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Ronan Terpereau. Invariant Hilbert schemes and desingularizations of symplectic reductions for classical groups. 2013. hal-00799369v1

**HAL Id: hal-00799369**

**<https://hal.science/hal-00799369v1>**

Preprint submitted on 12 Mar 2013 (v1), last revised 21 Dec 2013 (v3)

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# INVARIANT HILBERT SCHEMES AND DESINGULARIZATIONS OF SYMPLECTIC REDUCTIONS FOR CLASSICAL GROUPS

RONAN TERPEREAU

ABSTRACT. Let  $G \subset GL(V)$  be a classical group acting on the symplectic vector space  $W = V^{\oplus n'} \oplus V^{*\oplus n'}$ , and  $\mu : W \rightarrow Lie(G)^*$  the corresponding moment map. In this article, we use the theory of invariant Hilbert schemes to construct canonical desingularizations of the irreducible components of the symplectic reduction  $\mu^{-1}(0)//G$  for classes of examples where  $G = GL(V)$ ,  $O(V)$ , or  $Sp(V)$ . It is known that such components are isomorphic to closures of nilpotent orbits in simple Lie algebras, and we determine all the cases where the Hilbert-Chow morphism is a symplectic desingularization.

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## 1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

We work over the field of complex numbers  $\mathbb{C}$ . Let  $V$  and  $V'$  be two vector spaces of dimensions  $n$  and  $n'$  respectively, and let

$$(1) \quad W := \text{Hom}(V', V) \times \text{Hom}(V, V'),$$

which is naturally equipped with an action of  $GL(V') \times GL(V)$ :

$$\forall (g', g) \in GL(V') \times GL(V), \forall (u_1, u_2) \in W, (g', g) \cdot (u_1, u_2) := (g \circ u_1 \circ g'^{-1}, g' \circ u_2 \circ g^{-1}).$$

The  $GL(V)$ -module  $W$  is *symplectic*, that is, equipped with a  $GL(V)$ -invariant non-degenerate skew-symmetric bilinear form  $\Omega$  defined by:

$$(2) \quad \forall (u_1, u_2), (u'_1, u'_2) \in W, \Omega((u_1, u_2), (u'_1, u'_2)) := \text{tr}(u'_1 \circ u_2) - \text{tr}(u_1 \circ u'_2),$$

where  $\text{tr}(\cdot)$  denotes the trace. The symplectic  $GL(V)$ -module  $W$  has a  $GL(V)$ -equivariant *moment map* defined by:

$$(3) \quad \mu_{GL(V)} : \begin{array}{ccc} W & \rightarrow & \mathfrak{gl}(V)^* \\ (u_1, u_2) & \mapsto & (h \mapsto \text{tr}(u_2 \circ h \circ u_1)) \end{array}$$

where  $\mathfrak{gl}(V)$  denotes the Lie algebra of  $GL(V)$ . If  $G \subset GL(V)$  is any algebraic subgroup, and  $\mathfrak{g}$  is the Lie algebra of  $G$ , then the moment map  $\mu_G : W \rightarrow \mathfrak{g}^*$  for the action of  $G$  on  $W$  is obtained by composing  $\mu_{GL(V)}$  with the linear projection  $\mathfrak{gl}(V)^* \rightarrow \mathfrak{g}^*$ . To simplify the notation, we will denote  $\mu$  instead of  $\mu_G$ .

The map  $\mu$  is  $G$ -equivariant, and thus the set-theoretic fiber  $\mu^{-1}(0)$  is a  $G$ -stable closed subset of  $W$ . From now on, we suppose that  $G$  is reductive, and we denote  $W//G$  the *symplectic reduction* of  $W$  by  $G$ , that is, the categorical quotient  $\mu^{-1}(0)//G := \text{Spec}(\mathbb{C}[\mu^{-1}(0)]^G)$ . Let us mention that if  $G'$  is an algebraic subgroup of the  $G$ -automorphism group  $\text{Aut}^G(\mu^{-1}(0))$ , then  $G'$  acts on  $W//G$  and the quotient morphism  $\mu^{-1}(0) \rightarrow W//G$  is  $G'$ -equivariant. **The aim of this article** is to construct canonical desingularizations of the irreducible components of  $W//G$  for classes of examples where  $G = GL(V)$ ,  $O(V)$ , or  $Sp(V)$ . In these three cases, it is known that the irreducible components of  $W//G$  are closures of nilpotent orbits in simple Lie algebras of type A, C and D respectively. The geometry of such closures has been extensively studied ([KP79, KP81, KP82, Fu03a, Fu03b, Fu06b, Nam06]). In particular, the normalizations of such closures are *symplectic varieties* (as defined by Beauville in [Bea00]) whose symplectic desingularizations are the so-called *Springer desingularizations*, obtained by collapsing the cotangent bundle over some Grassmannians (see Section 2 for details).

Before stating our results, we need to recall briefly the definition of the *invariant Hilbert scheme*, constructed by Alexeev and Brion (see [AB05, Bri] for more details), in the particular case we are interested. Let  $G \subset GL(V)$  be a reductive algebraic subgroup,  $\text{Irr}(G)$  the set of isomorphism classes of irreducible representations of  $G$ , and  $h : \text{Irr}(G) \rightarrow \mathbb{N}$  a *Hilbert function*. If  $X \subset W$  a  $G$ -stable closed subscheme, then the invariant Hilbert scheme  $\text{Hilb}_h^G(X)$  is the moduli space that parametrizes the  $G$ -stable closed subschemes  $Z$  of  $X$  such that

$$\mathbb{C}[Z] \cong \bigoplus_{M \in \text{Irr}(G)} M^{\oplus h(M)}$$

as a  $G$ -module. We now suppose that  $X$  is reduced and that  $X//G$  is irreducible. If  $h = h_0$  is the Hilbert function of the general fiber of the quotient morphism  $\nu : X \rightarrow X//G$ , then there exists a projective morphism

$$\gamma : \text{Hilb}_{h_0}^G(X) \rightarrow X//G,$$

called the *Hilbert-Chow morphism*, that sends a closed subscheme  $Z \subset X$  to the point  $Z//G \subset X//G$ . If  $G' \subset \text{Aut}^G(X)$  is any algebraic subgroup, then it is known that  $G'$  acts on  $X//G$  and on  $\text{Hilb}_{h_0}^G(X)$  such that  $\nu$  and  $\gamma$  are  $G'$ -equivariant. The Hilbert-Chow morphism induces an isomorphism over the flat locus  $U \subset X//G$  of  $\nu$ . The *main component* of  $\text{Hilb}_{h_0}^G(X)$  is the irreducible component defined by

$$\text{Hilb}_{h_0}^G(X)^{\text{main}} := \overline{\gamma^{-1}(U)}.$$

Then the restriction  $\gamma : \text{Hilb}_{h_0}^G(X)^{\text{main}} \rightarrow X//G$  is a projective birational morphism, and thus  $\gamma$  is a candidate for a canonical desingularization of  $X//G$ .

We can now state the main result of this paper:

**Theorem A.** [Sections 3.3 and 4.3] *With the above notation, let  $G = GL(V)$ ,  $O(V)$  or  $Sp(V)$ , then  $W//G$  is*

- (1) *irreducible if  $G = GL(V)$ ,  $O(V)$ , or  $Sp(V)$  with  $(n' > n$  or  $n'$  odd);*
- (2) *the union of two isomorphic irreducible components if  $G = Sp(V)$  with  $(n' \leq n$  and  $n'$  even).*

*Let  $X$  denote  $\mu^{-1}(0)$  in Case (1) resp. the preimage of one of the two irreducible components of  $W//G$  by the quotient morphism  $\mu^{-1}(0) \rightarrow W//G$  in Case (2). Then*

the Hilbert-Chow morphism  $\gamma : \text{Hilb}_{h_0}^G(X)^{\text{main}} \rightarrow X//G$  is a Springer desingularization (and the unique one) if and only if

- $G = GL(V)$ ,  $n \geq n' - 1$ , and  $n'$  is even; or
- $G = O(V)$ , and  $n \geq 2n' - 1$ ; or
- $G = Sp(V)$ ,  $n$  and  $n'$  are even, and  $n \geq 2n' - 2$ .

Moreover,  $\gamma$  is still a desingularization (that strictly dominates the Springer desingularizations when they exist) if

- $G = GL(V)$  and  $((n = 1$  and  $n' \geq 3)$  or  $(n = 2$  and  $n' \geq 4))$ ; or
- $G = O(V)$  and  $(n = 1 < n'$  or  $n = 2 \leq n')$ ; or
- $G = Sp(V)$  and  $(n = 2 < n'$  or  $n = 4 \leq n')$ .

With the notation of Theorem A, we have no idea whether the Hilbert-Chow morphism  $\gamma : \text{Hilb}_{h_0}^G(X)^{\text{main}} \rightarrow X//G$  is always a desingularization or not. Let us mention that the case  $G = Sp(V)$  with  $(n = 2$  and  $n' = 3)$  was already handled by Becker in [Bec11]. In this case,  $W///G$  is a closure of a nilpotent orbit that admits two Springer desingularizations, and Becker showed that  $\gamma : \text{Hilb}_{h_0}^G(\mu^{-1}(0)) \rightarrow W///G$  is a desingularization that dominates them both. To obtain this result, Becker first used the existence of natural morphisms from the invariant Hilbert scheme to Grassmannians to identify  $\text{Hilb}_{h_0}^G(\mu^{-1}(0))^{\text{main}}$  with the total space of a homogeneous line bundle over a Grassmannian, and then she showed that  $\text{Hilb}_{h_0}^G(\mu^{-1}(0)) = \text{Hilb}_{h_0}^G(\mu^{-1}(0))^{\text{main}}$  by computing the tangent space of  $\text{Hilb}_{h_0}^G(\mu^{-1}(0))$  at every point of  $\text{Hilb}_{h_0}^G(\mu^{-1}(0))^{\text{main}}$ . In the present article, our most important theoretical result is a *reduction principle* that generalizes the first step of Becker's method.

**Reduction Principle.** [Propositions 3.8 and 4.8] *We take the notation of Theorem A, and we make the extra assumption that  $n' \geq 2n$  or  $(n' < 2n$  and  $n'$  is even) when  $G = GL(V)$ , and  $n' \geq n$  or  $(n' < n$  and  $n', n$  are even) when  $G = Sp(V)$ .*

*Then the main component  $\text{Hilb}_{h_0}^G(X)^{\text{main}}$  is the total space of a homogeneous fiber bundle over a flag variety whose fiber is isomorphic to  $\text{Hilb}_{h_{W'}}^G(W')^{\text{main}}$ , where  $W' = V^{\oplus N} \oplus V^{*\oplus N}$  with  $N := h_0(V^*)$ , and  $h_{W'}$  is the Hilbert function of the general fiber of the quotient morphism  $W' \rightarrow W'//G$ .*

A similar reduction principle (but in a different setting) was already obtained and used in [Tera]. Once we have performed the reduction step, we use [Tera, §1, Theorem] to identify the fiber  $\text{Hilb}_{h_{W'}}^G(W')^{\text{main}}$ . Unfortunately, the results that we get in [Tera] are very partial and that is why we are able to determine  $\text{Hilb}_{h_0}^G(X)^{\text{main}}$  only in the cases given by Theorem A. The last step to prove Theorem A is then to compare the Hilbert-Chow morphism  $\gamma : \text{Hilb}_{h_0}^G(X)^{\text{main}} \rightarrow X//G$  with the Springer desingularizations of  $X//G$ . We thus have a general method to determine  $\text{Hilb}_{h_0}^G(X)^{\text{main}}$  that will certainly be helpful to determine further examples when  $G = GL(V)$ ,  $O(V)$  or  $Sp(V)$ . In addition, the author thinks that the same method can be used to treat the case of  $G = SL(V)$ . However, when  $G = SO(V)$ , we do not know how to obtain a reduction principle as above.

