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INVARIANT HILBERT SCHEMES AND DESINGULARIZATIONS OF SYMPLECTIC REDUCTIONS FOR CLASSICAL GROUPS

RONAN TERPEREAU

ABSTRACT. Let $G \subset GL(V)$ be a reductive algebraic subgroup acting on the symplectic vector space $W = (V \oplus V^*)^{\oplus m}$, and let $\mu : W \rightarrow Lie(G)^*$ be the corresponding moment map. In this article, we use the theory of invariant Hilbert schemes to construct a canonical desingularization of the symplectic reduction $\mu^{-1}(0)//G$ for classes of examples where $G = GL(V)$, $O(V)$, or $Sp(V)$. For these classes of examples, $\mu^{-1}(0)//G$ is isomorphic to the closure of a nilpotent orbit in a simple Lie algebra, and we compare the Hilbert-Chow morphism with the (well-known) symplectic desingularizations of $\mu^{-1}(0)//G$.

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1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

First of all, let us recall briefly the definition of the *invariant Hilbert scheme*, constructed by Alexeev and Brion (see [AB05, Bri13] for more details). We work over the field of complex numbers \mathbb{C} . Let G be a reductive algebraic group, and let $h : \text{Irr}(G) \rightarrow \mathbb{N}$ be a *Hilbert function* which assigns to every irreducible representation of G a nonnegative integer. If X is an (possibly reducible) affine G -variety, then the invariant Hilbert scheme $\text{Hilb}_h^G(X)$ is the moduli space that parametrizes the G -stable closed subschemes Z of X such that

$$\mathbb{C}[Z] \cong \bigoplus_{M \in \text{Irr}(G)} M^{\oplus h(M)}$$

as a G -module. Let us now suppose that the categorical quotient

$$X//G := \text{Spec}(\mathbb{C}[X]^G)$$

is an irreducible variety. If $h = h_0$ is the Hilbert function of the *general fibers* of the quotient morphism $\nu : X \rightarrow X//G$ (that is, the fibers over a nonempty open subset of $X//G$), then there exists a projective morphism

$$\gamma : \text{Hilb}_{h_0}^G(X) \rightarrow X//G,$$

called the *Hilbert-Chow morphism*, that sends a closed subscheme $Z \subset X$ to the point $Z//G \subset X//G$. The Hilbert-Chow morphism induces an isomorphism over the

flat locus $U \subset X//G$ of ν . The *main component* of $\text{Hilb}_{h_0}^G(X)$ is the irreducible component defined by

$$\text{Hilb}_{h_0}^G(X)^{\text{main}} := \overline{\gamma^{-1}(U)}.$$

Then the restriction $\gamma: \text{Hilb}_{h_0}^G(X)^{\text{main}} \rightarrow X//G$ is a projective birational morphism, and thus γ is a candidate for a canonical desingularization of $X//G$. It is an open problem to determine whether this restriction is always a desingularization or not. Last, but not least, if H is any algebraic subgroup of the G -equivariant automorphism group $\text{Aut}^G(X)$, then H acts on $X//G$ and $\text{Hilb}_{h_0}^G(X)$, and the quotient morphism $\nu: X \rightarrow X//G$ and the Hilbert-Chow morphism $\gamma: \text{Hilb}_{h_0}^G(X) \rightarrow X//G$ are H -equivariant.

Let now G be an algebraic group, let \mathfrak{g} be the Lie algebra of G , and let W be a *symplectic G -module*, that is, a G -module equipped with a G -invariant non-degenerate skew-symmetric bilinear form. Then W has a G -equivariant *moment map*

$$\mu_G: W \rightarrow \mathfrak{g}^*,$$

which is defined in the usual way. To simplify the notation, we will use μ instead of μ_G . The map μ being G -equivariant, the set-theoretic fiber $\mu^{-1}(0)$ is a G -stable subvariety of W . From now on, we suppose that G is reductive. The categorical quotient $\mu^{-1}(0)//G$ is called the *symplectic reduction* of W by G and plays a central role in the study of W . It is an important problem to determine whether $\mu^{-1}(0)//G$ admits a *symplectic desingularization* (which is a distinguished desingularization, see Section 2 for details); a candidate for such a desingularization is given by the Hilbert-Chow morphism $\gamma: \text{Hilb}_{h_0}^G(\mu^{-1}(0))^{\text{main}} \rightarrow \mu^{-1}(0)//G$.

Let us take V a finite dimensional vector space, and m a nonnegative integer. In this paper, we are interested in the cases where

$$W := (V \oplus V^*)^{\oplus m},$$

on which $GL(V)$ acts naturally, and $G = GL(V)$, $O(V)$, or $Sp(V)$. In this situation, we can find a classical algebraic subgroup $H \subset \text{Aut}^G(\mu^{-1}(0))$, namely

- (1) $H = GL_m$ for $G = GL(V)$;
- (2) $H = Sp_{2m}$ for $G = O(V)$; and
- (3) $H = SO_{2m}$ for $G = Sp(V)$.

In each case, we will see that the symplectic reduction $\mu^{-1}(0)//G$ identifies with a nilpotent orbit closure in the Lie algebra \mathfrak{h} of H , except in Case (3) for m even and $m \leq \dim(V)$ where it is the union of two such orbit closures. In Case (3), if $\mu^{-1}(0)//G = Y_1 \cup Y_2$ is reducible, then **we always consider only one component** to simplify the statements (that is, $\mu^{-1}(0)//G$ should be replaced by Y_i and $\mu^{-1}(0)$ by $\nu^{-1}(Y_i)$). The geometry of nilpotent orbits has been extensively studied by Fu, Kraft, Namikawa, Procesi... ([KP79, KP81, KP82, Fu03a, Fu03b, Fu06b, Na06]). In particular, the normalizations of such closures are *symplectic varieties* (as defined by Beauville in [Bea00]) whose symplectic desingularizations are the so-called *Springer desingularizations*, obtained by collapsing the cotangent bundle over some flag varieties (see Section 2 for details).

In [Terb, Tera], we studied the invariant Hilbert scheme for classical groups acting on classical representations. We obtained classes of examples where the

Hilbert-Chow morphism is a desingularization of the categorical quotient, and further examples where it is not. In this article, we use the results of [Tera] to prove the following statements:

Theorem A. ([Sections 3.3 and 4.3]) *With the above notation, let $G = GL(V)$, $O(V)$, or $Sp(V)$, then the Hilbert-Chow morphism $\gamma : \text{Hilb}_{h_0}^G(\mu^{-1}(0))^{\text{main}} \rightarrow \mu^{-1}(0)//G$ is a symplectic desingularization (and the unique one) if and only if*

- $G = GL(V)$, $\dim(V) \geq m - 1$, and m is even; or
- $G = O(V)$, and $\dim(V) \geq 2m - 1$; or
- $G = Sp(V)$, $\dim(V)$ and m are even, and $\dim(V) \geq 2m - 2$.

Theorem B. ([Sections 3.3 and 4.3]) *With the above notation, let $G = GL(V)$, $O(V)$, or $Sp(V)$, then the Hilbert-Chow morphism $\gamma : \text{Hilb}_{h_0}^G(\mu^{-1}(0))^{\text{main}} \rightarrow \mu^{-1}(0)//G$ is a desingularization that strictly dominates the symplectic desingularizations (when they exist) in the following cases:*

- $G = GL(V)$ and either $\dim(V) = 1$, $m \geq 3$ or $\dim(V) = 2$, $m \geq 4$; or
- $G = O(V)$ and either $\dim(V) = 1 < m$ or $\dim(V) = 2 \leq m$; or
- $G = Sp(V)$ and either $\dim(V) = 2 < m$ or $\dim(V) = 4 \leq m$.

If $G \subset GL(V)$ is any reductive algebraic subgroup, then it is generally a difficult problem to determine whether $\text{Hilb}_{h_0}^G(\mu^{-1}(0))$ is irreducible, that is, equals its main component. In this direction, we obtain

Proposition C. ([Propositions 3.17 and 4.10]) *With the above notation, if $G = GL(V)$ and $m \geq 2\dim(V)$, then the invariant Hilbert scheme $\text{Hilb}_{h_0}^G(\mu^{-1}(0))$ has at least two irreducible components (and exactly two when $\dim(V) = 1$). On the other hand, if $G = O(V)$ or $Sp(V)$, and $m \geq \dim(V) = 2$, then $\text{Hilb}_{h_0}^G(\mu^{-1}(0))$ is irreducible.*

In Section 2, we recall some basic facts about symplectic varieties and closures of nilpotent orbits in simple Lie algebras. The case of $GL(V)$ is treated in Section 3, and the case of $Sp(V)$ is treated in Section 4. The case of $O(V)$ is quite similar to the case of $GL(V)$, and details can be found in the thesis [Terb, §3.4] from which this article is extracted. Besides, we think that our methods also apply when $G = SL(V)$, while the case $G = SO(V)$ should be more involved.

2. GENERALITIES ON SYMPLECTIC VARIETIES AND CLOSURES OF NILPOTENT ORBITS

2.1. Symplectic varieties and symplectic desingularizations. Let us first recall the definitions of symplectic variety and symplectic desingularization (see [Bea00] or the survey [Fu06a] for more details). Let Y be a normal variety whose regular locus Y_{reg} admits a *symplectic form* Ω (that is, Ω is a holomorphic 2-form which is closed and non-degenerate at every point of Y_{reg}) such that, for any desingularization $f : \tilde{Y} \rightarrow Y$, the 2-form $f^*(\Omega)$ extends to a 2-form on the whole \tilde{Y} , then we say that Y is a *symplectic variety*. Moreover, if $f : \tilde{Y} \rightarrow Y$ is a desingularization such that $f^*(\Omega)$ extends to a symplectic form on \tilde{Y} , then we say that f is a *symplectic desingularization* of Y . It must be emphasized that symplectic varieties do not always admit symplectic desingularizations, and when they do, there may be several of them.

As in the introduction, we denote $W = (V \oplus V^*)^{\oplus m}$, we take a reductive algebraic subgroup $G \subset GL(V)$ acting naturally on W , and we consider the symplectic reduction $\mu^{-1}(0)//G$. The following conjecture motivates the study (and the name!) of $\mu^{-1}(0)//G$:

Conjecture 2.1 (Kaledin, Lehn, Sorger). With the above notation, the irreducible components Y_1, \dots, Y_r of $\mu^{-1}(0)//G$ are symplectic varieties. Moreover, if every Y_i admits a symplectic desingularization, then the quotient $V^{\oplus m}//G$ is smooth.

When G is a finite group, Conjecture 2.1 was proved by Kaledin and Verbitsky, but the general case remains open. Let us mention that Becker showed in [Bec09] that the converse of the second part of Conjecture 2.1 holds for $G = Sp(V)$ with $\dim(V) = 2$. In our setting, that is when $G = GL(V)$, $O(V)$, or $Sp(V)$, one easily checks that Conjecture 2.1 holds (see [Terb, §A.2] for details).

