X_I and X_{II} , we can suppose that $Y_I = \overline{U_{n'}^I}$ and $Y_{II} = \overline{U_{n'}^{II}}$. The orthogonal group $O(E)$ acts transitively on $U_{n'}^I \cup U_{n'}^{II}$, hence the general fibers of ν_I and ν_{II} are isomorphic. In particular, these fibers have the same Hilbert function, denoted h_0 , and the same dimension, which is $n'n - \frac{1}{2}n'(n'-1)$.
- If $n' < n$ and n' is odd, then $W///G$ is irreducible, and we denote h_0 the Hilbert function of the general fibers of the quotient morphism $\nu : \mu^{-1}(0) \rightarrow W///G$. These fibers being reducible, determining h_0 is more complicated than in the previous cases (except the case $n' = 1$ which is trivial). From now on, we will always exclude the case $n' < n$ and n' is odd.

If $n' < n$ and n' is even, then we denote

$$(16) \quad H := \left\{ \left[\begin{array}{cc} M & 0_{n-n', n'} \\ 0_{n', n-n'} & I_{n'} \end{array} \right], M \in Sp_{n-n'} \right\} \cong Sp_{n-n'},$$

which is a reductive algebraic subgroup of $G \cong Sp_n$. The proof of the next proposition is analogous to the proof of Proposition 3.5:

Proposition 4.5. *If $n' > n$, then the general fibers of the quotient morphism $\nu : \mu^{-1}(0) \rightarrow W///G$ are isomorphic to G .*

If $n' = n$, then the general fibers of the quotient morphisms $\nu_I : X_I \rightarrow Y_I$ and $\nu_{II} : X_{II} \rightarrow Y_{II}$ are isomorphic to G .

If $n' < n$ and n' is even, then the general fibers of ν_I and ν_{II} are isomorphic to G/H , where $H \subset G$ is the subgroup defined by (16).

Corollary 4.6. *The Hilbert function h_0 defined above is given by:*

$$\forall M \in \text{Irr}(G), h_0(M) = \begin{cases} \dim(M) & \text{if } n' \geq n; \\ \dim(M^H) & \text{if } n' < n \text{ and } n' \text{ is even;} \end{cases}$$

where $H \subset G$ is the subgroup defined by (16).

4.2. The reduction principle for the main component. In this section, we give the guidelines to prove the reduction principle when $G = Sp(V)$ (Proposition 4.8). The strategy is the same as for $GL(V)$ (see Section 3.2), but as the symplectic reduction $W///G$ is reducible when $n' \leq n$ and n' is even, it seems necessary to give some additional details.

As $\mu^{-1}(0)$ is a $G' \times G$ -stable closed subset of W , it follows from [Bri, Lemma 3.3] that the invariant Hilbert scheme

$$\mathcal{H} := \text{Hilb}_{h_0}^G(\mu^{-1}(0))$$

is a G' -stable closed subscheme of $\text{Hilb}_{h_0}^G(W)$. As we aim at constructing canonical desingularizations of the irreducible components of $W///G$, we consider the two G' -stable closed subschemes $\mathcal{H}_I := \text{Hilb}_{h_0}^G(X_I)$ and $\mathcal{H}_{II} := \text{Hilb}_{h_0}^G(X_{II})$ instead of \mathcal{H} when $n' \leq n$ and n' is even. Let us note that if we fix $g'_0 \in O(E) \setminus SO(E)$ and make G' act on X_{II} by $(g'_0 g' g'_0^{-1}) \cdot x$ for every $g' \in G'$ and every $x \in X_{II}$, then $\phi: X_I \rightarrow X_{II}$, $x \mapsto g'_0 \cdot x$ is a $G' \times G$ -equivariant isomorphism, and thus $\mathcal{H}_I \cong \mathcal{H}_{II}$ as a G' -scheme. We denote $\mathcal{H}_I^{\text{main}}$ resp. $\mathcal{H}_{II}^{\text{main}}$, the main component of \mathcal{H}_I resp. of \mathcal{H}_{II} . We always have the (set-theoretic) inclusion $\mathcal{H}_I \cup \mathcal{H}_{II} \subset \mathcal{H}$, but this may not be an equality. If $n' > n$, then $W///G$ is irreducible, and we denote $\mathcal{H}^{\text{main}}$ the main component of \mathcal{H} .