Let now  $G \subset GL(V)$  be any reductive algebraic subgroup,  $X \subset \mu^{-1}(0)$  a maximal  $G$ -stable closed subset such that  $X//G$  is an irreducible component of  $W///G$ , and  $h_0$  the Hilbert function of the general fiber of the quotient morphism  $X \rightarrow X//G$ . Then it is generally a difficult problem to determine whether the main component

$\text{Hilb}_{h_0}^G(X)^{\text{main}}$  coincides with the whole invariant Hilbert scheme  $\text{Hilb}_{h_0}^G(X)$  or not. In this direction, we obtain

**Proposition B.** [Propositions 3.18 and 4.10] *With the notation of Theorem A, if  $G = GL(V)$  and  $n' \geq 2n$ , then the invariant Hilbert scheme  $\text{Hilb}_{h_0}^G(X)$  has at least two irreducible components (and exactly two when  $n = 1$ ). On the other hand, if  $G = O(V)$  or  $Sp(V)$ , and  $n' \geq n = 2$ , then  $\text{Hilb}_{h_0}^G(X)$  is irreducible.*

In Section 2, we recall some basic facts about symplectic varieties and closures of nilpotent orbits in simple Lie algebras. The case of  $GL(V)$  is treated in Section 3, and the case of  $Sp(V)$  is treated in Section 4. The case of  $O(V)$  is quite similar to the case of  $GL(V)$ , and details can be found in the thesis [Terb] from which this article is extracted: the Reduction Principle is [Terb, Proposition 3.4.10], Theorem A is [Terb, Corollaire 3.4.11 and Proposition A.2.5], and Proposition B is [Terb, Proposition 3.4.17].

**Acknowledgments:** I am deeply thankful to Michel Brion for proposing this subject to me, for a lot of helpful discussions, and for his patience. I thank Tanja Becker for exchange of knowledge on invariant Hilbert schemes by e-mail and during her stay in Grenoble in October 2010. I also thank Bart Van Steirteghem for helpful discussions during his stay in Grenoble in Summer 2011.

## 2. GENERALITIES ON SYMPLECTIC VARIETIES AND CLOSURES OF NILPOTENT ORBITS

**2.1. Symplectic varieties and symplectic desingularizations.** Let us first recall the definitions of symplectic variety and symplectic desingularization (see [Bea00] or the survey [Fu06a] for more details). Let  $Y$  be a normal variety whose regular part  $Y_{\text{reg}}$  admits a *symplectic form*  $\Omega$  (that is,  $\Omega$  is a holomorphic 2-form which is closed and non-degenerate at every point of  $Y_{\text{reg}}$ ) such that, for any desingularization  $f : \tilde{Y} \rightarrow Y$ , the 2-form  $f^*(\Omega)$  extends to a 2-form on the whole  $\tilde{Y}$ , then we say that  $Y$  is a *symplectic variety*. Moreover, if  $f : \tilde{Y} \rightarrow Y$  is a desingularization such that  $f^*(\Omega)$  extends to a symplectic form on  $\tilde{Y}$ , then we say that  $f$  is a *symplectic desingularization* of  $Y$ . It must be underlined that symplectic varieties do not always admit symplectic desingularizations, and when they do, there may be several of them.

Let  $W = \text{Hom}(V', V) \times \text{Hom}(V, V')$ , and  $G \subset GL(V)$  a reductive algebraic subgroup. We consider the symplectic reduction  $W///G$  as in the introduction. The following conjecture motivates the study (and the name!) of  $W///G$ :

**Conjecture 2.1** (Kaledin, Lehn, Sorger). With the above notation, the irreducible components  $Y_1, \dots, Y_r$  of  $W///G$  are symplectic varieties. Moreover, if every  $Y_i$  admits a symplectic desingularization, then the quotient  $\text{Hom}(V', V)//G$  is smooth.

When  $G$  is a finite group, Conjecture 2.1 was proved by Kaledin and Verbitsky, but the general case remains open. Let us mention that Becker showed in [Bec09] that the converse of the second part of Conjecture 2.1 holds for  $G = Sp(V)$  with  $n = 2$ . In our setting, that is when  $G = GL(V)$ ,  $O(V)$  or  $Sp(V)$ , one easily checks that Conjecture 2.1 holds (see [Terb, §A.2] for details).

**2.2. Closures of nilpotent orbits.** We now recall some basic facts concerning the closures of nilpotent orbits (see [CM93, Fu03a] for more details). Let  $\mathfrak{g}'$  be a semi-simple Lie algebra, and  $G'$  the adjoint group of  $\mathfrak{g}'$ . The following (well-known) result is due to Kostant, Kirillov, Souriau, and Panyushev:

**Theorem 2.2.** *The normalization of the closure of an adjoint orbit in a semi-simple Lie algebra is a symplectic variety.*

When  $\mathfrak{g}'$  is a simple Lie algebra of type A, C or D, one can associate partitions to nilpotent orbits. More precisely:

- If  $\mathfrak{g}' = \mathfrak{sl}_{n'}$ , then there exists a one-to-one correspondence between the partitions  $\mathbf{d} = (d_1 \geq \dots \geq d_k)$  of  $n'$  and the nilpotent orbits  $\mathcal{O}_{[d_1, \dots, d_k]}$  of  $\mathfrak{g}'$  ([CM93, §3.1]).
- If  $\mathfrak{g}' = \mathfrak{sp}_{2n'}$ , then there exists a one-to-one correspondence between the partitions  $\mathbf{d} = (d_1 \geq \dots \geq d_k)$  of  $2n'$ , in which every odd  $d_i$  occurs with even multiplicity, and the nilpotent orbits  $\mathcal{O}_{[d_1, \dots, d_k]}$  of  $\mathfrak{g}'$  ([CM93, §5.1]).
- If  $\mathfrak{g}' = \mathfrak{so}_{2n'}$ , then there exists a correspondence (which is not bijective anymore) between some partitions of  $2n'$  and the nilpotent orbits of  $\mathfrak{g}'$  ([CM93, §5.1]). Let  $\mathbf{d} = (d_1 \geq \dots \geq d_k)$  be a partition of  $2n'$  such that each even  $d_i$  occurs with even multiplicity. If at least one of the  $d_i$  is odd, then one associates to  $\mathbf{d}$  a unique nilpotent orbit  $\mathcal{O}_{[d_1, \dots, d_k]}$ ; else, one associates to  $\mathbf{d}$  two distinct nilpotent orbits  $\mathcal{O}_{[d_1, \dots, d_k]}^I$  and  $\mathcal{O}_{[d_1, \dots, d_k]}^{II}$ .

If  $\mathfrak{g}'$  is of type A, then  $\overline{\mathcal{O}_{\mathbf{d}}}$  is always normal ([KP79]). In types B,C and D, the geometry of  $\overline{\mathcal{O}_{\mathbf{d}}}$  was studied in [KP82]; in particular, if  $\mathfrak{g}'$  is of type C and  $d_1 + d_2 \leq 4$  resp. of type D and  $d_1 \leq 2$ , then  $\overline{\mathcal{O}_{\mathbf{d}}}$  is normal. From now on, we only consider partitions  $\mathbf{d}$  such that each  $d_i \leq 2$ . By Theorem 2.2, the variety  $\overline{\mathcal{O}_{\mathbf{d}}}$  is symplectic, and we are going to describe its symplectic desingularizations (see [Fu03a, Fu06b, FN04] for details).

Let  $f : Z \rightarrow \overline{\mathcal{O}_{\mathbf{d}}}$  be a symplectic desingularization. Then, by [Fu03a, Proposition 3.1], the group  $G'$  acts naturally on  $Z$  in such a way that  $f$  is  $G'$ -equivariant. One says that  $f$  is a *Springer desingularization* if there exists a parabolic subgroup  $P \subset G'$  and a  $G'$ -equivariant isomorphism between  $Z$  and the total space of the cotangent bundle over  $G'/P$ , denoted  $\mathcal{T}^*(G'/P)$ . Then, under this isomorphism, the map  $f$  becomes

$$\mathcal{T}^*(G'/P) \cong G' \times^P \mathfrak{u} \longrightarrow \mathfrak{g}', \quad (g', x) \longmapsto Ad(g').x,$$

where  $\mathfrak{u}$  is the nilradical of the Lie algebra of  $P$ , and  $G' \times^P \mathfrak{u} := (G' \times \mathfrak{u}) / \equiv$  with  $(g', u) \equiv (g' \circ p^{-1}, Ad(p).u)$ .

**Theorem 2.3.** ([Fu03a, Theorem 3.3]) *With the above notation, if  $f : Z \rightarrow \overline{\mathcal{O}_{\mathbf{d}}}$  is a symplectic desingularization, then  $f$  is a Springer desingularization.*

When  $\mathfrak{g}'$  is of type A, C or D, the Springer desingularizations of  $\overline{\mathcal{O}_{\mathbf{d}}}$  are known (up to isomorphism):

- Let  $\mathfrak{g}' = \mathfrak{sl}_{n'}$  and  $\mathbf{d} = [2^N, 1^{n'-2N}]$ . We denote  $\text{Gr}(p, \mathbb{C}^{n'})$  the Grassmannian of  $p$ -dimensional subspaces of  $\mathbb{C}^{n'}$ , and  $\mathcal{T}_1^*$  resp.  $\mathcal{T}_2^*$ , the cotangent bundle over  $\text{Gr}(N, \mathbb{C}^{n'})$  resp. over  $\text{Gr}(n' - N, \mathbb{C}^{n'})$ . By [Fu06b, §2], if  $N < \frac{n'}{2}$ , then  $\mathcal{T}_1^*$  and  $\mathcal{T}_2^*$  are the two Springer desingularizations of  $\overline{\mathcal{O}_{\mathbf{d}}}$ ; else,  $\mathcal{T}_1^* = \mathcal{T}_2^*$  is the unique Springer desingularization of  $\overline{\mathcal{O}_{\mathbf{d}}}$ .

- Let  $\mathfrak{g}' = \mathfrak{sp}_{2n'}$  and  $\mathbf{d} = [2^N, 1^{2(n'-N)}]$ . Then  $\overline{\mathcal{O}_{\mathbf{d}}}$  admits a Springer desingularization if and only if  $N = n'$  ([Fu03a, Proposition 3.19]). We denote  $\text{IGr}(p, \mathbb{C}^{2n'})$  the Grassmannian of isotropic  $p$ -dimensional subspaces of  $\mathbb{C}^{2n'}$ , and  $\mathcal{T}^*$  the cotangent bundle over  $\text{IGr}(n', \mathbb{C}^{2n'})$ . By [FN04, Proposition 3.5], if  $N = n'$ , then  $\mathcal{T}^*$  is the unique Springer desingularization of  $\overline{\mathcal{O}_{\mathbf{d}}}$ .
- Let  $\mathfrak{g}' = \mathfrak{so}_{2n'}$  and  $\mathbf{d} = [2^N, 1^{2(n'-N)}]$  (with  $N$  even). We recall that if  $d_1 = \dots = d_k = 2$ , then one associates to  $\mathbf{d}$  two distinct nilpotent orbits  $\mathcal{O}_{\mathbf{d}}^I$  and  $\mathcal{O}_{\mathbf{d}}^{II}$ . By [Fu03a, Proposition 3.20], the variety  $\overline{\mathcal{O}_{\mathbf{d}}}$  admits a Springer desingularization if and only if  $N \in \{n' - 1, n'\}$ . We denote  $\text{OGr}(p, \mathbb{C}^{2n'})$  the Grassmannian of isotropic  $p$ -dimensional subspaces of  $\mathbb{C}^{2n'}$ . The Grassmannian  $\text{OGr}(p, \mathbb{C}^{2n'})$  is irreducible except if  $p = n'$ , in which case  $\text{OGr}(n', \mathbb{C}^{2n'}) = \text{OG}^I \cup \text{OG}^{II}$  is the union of two irreducible components (exchanged by the natural action of the orthogonal group  $O(\mathbb{C}^{2n'})$ ). We denote  $\mathcal{T}_I^*$  resp.  $\mathcal{T}_{II}^*$ , the cotangent bundle over  $\text{OG}^I$ , resp. over  $\text{OG}^{II}$ . If  $N = n' - 1$ , then  $\mathcal{T}_I^*$  and  $\mathcal{T}_{II}^*$  are the two Springer desingularizations of  $\overline{\mathcal{O}_{[2^{n'-1}, 1^2]}}$  by [Fu06b, §2]. If  $N = n'$ , then  $\mathcal{T}_I^*$  resp.  $\mathcal{T}_{II}^*$ , is the unique Springer desingularization of  $\overline{\mathcal{O}_{[2^{n'}]}}^I$  resp. of  $\overline{\mathcal{O}_{[2^{n'}]}}^{II}$ , by [FN04, Proposition 3.5].