2.2. Closures of nilpotent orbits. We now recall some basic facts concerning the closures of nilpotent orbits (see [CM93, Fu03a] for more details). The following well-known result is due to Kostant, Kirillov, Souriau, and Panyushev:

Theorem 2.2. *The normalization of the closure of an adjoint orbit in a semi-simple Lie algebra is a symplectic variety.*

Let now \mathfrak{h} be a simple Lie algebra of classical type. If $\mathfrak{h} = \mathfrak{sl}_m$, then every nilpotent element $f \in \mathfrak{sl}_m$ is conjugate to an element of the form $\text{diag}(J_{d_1}, \dots, J_{d_k})$, where each J_{d_i} is a Jordan block of size d_i , and $\mathbf{d} = [d_1, \dots, d_k]$ is a partition of m . Then there exists a one-to-one correspondence between the partitions $\mathbf{d} = (d_1 \geq \dots \geq d_k)$ of m and the nilpotent orbits $\mathcal{O}_{\mathbf{d}} \subset \mathfrak{sl}_m$ (see [CM93, §3.1]). Now if $\mathfrak{h} = \mathfrak{sp}_{2m}$ resp. if $\mathfrak{h} = \mathfrak{so}_{2m}$, then a similar description exists (see [CM93, §5.1]), it is obtained by cutting \mathfrak{h} with a SL_{2m} -orbit $\mathcal{O}_{\mathbf{d}} \subset \mathfrak{sl}_{2m}$, which gives a unique Sp_{2m} -orbit resp. a unique O_{2m} -orbit, if it is not empty. Let us note that an O_{2m} -orbit can be non-connected giving rise to two SO_{2m} -orbits that we will denote $\mathcal{O}_{\mathbf{d}}^I$ and $\mathcal{O}_{\mathbf{d}}^II$.

If $\mathfrak{h} = \mathfrak{sl}_m$, then $\overline{\mathcal{O}_{\mathbf{d}}}$ is always normal ([KP79]). For the other classical types, the geometry of $\overline{\mathcal{O}_{\mathbf{d}}}$ was studied in [KP82]; in particular, if $\mathfrak{h} = \mathfrak{sp}_{2m}$ and $d_1 + d_2 \leq 4$ resp. if $\mathfrak{h} = \mathfrak{so}_{2m}$ and $d_1 \leq 2$, then $\overline{\mathcal{O}_{\mathbf{d}}}$ is normal. In the next sections, we will be interested only by conjugacy classes of elements $f \in \mathfrak{h}$ with $f^2 = 0$. Hence, from now on, we only consider partitions \mathbf{d} such that each $d_i \leq 2$. By Theorem 2.2, the variety $\overline{\mathcal{O}_{\mathbf{d}}}$ is symplectic, and we are going to describe its symplectic desingularizations (see [Fu03a, Fu06b, FN04] for details).

As before, let \mathfrak{h} be a simple Lie algebra of classical type, and let H be the adjoint group of \mathfrak{h} . We consider $f : Z \rightarrow \overline{\mathcal{O}_{\mathbf{d}}}$ a symplectic desingularization. Then, by [Fu03a, Proposition 3.1], the group H acts naturally on Z in such a way that f is H -equivariant. One says that f is a *Springer desingularization* if there exists a parabolic subgroup $P \subset H$ and a H -equivariant isomorphism between Z and the total space of the cotangent bundle over H/P , denoted by $\mathcal{T}^*(H/P)$. Then, under this isomorphism, the map f becomes

$$\mathcal{T}^*(H/P) \cong H \times^P \mathfrak{u} \longrightarrow \mathfrak{h}, \quad (h, x) \longmapsto \text{Ad}(h).x,$$

where \mathfrak{u} is the nilradical of the Lie algebra of P , and $H \times^P \mathfrak{u}$ denotes the quotient $(H \times \mathfrak{u})/P$ under the (free) action of P given by $p.(h, u) = (h \circ p^{-1}, \text{Ad}(p).u)$.

Theorem 2.3. ([Fu03a, Theorem 3.3]) *With the above notation, if $f : Z \rightarrow \overline{\mathcal{O}_{\mathbf{d}}}$ is a symplectic desingularization, then f is a Springer desingularization.*

Thanks to the work of Fu and Namikawa, the Springer desingularizations of $\overline{\mathcal{O}_{\mathbf{d}}}$ are known (up to isomorphism). In particular:

- Let $\mathfrak{h} = \mathfrak{sl}_m$ and $\mathbf{d} = [2^N, 1^{m-2N}]$ for some $0 \leq N \leq \frac{m}{2}$. We denote by $\text{Gr}(p, \mathbb{C}^m)$ the Grassmannian of p -dimensional subspaces of \mathbb{C}^m , and by \mathcal{T}_1^* resp. by \mathcal{T}_2^* , the cotangent bundle over $\text{Gr}(N, \mathbb{C}^m)$ resp. over $\text{Gr}(m-N, \mathbb{C}^m)$. By [Fu06b, §2], if $N < \frac{m}{2}$, then \mathcal{T}_1^* and \mathcal{T}_2^* are the two Springer desingularizations of $\overline{\mathcal{O}_{\mathbf{d}}}$; else, $\mathcal{T}_1^* = \mathcal{T}_2^*$ is the unique Springer desingularization of $\overline{\mathcal{O}_{\mathbf{d}}}$.
- Let $\mathfrak{h} = \mathfrak{sp}_{2m}$ and $\mathbf{d} = [2^N, 1^{2(m-N)}]$ for some $0 \leq N \leq m$. Then $\overline{\mathcal{O}_{\mathbf{d}}}$ admits a Springer desingularization if and only if $N = m$ ([Fu03a, Proposition 3.19]). We denote by $\text{IGr}(p, \mathbb{C}^{2m})$ the Grassmannian of isotropic p -dimensional subspaces of \mathbb{C}^{2m} , and by \mathcal{T}^* the cotangent bundle over $\text{IGr}(m, \mathbb{C}^{2m})$. By [FN04, Proposition 3.5], if $N = m$, then \mathcal{T}^* is the unique Springer desingularization of $\overline{\mathcal{O}_{\mathbf{d}}}$.
- Let $\mathfrak{h} = \mathfrak{so}_{2m}$ and $\mathbf{d} = [2^N, 1^{2(m-N)}]$ for some $0 \leq N \leq m$ with N even. If $N = m$, then one associates to \mathbf{d} two distinct nilpotent orbits $\mathcal{O}_{\mathbf{d}}^I$ and $\mathcal{O}_{\mathbf{d}}^{II}$. By [Fu03a, Proposition 3.20], the variety $\overline{\mathcal{O}_{\mathbf{d}}}$ admits a Springer desingularization if and only if $N \in \{m-1, m\}$. We denote by $\text{OGr}(p, \mathbb{C}^{2m})$ the Grassmannian of isotropic p -dimensional subspaces of \mathbb{C}^{2m} . The Grassmannian $\text{OGr}(p, \mathbb{C}^{2m})$ is irreducible except if $p = m$, in which case $\text{OGr}(m, \mathbb{C}^{2m}) = \text{OG}^I \cup \text{OG}^{II}$ is the union of two irreducible components (exchanged by the natural action of O_{2m}). We denote by \mathcal{T}_I^* resp. by \mathcal{T}_{II}^* , the cotangent bundle over OG^I resp. over OG^{II} . If $N = m-1$, then \mathcal{T}_I^* and \mathcal{T}_{II}^* are the two Springer desingularizations of $\overline{\mathcal{O}_{[2^{m-1}, 1^2]}}$ by [Fu06b, §2]. If $N = m$, then \mathcal{T}_I^* resp. \mathcal{T}_{II}^* , is the unique Springer desingularization of $\overline{\mathcal{O}_{[2^m]}^I}$ resp. of $\overline{\mathcal{O}_{[2^m]}^{II}}$, by [FN04, Proposition 3.5].

3. CASE OF GL_n

In this section, we denote V and V' two finite dimensional vector spaces, and we take $G = GL(V)$ and $H = GL(V')$, both acting on

$$W := \text{Hom}(V', V) \times \text{Hom}(V, V')$$

as follows:

$$\forall (g, h) \in G \times H, \forall (u_1, u_2) \in W, (g, h) \cdot (u_1, u_2) := (g \circ u_1 \circ h^{-1}, h \circ u_2 \circ g^{-1}).$$

We denote by \mathfrak{g} resp. by \mathfrak{h} , the Lie algebra of G resp. of H , and $N := \min(\lfloor \frac{m}{2} \rfloor, n)$, where $n := \dim(V)$, $m := \dim(V')$, and $\lfloor \cdot \rfloor$ is the lower integer part.

3.1. The quotient morphism. The two main results of this section are Proposition 3.3, which describes the symplectic reduction $\mu^{-1}(0)//G$, and Corollary 3.6, which gives the Hilbert function h_0 of the general fibers of the quotient morphism $\nu: \mu^{-1}(0) \rightarrow \mu^{-1}(0)//G$.

We recall that W is equipped with a G -invariant symplectic form Ω defined by:

$$(1) \quad \forall (u_1, u_2), (u'_1, u'_2) \in W, \Omega((u_1, u_2), (u'_1, u'_2)) := \text{tr}(u'_1 \circ u_2) - \text{tr}(u_1 \circ u'_2),$$

where $\text{tr}(\cdot)$ denotes the trace. The corresponding moment map is given by:

$$\begin{aligned} \mu: \quad W &\rightarrow \mathfrak{g}^* \\ (u_1, u_2) &\mapsto (f \mapsto \text{tr}(u_2 \circ f \circ u_1)) \end{aligned}$$

and thus the zero fiber of μ is the $G \times H$ -stable subvariety defined by:

$$\mu^{-1}(0) = \{(u_1, u_2) \in W \mid u_1 \circ u_2 = 0\}.$$

Let us determine the irreducible components of $\mu^{-1}(0)$ as well as their dimensions. Let $p \in \{0, \dots, m\}$; we define the subvariety

$$(2) \quad X_p := \left\{ (u_1, u_2) \in W \mid \begin{array}{l} \text{Im}(u_2) \subset \text{Ker}(u_1); \\ \text{rk}(u_2) \leq \min(n, p); \\ \dim(\text{Ker}(u_1)) \geq \max(m-n, p). \end{array} \right\} \subset \mu^{-1}(0),$$

and we consider the diagram

$$\begin{array}{ccc} Z_p := \{(u_1, u_2, L) \in W \times \text{Gr}(p, V') \mid \text{Im}(u_2) \subset L \subset \text{Ker}(u_1)\} & & \\ \downarrow p_1 & \searrow p_2 & \\ X_p & & \text{Gr}(p, V') \end{array}$$

where the p_i are the natural projections. We fix $L_0 \in \text{Gr}(p, V')$; the second projection equips Z_p with a structure of homogeneous vector bundle over $\text{Gr}(p, V')$ whose fiber over L_0 is isomorphic to $F_p := \text{Hom}(V'/L_0, V) \times \text{Hom}(V, L_0)$. Hence, Z_p is a smooth variety of dimension $p(m-p) + mn$.

Proposition 3.1. *The irreducible components of $\mu^{-1}(0)$ are*

$$\begin{cases} X_0, \dots, X_m & \text{if } m \leq n; \\ X_{m-n}, \dots, X_n & \text{if } n < m < 2n; \\ X_n & \text{if } m \geq 2n; \end{cases}$$

where X_p is defined by (2).