The scheme $\text{Hilb}_{h_0}^G(W)$ was studied in [Terb]. In particular, we obtained

Proposition 4.7. ([Terb, §1.5.1]) *Let h_0 be the Hilbert function given by Corollary 4.6, and $G' = SO(E)$ acting naturally on $\text{Gr}(2n' - h_0(V^*), E)$. Then there exists a G' -equivariant morphism*

$$\rho: \text{Hilb}_{h_0}^G(W) \rightarrow \text{Gr}(2n' - h_0(V^*), E)$$

given on closed points by $Z \mapsto \text{Ker}(f_Z)$, where $f_Z: E \cong \text{Mor}^G(W, V^*) \rightarrow \text{Mor}^G(Z, V^*)$ is the restriction map.

We identify $\text{Gr}(2n' - h_0(V^*), E)$ with $\text{Gr}(h_0(V^*), E^*)$. By Corollary 4.6, if $n' > n$ or ($n' \leq n$ and n' is even), then $h_0(V^*) = N := \min(n', n)$. We have a canonical isomorphism $E \cong E^*$, and thus the non-degenerate quadratic form q on E identifies with a non-degenerate quadratic form on E^* . For $i = 0, \dots, N$, we denote

$$A_i := \{L \in \text{Gr}(N, E^*) \mid q|_L \text{ is of rank } i\}.$$

If $n' > n$, then the A_i are the $n+1$ orbits for the action of G' on $\text{Gr}(n, E^*)$. However, if $n' \leq n$, then the A_i are G' -orbits for $i = 1, \dots, n'$, but the isotropic Grassmannian $A_0 = \text{OGr}(n', E^*)$ is the union of two G' -orbits, denoted OGr^I and OGr^{II} , which are exchanged by the action of any element of $O(E) \setminus SO(E)$. In any case, we have

$$\text{OGr}(N, E^*) = \overline{A_0} \subset \overline{A_1} \subset \dots \subset \overline{A_N} = \text{Gr}(N, E^*).$$

Let us now fix some notation:

- $L_0 \in A_0$, and P the parabolic subgroup of G' stabilizing L_0 ;

- $W' := \text{Hom}(E/L_0^\perp, V)$, which identifies with a $P \times G$ -module contained in $\mu^{-1}(0)$;
- $\mathcal{H}' := \text{Hilb}_{h_0}^G(W')$, and $\mathcal{H}'^{\text{main}}$ its main component.

It must be emphasized that, if $n' > n$ or ($n' \leq n$ and n' is even), then the Hilbert function of the general fibers of the quotient morphism $W' \rightarrow W'//G$ coincides with the Hilbert function h_0 of Corollary 4.6 (in particular, $\mathcal{H}'^{\text{main}}$ is well defined).

Proceeding as for Lemma 3.9, one may check that, if $n' > n$ resp. if ($n' \leq n$ and n' is even), then the morphism ρ of Proposition 4.7 sends $\mathcal{H}^{\text{main}}$ resp. $\mathcal{H}_I^{\text{main}}$ and $\mathcal{H}_{II}^{\text{main}}$, onto A_0 . More precisely, if $n' \leq n$ and n' is even, then ρ sends $\mathcal{H}_I^{\text{main}}$ onto one of the irreducible component of A_0 , and $\mathcal{H}_{II}^{\text{main}}$ onto the other component. Up to the exchange of these two components, we can suppose that ρ sends $\mathcal{H}_I^{\text{main}}$ onto OGr^I , and $\mathcal{H}_{II}^{\text{main}}$ onto OGr^{II} .