*Remark 2.4.* We will see in Sections 3 and 4 that the irreducible components of the symplectic reduction  $W//G$ , when  $G = GL(V)$  or  $Sp(V)$ , are closures of nilpotent orbits  $\mathcal{O}_{\mathbf{d}}$  in simple Lie algebras of type A and C respectively, where every  $d_i \leq 2$ . If  $G = O(V)$ , then it is shown in [Terb, §3.4] that  $W//G$  is the closure of the nilpotent orbit  $\mathcal{O}_{[2^N, 1^{2(n'-N)}]}$  in type D, where  $N := \min(n', n)$ .

### 3. CASE OF $GL_n$

In this section, we take  $G = GL(V)$  and  $G' = GL(V')$ , both acting on  $W = \text{Hom}(V', V) \times \text{Hom}(V, V')$  as in the introduction. We denote  $\mathfrak{g}$  resp.  $\mathfrak{g}'$ , the Lie algebra of  $G$  resp. of  $G'$ , and  $N := \min(E(\frac{n'}{2}), n)$ , where  $n := \dim(V)$ ,  $n' := \dim(V')$ , and  $E(\cdot)$  is the lower integer part.

**3.1. The quotient morphism.** The two main results of this section are Proposition 3.3, which describes the symplectic reduction  $W//G := \mu^{-1}(0)//G$ , and Corollary 3.6, which gives the Hilbert function  $h_0$  of the general fiber of the quotient morphism  $\nu: \mu^{-1}(0) \rightarrow W//G$ .

By (3), the zero fiber of the moment map  $\mu: W \rightarrow \mathfrak{g}^*$  is

$$\mu^{-1}(0) = \{(u_1, u_2) \in W \mid u_1 \circ u_2 = 0\},$$

which is a  $G' \times G$ -stable closed subset of  $W$ . Let us determine the irreducible components of  $\mu^{-1}(0)$  as well as their dimensions. Let  $m \in \{0, \dots, n'\}$ ; we define the closed subset

$$(4) \quad X_m := \left\{ (u_1, u_2) \in W \mid \begin{array}{l} \text{Im}(u_2) \subset \text{Ker}(u_1); \\ \text{rk}(u_2) \leq \min(n, m); \\ \dim(\text{Ker}(u_1)) \geq \max(n' - n, m). \end{array} \right\} \subset \mu^{-1}(0),$$

and we consider the diagram

$$\begin{array}{ccc}
 Z_m := \{(u_1, u_2, L) \in W \times \text{Gr}(m, V') \mid \text{Im}(u_2) \subset L \subset \text{Ker}(u_1)\} & & \\
 p_1 \downarrow & \searrow p_2 & \\
 X_m & & \text{Gr}(m, V')
 \end{array}$$

where the  $p_i$  are the natural projections. We fix  $L_0 \in \text{Gr}(m, V')$ ; the second projection equips  $Z_m$  with a structure of homogeneous vector bundle over  $\text{Gr}(m, V')$  whose fiber over  $L_0$  is isomorphic to  $F_m := \text{Hom}(V'/L_0, V) \times \text{Hom}(V, L_0)$ . Hence,  $Z_m$  is a smooth variety of dimension  $m(n' - m) + n'n$ .

**Proposition 3.1.** *The irreducible components of  $\mu^{-1}(0)$  are*

$$\begin{cases} X_0, \dots, X_{n'} & \text{if } n' \leq n; \\ X_{n'-n}, \dots, X_n & \text{if } n < n' < 2n; \\ X_n & \text{if } n' \geq 2n; \end{cases}$$

where  $X_m$  is defined by (4).

*Proof.* We have

$$\mu^{-1}(0) = \{(u_1, u_2) \in W \mid \text{Im}(u_2) \subset \text{Ker}(u_1)\} = \bigcup_{i=0}^{n'} X_i.$$

Furthermore, for every  $m \in \{0, \dots, n'\}$ , the morphism  $p_1$  is surjective and  $Z_m$  is irreducible, hence  $X_m$  is irreducible.

If  $n' \geq 2n$ , then

$$\begin{cases} X_0 \subset \dots \subset X_n; \\ X_n = \dots = X_{n'-n}; \\ X_{n'-n} \supset \dots \supset X_{n'}; \end{cases}$$

and thus  $\mu^{-1}(0) = X_n$ .

If  $n' < 2n$ , then

$$\begin{cases} X_0 \subset \dots \subset X_{\max(0, n'-n)}; \\ X_{\min(n', n)} \supset \dots \supset X_{n'}; \end{cases}$$

and one easily checks that there is no other inclusion relation between the  $X_m$ .  $\square$

**Corollary 3.2.** *The dimension of  $\mu^{-1}(0)$  is*

$$\dim(\mu^{-1}(0)) = \begin{cases} nn' + \frac{1}{4}n'^2 & \text{if } n' < 2n \text{ and } n' \text{ is even;} \\ nn' + \frac{1}{4}(n'^2 - 1) & \text{if } n' < 2n \text{ and } n' \text{ is odd;} \\ 2nn' - n^2 & \text{if } n' \geq 2n. \end{cases}$$

*Proof.* By Proposition 3.1, it is sufficient to compute the dimension of  $X_m$  for some  $m$ . If  $m \leq n$  or  $m \geq n' - n$ , one may check that the map  $p_1 : Z_m \rightarrow X_m$  is birational, and thus  $P(m) := \dim(X_m) = \dim(Z_m) = m(n' - m) + n'n$ . It remains simply to study the variations of the polynomial  $P$  to obtain the result.  $\square$



Let  $l \in \{0, \dots, N\}$ . We fix a basis  $\mathcal{B}$  of  $V$  resp.  $\mathcal{B}'$  of  $V'$ , and we introduce some notation that we will use in the proofs of Proposition 3.3 and Lemma 3.5:

$$(5) \quad \bullet \quad (u_1^l, u_2^l) := \left( \begin{bmatrix} 0_{l, n'-l} & I_l \\ 0_{n-l, n'-l} & 0_{n-l, l} \end{bmatrix}, \begin{bmatrix} I_l & 0_{l, n-l} \\ 0_{n'-l, l} & 0_{n'-l, n-l} \end{bmatrix} \right) \in W;$$

$$(6) \quad \bullet \quad f_l := \begin{bmatrix} 0_{l, n'-l} & I_l \\ 0_{n'-l, n'-l} & 0_{n'-l, l} \end{bmatrix} \in \mathfrak{g}'.$$

**Proposition 3.3.** *Let  $\mathcal{O}_{\mathbf{d}} \subset \mathfrak{g}' \cong \mathfrak{gl}_{n'}$  denote the nilpotent orbit associated to a partition  $\mathbf{d}$  of  $n'$ . Then the symplectic reduction of  $W$  by  $G$  is  $W///G = \overline{\mathcal{O}_{[2N, 1^{n'-2N}]}}$ .*

*Proof.* If  $f \in W///G$ , then there exists  $(u_1, u_2) \in \mu^{-1}(0)$  such that  $f = u_2 \circ u_1$ , and thus  $f \circ f = (u_2 \circ u_1) \circ (u_2 \circ u_1) = u_2 \circ (u_1 \circ u_2) \circ u_1 = 0$ , whence the inclusion " $\subset$ ". Now, let  $f \in \overline{\mathcal{O}_{[2N, 1^{n'-2N}]}}$ . Up to conjugation by an element of  $G'$ , we can suppose that  $f = f_l$  for some  $l \leq N$ , where  $f_l$  is defined by (6). But then  $u_2^l \circ u_1^l = f_l$  and  $u_1^l \circ u_2^l = 0$ , where  $u_1^l$  and  $u_2^l$  are defined by (5), and thus  $f \in W///G$ .  $\square$

**Corollary 3.4.** *The symplectic reduction  $W///G \subset \mathfrak{g}'$  decomposes into  $N+1$  orbits for the adjoint action of  $G'$ :*

$$U_i := \mathcal{O}_{[2^i, 1^{n'-2^i}]}, \quad \text{for } i = 0, \dots, N.$$

The closures of the nilpotent orbits  $U_i$  are nested in the following way:

$$\{0\} = \overline{U_0} \subset \dots \subset \overline{U_N} = W///G.$$

Hence,  $W///G$  is a symplectic variety (see Section 2), of dimension  $2N(n' - N)$  ([CM93, Corollary 6.1.4]), and whose singular locus is  $\overline{U_{N-1}}$  ([KP81, §3.2]).

By Corollary 3.2, the dimension of the general fiber of the quotient morphism  $\nu$  is

$$(7) \quad \begin{cases} nn' - \frac{1}{4}n'^2 & \text{if } n' < 2n \text{ and } n' \text{ is even;} \\ nn' - \frac{1}{4}(n'^2 - 1) & \text{if } n' < 2n \text{ and } n' \text{ is odd;} \\ n^2 & \text{if } n' \geq 2n. \end{cases}$$

If  $n' < 2n$ , then  $N = E(\frac{n'}{2})$ , and we denote

$$(8) \quad H := \left\{ \begin{bmatrix} M & 0_{n-N, N} \\ 0_{N, n-N} & I_N \end{bmatrix}, M \in GL_{n-N} \right\} \cong GL_{n-N},$$

which is a reductive algebraic subgroup of  $G \cong GL_n$ .

**Proposition 3.5.** *The general fiber of the quotient morphism  $\nu : \mu^{-1}(0) \rightarrow W///G$  is isomorphic to*

$$\begin{cases} G & \text{if } n' \geq 2n; \\ G/H & \text{if } n' < 2n \text{ and } n' \text{ is even;} \end{cases}$$

where  $H \subset G$  is the subgroup defined by (8).

*Proof.* We first suppose that  $n' < 2n$  and  $n'$  is even (that is,  $N = \frac{n'}{2}$ ). With the notation (5), and by [SB00, §I.6.2.5, Theorem 10], we have the equivalence

$$G.(u_1^N, u_2^N) \text{ is closed in } \mu^{-1}(0) \Leftrightarrow C_G(H).(u_1^N, u_2^N) \text{ is closed in } \mu^{-1}(0).$$

We have  $C_G(H) = \left\{ \begin{bmatrix} M & 0 \\ 0 & \lambda I_{n-N} \end{bmatrix}, M \in GL_N, \lambda \in \mathbb{G}_m \right\}$ , where  $\mathbb{G}_m$  denotes the multiplicative group. Hence

$$C_G(H) \cdot (u_1^N, u_2^N) = \left\{ \left( \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} M^{-1} & 0 \\ 0 & 0 \end{bmatrix} \right), M \in GL_N \right\} \subset \mu^{-1}(0)$$

is a closed subset, and thus  $G \cdot (u_1^N, u_2^N)$  is the unique closed orbit contained in the fiber  $\nu^{-1}(f_N)$ , where  $f_N$  is defined by (6). One may check that  $\text{Stab}_G((u_1^N, u_2^N)) = H$ . Furthermore,  $\dim(G/H) = N(2n - N)$ , which is also the dimension of the general fiber of  $\nu$  by (7), and thus  $\nu^{-1}(f_N) \cong G/H$ .

We now suppose that  $n' \geq 2n$  (that is,  $N = n$ ). One may check that  $\text{Stab}_G((u_1^n, u_2^n)) = Id$ , and thus the fiber  $\nu^{-1}(f_n)$  contains a unique closed orbit isomorphic to  $G$ . But  $\dim(G) = n^2$  is the dimension of the general fiber of  $\nu$  by (7), hence  $\nu^{-1}(f_n) \cong G$ .  $\square$

If  $n' < 2n$  and  $n'$  is odd, then the situation is more complicated (except the case  $n' = 1$  which is trivial) because the general fiber of the quotient morphism  $\nu$  is reducible. From now on, we will only consider the cases where  $n' \geq 2n$  or ( $n' < 2n$  and  $n'$  is even).

**Corollary 3.6.** *The Hilbert function  $h_0$  of the general fiber of the quotient morphism  $\nu: \mu^{-1}(0) \rightarrow W//G$  is given by:*

$$\forall M \in \text{Irr}(G), h_0(M) = \begin{cases} \dim(M) & \text{if } n' \geq 2n; \\ \dim(M^H) & \text{if } n' < 2n \text{ and } n' \text{ is even;} \end{cases}$$

where  $H \subset G$  is the subgroup defined by (8).