Proof. We have

$$\mu^{-1}(0) = \{(u_1, u_2) \in W \mid \text{Im}(u_2) \subset \text{Ker}(u_1)\} = \bigcup_{i=0}^m X_i.$$

Furthermore, for every $p \in \{0, \dots, m\}$, the morphism p_1 is surjective and Z_p is irreducible, hence X_p is irreducible.

If $m \geq 2n$, then

$$\begin{cases} X_0 \subset \dots \subset X_n; \\ X_n = \dots = X_{m-n}; \\ X_{m-n} \supset \dots \supset X_m; \end{cases}$$

and thus $\mu^{-1}(0) = X_n$.

If $m < 2n$, then

$$\begin{cases} X_0 \subset \dots \subset X_{\max(0, m-n)}; \\ X_{\min(m, n)} \supset \dots \supset X_m; \end{cases}$$

and one easily checks that there is no other inclusion relation between the X_p . \square

Corollary 3.2. *The dimension of $\mu^{-1}(0)$ is*

$$\dim(\mu^{-1}(0)) = \begin{cases} nm + \frac{1}{4}m^2 & \text{if } m < 2n \text{ and } m \text{ is even;} \\ nm + \frac{1}{4}(m^2 - 1) & \text{if } m < 2n \text{ and } m \text{ is odd;} \\ 2nm - n^2 & \text{if } m \geq 2n. \end{cases}$$

Proof. By Proposition 3.1, it suffices to compute the dimension of X_p for some p . If $p \leq n$ or $p \geq m - n$, then one may check that the map $p_1 : Z_p \rightarrow X_p$ is birational, and thus $Q(p) := \dim(X_p) = \dim(Z_p) = p(m - p) + mn$. It remains simply to study the variations of the polynomial Q to obtain the result. \square

We recall that the quotient morphism $W \rightarrow W//G$ is given by $(u_1, u_2) \mapsto u_2 \circ u_1 \in \text{End}(V') = \mathfrak{h}$, by classical invariant theory (see [Pro07, §9.1.4] for instance). Let us now fix $l \in \{0, \dots, N\}$. We also fix a basis \mathcal{B} of V resp. \mathcal{B}' of V' , and we introduce some notation that we will use in the proofs of Proposition 3.3 and Lemma 3.5:

$$(3) \quad \bullet (u_1^l, u_2^l) := \left(\begin{bmatrix} 0_{l, m-l} & I_l \\ 0_{n-l, m-l} & 0_{n-l, l} \end{bmatrix}, \begin{bmatrix} I_l & 0_{l, n-l} \\ 0_{m-l, l} & 0_{m-l, n-l} \end{bmatrix} \right) \in W;$$

$$(4) \quad \bullet f_l := \begin{bmatrix} 0_{l, m-l} & I_l \\ 0_{m-l, m-l} & 0_{m-l, l} \end{bmatrix} \in \mathfrak{h}.$$

If \mathbf{d} is a partition of m , then we denote by $\mathcal{O}_{\mathbf{d}} \subset \mathfrak{h} \cong \mathfrak{gl}_m$ the corresponding nilpotent orbit (see Section 2.2).

Proposition 3.3. *The symplectic reduction of W by G is $\mu^{-1}(0)//G = \overline{\mathcal{O}_{[2N, 1^{m-2N}]}}$.*

Proof. If $f \in \mu^{-1}(0)//G$, then there exists $(u_1, u_2) \in \mu^{-1}(0)$ such that $f = u_2 \circ u_1$, and thus $f \circ f = (u_2 \circ u_1) \circ (u_2 \circ u_1) = u_2 \circ (u_1 \circ u_2) \circ u_1 = 0$, whence the inclusion " \subset ". Now, let $f \in \overline{\mathcal{O}_{[2N, 1^{m-2N}]}}$. Up to conjugation by an element of H , we can suppose that $f = f_l$ for some $l \leq N$, where f_l is defined by (4). But then $u_2^l \circ u_1^l = f_l$ and $u_1^l \circ u_2^l = 0$, where u_1^l and u_2^l are defined by (3), and thus $f \in \mu^{-1}(0)//G$. \square

Corollary 3.4. *The symplectic reduction $\mu^{-1}(0)//G \subset \mathfrak{h}$ is irreducible and decomposes into $N + 1$ orbits for the adjoint action of H :*

$$U_i := \mathcal{O}_{[2^i, 1^{m-2i}]}, \quad \text{for } i = 0, \dots, N.$$

The closures of the nilpotent orbits U_i are nested in the following way:

$$\{0\} = \overline{U_0} \subset \dots \subset \overline{U_N} = \mu^{-1}(0)//G.$$

Hence, $\mu^{-1}(0)//G$ is a symplectic variety (see Section 2), of dimension $2N(m - N)$ ([CM93, Corollary 6.1.4]), and whose singular locus is $\overline{U_{N-1}}$ ([KP81, §3.2]).

By Corollary 3.2, the dimension of the general fibers of the quotient morphism ν is

$$(5) \quad \begin{cases} nm - \frac{1}{4}m^2 & \text{if } m < 2n \text{ and } m \text{ is even;} \\ nm - \frac{1}{4}(m^2 - 1) & \text{if } m < 2n \text{ and } m \text{ is odd;} \\ n^2 & \text{if } m \geq 2n. \end{cases}$$

If $m < 2n$, then $N = \lfloor \frac{m}{2} \rfloor$, and we denote

$$(6) \quad G' := \left\{ \begin{bmatrix} M & 0_{n-N, N} \\ 0_{N, n-N} & I_N \end{bmatrix}, M \in GL_{n-N} \right\} \cong GL_{n-N},$$

which is a reductive algebraic subgroup of $G \cong GL_n$.

Proposition 3.5. *The general fibers of the quotient morphism $\nu : \mu^{-1}(0) \rightarrow \mu^{-1}(0)//G$ are isomorphic to*

$$\begin{cases} G & \text{if } m \geq 2n; \\ G/G' & \text{if } m < 2n \text{ and } m \text{ is even;} \end{cases}$$

where $G' \subset G$ is the subgroup defined by (6).

Proof. We first suppose that $m < 2n$ and m is even (that is, $N = \frac{m}{2}$). With the notation (3), and by a result of Luna (see [SB00, §I.6.2.5, Theorem 10]), we have the equivalence

$$G.(u_1^N, u_2^N) \text{ is closed in } \mu^{-1}(0) \Leftrightarrow C_G(G').(u_1^N, u_2^N) \text{ is closed in } \mu^{-1}(0).$$

Now $C_G(G') = \left\{ \begin{bmatrix} M & 0 \\ 0 & \lambda I_{n-N} \end{bmatrix}, M \in GL_N, \lambda \in \mathbb{G}_m \right\}$, where \mathbb{G}_m denotes the multiplicative group. Hence

$$C_G(G').(u_1^N, u_2^N) = \left\{ \left(\begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} M^{-1} & 0 \\ 0 & 0 \end{bmatrix} \right), M \in GL_N \right\} \subset \mu^{-1}(0)$$

is a closed subset, and thus $G.(u_1^N, u_2^N)$ is the unique closed orbit contained in the fiber $\nu^{-1}(f_N)$, where f_N is defined by (4). One may check that $\text{Stab}_G((u_1^N, u_2^N)) = G'$. Furthermore, $\dim(G/G') = N(2n - N)$, which is also the dimension of the general fibers of ν by (5), and thus $\nu^{-1}(f_N) \cong G/G'$.

We now suppose that $m \geq 2n$ (that is, $N = n$). One may check that $\text{Stab}_G((u_1^n, u_2^n)) = Id$, and thus the fiber $\nu^{-1}(f_n)$ contains a unique closed orbit isomorphic to G . But $\dim(G) = n^2$ is the dimension of the general fibers of ν by (5), hence $\nu^{-1}(f_n) \cong G$. \square

Corollary 3.6. *The Hilbert function h_0 of the general fibers of the quotient morphism $\nu : \mu^{-1}(0) \rightarrow \mu^{-1}(0)//G$ is given by:*

$$\forall M \in \text{Irr}(G), h_0(M) = \begin{cases} \dim(M) & \text{if } m \geq 2n; \\ \dim(M^{G'}) & \text{if } m < 2n \text{ and } m \text{ is even;} \end{cases}$$

where $G' \subset G$ is the subgroup defined by (6).

If $m < 2n$ and m is odd, then the situation is more complicated (except the case $m = 1$ which is trivial) because the general fibers of the quotient morphism ν are reducible. From now on, we will only consider the cases where either $m \geq 2n$ or $m < 2n$, m is even.

3.2. The reduction principle for the main component. In this section we prove our most important theoretical result, which is the *reduction principle* (Proposition 3.8). Let us mention that a similar reduction principle (but in a different setting) was already obtained in [Tera].

The subvariety $\mu^{-1}(0) \subset W$ being $G \times H$ -stable, it follows from [Bri13, Lemma 3.3] that the invariant Hilbert scheme

$$\mathcal{H} := \text{Hilb}_{h_0}^G(\mu^{-1}(0))$$

is a H -stable closed subscheme of $\text{Hilb}_{h_0}^G(W)$. We denote by $\mathcal{H}^{\text{main}}$ the main component of \mathcal{H} . The scheme $\text{Hilb}_{h_0}^G(W)$ was studied in [Tera]; let us recall

Proposition 3.7. ([Tera, §4.4]) *Let h_0 be the Hilbert function given by Corollary 3.6, and let $H = GL(V')$ acting naturally on $\text{Gr}(m - h_0(V), V'^*) \times \text{Gr}(m - h_0(V^*), V')$. Then there exists a H -equivariant morphism*

$$\rho : \text{Hilb}_{h_0}^G(W) \rightarrow \text{Gr}(m - h_0(V), V'^*) \times \text{Gr}(m - h_0(V^*), V')$$

given on closed points by $[Z] \mapsto (\text{Ker}(f_Z^1), \text{Ker}(f_Z^2))$, where $f_Z^1 : V'^* \cong \text{Mor}^G(W, V) \rightarrow \text{Mor}^G(Z, V)$ and $f_Z^2 : V' \cong \text{Mor}^G(W, V^*) \rightarrow \text{Mor}^G(Z, V^*)$ are the restriction maps.

By Corollary 3.6, we have $h_0(V) = h_0(V^*) = N$. We identify $\mathrm{Gr}(m - N, V'^*)$ with $\mathrm{Gr}(N, V')$, and we denote

$$A_i := \{(L_1, L_2) \in \mathrm{Gr}(N, V') \times \mathrm{Gr}(m - N, V') \mid \dim(L_1 \cap L_2) = N - i\}, \text{ for } i = 0, \dots, N.$$

The A_i are the $N + 1$ orbits for the action of H on $\mathrm{Gr}(N, V') \times \mathrm{Gr}(m - N, V')$, and

$$A_0 = \overline{A_0} \subset \overline{A_1} \subset \dots \subset \overline{A_N} = \mathrm{Gr}(N, V') \times \mathrm{Gr}(m - N, V').$$

In particular, A_N is the unique open orbit and

$$(7) \quad A_0 = \mathcal{F}_{N, m-N} := \{(L_1, L_2) \in \mathrm{Gr}(N, V') \times \mathrm{Gr}(m - N, V') \mid L_1 \subset L_2\},$$

which is a partial flag variety, is the unique closed orbit. Let

- $a_0 := (L_1, L_2) \in A_0$, and P the parabolic subgroup of H stabilizing a_0 ;
- $W' := \{(u_1, u_2) \in W \mid L_2 \subset \mathrm{Ker}(u_1) \text{ and } \mathrm{Im}(u_2) \subset L_1\}$, which is a $G \times P$ -module contained in $\mu^{-1}(0)$; and
- $\mathcal{H}' := \mathrm{Hilb}_{h_0}^G(W')$, and $\mathcal{H}'^{\mathrm{main}}$ its main component.