It follows that the restriction of ρ equips $\mathcal{H}^{\text{main}}$ resp. $\mathcal{H}_I^{\text{main}}$, resp. $\mathcal{H}_{II}^{\text{main}}$, with a structure of a G' -homogeneous fiber bundle over A_0 resp. over OGr^I , resp. over OGr^{II} . Hence, it is enough to determine the fiber F_0 over L_0 to determine $\mathcal{H}^{\text{main}}$ resp. $\mathcal{H}_I^{\text{main}}$, resp. $\mathcal{H}_{II}^{\text{main}}$. Proceeding as in Section 3.2, we obtain that F_0 is isomorphic to $\mathcal{H}'^{\text{main}}$ as a P -scheme. We deduce

Proposition 4.8. *With the above notation, we have the following G' -equivariant isomorphisms:*

- If $n' > n$, then

$$\mathcal{H}^{\text{main}} \cong G' \times^P \mathcal{H}'^{\text{main}}.$$

- If ($n' \leq n$ and n' is even) and $L_0 \in \text{OGr}^I$ resp. $L_0 \in \text{OGr}^{II}$, then

$$\mathcal{H}_I^{\text{main}} \cong G' \times^P \mathcal{H}'^{\text{main}} \quad \text{resp.} \quad \mathcal{H}_{II}^{\text{main}} \cong G' \times^P \mathcal{H}'^{\text{main}}.$$

4.3. Proof of Theorem A for $\text{Sp}(\mathbf{V})$. Let us start by recalling

Theorem 4.9. [Tera, §1, Theorem] *Let $G = \text{Sp}(V)$, $W = \text{Hom}(E, V)$, and h_W the Hilbert function of the general fibers of the quotient morphism $W \rightarrow W//G$. We denote $n := \dim(V)$, $e := \dim(E)$, and Y_0 the blow-up of $W//G = \Lambda^2(E^*)^{\leq n} := \{Q \in \Lambda^2(E^*) \mid \text{rk}(Q) \leq n\}$ at 0. In the following cases, the invariant Hilbert scheme $\mathcal{H}' := \text{Hilb}_{h_W}^G(W)$ is a smooth variety, and the Hilbert-Chow morphism is the succession of blow-up described as follows:*

- if $n \geq 2e - 2$, then $\mathcal{H}' \cong W//G = \Lambda^2(E^*)$;
- if $e > n = 2$ or $e = n = 4$, then $\mathcal{H}' \cong Y_0$;
- if $e > n = 4$, then \mathcal{H}' is isomorphic to the blow-up of Y_0 along the strict transform of $\Lambda^2(E^*)^{\leq 2}$.

If $n' > n$, then we denote T the tautological bundle over $A_0 = \text{OGr}(n, E^*)$. If $n' \leq n$ and n' is even, then we denote T_I resp. T_{II} , the tautological bundle over OGr^I resp. over OGr^{II} . We deduce from Proposition 4.8 and Theorem 4.9 that we have the following G' -equivariant isomorphisms

$$(17) \quad \mathcal{H}^{\text{main}} \cong \begin{cases} \Lambda^2(T) & \text{if } n' > n = 2; \\ \text{Bl}_0(\Lambda^2(T)) & \text{if } n' > n = 4; \end{cases}$$

$$(18) \quad \mathcal{H}_\bullet^{\text{main}} \cong \begin{cases} \Lambda^2(T_\bullet) & \text{if } n \geq 2n' - 2 \text{ and } n' \text{ is even;} \\ \text{Bl}_0(\Lambda^2(T_\bullet)) & \text{if } n' = n = 4; \end{cases}$$

where \bullet stands for I or II , and $\text{Bl}_0(\cdot)$ denotes the blow-up along the zero section. In all these cases, the main component of the invariant Hilbert scheme is smooth,

and thus the Hilbert-Chow morphism $\gamma: \mathcal{H}^{\text{main}} \rightarrow W///G$ resp. $\gamma: \mathcal{H}_{\bullet}^{\text{main}} \rightarrow Y_{\bullet}$, is a canonical desingularization.

It remains to compare γ with the Springer desingularizations (when they exist) of the irreducible components of $W///G$. We saw in Section 2 that the irreducible components of $W///G$ have Springer desingularizations if and only if $n' \leq n+1$. We then distinguish between the following cases:

- (1) If $n' \leq n+1$ and n' is odd, then $W///G$ admits two Springer desingularizations, which are given by the cotangent bundles \mathcal{T}_I^* and \mathcal{T}_{II}^* over OGr^I and OGr^{II} respectively. The natural action of the orthogonal group $O(E)$ on $\text{OGr}(n', E^*)$ induces an action on the cotangent bundle $\mathcal{T}^*\text{OGr}(n', E^*)$ that exchanges \mathcal{T}_I^* and \mathcal{T}_{II}^* . On the other hand, it follows from Remark 4.1, that the group $O(E)$ stabilizes $\mathcal{H}^{\text{main}}$, and thus $\gamma: \mathcal{H}^{\text{main}} \rightarrow W///G$ cannot be a Springer desingularization.