**3.2. The reduction principle for the main component.** In this section we prove the reduction principle when  $G = GL(V)$  (Proposition 3.8).

The subset  $\mu^{-1}(0) \subset W$  being  $G' \times G$ -stable, it follows from [Bri, Lemma 3.3] that the invariant Hilbert scheme

$$\mathcal{H} := \text{Hilb}_{h_0}^G(\mu^{-1}(0))$$

is a  $G'$ -stable closed subscheme of  $\text{Hilb}_{h_0}^G(W)$ . The scheme  $\text{Hilb}_{h_0}^G(W)$  was studied in [Tera]; let us recall

**Proposition 3.7.** ([Tera, §4.4]) *Let  $h_0$  be the Hilbert function given by Corollary 3.6, and  $G' = GL(V')$  acting naturally on  $\text{Gr}(n' - h_0(V), V'^*) \times \text{Gr}(n' - h_0(V^*), V')$ . Then there exists a  $G'$ -equivariant morphism*

$$\begin{array}{ccc} \rho : \text{Hilb}_{h_0}^G(W) & \rightarrow & \text{Gr}(n' - h_0(V), V'^*) \times \text{Gr}(n' - h_0(V^*), V') \\ Z & \mapsto & (\text{Ker}(f_Z^1), \text{Ker}(f_Z^2)) \end{array}$$

where  $f_Z^1: V'^* \cong \text{Mor}^G(W, V) \rightarrow \text{Mor}^G(Z, V)$  and  $f_Z^2: V' \cong \text{Mor}^G(W, V^*) \rightarrow \text{Mor}^G(Z, V^*)$  are the restriction maps.

By Corollary 3.6, we have  $h_0(V) = h_0(V^*) = N$ . We identify  $\text{Gr}(n' - N, V'^*)$  with  $\text{Gr}(N, V')$ , and we denote

$$A_i := \{(L_1, L_2) \in \text{Gr}(N, V') \times \text{Gr}(n' - N, V') \mid \dim(L_1 \cap L_2) = N - i\}, \text{ for } i = 0, \dots, N.$$

The  $A_i$  are the  $N + 1$  orbits for the action of  $G'$  on  $\text{Gr}(N, V') \times \text{Gr}(n' - N, V')$ , and

$$A_0 = \overline{A_0} \subset \overline{A_1} \subset \dots \subset \overline{A_N} = \text{Gr}(N, V') \times \text{Gr}(n' - N, V').$$

In particular,  $A_N$  is the unique open orbit and

$$(9) \quad A_0 = \mathcal{F}_{N, n'-N} := \{(L_1, L_2) \in \text{Gr}(N, V') \times \text{Gr}(n' - N, V') \mid L_1 \subset L_2\},$$

which is a partial flag variety, is the unique closed orbit. Let

- $a_0 := (L_1, L_2) \in A_0$ , and  $P$  the parabolic subgroup of  $G'$  stabilizing  $a_0$ ;
- $W' := \{(u_1, u_2) \in W \mid L_2 \subset \text{Ker}(u_1) \text{ and } \text{Im}(u_2) \subset L_1\}$ , which is a  $P \times G'$ -module contained in  $\mu^{-1}(0)$ ;
- $\mathcal{H}' := \text{Hilb}_{h_0}^G(W')$ , and  $\mathcal{H}'^{\text{main}}$  its main component.

Let us note that, if  $n' \geq 2n$  or ( $n' < 2n$  and  $n'$  is even), then  $h_0$  coincides with the Hilbert function of the general fiber of the quotient morphism  $W' \rightarrow W'//G$  by [Tera, Proposition 4.13]; in particular,  $\mathcal{H}'^{\text{main}}$  is well-defined. We are going to show

**Proposition 3.8.** *If  $n' \geq 2n$  or ( $n' < 2n$  and  $n'$  is even), and with the above notation, we have a  $G'$ -equivariant isomorphism*

$$\mathcal{H}^{\text{main}} \cong G' \times^P \mathcal{H}'^{\text{main}}.$$

First of all, we need

**Lemma 3.9.** *If  $n' \geq 2n$  or ( $n' < 2n$  and  $n'$  is even), then the morphism  $\rho$  of Proposition 3.7 sends  $\mathcal{H}^{\text{main}}$  onto  $A_0$ , the  $G'$ -variety defined by (9).*

*Proof.* As the quotient morphism  $\nu : \mu^{-1}(0) \rightarrow W'//G$  is flat over the open orbit  $U_N$ , the restriction of the Hilbert-Chow morphism  $\gamma$  to  $\gamma^{-1}(U_N)$  is an isomorphism. We fix  $f_N \in U_N$ , and we denote  $Q := \text{Stab}_{G'}(f_N)$ , and  $Z_N$  the unique point of  $\mathcal{H}$  such that  $\gamma(Z_N) = f_N$ . As  $\gamma$  is  $G'$ -equivariant,  $Z_N$  is  $Q$ -stable. In addition,  $\rho$  is also  $G'$ -equivariant, hence  $\rho(Z_N)$  is a fixed point for the action of  $Q$ . But one may check that  $\text{Gr}(N, V') \times \text{Gr}(n' - N, V')$  has a unique fixed point for  $Q$ , which is contained in  $A_0$ . Then, as  $A_0$  is  $G'$ -stable, we have  $\rho(Z) \in A_0$ , for every  $Z \in \gamma^{-1}(U_N)$ . Hence,  $\rho^{-1}(A_0)$  is a closed subset of  $\text{Hilb}_{h_0}^G(W)$  containing  $\gamma^{-1}(U_N)$ , and the result follows.  $\square$

The restriction  $\rho|_{\mathcal{H}^{\text{main}}} : \mathcal{H}^{\text{main}} \rightarrow A_0$  is  $G'$ -equivariant, hence  $\mathcal{H}^{\text{main}}$  is the total space of a  $G'$ -homogeneous fiber bundle over  $A_0$ . Let  $F$  be the scheme-theoretic fiber of  $\rho|_{\mathcal{H}^{\text{main}}}$  over  $a_0$ . The action of  $P$  on  $\mathcal{H}^{\text{main}}$ , induced by the action of  $G'$ , stabilizes  $F$ , and we have a  $G'$ -equivariant isomorphism

$$(10) \quad \mathcal{H}^{\text{main}} \cong G' \times^P F.$$

Hence, to show Proposition 3.8, we have to determine  $F$  as a  $P$ -scheme. We start by considering  $F'$ , the scheme-theoretic fiber of the restriction  $\rho|_{\mathcal{H}} : \mathcal{H} \rightarrow \text{Gr}(N, V') \times \text{Gr}(n' - N, V')$  over  $a_0$ , as a  $P$ -scheme. The proof of the next lemma is analogous to the proof of [Tera, Lemma 3.7].

**Lemma 3.10.** *With the above notation, we have a  $P$ -equivariant isomorphism*

$$F' \cong \mathcal{H}',$$

where  $P$  acts on  $\mathcal{H}'$  via its action on  $W'$ .

As  $\mathcal{H}^{\text{main}}$  is a variety of dimension  $2N(n' - N)$ , we deduce that  $F$  is a variety of dimension  $N^2$ . By Lemma 3.10, the fiber  $F$  is isomorphic to a subvariety of  $\mathcal{H}'^{\text{main}}$ , but  $\dim(\mathcal{H}'^{\text{main}}) = N^2$ , and thus we have a  $P$ -equivariant isomorphism

$$(11) \quad F \cong \mathcal{H}'^{\text{main}},$$

and Proposition 3.8 follows.

*Remark 3.11.* The scheme  $\mathcal{H}'$  is  $P$ -stable and identifies with a closed subscheme of  $\mathcal{H}$ , hence we have an inclusion of  $G'$ -schemes  $G' \times^P \mathcal{H}' \subset \mathcal{H}$ .

**3.3. Proof of Theorem A for  $GL(V)$ .** First, we recall the following result:

**Theorem 3.12.** [Tera, §1, Theorem] *Let  $G = GL(V)$ ,  $W = \text{Hom}(V', V) \times \text{Hom}(V, V')$ , and  $h_W$  the Hilbert function of the general fiber of the quotient morphism  $W \rightarrow W//G$ . We denote  $n := \dim(V)$ ,  $n' := \dim(V')$ , and  $Y_0$  the blow-up of  $W//G = \text{End}(V')^{\leq n}$  at 0. In the following cases, the invariant Hilbert scheme  $\mathcal{H}' := \text{Hilb}_{h_W}^G(W)$  is a smooth variety and the Hilbert-Chow morphism is the succession of blow-up described as follows:*

- if  $n \geq 2n' - 1$ , then  $\mathcal{H}' \cong W//G = \text{End}(V')$ ;
- if  $n' > n = 1$  or  $n' = n = 2$ , then  $\mathcal{H}' \cong Y_0$ ;
- if  $n' > n = 2$ , then  $\mathcal{H}'$  is isomorphic to the blow-up of  $Y_0$  along the strict transform of  $\text{End}(V')^{\leq 1}$ .

Let us now consider the following diagram

$$(12) \quad \begin{array}{ccc} & \mathcal{F}_{N, n'-N} & \\ p_1 \swarrow & & \searrow p_2 \\ \text{Gr}(N, V') & & \text{Gr}(n'-N, V') \end{array}$$

where  $\mathcal{F}_{N, n'-N}$  is defined by (9),  $p_1$  and  $p_2$  being the natural projections. We denote  $\underline{V}'$  the constant vector bundle over  $\mathcal{F}_{N, n'-N}$  with fiber  $V'$ , and  $T_1$  resp.  $T_2$ , the pull-back of the tautological bundle over  $\text{Gr}(N, V')$  by  $p_1$ , resp. over  $\text{Gr}(n'-N, V')$  by  $p_2$ . In particular, if  $N = \frac{n'}{2}$ , then  $\mathcal{F}_{N, n'-N} = \text{Gr}(N, V')$  and  $T := T_1 = T_2$  is the tautological bundle over  $\text{Gr}(N, V')$ .

We deduce from Proposition 3.8 and Theorem 3.12 that we have the following  $G'$ -equivariant isomorphisms

$$(13) \quad \mathcal{H}^{\text{main}} \cong \begin{cases} \text{Hom}(\underline{V}'/T, T) & \text{if } n \geq n' - 1 \text{ and } n' \text{ is even;} \\ \text{Hom}(\underline{V}'/T_2, T_1) & \text{if } n = 1 \text{ and } n' \geq 3; \\ \text{Bl}_0(\text{Hom}(\underline{V}'/T_2, T_1)) & \text{if } n = 2 \text{ and } n' \geq 4; \end{cases}$$

where  $\text{Bl}_0(\cdot)$  denotes the blow-up along the zero section. In all these cases,  $\mathcal{H}^{\text{main}}$  is smooth, and thus the Hilbert-Chow morphism  $\gamma : \mathcal{H}^{\text{main}} \rightarrow W//G$  is a (distinguished) desingularization.

On the other hand, we saw in Section 2 that the Springer desingularizations of  $W//G$  are the cotangent bundles  $\mathcal{T}_1^* := \mathcal{T}^* \text{Gr}(N, V')$  and  $\mathcal{T}_2^* := \mathcal{T}^* \text{Gr}(n'-N, V') \cong \mathcal{T}^* \text{Gr}(N, V'^*)$ . We then distinguish between two cases:

- (1) If  $N < \frac{n'}{2}$ , then let us show by contradiction that  $\gamma : \mathcal{H}^{\text{main}} \rightarrow W//G$  can not be a Springer desingularization. First, we consider the isomorphism of  $G' \times G$ -modules

$$W \rightarrow \text{Hom}(V'^*, V^*) \times \text{Hom}(V^*, V'^*), \quad (u_1, u_2) \mapsto (u_2^*, u_1^*),$$

where  $u_\bullet^*$  denotes the pull-back of  $u_\bullet$ . Denoting  $\mathcal{H}_0 := \text{Hilb}_{h_0}^G(\mu'^{-1}(0))$ , where  $\mu'$  is the moment map for the natural action of  $G$  on  $\text{Hom}(V'^*, V^*) \times \text{Hom}(V^*, V'^*)$ , we have an isomorphism of  $G'$ -varieties  $\mathcal{H}^{\text{main}} \cong \mathcal{H}_0^{\text{main}}$ . Now if we suppose that (say)  $\mathcal{H}^{\text{main}} \cong \mathcal{T}_1^*$ , then we get that  $\mathcal{H}_0^{\text{main}} \cong \mathcal{T}_2^*$ , and thus  $\mathcal{T}_1^* \cong \mathcal{T}_2^*$  as a  $G'$ -variety, which is absurd.