If either $m \geq 2n$ or $m < 2n$, m even, then h_0 coincides with the Hilbert function of the general fibers of the quotient morphism $W' \rightarrow W'//G$ by [Tera, Proposition 4.13]; in particular, $\mathcal{H}'^{\mathrm{main}}$ is well-defined. We are going to prove

Proposition 3.8. *If either $m \geq 2n$ or $m < 2n$, m even, and with the above notation, there is a H -equivariant isomorphism*

$$\mathcal{H}^{\mathrm{main}} \cong H \times^P \mathcal{H}'^{\mathrm{main}}.$$

First of all, we need

Lemma 3.9. *If either $m \geq 2n$ or $m < 2n$ with m even, then the morphism ρ of Proposition 3.7 sends $\mathcal{H}^{\mathrm{main}}$ onto A_0 , the H -variety defined by (7).*

Proof. As the quotient morphism $\nu : \mu^{-1}(0) \rightarrow \mu^{-1}(0)//G$ is flat over the open orbit U_N , the restriction of the Hilbert-Chow morphism γ to $\gamma^{-1}(U_N)$ is an isomorphism. We fix $f_N \in U_N$, and we denote $Q := \mathrm{Stab}_H(f_N)$, and $[Z_N]$ the unique point of \mathcal{H} such that $\gamma([Z_N]) = f_N$. As γ is H -equivariant, $[Z_N]$ is Q -stable. In addition, ρ is also H -equivariant, hence $\rho([Z_N])$ is a fixed point for the action of Q . But one may check that $\mathrm{Gr}(N, V') \times \mathrm{Gr}(m - N, V')$ has a unique fixed point for Q , which is contained in A_0 . Then, as A_0 is H -stable, we have $\rho([Z]) \in A_0$, for every $[Z] \in \gamma^{-1}(U_N)$. Hence, $\rho^{-1}(A_0)$ is a closed subscheme of $\mathrm{Hilb}_{h_0}^G(W)$ containing $\gamma^{-1}(U_N)$, and the result follows. \square

The restriction $\rho|_{\mathcal{H}^{\mathrm{main}}} : \mathcal{H}^{\mathrm{main}} \rightarrow A_0$ is H -equivariant, hence $\mathcal{H}^{\mathrm{main}}$ is the total space of a H -homogeneous fiber bundle over A_0 . Let F be the scheme-theoretic fiber of $\rho|_{\mathcal{H}^{\mathrm{main}}}$ over a_0 . The action of P on $\mathcal{H}^{\mathrm{main}}$, induced by the action of H , stabilizes F , and there is a H -equivariant isomorphism

$$(8) \quad \mathcal{H}^{\mathrm{main}} \cong H \times^P F.$$

Hence, to prove Proposition 3.8, we have to determine F as a P -scheme. We start by considering F' , the scheme-theoretic fiber of the restriction $\rho|_{\mathcal{H}} : \mathcal{H} \rightarrow \mathrm{Gr}(N, V') \times \mathrm{Gr}(m - N, V')$ over a_0 , as a P -scheme. The proof of the next lemma is analogous to the proof of [Tera, Lemma 3.7].

Lemma 3.10. *With the above notation, there is a P -equivariant isomorphism*

$$F' \cong \mathcal{H}',$$

where P acts on \mathcal{H}' via its action on W' .

As $\mathcal{H}^{\text{main}}$ is an irreducible variety of dimension $2N(m-N)$, we deduce from (8) that F is an irreducible variety of dimension N^2 . By Lemma 3.10, the fiber F is isomorphic to a subvariety of $\mathcal{H}'^{\text{main}}$, but $\dim(\mathcal{H}'^{\text{main}}) = N^2$, and thus there is a P -equivariant isomorphism

$$(9) \quad F \cong \mathcal{H}'^{\text{main}},$$

and Proposition 3.8 follows.

Remark 3.11. The scheme \mathcal{H}' is P -stable and identifies with a closed subscheme of \mathcal{H} , hence there is an inclusion of H -schemes $H \times^P \mathcal{H}' \subset \mathcal{H}$.

3.3. Proofs of Theorems A and B for $GL(V)$. Our strategy to prove Theorems A and B is the following: first we perform a reduction step (Proposition 3.8), then we use [Tera, §1, Theorem] to identify $\text{Hilb}_{h_{W'}}^G(W')^{\text{main}}$, and finally we compare the Hilbert-Chow morphism $\gamma: \text{Hilb}_{h_0}^G(\mu^{-1}(0))^{\text{main}} \rightarrow \mu^{-1}(0)//G$ with the Springer desingularizations of $\mu^{-1}(0)//G$. Let us start by recalling

Theorem 3.12. ([Tera, §1, Theorem]) *Let $G = GL(V)$, let $W = \text{Hom}(V', V) \times \text{Hom}(V, V')$, and let h_W be the Hilbert function of the general fibers of the quotient morphism $W \rightarrow W//G$. We denote $n := \dim(V)$, $m := \dim(V')$, and by Y_0 the blow-up of $W//G = \text{End}(V')^{\leq n} := \{f \in \text{End}(V') \mid \text{rk}(f) \leq n\}$ at 0. In the following cases, the invariant Hilbert scheme $\mathcal{H}' := \text{Hilb}_{h_W}^G(W)$ is a smooth variety and the Hilbert-Chow morphism is the succession of blow-up described as follows:*

- if $n \geq 2m - 1$, then $\mathcal{H}' \cong W//G = \text{End}(V')$;
- if $m > n = 1$ or $m = n = 2$, then $\mathcal{H}' \cong Y_0$;
- if $m > n = 2$, then \mathcal{H}' is isomorphic to the blow-up of Y_0 along the strict transform of $\text{End}(V')^{\leq 1}$.

Let us now consider the following diagram

$$\begin{array}{ccc} & \mathcal{F}_{N, m-N} & \\ p_1 \swarrow & & \searrow p_2 \\ \text{Gr}(N, V') & & \text{Gr}(m-N, V') \end{array}$$

where $\mathcal{F}_{N, m-N}$ is defined by (7), p_1 and p_2 being the natural projections. We denote by \underline{V}' the constant vector bundle over $\mathcal{F}_{N, m-N}$ with fiber V' , and by T_1 resp. by T_2 , the pull-back of the tautological bundle over $\text{Gr}(N, V')$ by p_1 , resp. over $\text{Gr}(m-N, V')$ by p_2 . In particular, if $N = \frac{m}{2}$, then $\mathcal{F}_{N, m-N} = \text{Gr}(N, V')$ and $T := T_1 = T_2$ is the tautological bundle over $\text{Gr}(N, V')$.

We deduce from Proposition 3.8 and Theorem 3.12 the following H -equivariant isomorphisms

$$(10) \quad \mathcal{H}^{\text{main}} \cong \begin{cases} \text{Hom}(\underline{V}'/T, T) & \text{if } n \geq m-1 \text{ and } m \text{ is even;} \\ \text{Hom}(\underline{V}'/T_2, T_1) & \text{if } n = 1 \text{ and } m \geq 3; \\ \text{Bl}_0(\text{Hom}(\underline{V}'/T_2, T_1)) & \text{if } n = 2 \text{ and } m \geq 4; \end{cases}$$

where $\text{Bl}_0(\cdot)$ denotes the blow-up along the zero section. In all these cases, $\mathcal{H}^{\text{main}}$ is smooth, and thus the Hilbert-Chow morphism $\gamma: \mathcal{H}^{\text{main}} \rightarrow \mu^{-1}(0)//G$ is a desingularization.

On the other hand, we saw in Section 2 that the Springer desingularizations of $\mu^{-1}(0)//G$ are the cotangent bundles $\mathcal{T}_1^* := \mathcal{T}^*\text{Gr}(N, V')$ and $\mathcal{T}_2^* := \mathcal{T}^*\text{Gr}(m-N, V') \cong \mathcal{T}^*\text{Gr}(N, V'^*)$. We then distinguish between two cases:

- (1) If $N < \frac{m}{2}$, then let us prove by contradiction that $\gamma : \mathcal{H}^{\text{main}} \rightarrow \mu^{-1}(0)//G$ cannot be a Springer desingularization. First, we consider the isomorphism of $G \times H$ -modules $W \cong W^*$. Denoting $\mathcal{H}^* := \text{Hilb}_{h_0}^G(\mu^{*-1}(0))$, where μ^* is the moment map for the natural action of G on W^* , there is an isomorphism of H -varieties $\mathcal{H}^{\text{main}} \cong \mathcal{H}^{*\text{main}}$. Now if we suppose that (say) $\mathcal{H}^{\text{main}} \cong \mathcal{T}_1^*$, then we get that $\mathcal{H}^{*\text{main}} \cong \mathcal{T}_2^*$, and thus $\mathcal{T}_1^* \cong \mathcal{T}_2^*$ as a H -variety, which is absurd. However, one easily checks that if $n \in \{1, 2\}$ and $m \geq 2n + 1$, then $\gamma : \mathcal{H}^{\text{main}} \rightarrow \mu^{-1}(0)//G$ dominates the two Springer desingularizations \mathcal{T}_1^* and \mathcal{T}_2^* (see [Terb, §A.2.2] for details).
- (2) If $N = \frac{m}{2}$, then $\mathcal{T}^* := \mathcal{T}_1^* = \mathcal{T}_2^*$ is the unique Springer desingularization of $\mu^{-1}(0)//G$. Let us show that $\gamma : \mathcal{H}^{\text{main}} \rightarrow \mu^{-1}(0)//G$ is the Springer desingularization if and only if $n \geq m - 1$. The implication " \Leftarrow " is given by (10) since $\mathcal{T}^* \cong \text{Hom}(V'/T, T)$. The other implication is given by:

Lemma 3.13. *If $N = \frac{m}{2}$ and the Hilbert-Chow morphism $\gamma : \mathcal{H}^{\text{main}} \rightarrow \mu^{-1}(0)//G$ is the Springer desingularization, then $n \geq m - 1$.*

Proof. We suppose that $\gamma : \mathcal{H}^{\text{main}} \rightarrow \mu^{-1}(0)//G$ is the Springer desingularization, that is, $\mathcal{H}^{\text{main}} \cong \mathcal{T}^*$ as a H -variety. We fix $L \in \text{Gr}(N, V')$, and we define $P \subset H$, W' , and $\mathcal{H}'^{\text{main}}$ as in Section 3.2. We have $\mathcal{T}^* \cong H \times^P \text{Hom}(V'/L, L)$, and it follows from (8) and (9) that $\mathcal{H}^{\text{main}} \cong H \times^P \mathcal{H}'^{\text{main}}$. Hence, $\mathcal{H}'^{\text{main}} \cong \text{Hom}(V'/L, L)$ as a P -variety. We denote by $\gamma' : \mathcal{H}'^{\text{main}} \rightarrow W'//G$ the restriction of the Hilbert-Chow morphism. As γ' is projective and birational, and $W'//G = \text{Hom}(V'/L, L)$ is smooth, Zariski's Main Theorem implies that γ' is an isomorphism. It follows that the quotient morphism $\nu' : W' \rightarrow W'//G$ is flat, and thus $n \geq 2N - 1$ by [Tera, Corollary 4.12]. \square

In addition, if $m = 4$ and $n = 2$, then by (10) we have $\mathcal{H}^{\text{main}} \cong \text{Bl}_0(\mathcal{T}^*)$, and thus γ dominates the unique Springer desingularization of $\mu^{-1}(0)//G$.