However, if $n \in \{2, 4\}$ and $n' = n+1$, then one may show that γ dominates the two Springer desingularizations of $W///G$ (see [Bec11, Introduction] for the case $n=2$, the case $n=4$ being analogous).

- (2) If $n' \leq n$ and n' is even, then Y_{\bullet} has a unique Springer desingularization, which is given by the cotangent bundle $\mathcal{T}_{\bullet}^* \cong \Lambda^2(T_{\bullet})$ over OGr_{\bullet} . Proceeding as we did for $GL(V)$ in Section 3.3, one may show that $\gamma: \mathcal{H}_{\bullet}^{\text{main}} \rightarrow Y_{\bullet}$ is the Springer desingularization if and only if $n \geq 2n' - 2$.

In addition, if $n' = n = 4$, then by (18) we have $\mathcal{H}_{\bullet}^{\text{main}} \cong \text{Bl}_0(\mathcal{T}_{\bullet}^*)$, and thus γ dominates the unique Springer desingularization of Y_{\bullet} .

4.4. Study of the case $n=2$. In this section, we suppose that $n' \geq n=2$ (the case $n'=1$ being trivial). We will show that if $n' \geq 3$ resp. if $n'=2$, then \mathcal{H} resp. \mathcal{H}_{\bullet} (where \bullet stands for I or II), is irreducible. In particular, the geometric properties of the invariant Hilbert scheme for $G = Sp(V)$ are quite different from the case of $G = GL(V)$ studied in Section 3. Let us recall that the case ($n'=3$ and $n=2$) was treated by Becker in [Bec11]; she showed that \mathcal{H} is the total space of a line bundle over $\text{OGr}(2, E^*)$.

Then $G \cong Sp_2 = SL_2$, and the morphism of Proposition 4.7 is $\rho: \text{Hilb}_{h_0}^G(W) \rightarrow \text{Gr}(2, E^*)$. Denoting $\mathfrak{g}'^{\leq 2} := \{f \in \mathfrak{g}' \mid \text{rk}(f) \leq 2\}$, we have a G' -equivariant isomorphism

$$(19) \quad \mathbb{P}(\mathfrak{g}'^{\leq 2}) \cong \text{Gr}(2, E^*),$$

and thus we can consider the morphism $\rho': \text{Hilb}_{h_0}^G(W) \rightarrow \mathbb{P}(\mathfrak{g}'^{\leq 2})$ induced by ρ . By Proposition 4.3, we have

$$W///G = \begin{cases} \overline{\mathcal{O}_{[2^2, 1^{2n'-4}]}} & \text{if } n' \geq 3; \\ \overline{\mathcal{O}_{[2^2]}^I} \cup \overline{\mathcal{O}_{[2^2]}^{II}} & \text{if } n' = 2. \end{cases}$$

Proposition 4.10. *We equip all the invariant Hilbert schemes with their reduced structures. If $n' > n=2$, then $\mathcal{H} = \mathcal{H}^{\text{main}}$ is a smooth variety isomorphic to*

$$\text{Bl}_0(\overline{\mathcal{O}_{[2^2, 1^{2n'-4}]}}) := \{(f, L) \in \overline{\mathcal{O}_{[2^2, 1^{2n'-4}]}} \times \mathbb{P}(\overline{\mathcal{O}_{[2^2, 1^{2n'-4}]}}) \mid f \in L\},$$

and the Hilbert-Chow morphism $\gamma: \mathcal{H} \rightarrow W///G$ is the blow-up of $\overline{\mathcal{O}_{[2^2, 1^{2n'-4}]}}$ at 0. If $n' = n=2$, then $\text{Hilb}_{h_0}^G(\mu^{-1}(0)) = \mathcal{H}_I \cup \mathcal{H}_{II}$ is the union of two smooth varieties isomorphic to $\text{Bl}_0(\overline{\mathcal{O}_{[2^2]}^I})$ and $\text{Bl}_0(\overline{\mathcal{O}_{[2^2]}^{II}})$ respectively, and the set-theoretic intersection $\mathcal{H}_I \cap \mathcal{H}_{II}$ is formed by the homogeneous ideals of $\mathbb{C}[\mu^{-1}(0)]$. Moreover, the

Hilbert-Chow morphism $\gamma : \mathcal{H}_I \rightarrow \overline{\mathcal{O}}_{[2^2]}^I$ resp. $\gamma : \mathcal{H}_{II} \rightarrow \overline{\mathcal{O}}_{[2^2]}^{II}$, is the blow-up of $\overline{\mathcal{O}}_{[2^2]}^I$ resp. of $\overline{\mathcal{O}}_{[2^2]}^{II}$, at 0.