However, one easily checks that if  $n \in \{1, 2\}$  and  $n' \geq 2n + 1$ , then  $\gamma :$

$\mathcal{H}^{\text{main}} \rightarrow W//G$  dominates the two Springer desingularizations  $\mathcal{T}_1^*$  and  $\mathcal{T}_2^*$  (see [Terb, §A.2.2] for details).

- (2) If  $N = \frac{n'}{2}$ , then  $\mathcal{T}^* := \mathcal{T}_1^* = \mathcal{T}_2^*$  is the unique Springer desingularization of  $W//G$ . Let us show that  $\gamma: \mathcal{H}^{\text{main}} \rightarrow W//G$  is a Springer desingularization if and only if  $n \geq n' - 1$ . The implication " $\Leftarrow$ " is given by (13) since  $\mathcal{T}^* \cong \text{Hom}(\underline{V}'/T, T)$ . The other implication is given by:

**Lemma 3.13.** *If  $N = \frac{n'}{2}$  and the Hilbert-Chow morphism  $\gamma: \mathcal{H}^{\text{main}} \rightarrow W//G$  is a Springer desingularization, then  $n \geq n' - 1$ .*

*Proof.* We suppose that  $\gamma: \mathcal{H}^{\text{main}} \rightarrow W//G$  is a Springer desingularization, that is,  $\mathcal{H}^{\text{main}} \cong \mathcal{T}^*$  as a  $G'$ -variety. We fix  $L \in \text{Gr}(N, V')$ , and we define  $P \subset G'$ ,  $W'$ , and  $\mathcal{H}'^{\text{main}}$  as in Section 3.2. We have  $\mathcal{T}^* \cong G' \times^P \text{Hom}(V'/L, L)$ , and it follows from (10) and (11) that  $\mathcal{H}^{\text{main}} \cong G' \times^P \mathcal{H}'^{\text{main}}$ . Hence,  $\mathcal{H}'^{\text{main}} \cong \text{Hom}(V'/L, L)$  as a  $P$ -variety. We denote  $\gamma': \mathcal{H}'^{\text{main}} \rightarrow W'//G$  the restriction of the Hilbert-Chow morphism. As  $\gamma'$  is projective and birational, and  $W'//G = \text{Hom}(V'/L, L)$  is smooth, the Zariski's Main Theorem implies that  $\gamma'$  is an isomorphism. It follows that the quotient morphism  $\nu': W' \rightarrow W'//G$  is flat, and thus  $n \geq 2N - 1$  by [Tera, Corollary 4.12].  $\square$

In addition, if  $n' = 4$  and  $n = 2$ , then by (13) we have  $\mathcal{H}^{\text{main}} \cong \text{Bl}_0(\mathcal{T}^*)$ , and thus  $\gamma$  dominates the unique Springer desingularization of  $W//G$ .

**3.4. Reducibility of the invariant Hilbert scheme.** The aim of this section is to show Proposition B for  $GL(V)$ , that is, to show

**Proposition 3.14.** *If  $n' \geq 2n$ , then the invariant Hilbert scheme  $\mathcal{H}$  is reducible.*

We suppose that  $n' \geq 2n$ , then  $N = n$ . We fix

$$(14) \quad a_n = (L'_1, L'_2) \in A_n$$

a point of the open  $G'$ -orbit of  $\text{Gr}(n, V') \times \text{Gr}(n' - n, V')$ , and we consider

$$\begin{aligned} W'' &:= \{(u_1, u_2) \in W \mid L'_2 \subset \text{Ker}(u_1) \text{ and } \text{Im}(u_2) \subset L'_1\} \\ &\cong \text{Hom}(V'/L'_2, V) \times \text{Hom}(V, L'_1), \end{aligned}$$

which is a  $G$ -submodule of  $W$ . As  $V' = L'_1 \oplus L'_2$ , we have a natural identification  $W'' \cong \text{Hom}(L'_1, V) \times \text{Hom}(V, L'_1)$  as a  $G$ -module. We saw at the beginning of Section 1 that the  $G$ -module  $W''$  is symplectic and has a  $G$ -equivariant moment map that we denote  $\mu': W'' \rightarrow \mathfrak{g}^*$ . The proof of the next lemma is analogous to the proof of [Tera, Lemma 3.7].

**Lemma 3.15.** *We suppose that  $n' \geq 2n$ , and let  $\rho: \mathcal{H} \rightarrow \text{Gr}(n, V') \times \text{Gr}(n' - n, V')$  be the morphism of Proposition 3.7. The scheme-theoretic fiber  $F''$  of  $\rho$  over the point  $a_n$ , defined by (14), is isomorphic to the invariant Hilbert scheme  $\text{Hilb}_{h_0}^G(\mu'^{-1}(0))$ , where  $h_0$  is the Hilbert function defined by  $h_0(M) = \dim(M)$ , for every  $M \in \text{Irr}(G)$ , and  $\mu': W'' \rightarrow \mathfrak{g}^*$  is the moment map defined above.*

*Remark 3.16.* The Hilbert function  $h_0$  of Lemma 3.15 does not generally coincide with the Hilbert function of the general fiber of the quotient morphism  $\mu'^{-1}(0) \rightarrow \mu'^{-1}(0)//G$ .

By Lemma 3.9, the morphism  $\rho : \text{Hilb}_{h_0}^G(W) \rightarrow \text{Gr}(n, V') \times \text{Gr}(n' - n, V')$  of Proposition 3.7 sends  $\mathcal{H}^{\text{main}}$  onto  $A_0$ . Hence, to show Proposition 3.14, it is enough, by Lemma 3.15, to show that  $\text{Hilb}_{h_0}^G(\mu'^{-1}(0))$  is non-empty. We denote  $V'' := L'_1$ , and we equip  $W'' \cong \text{Hom}(V'', V) \times \text{Hom}(V, V'')$  with the natural action of  $G'' := GL(V'')$ . Then we have

$$(15) \quad \begin{aligned} \mathbb{C}[W'']_2 \cong & (S^2(V'') \otimes S^2(V^*)) \oplus (S^2(V''^*) \otimes S^2(V)) \\ & \oplus (\Lambda^2(V'') \otimes \Lambda^2(V^*)) \oplus (\Lambda^2(V''^*) \otimes \Lambda^2(V)) \\ & \oplus ((sl(V'') \oplus M_0) \otimes (sl(V) \oplus V_0)) \text{ as a } G'' \times G\text{-module,} \end{aligned}$$

where  $M_0$  resp.  $V_0$ , is the trivial  $G''$ -module resp. the trivial  $G$ -module, and  $sl(V'') := \{f \in \text{End}(V'') \mid \text{tr}(f) = 0\}$ .

We denote  $I_0$  the ideal of  $\mathbb{C}[W'']$  generated by  $(sl(V'') \otimes V_0) \oplus (M_0 \otimes V_0) \oplus (M_0 \otimes sl(V)) \subset \mathbb{C}[W'']_2$ . The ideal  $I_0$  is homogeneous,  $G'' \times G$ -stable, and contains the ideal generated by the homogeneous  $G''$ -invariants of positive degree of  $\mathbb{C}[W'']$ . In particular,  $I_0$  identifies with an ideal of  $\mathbb{C}[\mu'^{-1}(0)]$ .

**Proposition 3.17.** *Let  $I_0 \subset \mathbb{C}[W'']$  be the ideal defined above, then  $I_0$  is a point of the invariant Hilbert scheme  $\text{Hilb}_{h_0}^G(\mu'^{-1}(0))$  defined in Lemma 3.15.*

*Proof.* We have to check that the ideal  $I_0$  has the good Hilbert function, that is,

$$\mathbb{C}[W'']/I_0 \cong \bigoplus_{M \in \text{Irr}(G)} M^{\oplus \dim(M)}$$

as a  $G$ -module. To do that, we are going to adapt the method used by Kraft and Schwarz to prove [KS, Theorem 9.1].

We denote  $R := V''^* \otimes V$ , which is an irreducible  $G'' \times G$ -submodule of  $W''^* \cong R \oplus R^*$ . Then  $R$  and  $R^*$  are orthogonal modulo  $I_0$ , which means that the image of the  $G'' \times G$ -submodule  $R \otimes R^* \subset \mathbb{C}[W'']_2$  in  $\mathbb{C}[W'']/I_0$  is isomorphic to the highest weight component of  $R \otimes R^*$  (that is,  $sl(V'') \otimes sl(V)$ ). Then, by [Bri85, Lemme 4.1], any irreducible  $G'' \times G$ -submodule of  $\mathbb{C}[R]$  is orthogonal to any irreducible  $G'' \times G$ -submodule of  $\mathbb{C}[R^*]$ , and thus the natural morphism

$$\phi : \mathbb{C}[R]^{U'' \times U} \otimes \mathbb{C}[R^*]^{U'' \times U} \rightarrow (\mathbb{C}[W'']/I_0)^{U'' \times U}$$

is surjective, where  $U''$  resp.  $U$ , denotes the unipotent radical of a Borel subgroup  $B'' \subset G''$  resp.  $B \subset G$ . Furthermore, if  $T'' \subset B''$  resp.  $T \subset B$ , is a maximal torus, then  $\phi$  is  $T'' \times T$ -equivariant.

Now by [Pro07, §13.5.1] we have the following isomorphisms of  $T'' \times T$ -algebras  $\mathbb{C}[R]^{U'' \times U} \cong \mathbb{C}[x_1, \dots, x_n]$ , where  $x_i \in \Lambda^i V'' \otimes \Lambda^i V^*$  is a highest weight vector, and  $\mathbb{C}[R^*]^{U'' \times U} \cong \mathbb{C}[y_1, \dots, y_n]$ , where  $y_j \in \Lambda^j V''^* \otimes \Lambda^j V$  is a highest weight vector. Hence, we have an exact sequence

$$0 \rightarrow K_0 \rightarrow \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n] \rightarrow (\mathbb{C}[W'']/I_0)^{U'' \times U} \rightarrow 0,$$

where  $K_0$  is the kernel of  $\phi$ . One may check that the ideal  $K_0$  is generated by the products  $x_r y_s$  with  $r + s > n$  (see [KS, §9, Proof of Theorem 9.1(1)]).

We denote  $\Lambda = \langle \epsilon_1, \dots, \epsilon_n \rangle$  the weight lattice of the linear group  $GL_n$ , and  $\Lambda_+ \subset \Lambda$  the subset of dominant weights, that is, weights of the form  $r_1 \epsilon_1 + \dots + r_n \epsilon_n$ , with  $r_1 \geq \dots \geq r_n$ . If  $\lambda \in \Lambda_+$ , then we denote  $S^\lambda(\mathbb{C}^n)$  the irreducible  $GL_n$ -module of highest weight  $\lambda$ . We fix  $\lambda = k_1 \epsilon_1 + \dots + k_t \epsilon_t - k_{t+1} \epsilon_{t+1} - \dots - k_n \epsilon_n \in \Lambda_+$ , where each

$k_i$  is a nonnegative integer. One easily checks that the weight of the monomial

$$x_{n-t}^{k_{t+1}} x_{n-t-1}^{k_{t+2}-k_{t+1}} x_{n-t-2}^{k_{t+3}-k_{t+2}} \dots x_1^{k_n-k_{n-1}} y_t^{k_t} y_{t-1}^{k_{t-1}-k_t} y_{t-2}^{k_{t-2}-k_{t-1}} \dots y_1^{k_1-k_2}$$

for the action of  $T'' \times T$  is  $(\lambda^*, \lambda)$ , where  $\lambda^*$  denotes the highest weight of the  $GL_n$ -module  $S^\lambda(\mathbb{C}^{n^*})$ , and that  $\lambda$  uniquely determines this monomial. We get that the isotypic component of the  $G$ -module  $S^\lambda(V)$  in  $\mathbb{C}[W'']/I_0$  is the  $G'' \times G$ -module  $S^\lambda(V''^*) \otimes S^\lambda(V)$ . As  $\dim(V) = \dim(V'') = n$ , we have  $\dim(S^\lambda(V)) = \dim(S^\lambda(V''^*))$  for every  $\lambda \in \Lambda^+$ . In other words, each irreducible  $G$ -module  $M$  occurs in  $\mathbb{C}[W'']/I_0$  with multiplicity  $\dim(M)$ .  $\square$

By Proposition 3.17, the scheme  $\text{Hilb}_{h_0}^G(\mu'^{-1}(0))$  is non-empty, and thus  $\mathcal{H}$  has an irreducible component, different from  $\mathcal{H}^{\text{main}}$ , of dimension greater or equal to  $\dim(A_n) = 2n(n' - n)$ , which implies Proposition 3.14.