3.4. Reducibility of the invariant Hilbert scheme. The aim of this section is to prove Proposition C from the introduction, for $G = GL(V)$. We suppose that $m \geq 2n$, then $N = n$. We fix

$$(11) \quad a_n = (L'_1, L'_2) \in A_n$$

a point of the open H -orbit of $\text{Gr}(n, V') \times \text{Gr}(m - n, V')$, and we consider

$$\begin{aligned} W'' &:= \{(u_1, u_2) \in W \mid L'_2 \subset \text{Ker}(u_1) \text{ and } \text{Im}(u_2) \subset L'_1\} \\ &\cong \text{Hom}(V'/L'_2, V) \times \text{Hom}(V, L'_1), \end{aligned}$$

which is a G -submodule of W . As $V' = L'_1 \oplus L'_2$, there is a natural identification $W'' \cong \text{Hom}(L'_1, V) \times \text{Hom}(V, L'_1)$ as a G -module. Hence, the G -module W'' is symplectic and we denote by $\mu'' : W'' \rightarrow \mathfrak{g}^*$ the corresponding G -equivariant moment map (see the beginning of Section 3.1 for details). The proof of the next lemma is analogous to the proof of [Tera, Lemma 3.7].

Lemma 3.14. *We suppose that $m \geq 2n$, and let $\rho : \mathcal{H} \rightarrow \text{Gr}(n, V') \times \text{Gr}(m - n, V')$ be the morphism of Proposition 3.7. The scheme-theoretic fiber F'' of ρ over the point a_n , defined by (11), is isomorphic to the invariant Hilbert scheme $\text{Hilb}_{h_0}^G(\mu''^{-1}(0))$,*

where h_0 is the Hilbert function defined by $h_0(M) = \dim(M)$, for every $M \in \text{Irr}(G)$, and $\mu'' : W'' \rightarrow \mathfrak{g}^*$ is the moment map defined above.

Remark 3.15. The Hilbert function h_0 of Lemma 3.14 does not generally coincide with the Hilbert function of the general fibers of the quotient morphism $\mu''^{-1}(0) \rightarrow \mu''^{-1}(0)//G$.

By Lemma 3.9, the morphism $\rho : \text{Hilb}_{h_0}^G(W) \rightarrow \text{Gr}(n, V') \times \text{Gr}(m-n, V')$ of Proposition 3.7 sends $\mathcal{H}^{\text{main}}$ onto A_0 . Hence, to prove Proposition C for $G = GL(V)$, it is enough, by Lemma 3.14, to prove that $\text{Hilb}_{h_0}^G(\mu''^{-1}(0))$ is non-empty. We denote $V'' := L'_1$, and we equip $W'' \cong \text{Hom}(V'', V) \times \text{Hom}(V, V'')$ with the natural action of $H' := GL(V'')$. Then

$$\begin{aligned} \mathbb{C}[W'']_2 \cong & (S^2(V'') \otimes S^2(V^*)) \oplus (S^2(V''^*) \otimes S^2(V)) \\ & \oplus (\Lambda^2(V'') \otimes \Lambda^2(V^*)) \oplus (\Lambda^2(V''^*) \otimes \Lambda^2(V)) \\ & \oplus ((sl(V'') \oplus M_0) \otimes (sl(V) \oplus V_0)) \text{ as a } G \times H' \text{-module,} \end{aligned}$$

where V_0 is the trivial G -module resp. M_0 , is the trivial H' -module, and $sl(V'') := \{f \in \text{End}(V'') \mid \text{tr}(f) = 0\}$.

We denote by I_0 the ideal of $\mathbb{C}[W'']$ generated by $(sl(V'') \otimes V_0) \oplus (M_0 \otimes V_0) \oplus (M_0 \otimes sl(V)) \subset \mathbb{C}[W'']_2$. The ideal I_0 is homogeneous, $G \times H'$ -stable, and contains the ideal generated by the homogeneous H' -invariants of positive degree of $\mathbb{C}[W'']$. In particular, I_0 identifies with an ideal of $\mathbb{C}[\mu''^{-1}(0)]$.

Proposition 3.16. *Let $I_0 \subset \mathbb{C}[W'']$ be the ideal defined above, then I_0 is a point of the invariant Hilbert scheme $\text{Hilb}_{h_0}^G(\mu''^{-1}(0))$ defined in Lemma 3.14.*

Proof. We have to check that the ideal I_0 has the Hilbert function h_0 , that is,

$$\mathbb{C}[W'']/I_0 \cong \bigoplus_{M \in \text{Irr}(G)} M^{\oplus \dim(M)}$$

as a G -module. To do that that, we are going to adapt the method used by Kraft and Schwarz to prove [KS, Theorem 9.1]. The result [loc. cit.] was used in [Terb, §2.1.3 and §3.3.2].

We denote $R := V''^* \otimes V$, which is an irreducible $G \times H'$ -submodule of $W''^* \cong R \oplus R^*$. Then R and R^* are orthogonal modulo I_0 , which means that the image of the $G \times H'$ -submodule $R \otimes R^* \subset \mathbb{C}[W'']_2$ in $\mathbb{C}[W'']/I_0$ is isomorphic to the highest weight component of $R \otimes R^*$ (that is, $sl(V'') \otimes sl(V)$). Then, by [Bri85, Lemme 4.1], any irreducible $G \times H'$ -submodule of $\mathbb{C}[R]$ is orthogonal to any irreducible $G \times H'$ -submodule of $\mathbb{C}[R^*]$, and thus the natural morphism

$$\phi : \mathbb{C}[R]^{U \times U'} \otimes \mathbb{C}[R^*]^{U \times U'} \rightarrow (\mathbb{C}[W'']/I_0)^{U \times U'}$$

is surjective, where U resp. U' , denotes the unipotent radical of a Borel subgroup $B \subset G$ resp. $B' \subset H'$. Furthermore, if $T \subset B$ resp. if $T' \subset B'$, is a maximal torus, then ϕ is $T \times T'$ -equivariant.

Now by [Pro07, §13.5.1] we have the following isomorphisms of $T \times T'$ -algebras $\mathbb{C}[R]^{U \times U'} \cong \mathbb{C}[x_1, \dots, x_n]$, where $x_i \in \Lambda^i V'' \otimes \Lambda^i V^*$ is a highest weight vector, and $\mathbb{C}[R^*]^{U \times U'} \cong \mathbb{C}[y_1, \dots, y_n]$, where $y_j \in \Lambda^j V''^* \otimes \Lambda^j V$ is a highest weight vector. Hence, there is an exact sequence

$$0 \rightarrow K_0 \rightarrow \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n] \rightarrow (\mathbb{C}[W'']/I_0)^{U \times U'} \rightarrow 0,$$

where K_0 is the kernel of ϕ . One may check that the ideal K_0 is generated by the products $x_r y_s$ with $r + s > n$ (see [KS, §9, Proof of Theorem 9.1(1)]).

We denote $\Lambda = \langle \epsilon_1, \dots, \epsilon_n \rangle$ the weight lattice of the linear group GL_n with its natural basis, and $\Lambda_+ \subset \Lambda$ the subset of dominant weights, that is, weights of the form $r_1 \epsilon_1 + \dots + r_n \epsilon_n$, with $r_1 \geq \dots \geq r_n$. If $\lambda \in \Lambda_+$, then we denote by $S^\lambda(\mathbb{C}^n)$ the irreducible GL_n -module of highest weight λ . We fix $\lambda = k_1 \epsilon_1 + \dots + k_t \epsilon_t - k_{t+1} \epsilon_{t+1} - \dots - k_n \epsilon_n \in \Lambda_+$, where each k_i is a nonnegative integer. One easily checks that the weight of the monomial

$$x_{n-t}^{k_{t+1}} x_{n-t-1}^{k_{t+2}-k_{t+1}} x_{n-t-2}^{k_{t+3}-k_{t+2}} \dots x_1^{k_n-k_{n-1}} y_t^{k_t} y_{t-1}^{k_{t-1}-k_t} y_{t-2}^{k_{t-2}-k_{t-1}} \dots y_1^{k_1-k_2}$$

for the action of $T \times T'$ is (λ, λ^*) , where λ^* denotes the highest weight of the GL_n -module $S^\lambda(\mathbb{C}^{n^*})$, and that λ uniquely determines this monomial. We get that the isotypic component of the G -module $S^\lambda(V)$ in $\mathbb{C}[W'']/I_0$ is the $G \times H'$ -module $S^\lambda(V''^*) \otimes S^\lambda(V)$. As $\dim(V) = \dim(V'') = n$, we have $\dim(S^\lambda(V)) = \dim(S^\lambda(V''^*))$, for every $\lambda \in \Lambda_+$. In other words, each irreducible G -module M occurs in $\mathbb{C}[W'']/I_0$ with multiplicity $\dim(M)$. \square

By Proposition 3.16, the scheme $\text{Hilb}_{h_0}^G(\mu'^{-1}(0))$ is non-empty, and thus \mathcal{H} has an irreducible component, different from $\mathcal{H}^{\text{main}}$, of dimension greater or equal to $\dim(A_n) = 2n(m-n)$, which implies Proposition C for $G = GL(V)$.

3.5. Study of the case $n = 1$. We saw in Section 3.3 that $\mathcal{H}^{\text{main}}$ is a smooth variety, and in Section 3.4 that \mathcal{H} is always reducible. In this section, we determine the irreducible components of \mathcal{H} when $n = 1$.

We suppose that $m \geq 2$ (the case $m = 1$ being trivial). Then $G = \mathbb{G}_m$ is the multiplicative group, $\mu^{-1}(0)//G = \overline{\mathcal{O}_{[2,1^{m-2}]}} \subset \mathfrak{h}$ by Proposition 3.3, and the morphism of Proposition 3.7 is $\rho: \text{Hilb}_{h_0}^G(W) \rightarrow \mathbb{P}(V') \times \mathbb{P}(V'^*)$. The Segre embedding gives a H -equivariant isomorphism $\mathbb{P}(V') \times \mathbb{P}(V'^*) \cong \mathbb{P}(\mathfrak{h}^{\leq 1})$, where $\mathfrak{h}^{\leq 1} := \{f \in \mathfrak{h} \mid \text{rk}(f) \leq 1\}$, and thus we can consider $\rho': \text{Hilb}_{h_0}^G(W) \rightarrow \mathbb{P}(\mathfrak{h}^{\leq 1})$, the morphism induced by ρ .

