



**HAL**  
open science

# Spectral properties of Schrödinger operators on compact manifolds: rigidity, flows, interpolation and spectral estimates

Jean Dolbeault, Maria J. Esteban, Ari Laptev, Michael Loss

## ► To cite this version:

Jean Dolbeault, Maria J. Esteban, Ari Laptev, Michael Loss. Spectral properties of Schrödinger operators on compact manifolds: rigidity, flows, interpolation and spectral estimates. *Comptes rendus de l'Académie des sciences. Série I, Mathématique*, 2013, 351 (11-12), pp.437-440. hal-00799252v2

**HAL Id: hal-00799252**

**<https://hal.science/hal-00799252v2>**

Submitted on 23 Jul 2013

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Spectral properties of Schrödinger operators on compact manifolds: rigidity, flows, interpolation and spectral estimates

Jean Dolbeault<sup>a</sup> and Maria J. Esteban<sup>a</sup> and Ari Laptev<sup>b</sup> and Michael Loss<sup>c</sup>

<sup>a</sup>*Ceremade (UMR CNRS no. 7534), Université Paris-Dauphine, Place de Lattre de Tassigny, 75775 Paris 16, France*

<sup>b</sup>*Department of Mathematics, Imperial College London, Huxley Building, 180 Queen's Gate, SW7 2AZ, UK*

<sup>c</sup>*School of Mathematics, Skiles Building, Georgia Institute of Technology, Atlanta GA 30332-0160, USA*

---

## Abstract

This note is devoted to optimal spectral estimates for Schrödinger operators on compact connected Riemannian manifolds without boundary. These estimates are based on the use of appropriate interpolation inequalities and on some recent rigidity results for nonlinear elliptic equations on those manifolds.

**Keywords.** Sobolev inequality; interpolation; Gagliardo-Nirenberg inequalities; rigidity results; Lieb-Thirring inequalities; fast diffusion equation; Laplace-Beltrami operator; Schrödinger equation; eigenvalues; spectral estimates; optimal constants; compact manifolds; Ricci curvature; Ricci tensor

MSC (2010): 58J05; 58J35; 58J60; 53C21; 26D10

**Propriétés spectrales d'opérateurs de Schrödinger sur des variétés compactes : rigidité, flots, interpolation et estimations spectrales.**

**Résumé.** Cette note est consacrée à des estimations spectrales optimales pour des opérateurs de Schrödinger sur des variétés Riemanniennes compactes et simplement connexes, sans bord. Ces estimations sont basées sur certaines inégalités d'interpolation et sur un résultat récent de rigidité pour des équations elliptiques non linéaires sur ces variétés.

---

## 1. Spectral properties of Schrödinger operators on the sphere

We start by briefly reviewing some results that have been obtained in [4]. Let us define  $2^* := \frac{2d}{d-2}$  if  $d \geq 3$ , and  $2^* := \infty$  if  $d = 1$  or  $2$ . We denote by  $\Delta_g$  the Laplace-Beltrami operator on the unit sphere  $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ . It is well known (see [2]) that the equation

$$-\Delta_g u + \frac{\lambda}{q-2} u = u^{q-1}$$

has only constant solutions as long as  $q \in (2, 2^*)$  and  $\lambda \leq d$ . See [3] for a review and various related results. Assume that the measure on  $\mathbb{S}^d$  is the one induced by Lebesgue's measure on  $\mathbb{R}^{d+1}$ . This convention differs from the one of [4]. The inequality

$$\|\nabla u\|_{L^2(\mathbb{S}^d)}^2 + \alpha \|u\|_{L^2(\mathbb{S}^d)}^2 \geq \mu(\alpha) \|u\|_{L^q(\mathbb{S}^d)}^2 \quad \forall u \in H^1(\mathbb{S}^d),$$

---

*Email addresses:* [dolbeaul@ceremade.dauphine.fr](mailto:dolbeaul@ceremade.dauphine.fr) (Jean Dolbeault), [esteban@ceremade.dauphine.fr](mailto:esteban@ceremade.dauphine.fr) (Maria J. Esteban), [a.laptev@imperial.ac.uk](mailto:a.laptev@imperial.ac.uk) (Ari Laptev), [loss@math.gatech.edu](mailto:loss@math.gatech.edu) (Michael Loss).

for any  $q \in (2, 2^*)$  can be established by standard variational methods. According to [4], the optimal function  $\mu : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is concave, increasing, and such that  $\mu(\alpha) = \kappa \alpha$  for any  $\alpha \leq \frac{d}{q-2}$ ,  $\mu(\alpha) < \kappa \alpha$  for  $\alpha > \frac{d}{q-2}$  where  $\kappa := |\mathbb{S}^d|^{1-2/q}$  is a normalization factor and

$$\mu(\alpha) \sim \mathsf{K}_{q,d} \alpha^{1-\vartheta} \quad \text{as } \alpha \rightarrow +\infty, \quad \text{where } \vartheta := d \frac{q-2}{2q},$$

$$\mathsf{K}_{q,d} := \inf_{v \in \mathsf{H}^1(\mathbb{R}^d) \setminus \{0\}} \frac{\|\nabla v\|_{L^2(\mathbb{R}^d)}^2 + \|v\|_{L^2(\mathbb{R}^d)}^2}{\|v\|_{L^q(\mathbb{R}^d)}^2}. \quad (1)$$