**3.5. Study of the case  $n = 1$ .** In all this Section, we suppose that  $n = 1$  and  $n' \geq 2$  (the case  $n' = 1$  being trivial). Then  $G = \mathbb{G}_m$  is the multiplicative group,  $W//G = \overline{\mathcal{O}_{[2,1^{n'-2}]}} \subset \mathfrak{g}'$ , and the morphism of Proposition 3.7 is  $\rho : \text{Hilb}_{h_0}^G(W) \rightarrow \mathbb{P}(V') \times \mathbb{P}(V'^*)$ . The Segre embedding gives a  $G'$ -equivariant isomorphism  $\mathbb{P}(V') \times \mathbb{P}(V'^*) \cong \mathbb{P}(\mathfrak{g}'^{\leq 1})$ , where  $\mathfrak{g}'^{\leq 1} := \{f \in \mathfrak{g}' \mid \text{rk}(f) \leq 1\}$ , and thus we can consider  $\rho' : \text{Hilb}_{h_0}^G(W) \rightarrow \mathbb{P}(\mathfrak{g}'^{\leq 1})$ , the morphism induced by  $\rho$ .

We saw in Section 3.3 that  $\mathcal{H}^{\text{main}}$  is a smooth variety, and in Section 3.4 that  $\mathcal{H}$  is always reducible. In this section, we determine the irreducible components of  $\mathcal{H}$  when  $n = 1$ .

**Proposition 3.18.** *We equip  $\mathcal{H}$  with its reduced structure. If  $n' > n = 1$  and with the above notation, we have a  $G'$ -equivariant isomorphism*

$$\mathcal{H} \cong \left\{ (f, L) \in \overline{\mathcal{O}_{[2,1^{n'-2}]}} \times \mathbb{P}(\mathfrak{g}'^{\leq 1}) \mid f \in L \right\}.$$

In particular,  $\mathcal{H}$  is the union of two smooth irreducible components of dimension  $2n' - 2$  defined by:

- $C_1 := \left\{ (f, L) \in \overline{\mathcal{O}_{[2,1^{n'-2}]}} \times \mathbb{P}(\overline{\mathcal{O}_{[2,1^{n'-2}]}}) \mid f \in L \right\} = \mathcal{H}^{\text{main}}$ , and the Hilbert-Chow morphism  $\gamma : \mathcal{H}^{\text{main}} \rightarrow \overline{\mathcal{O}_{[2,1^{n'-2}]}}$  is the blow-up of  $\overline{\mathcal{O}_{[2,1^{n'-2}]}}$  at 0;
- $C_2 := \left\{ (0, L) \in \overline{\mathcal{O}_{[2,1^{n'-2}]}} \times \mathbb{P}(\mathfrak{g}'^{\leq 1}) \right\} \cong \mathbb{P}(\mathfrak{g}'^{\leq 1})$ .

*Proof.* By [Tera, §1, Theorem], we have a  $G'$ -equivariant isomorphism

$$\gamma \times \rho' : \text{Hilb}_{h_0}^G(W) \rightarrow \left\{ (f, L) \in \mathfrak{g}'^{\leq 1} \times \mathbb{P}(\mathfrak{g}'^{\leq 1}) \mid f \in L \right\}.$$

Then, by [Bri, Lemma 3.3], we have a  $G'$ -equivariant closed embedding

$$\gamma \times \rho' : \mathcal{H} \hookrightarrow \mathcal{Y} := \left\{ (f, L) \in \overline{\mathcal{O}_{[2,1^{n'-2}]}} \times \mathbb{P}(\mathfrak{g}'^{\leq 1}) \mid f \in L \right\}.$$

One may check that  $\mathcal{Y}$  is the union of the two closed subset  $C_1$  and  $C_2$ , both of dimension  $2n' - 2$ . The morphism  $\gamma \times \rho'$  sends  $\mathcal{H}^{\text{main}}$  into  $C_1$ ; the varieties  $\mathcal{H}^{\text{main}}$  and  $C_1$  have the same dimension, hence  $\gamma \times \rho' : \mathcal{H}^{\text{main}} \rightarrow C_1$  is an isomorphism. On the other hand, we saw in Section 3.4 that  $\mathcal{H}$  admits a second irreducible component, denoted  $\mathcal{H}_2$ , of dimension at least  $2n' - 2$ , which is the dimension of  $C_2$ , and thus  $\gamma \times \rho'$  is an isomorphism between  $\mathcal{H}_2$  and  $C_2$ .  $\square$

*Remark 3.19.* One may check that the component  $C_2$  of Proposition 3.18 is formed by the homogeneous ideals of  $\mathbb{C}[\mu^{-1}(0)]$ .

When  $n' \geq 2n \geq 4$ , great dimensional irreducible components (other than  $\mathcal{H}^{\text{main}}$ ) may appear. For instance, if  $n = 2$  and  $n' \geq 4$ , then one may check that the irreducible component formed by the homogeneous ideals of  $\mathbb{C}[\mu^{-1}(0)]$  is of dimension  $4n' - 5$ , whereas the main component  $\mathcal{H}^{\text{main}}$  is of dimension  $4n' - 8$ . In addition, we showed in Section 3.4 that  $\mathcal{H}$  has at least two components, but  $\mathcal{H}$  may have much more components.

#### 4. CASE OF $Sp_n$ ( $n$ EVEN)

In this section, we denote  $E := V' \oplus V'^*$ , and we take  $G = Sp(V)$  and  $G' = SO(E)$ . We identify  $W = \text{Hom}(V', V) \times \text{Hom}(V, V') \cong \text{Hom}(E, V)$ , equipped with the natural action of  $G' \times G$ . We denote  $\mathfrak{g}$  resp.  $\mathfrak{g}'$ , the Lie algebra of  $G$  resp. of  $G'$ , and we recall that we denote  $n := \dim(V)$  (which is even), and  $n' := \dim(V')$ .

**4.1. The quotient morphism.** The main results of this section are Proposition 4.3, which describes the irreducible components of the symplectic reduction  $W///G := \mu^{-1}(0)//G$ , and Corollary 4.6, which gives the Hilbert function of the general fiber of the quotient morphism for each irreducible component of  $W///G$ . Contrary to the case of  $GL(V)$  studied in Section 3, we will see that  $W///G$  is reducible when  $n' \leq n$  and  $n'$  is even.

As  $G$  resp.  $G'$ , preserves a non-degenerate bilinear form on  $V$  resp. on  $E$ , we have a canonical isomorphism  $V \cong V^*$  resp.  $E \cong E^*$ . If  $w \in \text{Hom}(E, V)$ , we denote  ${}^t w \in \text{Hom}(V^*, E^*) \cong \text{Hom}(V, E)$  the transpose of  $w$ . Then, by [Bec09, Proposition 3.1], the zero fiber of the moment map  $\mu : W \rightarrow \mathfrak{g}^*$  is

$$\mu^{-1}(0) = \{w \in W \mid w \circ {}^t w = 0\},$$

which is a  $G' \times G$ -stable closed subset of  $W$ .

*Remark 4.1.* One may check that the biggest subgroup of  $GL(E)$  that stabilizes  $\mu^{-1}(0)$  in  $W$  is the orthogonal group  $O(E)$ . However, for practical reasons (study of the nilpotent orbits of  $\mathfrak{g}'$ ), we prefer to consider the action of  $G' = SO(E)$ .

The proof of the next proposition is analogous to the one of Proposition 3.1 and Corollary 3.2.

**Proposition 4.2.** *The zero fiber of the moment map  $\mu : W \rightarrow \mathfrak{g}^*$  is*

- a variety of dimension  $2n'n - \frac{1}{2}n(n+1)$  if  $n' > n$ ;
- the union of two varieties of dimension  $n'n + \frac{1}{2}n'(n'-1)$  if  $n' \leq n$ .

The following result was proved by Becker ([Bec09, Proposition 3.6]):

**Proposition 4.3.** *Let  $\mathbf{d}$  be a partition of  $2n'$ , and  $\mathcal{O}_{\mathbf{d}}$  resp.  $\mathcal{O}_{\mathbf{d}}^I$  and  $\mathcal{O}_{\mathbf{d}}^{II}$ , denote the nilpotent orbit(s) of  $\mathfrak{g}' \cong \mathfrak{so}_{2n'}$  associated to  $\mathbf{d}$ . Then the symplectic reduction of  $W$  by  $G$  is*

$$W///G = \begin{cases} \overline{\mathcal{O}_{[2^n, 1^{2(n'-n)]}}} & \text{if } n' > n; \\ \overline{\mathcal{O}_{[2^{n'-1}, 1^2]}} & \text{if } n' < n \text{ and } n' \text{ is odd}; \\ \overline{\mathcal{O}_{[2^{n'}]}^I \cup \mathcal{O}_{[2^{n'}]}^{II}} & \text{if } n' \leq n \text{ and } n' \text{ is even.} \end{cases}$$

**Corollary 4.4.** *The orbits for the adjoint action of  $G'$  on  $W///G$  are*

- $U_i := \mathcal{O}_{[2^i, 1^{2(n'-i)]}}$ , for  $i = 0, 2, \dots, n$ , if  $n' > n$ ;
- $U_i := \mathcal{O}_{[2^i, 1^{2(n'-i)]}}$ , for  $i = 0, 2, \dots, n'-1$ , if  $n' < n$  and  $n'$  is odd;



- $U_i := \mathcal{O}_{[2^i, 12(n'-i)]}$ , for  $i = 0, 2, \dots, n' - 2$ , and  $U_{n'}^I := \mathcal{O}_{[2n']}$ ,  $U_{n'}^{II} := \mathcal{O}_{[2n']}$ , if  $n' \leq n$  and  $n'$  is even.

The closures of the nilpotent orbits  $U_i$  are nested in the following way:

$$\begin{cases} \{0\} = \overline{U_0} \subset \overline{U_2} \subset \dots \subset \overline{U_n} & \text{if } n' > n; \\ \{0\} = \overline{U_0} \subset \overline{U_2} \subset \dots \subset \overline{U_{n'-1}} & \text{if } n' < n \text{ and } n' \text{ is odd}; \\ \{0\} = \overline{U_0} \subset \overline{U_2} \subset \dots \subset \overline{U_{n'-2}} = \overline{U_{n'}^I} \cap \overline{U_{n'}^{II}} & \text{if } n' \leq n \text{ and } n' \text{ is even.} \end{cases}$$

If ( $n' > n$  or  $n'$  is odd) resp. if ( $n' \leq n$  and  $n'$  is even), then the symplectic reduction  $W///G$  is the closure of a nilpotent orbit resp. is the union of two closures of nilpotent orbits, and thus the irreducible components of  $W///G$  are symplectic varieties (see Section 2). If  $n' > n$ , then  $W///G$  is of dimension  $2n'n - n(n+1)$  ([CM93, Corollary 6.1.4]), and its singular locus is  $\overline{U_{n-2}}$  ([KP82, Theorem 2]). On the other hand, if  $n' \leq n$ , then each irreducible component of  $W///G$  is of dimension  $n'(n'-1)$ , and the singular locus of  $W///G$  is  $\overline{U_{n'-2}}$  resp.  $\overline{U_{n'-3}}$ , when  $n'$  is even resp. when  $n'$  is odd.