Proof. The proofs for the cases $n' = 2$ and $n' \geq 3$ are quite similar, and thus we will only consider the case $n' \geq 3$ (which is simpler in terms of notation!). Using arguments similar to those used to show Proposition 3.18, we obtain a closed embedding

$$\gamma \times \rho' : \mathcal{H} \hookrightarrow \mathcal{Y} := \left\{ (f, L) \in \overline{\mathcal{O}}_{[2^2, 1^{2n'-4}]} \times \mathbb{P}(\mathfrak{g}'^{\leq 2}) \mid f \in L \right\}.$$

One may check that \mathcal{Y} is the union of the two subvarieties C_1 and C_2 defined by:

- $C_1 := \text{Bl}_0(\overline{\mathcal{O}}_{[2^2, 1^{2n'-4}]})$;
- $C_2 := \left\{ (0, L) \in \overline{\mathcal{O}}_{[2^2, 1^{2n'-4}]} \times \mathbb{P}(\mathfrak{g}'^{\leq 2}) \right\} \cong \mathbb{P}(\mathfrak{g}'^{\leq 2})$.

The subvarieties C_1 and C_2 are of dimension $4n' - 6$ and $4n' - 4$ respectively. The morphism $\gamma \times \rho'$ sends $\mathcal{H}^{\text{main}}$ into C_1 ; the varieties $\mathcal{H}^{\text{main}}$ and C_1 have the same dimension, hence $\gamma \times \rho' : \mathcal{H}^{\text{main}} \rightarrow C_1$ is an isomorphism.

Now it follows from [Terb, Proposition 3.3.13] that the component C_2 identifies with the closed subset of $\text{Hilb}_{n_0}^G(W)$ formed by the homogeneous ideals of $\mathbb{C}[W]$. Let us describe this identification. If $L \in C_2 \cong \mathbb{P}(\mathfrak{g}'^{\leq 2})$, then we denote I_L the ideal of $\mathbb{C}[W]$ generated by the homogeneous G -invariants of positive degree of $\mathbb{C}[W]$, and by the G -module $L^\perp \otimes V \subset \mathbb{C}[W]_1 \cong E \otimes V$, where L is identified with a 2-dimensional subspace of E^* via (19). Let us show that I_L is a point of \mathcal{H} if and only if $L \in \text{OGr}(2, E^*)$; the result will follow since $\mathbb{P}(\overline{\mathcal{O}}_{[2^2, 1^{2n'-4}]})$ identifies with $\text{OGr}(2, E^*)$ via the isomorphism (19), and since $\left\{ (0, L) \in \overline{\mathcal{O}}_{[2^2, 1^{2n'-4}]} \times \mathbb{P}(\overline{\mathcal{O}}_{[2^2, 1^{2n'-4}]}) \right\}$ is a subvariety of C_1 .

We denote $W' := \text{Hom}(E/L^\perp, V)$, then

$$\mathbb{C}[W']_2 \cong (S^2(E/L^\perp) \otimes S^2(V)) \oplus (\Lambda^2(E/L^\perp) \otimes \Lambda^2(V))$$

as a G -module. Let I'_L be the ideal of $\mathbb{C}[W']$ generated by $\Lambda^2(E/L^\perp) \otimes \Lambda^2(V) \subset \mathbb{C}[W']_2$, then one may check (using [Terb, Proposition 3.3.13]) that

$$\mathbb{C}[W]/I_L \cong \mathbb{C}[W']/I'_L \cong \bigoplus_{M \in \text{Irr}(G)} M^{\oplus \dim(M)}$$

as a G -module. Hence

$$\begin{aligned} I_L \in \mathcal{H} &\Leftrightarrow I_L \cap \mathbb{C}[W]_2 \supset E_0 \otimes S^2(V), \text{ where } E_0 \text{ is the trivial representation of } G'; \\ &\Leftrightarrow q|_L = 0, \text{ where } q \text{ is the quadratic form preserved by } G'; \\ &\Leftrightarrow L \in \text{OGr}(2, E^*). \end{aligned}$$