Let us define  $p = \frac{q}{q-2}$  so that  $p \in (1, +\infty)$  if  $d = 1$  and  $p \in (\frac{d}{2}, +\infty)$  if  $d \geq 2$ . If we denote by  $\mu \mapsto \alpha(\mu)$  the inverse function of  $\alpha \mapsto \mu(\alpha)$  and by  $\lambda_1(-\Delta_g - V)$  the lowest (nonpositive) eigenvalue of  $-\Delta_g - V$ , then we have the estimate

$$|\lambda_1(-\Delta_g - V)| \leq \alpha(\|V\|_{L^p(\mathbb{S}^d)}) \quad \forall V \in L^p(\mathbb{S}^d).$$

for any nonnegative  $V \in L^p(\mathbb{S}^d)$ . Moreover we have  $\alpha(\mu)^{p-d/2} = L_{p-\frac{d}{2},d}^1 \mu^p (1 + o(1))$  as  $\mu \rightarrow +\infty$  where  $L_{\gamma,d}^1 := (\mathsf{K}_{q,d})^{-(\gamma+d/2)}$ . Equality is achieved for any  $\mu > 0$  by some nonnegative  $V$ , which is constant if and only if  $\mu \leq \frac{d}{2}(p-1)$ .

The case  $q \in (1, 2)$  can also be covered and we refer to [4] for further details. This case leads to estimates from below for the first eigenvalue of the operator  $-\Delta_g + W$ , where  $W$  is a positive potential.

## 2. A rigidity result on compact manifolds and a subcritical interpolation inequality

From here on we shall assume that  $(\mathfrak{M}, g)$  is a smooth compact connected Riemannian manifold of dimension  $d \geq 1$ , without boundary, and let  $\Delta_g$  be the Laplace-Beltrami operator on  $\mathfrak{M}$ . We shall denote by  $dv_g$  the volume element and by  $\mathfrak{R}$  the Ricci tensor. Let  $\lambda_1$  be the lowest positive eigenvalue of  $-\Delta_g$ .

For such manifolds a new rigidity result has been recently established in [5], thus extending a series of results of [6,2,8,1,9]. In order to state this result let us define the quantities:

$$\theta = \frac{(d-1)^2(p-1)}{d(d+2)+p-1} \quad \text{and} \quad \mathsf{Q}_g u := \mathsf{H}_g u - \frac{g}{d} \Delta_g u - \frac{(d-1)(p-1)}{\theta(d+3-p)} \left[ \frac{\nabla u \otimes \nabla u}{u} - \frac{g}{d} \frac{|\nabla u|^2}{u} \right]$$

where  $\mathsf{H}_g u$  denotes Hessian of  $u$ , and

$$\Lambda_\star := \inf_{u \in \mathsf{H}^2(\mathfrak{M}) \setminus \{0\}} \frac{(1-\theta) \int_{\mathfrak{M}} (\Delta_g u)^2 dv_g + \frac{\theta d}{d-1} \int_{\mathfrak{M}} [\|\mathsf{Q}_g u\|^2 + \mathfrak{R}(\nabla u, \nabla u)] dv_g}{\int_{\mathfrak{M}} |\nabla u|^2 dv_g}. \quad (2)$$

It is not difficult to see that  $\Lambda_\star \leq \lambda_1$ .

**Theorem 1** [5, cf. Theorem 3] *Assume that  $\Lambda_\star$  is strictly positive. Then for any  $q \in (1, 2) \cup (2, 2^*)$  and any  $\lambda \in (0, \Lambda_\star)$ , the equation*

$$-\Delta_g v + \frac{\lambda}{q-2} (v - v^{q-1}) = 0$$

*has 1 as its unique positive solution in  $C^2(\mathfrak{M})$ .*

Note that in the particular case  $\mathfrak{M} = \mathbb{S}^d$ ,  $\Lambda_\star = \lambda_1(-\Delta_g) = d$ . The proof relies on the nonlinear flow

$$u_t = u^{2-2\beta} \left( \Delta_g u + (1 + \beta(q-2)) \frac{|\nabla u|^2}{u} \right), \quad \beta = \frac{(d+2)(d+3-p)\theta}{(d-1)^2(p-1)^2 - (d+2)^2(p-2)\theta}, \quad (3)$$

that can also be used to prove the following  $A$ - $B$  type interpolation inequality (see [2,7]). Let us define

$$\kappa := \text{vol}_g(\mathfrak{M})^{1-2/q}.$$

**Theorem 2** [5, cf. Theorem 4] For any  $q \in (1, 2) \cup (2, 2^*)$  if  $d \geq 3$ ,  $q \in (1, 2) \cup (2, \infty)$  if  $d = 1$  or  $2$ , the inequality

$$\|\nabla v\|_{L^2(\mathfrak{M})}^2 \geq \frac{\Lambda}{q-2} \left[ \kappa \|v\|_{L^q(\mathfrak{M})}^2 - \|v\|_{L^2(\mathfrak{M})}^2 \right] \quad \forall v \in H^1(\mathfrak{M}).$$

holds for some optimal  $\Lambda \in [\Lambda_*, \lambda_1]$  if  $\Lambda_* > 0$ . Moreover, if  $\Lambda_* < \lambda_1$ , then we have  $\Lambda_* < \Lambda \leq \lambda_1$ .