We are now interested by the Hilbert function of the general fiber of the quotient morphism for each irreducible component of  $W///G$ . We will distinguish between the following cases:

- If  $n' > n$ , then  $W///G$  is irreducible, and we denote  $h_0$  the Hilbert function of the general fiber of the quotient morphism  $\nu : \mu^{-1}(0) \rightarrow W///G$ . By Proposition 4.2, the dimension of the general fiber of  $\nu$  is  $\frac{1}{2}n(n+1)$ .
- If  $n' \leq n$  and  $n'$  is even, then by Proposition 4.2, the zero fiber  $\mu^{-1}(0)$  is the union of two irreducible components that we denote  $X_I$  and  $X_{II}$ . Let  $\nu_I : X_I \rightarrow Y_I$  and  $\nu_{II} : X_{II} \rightarrow Y_{II}$  be the quotient morphisms. Up to the exchange of  $X_I$  and  $X_{II}$ , we can suppose that  $Y_I = \overline{U_{n'}^I}$  and  $Y_{II} = \overline{U_{n'}^{II}}$ . The orthogonal group  $O(E)$  acts transitively on  $U_{n'}^I \cup U_{n'}^{II}$ , hence the general fibers of  $\nu_I$  and  $\nu_{II}$  are isomorphic. In particular, the general fibers of  $\nu_I$  and  $\nu_{II}$  have the same Hilbert function, denoted  $h_0$ , and the same dimension, which is  $n'n - \frac{1}{2}n'(n'-1)$ .
- If  $n' < n$  and  $n'$  is odd, then  $W///G$  is irreducible, and we denote  $h_0$  the Hilbert function of the general fiber of the quotient morphism  $\nu : \mu^{-1}(0) \rightarrow W///G$ . The general fiber being reducible, determining  $h_0$  is more complicated than in the previous cases (except the case  $n' = 1$  which is trivial). From now on, we will always exclude the case  $n' < n$  and  $n'$  is odd.

If  $n' < n$  and  $n'$  is even, then we denote

$$(16) \quad H := \left\{ \begin{bmatrix} M & 0_{n-n', n'} \\ 0_{n', n-n'} & I_{n'} \end{bmatrix}, M \in Sp_{n-n'} \right\} \cong Sp_{n-n'},$$

which is a reductive algebraic subgroup of  $G \cong Sp_n$ . The proof of the next proposition is analogous to the proof of Proposition 3.5:

**Proposition 4.5.** *If  $n' > n$ , then the general fiber of the quotient morphism  $\nu : \mu^{-1}(0) \rightarrow W///G$  is isomorphic to  $G$ .*

*If  $n' = n$ , then the general fibers of the quotient morphisms  $\nu_I : X_I \rightarrow Y_I$  and  $\nu_{II} : X_{II} \rightarrow Y_{II}$  are isomorphic to  $G$ .*

*If  $n' < n$  and  $n'$  is even, then the general fibers of  $\nu_I$  and  $\nu_{II}$  are isomorphic to  $G/H$ , where  $H \subset G$  is the subgroup defined by (16).*

**Corollary 4.6.** *The Hilbert function  $h_0$  defined above is given by:*

$$\forall M \in \text{Irr}(G), h_0(M) = \begin{cases} \dim(M) & \text{if } n' \geq n; \\ \dim(M^H) & \text{if } n' < n \text{ and } n' \text{ is even;} \end{cases}$$

where  $H \subset G$  is the subgroup defined by (16).

**4.2. The reduction principle for the main component.** In this section, we give the guidelines to prove the reduction principle when  $G = Sp(V)$  (Proposition 4.8). The strategy is the same as for  $GL(V)$  (see Section 3.2), but as the symplectic reduction  $W///G$  is reducible when  $n' \leq n$  and  $n'$  is even, it seems necessary to give some details.

As  $\mu^{-1}(0)$  is a  $G' \times G$ -stable closed subset of  $W$ , it follows from [Bri, Lemma 3.3] that the invariant Hilbert scheme

$$\mathcal{H} := \text{Hilb}_{h_0}^G(\mu^{-1}(0))$$

is a  $G'$ -stable closed subscheme of  $\text{Hilb}_{h_0}^G(W)$ . As we aim at constructing canonical desingularizations of the irreducible components of  $W///G$ , we consider the two  $G'$ -stable closed subschemes  $\mathcal{H}_I := \text{Hilb}_{h_0}^G(X_I)$  and  $\mathcal{H}_{II} := \text{Hilb}_{h_0}^G(X_{II})$  instead of  $\mathcal{H}$  when  $n' \leq n$  and  $n'$  is even. Let us note that if we fix  $g'_0 \in O(E) \backslash SO(E)$  and make  $G'$  act on  $X_{II}$  by  $(g'_0 g' g'_0{}^{-1}) \cdot x$  for every  $g' \in G'$  and every  $x \in X_{II}$ , then  $\phi: X_I \rightarrow X_{II}$ ,  $x \mapsto g'_0 \cdot x$  is a  $G' \times G$ -equivariant isomorphism, and thus  $\mathcal{H}_I \cong \mathcal{H}_{II}$  as a  $G'$ -scheme. We denote  $\mathcal{H}_I^{\text{main}}$  resp.  $\mathcal{H}_{II}^{\text{main}}$ , the main component of  $\mathcal{H}_I$  resp. of  $\mathcal{H}_{II}$ . We always have the (set-theoretic) inclusion  $\mathcal{H}_I \cup \mathcal{H}_{II} \subset \mathcal{H}$ , but this may not be an equality. If  $n' > n$ , then  $W///G$  is irreducible, and we denote  $\mathcal{H}^{\text{main}}$  the main component of  $\mathcal{H}$ .

The scheme  $\text{Hilb}_{h_0}^G(W)$  was studied in [Terb]. In particular, we obtained

**Proposition 4.7.** ([Terb, §1.5.1]) *Let  $h_0$  be the Hilbert function given by Corollary 4.6, and  $G' = SO(E)$  acting naturally on  $\text{Gr}(2n' - h_0(V^*), E)$ . Then there exists a  $G'$ -equivariant morphism*

$$\begin{array}{ccc} \rho & : & \text{Hilb}_{h_0}^G(W) \rightarrow \text{Gr}(2n' - h_0(V^*), E) \\ & & Z \mapsto \text{Ker}(f_Z) \end{array}$$

where  $f_Z: E \cong \text{Mor}^G(W, V^*) \rightarrow \text{Mor}^G(Z, V^*)$  is the restriction map.

We identify  $\text{Gr}(2n' - h_0(V^*), E)$  with  $\text{Gr}(h_0(V^*), E^*)$ . By Corollary 4.6, if  $n' > n$  or  $(n' \leq n$  and  $n'$  is even), then  $h_0(V^*) = N := \min(n', n)$ . We denote  $q$  the non-degenerate quadratic form on  $E$  preserved by  $O(E)$ . We have a canonical isomorphism  $E \cong E^*$ , and thus  $q$  identifies with a non-degenerate quadratic form on  $E^*$ . For  $i = 0, \dots, N$ , we denote

$$A_i := \{L \in \text{Gr}(N, E^*) \mid q|_L \text{ is of rank } i\}.$$

If  $n' > n$ , then the  $A_i$  are the  $n+1$  orbits for the action of  $G'$  on  $\text{Gr}(n, E^*)$ . However, if  $n' \leq n$ , then the  $A_i$  are  $G'$ -orbits for  $i = 1, \dots, n'$ , but the isotropic Grassmannian  $A_0 = \text{OGr}(n', E^*)$  is the union of two  $G'$ -orbits, denoted  $\text{OGr}^I$  and  $\text{OGr}^{II}$ , which are exchanged by the action of any element of  $O(E) \backslash SO(E)$ . In any case, we have

$$\text{OGr}(N, E^*) = \overline{A_0} \subset \overline{A_1} \subset \dots \subset \overline{A_N} = \text{Gr}(N, E^*).$$

Let us now fix some notation:

- $L_0 \in A_0$ , and  $P$  the parabolic subgroup of  $G'$  stabilizing  $L_0$ ;

- $W' := \text{Hom}(E/L_0^\perp, V)$ , which identifies with a  $P \times G$ -module contained in  $\mu^{-1}(0)$ ;
- $\mathcal{H}' := \text{Hilb}_{h_0}^G(W')$ , and  $\mathcal{H}'^{\text{main}}$  its main component.

It must be underlined that, if  $n' > n$  or ( $n' \leq n$  and  $n'$  is even), then the Hilbert function of the general fiber of the quotient morphism  $W' \rightarrow W'//G$  coincides with the Hilbert function  $h_0$  of Corollary 4.6 (in particular,  $\mathcal{H}'^{\text{main}}$  is well defined).

Proceeding as for Lemma 3.9, one may check that, if  $n' > n$  resp. if ( $n' \leq n$  and  $n'$  is even), then the morphism  $\rho$  of Proposition 4.7 sends  $\mathcal{H}^{\text{main}}$  resp.  $\mathcal{H}_I^{\text{main}}$  and  $\mathcal{H}_{II}^{\text{main}}$ , onto  $A_0$ . More precisely, if  $n' \leq n$  and  $n'$  is even, then  $\rho$  sends  $\mathcal{H}_I^{\text{main}}$  onto one of the irreducible component of  $A_0$ , and  $\mathcal{H}_{II}^{\text{main}}$  onto the other component. Up to the exchange of these two components, we can suppose that  $\rho$  sends  $\mathcal{H}_I^{\text{main}}$  onto  $\text{OGr}^I$ , and  $\mathcal{H}_{II}^{\text{main}}$  onto  $\text{OGr}^{II}$ .

It follows that the restriction of  $\rho$  equips  $\mathcal{H}^{\text{main}}$  resp.  $\mathcal{H}_I^{\text{main}}$ , resp.  $\mathcal{H}_{II}^{\text{main}}$ , with a structure of a  $G'$ -homogeneous fiber bundle over  $A_0$  resp. over  $\text{OGr}^I$ , resp. over  $\text{OGr}^{II}$ . Hence, it is enough to determine the fiber  $F_0$  over  $L_0$  to determine  $\mathcal{H}^{\text{main}}$  resp.  $\mathcal{H}_I^{\text{main}}$ , resp.  $\mathcal{H}_{II}^{\text{main}}$ . Proceeding as in Section 3.2, we obtain that  $F_0$  is isomorphic to  $\mathcal{H}'^{\text{main}}$  as a  $P$ -scheme. We deduce

**Proposition 4.8.** *With the above notation, we have the following  $G'$ -equivariant isomorphisms:*

- If  $n' > n$ , then

$$\mathcal{H}^{\text{main}} \cong G' \times^P \mathcal{H}'^{\text{main}}.$$

- If ( $n' \leq n$  and  $n'$  is even) and  $L_0 \in \text{OGr}^I$  resp.  $L_0 \in \text{OGr}^{II}$ , then

$$\mathcal{H}_I^{\text{main}} \cong G' \times^P \mathcal{H}'^{\text{main}} \quad \text{resp.} \quad \mathcal{H}_{II}^{\text{main}} \cong G' \times^P \mathcal{H}'^{\text{main}}.$$

**4.3. Proof of Theorem A for  $\text{Sp}(\mathbf{V})$ .** Let us start by recalling

**Theorem 4.9.** [Tera, §1, Theorem] *Let  $G = \text{Sp}(V)$ ,  $W = \text{Hom}(E, V)$ , and  $h_W$  the Hilbert function of the general fiber of the quotient morphism  $W \rightarrow W//G$ . We denote  $n := \dim(V)$ ,  $e := \dim(E)$ , and  $Y_0$  the blow-up of  $W//G = \Lambda^2(E^*)^{\leq n} := \{Q \in \Lambda^2(E^*) \mid \text{rk}(Q) \leq n\}$  at 0. In the following cases, the invariant Hilbert scheme  $\mathcal{H}' := \text{Hilb}_{h_W}^G(W)$  is a smooth variety, and the Hilbert-Chow morphism is the succession of blow-up described as follows:*

- if  $n \geq 2e - 2$ , then  $\mathcal{H}' \cong W//G = \Lambda^2(E^*)$ ;
- if  $e > n = 2$  or  $e = n = 4$ , then  $\mathcal{H}' \cong Y_0$ ;
- if  $e > n = 4$ , then  $\mathcal{H}'$  is isomorphic to the blow-up of  $Y_0$  along the strict transform of  $\Lambda^2(E^*)^{\leq 2}$ .