Proposition 3.17. *We equip the invariant Hilbert scheme \mathcal{H} with its reduced structure. If $m > n = 1$, then there is a H -equivariant isomorphism*

$$\mathcal{H} \cong \{(f, L) \in \overline{\mathcal{O}_{[2,1^{m-2}]}} \times \mathbb{P}(\mathfrak{h}^{\leq 1}) \mid f \in L\}.$$

In particular, \mathcal{H} is the union of two smooth irreducible components of dimension $2m - 2$ defined by:

- $C_1 := \{(f, L) \in \overline{\mathcal{O}_{[2,1^{m-2}]}} \times \mathbb{P}(\overline{\mathcal{O}_{[2,1^{m-2}]}}) \mid f \in L\} = \mathcal{H}^{\text{main}}$, and the Hilbert-Chow morphism $\gamma: \mathcal{H}^{\text{main}} \rightarrow \overline{\mathcal{O}_{[2,1^{m-2}]}}$ is the blow-up of $\overline{\mathcal{O}_{[2,1^{m-2}]}}$ at 0;
- $C_2 := \{(0, L) \in \overline{\mathcal{O}_{[2,1^{m-2}]}} \times \mathbb{P}(\mathfrak{h}^{\leq 1})\} \cong \mathbb{P}(\mathfrak{h}^{\leq 1})$, and the Hilbert-Chow morphism is the zero map.

Proof. By [Tera, §1, Theorem], there is a H -equivariant isomorphism

$$\gamma \times \rho': \text{Hilb}_{h_0}^G(W) \rightarrow \{(f, L) \in \mathfrak{h}^{\leq 1} \times \mathbb{P}(\mathfrak{h}^{\leq 1}) \mid f \in L\}.$$

Since $\mathcal{H} \hookrightarrow \text{Hilb}_{h_0}^G(W)$, there is a H -equivariant closed embedding

$$\gamma \times \rho': \mathcal{H} \hookrightarrow \mathcal{Y} := \{(f, L) \in \overline{\mathcal{O}_{[2,1^{m-2}]}} \times \mathbb{P}(\mathfrak{h}^{\leq 1}) \mid f \in L\}.$$

One may check that \mathcal{Y} is the union of the two irreducible components C_1 and C_2 , both of dimension $2m - 2$. The morphism $\gamma \times \rho'$ sends $\mathcal{H}^{\text{main}}$ into C_1 ; the varieties $\mathcal{H}^{\text{main}}$ and C_1 have the same dimension, hence $\gamma \times \rho': \mathcal{H}^{\text{main}} \rightarrow C_1$ is an

isomorphism. On the other hand, we saw in Section 3.4 that \mathcal{H} admits another irreducible component, denoted by \mathcal{H}_2 , of dimension at least $2m - 2$, which is the dimension of C_2 , and thus $\gamma \times \rho'$ is an isomorphism between \mathcal{H}_2 and C_2 . \square

Remark 3.18. One may check that the component C_2 of Proposition 3.17 consists of the homogeneous ideals of $\mathbb{C}[\mu^{-1}(0)]$.

When $m \geq 2n \geq 4$, irreducible components of dimension greater than $\dim(\mathcal{H}^{\text{main}})$ may appear. For instance, if $n = 2$ and $m \geq 4$, then one may check that the irreducible component consisting of the homogeneous ideals of $\mathbb{C}[\mu^{-1}(0)]$ is of dimension $4m - 5$, whereas the main component $\mathcal{H}^{\text{main}}$ is of dimension $4m - 8$. In addition, we showed in Section 3.4 that \mathcal{H} has at least two components, but \mathcal{H} may have more components.

4. CASE OF Sp_n

Let V and V' be two vector spaces of dimension n (which is even) and m respectively, and let $W := \text{Hom}(V', V) \times \text{Hom}(V, V')$. We denote $E := V' \oplus V'^*$ on which we fix a non-degenerate quadratic form q , and we take $G = Sp(V)$ and $H = SO(E)$. As G resp. H , preserves a non-degenerate bilinear form on V resp. on E , we can identify $V \cong V^*$ as a G -module resp. $E \cong E^*$ as a H -module. It follows that

$$\begin{aligned} W &\cong \text{Hom}(V', V) \times \text{Hom}(V'^*, V^*) \\ &\cong \text{Hom}(V', V) \times \text{Hom}(V'^*, V) \\ &\cong \text{Hom}(E, V) \end{aligned}$$

as a G -module, and thus H acts naturally on W . We denote by \mathfrak{g} resp. by \mathfrak{h} , the Lie algebra of G resp. of H .

4.1. The quotient morphism. The main results of this section are Proposition 4.3, which describes the irreducible components of the symplectic reduction $\mu^{-1}(0)//G$, and Corollary 4.6, which gives the Hilbert function of the general fibers of the quotient morphism $\nu : \mu^{-1}(0) \rightarrow \mu^{-1}(0)//G$ for each irreducible component of $\mu^{-1}(0)//G$. Contrary to the case of $GL(V)$ studied in Section 3, we will see that $\mu^{-1}(0)//G$ is reducible when $m \leq n$ and m is even.

We have seen that W is equipped with a G -invariant symplectic form (see the beginning of Section 3.1 for details). If $w \in \text{Hom}(E, V)$, we denote the transpose of w by ${}^t w \in \text{Hom}(V^*, E^*) \cong \text{Hom}(V, E)$. Then, by [Bec09, Proposition 3.1], the zero fiber of the moment map $\mu : W \rightarrow \mathfrak{g}^*$ is the $G \times H$ -stable subvariety defined by:

$$\mu^{-1}(0) = \{w \in W \mid w \circ {}^t w = 0\}.$$

Remark 4.1. One may check that the biggest subgroup of $GL(E)$ that stabilizes $\mu^{-1}(0)$ in W is the orthogonal group $O(E)$. However, we prefer to consider the action of $H = SO(E)$ for practical reasons.

The proof of the next proposition is analogous to those of Proposition 3.1 and Corollary 3.2.

Proposition 4.2. *The zero fiber of the moment map $\mu : W \rightarrow \mathfrak{g}^*$ is*

- *an irreducible subvariety of dimension $2mn - \frac{1}{2}n(n+1)$ if $m > n$;*
- *the union of two irreducible components of dimension $mn + \frac{1}{2}m(m-1)$ if $m \leq n$.*

If \mathbf{d} is a partition of $2m$, then we denote by $\mathcal{O}_{\mathbf{d}}$ resp. by $\mathcal{O}_{\mathbf{d}}^I$ and $\mathcal{O}_{\mathbf{d}}^{II}$, the corresponding nilpotent orbit(s) of $\mathfrak{h} \cong \mathfrak{so}_{2m}$ associated to \mathbf{d} (see Section 2.2). The following result was proved by Becker:

Proposition 4.3. ([Bec09, Proposition 3.6]) *The symplectic reduction of W by G is*

$$\mu^{-1}(0)//G = \begin{cases} \overline{\mathcal{O}_{[2^n, 1^{2(m-n)}}]} & \text{if } m > n; \\ \overline{\mathcal{O}_{[2^{m-1}, 1^2]}} & \text{if } m < n \text{ and } m \text{ is odd}; \\ \overline{\mathcal{O}_{[2^m]}^I \cup \mathcal{O}_{[2^m]}^{II}} & \text{if } m \leq n \text{ and } m \text{ is even.} \end{cases}$$

Corollary 4.4. *The orbits for the adjoint action of H on $\mu^{-1}(0)//G$ are*

- $U_i := \mathcal{O}_{[2^i, 1^{2(m-i)}]}$, for $i = 0, 2, \dots, n$, if $m > n$;
- $U_i := \mathcal{O}_{[2^i, 1^{2(m-i)}]}$, for $i = 0, 2, \dots, m-1$, if $m < n$ and m is odd;
- $U_i := \mathcal{O}_{[2^i, 1^{2(m-i)}]}$, for $i = 0, 2, \dots, m-2$, and $U_m^I := \mathcal{O}_{[2^m]}^I$, $U_m^{II} := \mathcal{O}_{[2^m]}^{II}$, if $m \leq n$ and m is even.

The closures of the nilpotent orbits U_i are nested in the following way:

$$\begin{cases} \{0\} = \overline{U_0} \subset \overline{U_2} \subset \dots \subset \overline{U_n} & \text{if } m > n; \\ \{0\} = \overline{U_0} \subset \overline{U_2} \subset \dots \subset \overline{U_{m-1}} & \text{if } m < n \text{ and } m \text{ is odd}; \\ \{0\} = \overline{U_0} \subset \overline{U_2} \subset \dots \subset \overline{U_{m-2}} = \overline{U_m^I} \cap \overline{U_m^{II}} & \text{if } m \leq n \text{ and } m \text{ is even.} \end{cases}$$

If $m > n$ or m is odd resp. if $m \leq n$ and m is even, then the symplectic reduction $\mu^{-1}(0)//G$ is the closure of a nilpotent orbit resp. the union of two closures of nilpotent orbits, and thus the irreducible components of $\mu^{-1}(0)//G$ are symplectic varieties (see Section 2). If $m > n$, then $\mu^{-1}(0)//G$ is of dimension $2mn - n(n+1)$, and its singular locus is $\overline{U_{n-2}}$. On the other hand, if $m \leq n$, then each irreducible component of $\mu^{-1}(0)//G$ is of dimension $m(m-1)$, and the singular locus of $\mu^{-1}(0)//G$ is $\overline{U_{m-2}}$ resp. $\overline{U_{m-3}}$, when m is even resp. when m is odd. The dimension of the irreducible components of $\mu^{-1}(0)//G$ is given by [CM93, Corollary 6.1.4], and the singular locus of $\mu^{-1}(0)//G$ is given by [KP82, Theorem 2].

We are now interested in the Hilbert function of the general fibers of the quotient morphism for each irreducible component of $\mu^{-1}(0)//G$. We will distinguish between the following cases:

- If $m > n$, then $\mu^{-1}(0)//G$ is irreducible, and we denote by h_0 the Hilbert function of the general fibers of the quotient morphism $\nu : \mu^{-1}(0) \rightarrow \mu^{-1}(0)//G$. By Proposition 4.2, the dimension of these fibers is $\frac{1}{2}n(n+1)$.
- If $m \leq n$ and m is even, then by Proposition 4.2, the zero fiber $\mu^{-1}(0)$ is the union of two irreducible components that we denote by X_I and by X_{II} . Let $\nu_I : X_I \rightarrow Y_I$ and $\nu_{II} : X_{II} \rightarrow Y_{II}$ be the quotient morphisms. Up to the exchange of X_I and X_{II} , we can suppose that $Y_I = \overline{U_m^I}$ and $Y_{II} = \overline{U_m^{II}}$. The orthogonal group $O(E)$ acts transitively on $U_m^I \cup U_m^{II}$, hence the general fibers of ν_I and ν_{II} are isomorphic. In particular, these fibers have the same Hilbert function, denoted by h_0 , and the same dimension, which is $mn - \frac{1}{2}m(m-1)$.
- If $m < n$ and m is odd, then $\mu^{-1}(0)//G$ is irreducible, and we denote by h_0 the Hilbert function of the general fibers of the quotient morphism $\nu : \mu^{-1}(0) \rightarrow \mu^{-1}(0)//G$. These fibers being reducible, determining h_0 is more complicated than in the previous cases (except the case $m = 1$ which

is trivial). From now on, we will always exclude the case where $m < n$ and m is odd.