□

Remark 4.11. In the proof of Proposition 4.10, we showed that if $n' > n = 2$, then the homogeneous ideals of \mathcal{H} are contained in $\mathcal{H}^{\text{main}}$. Using analogous arguments, one may check that this statement is true more generally when $n' > n \geq 2$.

REFERENCES

- [AB05] V. Alexeev and M. Brion. Moduli of affine schemes with reductive group action. *J. Algebraic Geom.*, 14:83–117, 2005.
- [Bea00] A. Beauville. Symplectic singularities. *Invent. Math.*, 139:no.3, 541–549, 2000.
- [Bec09] T. Becker. On the existence of symplectic resolutions of symplectic reductions. *Mathematische Zeitschrift*, 265:343–363, 2009.
- [Bec11] T. Becker. An example of an SL_2 -Hilbert scheme with multiplicities. *Transform. Groups*, 16:no. 4, 915–938, 2011.
- [Bri] M. Brion. Invariant Hilbert schemes. arXiv: 1102.0198, to appear in "*Handbook of Moduli*".
- [Bri85] M. Brion. Représentations exceptionnelles des groupes semi-simples. *Ann. Sci. Ecole Norm. Sup.*, 2:345–387, 1985.
- [CM93] D. Collingwood and W. McGovern. *Nilpotent orbits in semisimple Lie algebras*. Van Nostrand Reinhold Mathematics Series, vol. 296. Van Nostrand Reinhold Co., New York, 1993.
- [FN04] B. Fu and Y. Namikawa. Uniqueness of crepant resolutions and symplectic singularities. *Ann. Inst. Fourier (Grenoble)*, 54:1–19, 2004.
- [Fu03a] B. Fu. Symplectic resolutions for nilpotent orbits. *Invent. Math.*, 151:167–186, 2003.
- [Fu03b] B. Fu. Symplectic resolutions for nilpotent orbits (II). *C. R. Acad. Sci. Paris*, 337:277–281, 2003.
- [Fu06a] B. Fu. A survey on symplectic singularities and resolutions. *Ann. Math. Blaise Pascal*, 13:209–236, 2006.
- [Fu06b] B. Fu. Symplectic resolutions for nilpotent orbits (III). *C. R. Acad. Sci. Paris*, 342:585–588, 2006.
- [KP79] H. Kraft and C. Procesi. Closures of conjugacy classes of matrices are normal. *Invent. Math.*, 53:227–247, 1979.
- [KP81] H. Kraft and C. Procesi. Minimal singularities in GL_n . *Invent. Math.*, 62:503–515, 1981.
- [KP82] H. Kraft and C. Procesi. On the geometry of conjugacy classes in classical groups. *Comment. Math. Helvetici*, 57:539–602, 1982.
- [KS] H. Kraft and G. W. Schwarz. Representations with a reduced null cone. arXiv: 1112.3634, to appear in *Progress in Mathematics* (Birkhäuser), a volume in honor of Nolan Wallach.
- [Nam06] Y. Namikawa. Birational geometry of symplectic resolutions of nilpotent orbits. *Moduli spaces and arithmetic geometry*, 45, Math. Soc. Japan, Tokyo:75–116, Adv. Stud. Pure Math., 2006.
- [Pro07] C. Procesi. *Lie Groups, an Approach through Invariants and Representations*. Universitext. Springer, New York, 2007.
- [SB00] G. W. Schwarz and M. Brion. *Théorie des invariants et géométrie des variétés quotients*. Travaux en cours, vol. 61. Hermann, Paris, 2000.
- [Tera] R. Terpereau. Invariant Hilbert schemes and desingularizations of quotients by classical groups. arXiv: 1301.4020.
- [Terb] R. Terpereau. Schémas de Hilbert invariants et théorie classique des invariants (Ph.D. thesis). arXiv: 1211.1472.

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