If  $n' > n$ , then we denote  $T$  the tautological bundle over  $A_0 = \text{OGr}(n, E^*)$ . If  $n' \leq n$  and  $n'$  is even, then we denote  $T_I$  resp.  $T_{II}$ , the tautological bundle over  $\text{OGr}^I$  resp. over  $\text{OGr}^{II}$ . We deduce from Proposition 4.8 and Theorem 4.9 that we have the following  $G'$ -equivariant isomorphisms

$$(17) \quad \mathcal{H}^{\text{main}} \cong \begin{cases} \Lambda^2(T) & \text{if } n' > n = 2; \\ \text{Bl}_0(\Lambda^2(T)) & \text{if } n' > n = 4; \end{cases}$$

$$(18) \quad \mathcal{H}_\bullet^{\text{main}} \cong \begin{cases} \Lambda^2(T_\bullet) & \text{if } n \geq 2n' - 2 \text{ and } n' \text{ is even;} \\ \text{Bl}_0(\Lambda^2(T_\bullet)) & \text{if } n' = n = 4; \end{cases}$$

where  $\bullet$  stands for  $I$  or  $II$ , and  $\text{Bl}_0(\cdot)$  denotes the blow-up along the zero section. In all these cases, the main component of the invariant Hilbert scheme is smooth,

and thus the Hilbert-Chow morphism  $\gamma: \mathcal{H}^{\text{main}} \rightarrow W///G$  resp.  $\gamma: \mathcal{H}_{\bullet}^{\text{main}} \rightarrow Y_{\bullet}$ , is a (distinguished) desingularization.

It remains to compare  $\gamma$  with the Springer desingularizations (when they exist) of the irreducible components of  $W///G$ . We saw in Section 2 that the irreducible components of  $W///G$  have Springer desingularizations if and only if  $n' \leq n+1$ . We then distinguish between the following cases:

- (1) If  $n' \leq n+1$  and  $n'$  is odd, then  $W///G$  admits two Springer desingularizations, which are given by the cotangent bundles  $\mathcal{T}_I^*$  and  $\mathcal{T}_{II}^*$  over  $\text{OGr}^I$  and  $\text{OGr}^{II}$  respectively. The natural action of the orthogonal group  $O(E)$  on  $\text{OGr}(n', E^*)$  induces an action on the cotangent bundle  $\mathcal{T}^*\text{OGr}(n', E^*)$  that exchanges  $\mathcal{T}_I^*$  and  $\mathcal{T}_{II}^*$ . On the other hand, it follows from Remark 4.1, that the group  $O(E)$  stabilizes  $\mathcal{H}^{\text{main}}$ , and thus  $\gamma: \mathcal{H}^{\text{main}} \rightarrow W///G$  can not be a Springer desingularization. However, if  $n \in \{2, 4\}$  and  $n' = n+1$ , then one may show that  $\gamma$  dominates the two Springer desingularizations of  $W///G$  (see [Bec11, Introduction] for the case  $n=2$ , the case  $n=4$  being analogous).
- (2) If  $n' \leq n$  and  $n'$  is even, then  $Y_{\bullet}$  has a unique Springer desingularization, which is given by the cotangent bundle  $\mathcal{T}_{\bullet}^* \cong \Lambda^2(T_{\bullet})$  over  $\text{OGr}_{\bullet}$ . Proceeding as we did for  $GL(V)$  in Section 3.3, one may show that  $\gamma: \mathcal{H}_{\bullet}^{\text{main}} \rightarrow Y_{\bullet}$  is the Springer desingularization if and only if  $n \geq 2n' - 2$ . In addition, if  $n' = n = 4$ , then by (18) we have  $\mathcal{H}_{\bullet}^{\text{main}} \cong \text{Bl}_0(\mathcal{T}_{\bullet}^*)$ , and thus  $\gamma$  dominates the unique Springer desingularization of  $Y_{\bullet}$ .

**4.4. Study of the case  $n=2$ .** In this section, we suppose that  $n' \geq n=2$  (the case  $n'=1$  being trivial). Then  $G \cong Sp_2$ , and the morphism of Proposition 4.7 is  $\rho: \text{Hilb}_{h_0}^G(W) \rightarrow \text{Gr}(2, E^*)$ . Denoting  $\mathfrak{g}'^{\leq 2} := \{f \in \mathfrak{g}' \mid \text{rk}(f) \leq 2\}$ , we have a  $G'$ -equivariant isomorphism

$$(19) \quad \mathbb{P}(\mathfrak{g}'^{\leq 2}) \cong \text{Gr}(2, E^*),$$

and thus we can consider  $\rho': \text{Hilb}_{h_0}^G(W) \rightarrow \mathbb{P}(\mathfrak{g}'^{\leq 2})$ , the morphism induced by  $\rho$ . By Proposition 4.3, we have

$$W///G = \begin{cases} \overline{\mathcal{O}_{[2^2, 1^{2n'-4}]}} & \text{if } n' \geq 3; \\ \overline{\mathcal{O}_{[2^2]}^I \cup \mathcal{O}_{[2^2]}^{II}} & \text{if } n' = 2. \end{cases}$$

We will show that if  $n' \geq 3$  resp. if  $n' = 2$ , then  $\mathcal{H}$  resp.  $\mathcal{H}_{\bullet}$  (where  $\bullet$  stands for  $I$  or  $II$ ), coincides with its main component. In particular, the geometric properties of the invariant Hilbert scheme for  $G = Sp(V)$  seem to be quite different from the case of  $G = GL(V)$  studied in Section 3. Let us recall that the case ( $n' = 3$  and  $n = 2$ ) was treated by Becker in [Bec11]; she showed that  $\mathcal{H}$  coincides with  $\mathcal{H}^{\text{main}}$  as a scheme, and is the total space of a line bundle over  $\text{OGr}(2, E^*)$ .

**Proposition 4.10.** *We equip all the invariant Hilbert schemes with their reduced structures. If  $n' > n = 2$ , then  $\mathcal{H} = \mathcal{H}^{\text{main}}$  is a smooth variety isomorphic to*

$$\text{Bl}_0(\overline{\mathcal{O}_{[2^2, 1^{2n'-4}]}}) := \{(f, L) \in \overline{\mathcal{O}_{[2^2, 1^{2n'-4}]}} \times \mathbb{P}(\overline{\mathcal{O}_{[2^2, 1^{2n'-4}]}}) \mid f \in L\},$$

and the Hilbert-Chow morphism  $\gamma: \mathcal{H} \rightarrow W///G$  is the blow-up of  $\overline{\mathcal{O}_{[2^2, 1^{2n'-4}]}}$  at 0. If  $n' = n = 2$ , then  $\text{Hilb}_{h_0}^G(\mu^{-1}(0)) = \mathcal{H}_I \cup \mathcal{H}_{II}$  is the union of two smooth varieties isomorphic to  $\text{Bl}_0(\overline{\mathcal{O}_{[2^2]}^I})$  and  $\text{Bl}_0(\overline{\mathcal{O}_{[2^2]}^{II}})$  respectively, and the intersection  $\mathcal{H}_I \cap \mathcal{H}_{II}$

is formed by the homogeneous ideals of  $\mathbb{C}[\overline{\mu^{-1}(0)}]$ . Moreover, the Hilbert-Chow morphism  $\gamma: \mathcal{H}_I \rightarrow \overline{\mathcal{O}_{[2^2]}}^I$  resp.  $\gamma: \mathcal{H}_{II} \rightarrow \overline{\mathcal{O}_{[2^2]}}^{II}$ , is the blow-up of  $\overline{\mathcal{O}_{[2^2]}}^{II}$  resp. of  $\overline{\mathcal{O}_{[2^2]}}^I$ , at 0.

*Proof.* The proofs for the cases  $n' = 2$  and  $n' \geq 3$  are quite similar, and thus we will only consider the case  $n' \geq 3$  (which is simpler in terms of notation!). It follows from [Bri, Lemma 3.3] and [Tera, §1, Theorem] that we have a closed embedding

$$\gamma \times \rho': \mathcal{H} \hookrightarrow \mathcal{Y} := \left\{ (f, L) \in \overline{\mathcal{O}_{[2^2, 1^{2n'-4}]}} \times \mathbb{P}(\mathfrak{g}'^{\leq 2}) \mid f \in L \right\}.$$

One may check that  $\mathcal{Y}$  is the union of the two subvarieties  $C_1$  and  $C_2$  defined by:

- $C_1 := \text{Bl}_0(\overline{\mathcal{O}_{[2^2, 1^{2n'-4}]}})$ ;
- $C_2 := \left\{ (0, L) \in \overline{\mathcal{O}_{[2^2, 1^{2n'-4}]}} \times \mathbb{P}(\mathfrak{g}'^{\leq 2}) \right\} \cong \mathbb{P}(\mathfrak{g}'^{\leq 2})$ .

The subvarieties  $C_1$  and  $C_2$  are of dimension  $4n' - 6$  and  $4n' - 4$  respectively. The morphism  $\gamma \times \rho'$  sends  $\mathcal{H}^{\text{main}}$  into  $C_1$ ; the varieties  $\mathcal{H}^{\text{main}}$  and  $C_1$  have the same dimension, hence  $\gamma \times \rho': \mathcal{H}^{\text{main}} \rightarrow C_1$  is an isomorphism.

Now it follows from [Terb, Proposition 3.3.13] that the component  $C_2$  identifies with the closed subset of  $\text{Hilb}_{h_0}^G(W)$  formed by the homogeneous ideals of  $\mathbb{C}[W]$ . Let us describe this correspondance. If  $L \in C_2 \cong \mathbb{P}(\mathfrak{g}'^{\leq 2})$ , then we define the ideal  $I_L \subset \mathbb{C}[W]$  generated by the homogeneous  $G$ -invariants of positive degree of  $\mathbb{C}[W]$ , and by the  $G$ -module  $L^\perp \otimes V \subset \mathbb{C}[W]_1 \cong E \otimes V$ , where  $L$  is identified with a 2-dimensional subspace of  $E^*$  with (19). Let us show that  $I_L$  is a point of  $\mathcal{H}$  if and only if  $L \in \text{OGr}(2, E^*)$ ; the result will follow since  $\mathbb{P}(\overline{\mathcal{O}_{[2^2, 1^{2n'-4}]}})$  identifies with  $\text{OGr}(2, E^*)$  via the isomorphism (19), and that  $\left\{ (0, L) \in \overline{\mathcal{O}_{[2^2, 1^{2n'-4}]}} \times \mathbb{P}(\overline{\mathcal{O}_{[2^2, 1^{2n'-4}]}}) \right\}$  is a subvariety of  $C_1$ .

We denote  $W' := \text{Hom}(E/L^\perp, V)$ , then

$$\mathbb{C}[W']_2 \cong (S^2(E/L^\perp) \otimes S^2(V)) \oplus (\Lambda^2(E/L^\perp) \otimes \Lambda^2(V))$$

as a  $G$ -module. Let  $I'_L$  be the ideal of  $\mathbb{C}[W']$  generated by  $\Lambda^2(E/L^\perp) \otimes \Lambda^2(V) \subset \mathbb{C}[W']_2$ , then one may check that

$$\mathbb{C}[W']/I'_L \cong \mathbb{C}[W]/I_L \cong \bigoplus_{M \in \text{Irr}(G)} M^{\oplus \dim(M)}$$

as a  $G$ -module. Hence

$$\begin{aligned} I_L \in \mathcal{H} &\Leftrightarrow I_L \cap \mathbb{C}[W]_2 \supset E_0 \otimes S^2(V), \text{ where } E_0 \text{ is the trivial representation of } G'; \\ &\Leftrightarrow q_L = 0, \text{ where } q \text{ is the quadratic form preserved by } G'; \\ &\Leftrightarrow L \in \text{OGr}(2, E^*). \end{aligned}$$

□

*Remark 4.11.* In the proof of Proposition 4.10, we show that if  $n' > n = 2$ , then the homogeneous ideals of  $\mathcal{H}$  are contained in  $\mathcal{H}^{\text{main}}$ . Using analogous arguments, one may check that this statement is true more generally when  $n' > n \geq 2$ .

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