If $m < n$ and m is even, then we denote

$$(12) \quad G' := \left\{ \begin{bmatrix} M & 0_{n-m,m} \\ 0_{m,n-m} & I_m \end{bmatrix}, M \in Sp_{n-m} \right\} \cong Sp_{n-m},$$

which is a reductive algebraic subgroup of $G \cong Sp_n$. The proof of the next proposition is analogous to the proof of Proposition 3.5:

Proposition 4.5. *If $m > n$, then the general fibers of the quotient morphism $\nu : \mu^{-1}(0) \rightarrow \mu^{-1}(0)//G$ are isomorphic to G .*

If $m = n$, then the general fibers of the quotient morphisms $\nu_I : X_I \rightarrow Y_I$ and $\nu_{II} : X_{II} \rightarrow Y_{II}$ are isomorphic to G .

If $m < n$ and m is even, then the general fibers of ν_I and ν_{II} are isomorphic to G/G' , where $G' \subset G$ is the subgroup defined by (12).

Corollary 4.6. *The Hilbert function h_0 defined above is given by:*

$$\forall M \in \text{Irr}(G), h_0(M) = \begin{cases} \dim(M) & \text{if } m \geq n; \\ \dim(M^{G'}) & \text{if } m < n \text{ and } m \text{ is even;} \end{cases}$$

where $G' \subset G$ is the subgroup defined by (12).

4.2. The reduction principle for the main component. In this section, we give the guidelines to prove the reduction principle when $G = Sp(V)$ (Proposition 4.8). The strategy is the same as for $GL(V)$ (see Section 3.2), but as the symplectic reduction $\mu^{-1}(0)//G$ is reducible when $m \leq n$ and m is even, it seems necessary to give some additional details.

As $\mu^{-1}(0)$ is a $G \times H$ -stable subvariety of W , it follows from [Bri13, Lemma 3.3] that the invariant Hilbert scheme

$$\mathcal{H} := \text{Hilb}_{h_0}^G(\mu^{-1}(0))$$

is a H -stable closed subscheme of $\text{Hilb}_{h_0}^G(W)$. As we aim at constructing desingularizations of the irreducible components of $\mu^{-1}(0)//G$, we consider the two H -stable closed subschemes $\mathcal{H}_I := \text{Hilb}_{h_0}^G(X_I)$ and $\mathcal{H}_{II} := \text{Hilb}_{h_0}^G(X_{II})$ instead of \mathcal{H} when $m \leq n$ and m is even. Let us note that if we fix $y_0 \in O(E) \setminus SO(E)$ and make H act on X_{II} by $(y_0 y y_0^{-1}) \cdot x$ for every $y \in H$ and every $x \in X_{II}$, then $\phi : X_I \rightarrow X_{II}$, $x \mapsto y_0 \cdot x$ is a $G \times H$ -equivariant isomorphism, and thus $\mathcal{H}_I \cong \mathcal{H}_{II}$ as a H -scheme. We denote by $\mathcal{H}_I^{\text{main}}$ resp. by $\mathcal{H}_{II}^{\text{main}}$, the main component of \mathcal{H}_I resp. of \mathcal{H}_{II} . We always have the (set-theoretic) inclusion $\mathcal{H}_I \cup \mathcal{H}_{II} \subset \mathcal{H}$, but this may not be an equality. If $m > n$, then $\mu^{-1}(0)//G$ is irreducible, and we denote by $\mathcal{H}^{\text{main}}$ the main component of \mathcal{H} .

The scheme $\text{Hilb}_{h_0}^G(W)$ was studied in [Terb]. In particular, we obtained

Proposition 4.7. ([Terb, §1.5.1]) *Let h_0 be the Hilbert function given by Corollary 4.6, and let $H = SO(E)$ acting naturally on $\text{Gr}(2m - h_0(V^*), E)$. Then there exists a H -equivariant morphism*

$$\rho : \text{Hilb}_{h_0}^G(W) \rightarrow \text{Gr}(2m - h_0(V^*), E)$$

given on closed points by $[Z] \mapsto \text{Ker}(f_Z)$, where $f_Z : E \cong \text{Mor}^G(W, V^*) \rightarrow \text{Mor}^G(Z, V^*)$ is the restriction map.

We identify $\mathrm{Gr}(2m - h_0(V^*), E)$ with $\mathrm{Gr}(h_0(V^*), E^*)$. By Corollary 4.6, if either $m > n$ or $m \leq n$, m even, then $h_0(V^*) = N := \min(m, n)$. The non-degenerate quadratic form q on E gives a canonical isomorphism $E \cong E^*$. In particular, q identifies with a non-degenerate quadratic form on E^* . For $i = 0, \dots, N$, we denote

$$A_i := \{L \in \mathrm{Gr}(N, E^*) \mid q|_L \text{ is of rank } i\}.$$

If $m > n$, then the A_i are the $n+1$ orbits for the action of H on $\mathrm{Gr}(n, E^*)$.

However, if $m \leq n$, then the A_i are H -orbits for $i = 1, \dots, m$, but the isotropic Grassmannian $A_0 = \mathrm{OGr}(m, E^*)$ is the union of two H -orbits, denoted by OGr^I and by OGr^{II} , which are exchanged by the action of any element of $O(E) \setminus SO(E)$. In any case, we have

$$\mathrm{OGr}(N, E^*) = \overline{A_0} \subset \overline{A_1} \subset \dots \subset \overline{A_N} = \mathrm{Gr}(N, E^*).$$

Let us now fix some notation:

- $L_0 \in A_0$, and P the parabolic subgroup of H stabilizing L_0 ;
- $W' := \mathrm{Hom}(E/L_0^\perp, V)$, which identifies with a $G \times P$ -module contained in $\mu^{-1}(0)$; and
- $\mathcal{H}' := \mathrm{Hilb}_{h_0}^G(W')$, and $\mathcal{H}'^{\mathrm{main}}$ its main component.

It must be emphasized that, if either $m > n$ or $m \leq n$, m even, then the Hilbert function of the general fibers of the quotient morphism $W' \rightarrow W'//G$ coincides with the Hilbert function h_0 of Corollary 4.6 (in particular, $\mathcal{H}'^{\mathrm{main}}$ is well defined).

Proceeding as for Lemma 3.9, one may check that, if $m > n$ resp. if $m \leq n$ with m even, then the morphism ρ of Proposition 4.7 sends $\mathcal{H}^{\mathrm{main}}$ resp. $\mathcal{H}_I^{\mathrm{main}}$ and $\mathcal{H}_{II}^{\mathrm{main}}$, onto A_0 . More precisely, if $m \leq n$ and m is even, then ρ sends $\mathcal{H}_I^{\mathrm{main}}$ onto one of the irreducible component of A_0 , and $\mathcal{H}_{II}^{\mathrm{main}}$ onto the other component. Up to the exchange of these two components, we can suppose that ρ sends $\mathcal{H}_I^{\mathrm{main}}$ onto OGr^I , and $\mathcal{H}_{II}^{\mathrm{main}}$ onto OGr^{II} .

It follows that the restriction of ρ equips $\mathcal{H}^{\mathrm{main}}$ resp. $\mathcal{H}_I^{\mathrm{main}}$, resp. $\mathcal{H}_{II}^{\mathrm{main}}$, with a structure of a H -homogeneous fiber bundle over A_0 resp. over OGr^I , resp. over OGr^{II} . Hence, it is enough to determine the fiber F_0 over L_0 to determine $\mathcal{H}^{\mathrm{main}}$ resp. $\mathcal{H}_I^{\mathrm{main}}$, resp. $\mathcal{H}_{II}^{\mathrm{main}}$. Proceeding as in Section 3.2, we obtain that F_0 is isomorphic to $\mathcal{H}'^{\mathrm{main}}$ as a P -scheme. We deduce

Proposition 4.8. *With the above notation, we have the following H -equivariant isomorphisms:*

- If $m > n$, then

$$\mathcal{H}^{\mathrm{main}} \cong H \times^P \mathcal{H}'^{\mathrm{main}}.$$

- If $m \leq n$, m even, and $L_0 \in \mathrm{OGr}^I$ resp. $L_0 \in \mathrm{OGr}^{II}$, then

$$\mathcal{H}_I^{\mathrm{main}} \cong H \times^P \mathcal{H}'^{\mathrm{main}} \quad \text{resp.} \quad \mathcal{H}_{II}^{\mathrm{main}} \cong H \times^P \mathcal{H}'^{\mathrm{main}}.$$

4.3. Proofs of Theorems A and B for $Sp(V)$. In this section, we proceed as in Section 3.3 to prove Theorems A and B when $G = Sp(V)$. Before going any further, let us mention that the case $n = 2$, $m = 3$ was already handled by Becker in [Bec11]. In this situation, $\mu^{-1}(0)//G$ is a closure of a nilpotent orbit that admits two Springer desingularizations, and Becker showed that $\gamma: \mathrm{Hilb}_{h_0}^G(\mu^{-1}(0)) \rightarrow \mu^{-1}(0)//G$ is a desingularization that dominates them both. To obtain this result, Becker first used the existence of natural morphisms from the invariant Hilbert scheme to Grassmannians to identify $\mathrm{Hilb}_{h_0}^G(\mu^{-1}(0))^{\mathrm{main}}$ with

the total space of a homogeneous line bundle over a Grassmannian, and then she showed that $\text{Hilb}_{h_0}^G(\mu^{-1}(0)) = \text{Hilb}_{h_0}^G(\mu^{-1}(0))^{\text{main}}$ by computing the tangent space of $\text{Hilb}_{h_0}^G(\mu^{-1}(0))$ at every point of the main component.

Let us now recall the following result:

Theorem 4.9. ([Tera, §1, Theorem]) *Let $G = \text{Sp}(V)$, let $W = \text{Hom}(E, V)$, and let h_W be the Hilbert function of the general fibers of the quotient morphism $W \rightarrow W//G$. We denote $n := \dim(V)$, $e := \dim(E)$, and we denote by Y_0 the blow-up of $W//G = \Lambda^2(E^*)^{\leq n} := \{Q \in \Lambda^2(E^*) \mid \text{rk}(Q) \leq n\}$ at 0. In the following cases, the invariant Hilbert scheme $\mathcal{H}' := \text{Hilb}_{h_W}^G(W)$ is a smooth variety, and the Hilbert-Chow morphism is the succession of blow-up described as follows:*

- if $n \geq 2e - 2$, then $\mathcal{H}' \cong W//G = \Lambda^2(E^*)$;
- if $e > n = 2$ or $e = n = 4$, then $\mathcal{H}' \cong Y_0$;
- if $e > n = 4$, then \mathcal{H}' is isomorphic to the blow-up of Y_0 along the strict transform of $\Lambda^2(E^*)^{\leq 2}$.

If $m > n$, then we denote by T the tautological bundle over $A_0 = \text{OGr}(n, E^*)$. If $m \leq n$ and m is even, then we denote by T_I resp. by T_{II} , the tautological bundle over OGr^I resp. over OGr^{II} . We deduce from Proposition 4.8 and Theorem 4.9 the following H -equivariant isomorphisms

$$(13) \quad \mathcal{H}^{\text{main}} \cong \begin{cases} \Lambda^2(T) & \text{if } m > n = 2; \\ \text{Bl}_0(\Lambda^2(T)) & \text{if } m > n = 4; \end{cases}$$

$$\mathcal{H}_{\bullet}^{\text{main}} \cong \begin{cases} \Lambda^2(T_{\bullet}) & \text{if } n \geq 2m - 2 \text{ and } m \text{ is even;} \\ \text{Bl}_0(\Lambda^2(T_{\bullet})) & \text{if } m = n = 4; \end{cases}$$

where \bullet stands for I or II , and $\text{Bl}_0(\cdot)$ denotes the blow-up along the zero section. In all these cases, the main component of the invariant Hilbert scheme is smooth, and thus the Hilbert-Chow morphism $\gamma: \mathcal{H}^{\text{main}} \rightarrow \mu^{-1}(0)//G$ resp. $\gamma: \mathcal{H}_{\bullet}^{\text{main}} \rightarrow Y_{\bullet}$, is a desingularization.

It remains to compare γ with the Springer desingularizations (when they exist) of the irreducible components of $\mu^{-1}(0)//G$. We saw in Section 2 that the irreducible components of $\mu^{-1}(0)//G$ have Springer desingularizations if and only if $m \leq n + 1$. We then distinguish between the following cases:

- (1) If $m \leq n + 1$ and m is odd, then $\mu^{-1}(0)//G$ admits two Springer desingularizations, which are given by the cotangent bundles \mathcal{T}_I^* and \mathcal{T}_{II}^* over OGr^I and OGr^{II} respectively. The natural action of the orthogonal group $O(E)$ on $\text{OGr}(m, E^*)$ induces an action on the cotangent bundle $\mathcal{T}^* \text{OGr}(m, E^*)$ that exchanges \mathcal{T}_I^* and \mathcal{T}_{II}^* . On the other hand, it follows from Remark 4.1, that the group $O(E)$ stabilizes $\mathcal{H}^{\text{main}}$, and thus $\gamma: \mathcal{H}^{\text{main}} \rightarrow \mu^{-1}(0)//G$ cannot be a Springer desingularization. However, if $n \in \{2, 4\}$ and $m = n + 1$, then one may prove that γ dominates the two Springer desingularizations of $\mu^{-1}(0)//G$ (see [Bec11, Introduction] for the case $n = 2$, the case $n = 4$ being analogous).
- (2) If $m \leq n$ and m is even, then Y_{\bullet} has a unique Springer desingularization, which is given by the cotangent bundle $\mathcal{T}_{\bullet}^* \cong \Lambda^2(T_{\bullet})$ over OGr^{\bullet} . Proceeding as we did for $GL(V)$ in Section 3.3, one may prove that $\gamma: \mathcal{H}_{\bullet}^{\text{main}} \rightarrow Y_{\bullet}$ is the Springer desingularization if and only if $n \geq 2m - 2$.

In addition, if $m = n = 4$, then by (13) we have $\mathcal{H}_\bullet^{\text{main}} \cong \text{Bl}_0(\mathcal{T}_\bullet^*)$, and thus γ dominates the unique Springer desingularization of Y_\bullet .

4.4. Study of the case $n = 2$. In this section, we suppose that $m \geq n = 2$ (the case $m = 1$ being trivial). We will prove that if $m \geq 3$ resp. if $m = 2$, then \mathcal{H} resp. \mathcal{H}_\bullet (where \bullet stands for I or II), is irreducible. In particular, the geometric properties of the invariant Hilbert scheme for $G = Sp(V)$ are quite different from the case of $G = GL(V)$ studied in Section 3. Let us recall that the case $m = 3, n = 2$ was treated by Becker in [Bec11]; she showed that \mathcal{H} is the total space of a line bundle over $\text{OGr}(2, E^*)$.

We have $G \cong Sp_2 = SL_2$, and the morphism of Proposition 4.7 is $\rho: \text{Hilb}_{h_0}^G(W) \rightarrow \text{Gr}(2, E^*)$. Denoting $\mathfrak{h}^{\leq 2} := \{f \in \mathfrak{h} \mid \text{rk}(f) \leq 2\}$, there is a H -equivariant isomorphism

$$(14) \quad \mathbb{P}(\mathfrak{h}^{\leq 2}) \cong \text{Gr}(2, E^*),$$

and thus we can consider the morphism $\rho': \text{Hilb}_{h_0}^G(W) \rightarrow \mathbb{P}(\mathfrak{h}^{\leq 2})$ induced by ρ . By Proposition 4.3, we have

$$\mu^{-1}(0) // G = \begin{cases} \overline{\mathcal{O}_{[2^2, 1^{2m-4}]}} & \text{if } m \geq 3; \\ \overline{\mathcal{O}_{[2^2]}^I \cup \mathcal{O}_{[2^2]}^{II}} & \text{if } m = 2. \end{cases}$$

Proposition 4.10. *We equip all the invariant Hilbert schemes with their reduced structures. If $m > n = 2$, then $\mathcal{H} = \mathcal{H}^{\text{main}}$ is a smooth variety isomorphic to*

$$\text{Bl}_0(\overline{\mathcal{O}_{[2^2, 1^{2m-4}]}}) := \{(f, L) \in \overline{\mathcal{O}_{[2^2, 1^{2m-4}]}} \times \mathbb{P}(\overline{\mathcal{O}_{[2^2, 1^{2m-4}]}}) \mid f \in L\},$$

and the Hilbert-Chow morphism $\gamma: \mathcal{H} \rightarrow \mu^{-1}(0) // G$ is the blow-up of $\overline{\mathcal{O}_{[2^2, 1^{2m-4}]}}$ at 0. If $m = n = 2$, then $\text{Hilb}_{h_0}^G(\mu^{-1}(0)) = \mathcal{H}_I \cup \mathcal{H}_{II}$ is the union of two smooth irreducible components isomorphic to $\text{Bl}_0(\overline{\mathcal{O}_{[2^2]}^I})$ and $\text{Bl}_0(\overline{\mathcal{O}_{[2^2]}^{II}})$ respectively, and the set-theoretic intersection $\mathcal{H}_I \cap \mathcal{H}_{II}$ is formed by the homogeneous ideals of $\mathbb{C}[\mu^{-1}(0)]$. Moreover, the Hilbert-Chow morphism $\gamma: \mathcal{H}_I \rightarrow \overline{\mathcal{O}_{[2^2]}^I}$ resp. $\gamma: \mathcal{H}_{II} \rightarrow \overline{\mathcal{O}_{[2^2]}^{II}}$, is the blow-up of $\overline{\mathcal{O}_{[2^2]}^I}$ resp. of $\overline{\mathcal{O}_{[2^2]}^{II}}$, at 0.

Proof. The proofs for the cases $m = 2$ and $m \geq 3$ are quite similar, and thus we will only consider the case $m \geq 3$ (which is simpler in terms of notation!). Using arguments similar to those used to prove Proposition 3.17, we obtain a closed embedding

$$\gamma \times \rho': \mathcal{H} \hookrightarrow \mathcal{Y} := \{(f, L) \in \overline{\mathcal{O}_{[2^2, 1^{2m-4}]}} \times \mathbb{P}(\mathfrak{h}^{\leq 2}) \mid f \in L\}.$$

One may check that \mathcal{Y} is the union of the two irreducible components C_1 and C_2 defined by:

- $C_1 := \text{Bl}_0(\overline{\mathcal{O}_{[2^2, 1^{2m-4}]}})$; and
- $C_2 := \{(0, L) \in \overline{\mathcal{O}_{[2^2, 1^{2m-4}]}} \times \mathbb{P}(\mathfrak{h}^{\leq 2})\} \cong \mathbb{P}(\mathfrak{h}^{\leq 2})$.

The components C_1 and C_2 are of dimension $4m - 6$ and $4m - 4$ respectively. The morphism $\gamma \times \rho'$ sends $\mathcal{H}^{\text{main}}$ into C_1 ; the varieties $\mathcal{H}^{\text{main}}$ and C_1 have the same dimension, hence $\gamma \times \rho': \mathcal{H}^{\text{main}} \rightarrow C_1$ is an isomorphism.

Now it follows from [Terb, Proposition 3.3.13] that the component C_2 identifies with the closed subset of $\text{Hilb}_{h_0}^G(W)$ formed by the homogeneous ideals of $\mathbb{C}[W]$. Let us describe this identification. If $L \in C_2 \cong \mathbb{P}(\mathfrak{h}^{\leq 2})$, then we denote by I_L the ideal of $\mathbb{C}[W]$ generated by the homogeneous G -invariants of positive degree

of $\mathbb{C}[W]$, and by the G -module $L^\perp \otimes V \subset \mathbb{C}[W]_1 \cong E \otimes V$, where L is identified with a 2-dimensional subspace of E^* via the isomorphism (14). Let us show that I_L is a point of \mathcal{H} if and only if $L \in \text{OGr}(2, E^*)$; the result will follow since $\mathbb{P}(\overline{\mathcal{O}_{[2^2, 1^{2m-4}]}})$ identifies with $\text{OGr}(2, E^*)$ via the isomorphism (14), and since $\{(0, L) \in \overline{\mathcal{O}_{[2^2, 1^{2m-4}]}} \times \mathbb{P}(\overline{\mathcal{O}_{[2^2, 1^{2m-4}]}})\}$ is a subvariety of C_1 . We denote $W' := \text{Hom}(E/L^\perp, V)$, then

$$\mathbb{C}[W']_2 \cong (S^2(E/L^\perp) \otimes S^2(V)) \oplus (\Lambda^2(E/L^\perp) \otimes \Lambda^2(V))$$

as a G -module. Let I'_L be the ideal of $\mathbb{C}[W']$ generated by $\Lambda^2(E/L^\perp) \otimes \Lambda^2(V) \subset \mathbb{C}[W']_2$, then one may check (using [Terb, Proposition 3.3.13]) that

$$\mathbb{C}[W]/I_L \cong \mathbb{C}[W']/I'_L \cong \bigoplus_{M \in \text{Irr}(G)} M^{\oplus \dim(M)}$$

as a G -module. Hence

$$\begin{aligned} I_L \in \mathcal{H} &\Leftrightarrow I_L \cap \mathbb{C}[W]_2 \supset E_0 \otimes S^2(V), \text{ where } E_0 \text{ is the trivial representation of } H; \\ &\Leftrightarrow q|_L = 0, \text{ where } q \text{ is the quadratic form preserved by } H; \\ &\Leftrightarrow L \in \text{OGr}(2, E^*). \end{aligned}$$

□

Remark 4.11. In the proof of Proposition 4.10, we showed that if $m > n = 2$, then the homogeneous ideals of \mathcal{H} are contained in $\mathcal{H}^{\text{main}}$. Using analogous arguments, one may check that this statement is true more generally when $m > n \geq 2$.

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