The above results hold true because the flow (3) contracts

$$\mathcal{F}[u] := \int_{\mathfrak{M}} |\nabla(u^\beta)|^2 dv_g + \frac{\Lambda_*}{q-2} \left[ \int_{\mathfrak{M}} u^{2\beta} dv_g - \kappa \left( \int_{\mathfrak{M}} u^{\beta q} dv_g \right)^{2/q} \right].$$

The above choices for  $\theta$  and  $\beta$  are optimal for this contraction property: see [5].

As a consequence and exactly as in the case of the sphere, we get the first result of this note.

**Proposition 3** Assume that  $q \in (2, 2^*)$  if  $d \geq 3$ , or  $q \in (2, \infty)$  if  $d = 1$  or  $2$ . There exists a concave increasing function  $\mu : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that  $\mu(\alpha) = \kappa \alpha$  for any  $\alpha \leq \frac{\Lambda}{q-2}$ ,  $\mu(\alpha) < \kappa \alpha$  for  $\alpha > \frac{\Lambda}{q-2}$  and

$$\|\nabla u\|_{L^2(\mathfrak{M})}^2 + \alpha \|u\|_{L^2(\mathfrak{M})}^2 \geq \mu(\alpha) \|u\|_{L^q(\mathfrak{M})}^2 \quad \forall u \in H^1(\mathfrak{M}).$$

The asymptotic behaviour of  $\mu$  is given by  $\mu(\alpha) \sim K_{q,d} \alpha^{1-\vartheta}$  as  $\alpha \rightarrow +\infty$ , with  $\vartheta = d \frac{q-2}{2q}$  and  $K_{q,d}$  defined by (1).

*Proof.* There is an optimal function for the interpolation inequality, as can be shown by standard variational techniques. Applying Theorem 1 to the solutions of the Euler-Lagrange equations completes the proof for fixed values of  $\alpha$ . As an infimum on  $u$  of affine functions with respect to  $\alpha$ , the function  $\alpha \rightarrow \mu(\alpha)$  is concave. It remains to establish the properties of  $\alpha$  for large values of  $\mu$ .

Using a well chosen test function obtained by scaling an optimal function for (1) on the tangent plane to an arbitrary point of  $\mathfrak{M}$ , one can prove that  $\limsup_{\alpha \rightarrow +\infty} \alpha^{\vartheta-1} \mu(\alpha) \leq K_{q,d}$ . Arguing by contradiction as in [4, Proposition 10], we can find a sequence  $(\alpha_n)_{n \in \mathbb{N}}$  such that  $\lim_{n \rightarrow +\infty} \alpha_n = +\infty$  and  $\lim_{n \rightarrow +\infty} \alpha_n^{\vartheta-1} \mu(\alpha_n) < K_{q,d}$ , and a sequence of optimal functions  $(u_n)_{n \in \mathbb{N}}$  such that  $\|u_n\|_{L^q(\mathbb{S}^d)} = 1$ , which concentrates because  $\limsup_{n \rightarrow +\infty} \alpha_n^{\vartheta} \|u_n\|_{L^2(\mathbb{S}^d)}^2 < K_{q,d}$ . Some classical surgery and a convexity inequality provide a contradiction by constructing a minimizing sequence for (1).  $\square$

### 3. Ground state estimates for Schrödinger operators on Riemannian manifolds

In this section, we keep using the notations of Section 2 and generalize to  $(\mathfrak{M}, g)$  the spectral results established for the sphere in [4]. By inverting the function  $\alpha \mapsto \mu(\alpha)$ , we see that  $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is increasing, convex and satisfies:  $\alpha(\mu) = \frac{\mu}{\kappa}$  for any  $\mu \in (0, \frac{\kappa \Lambda}{q-2})$ ,  $\alpha(\mu) > \frac{\mu}{\kappa}$  for  $\mu \in (\frac{\kappa \Lambda}{q-2}, +\infty)$ . With  $L_{\gamma,d}^1 := (K_{q,d})^{-p}$ ,  $\gamma = p - \frac{d}{2}$ , we obtain for a general manifold  $\mathfrak{M}$  the same behavior of  $\mu \mapsto \alpha(\mu)$  when  $\mu \rightarrow +\infty$  as in the case of a sphere.

Let  $\mu := \|V\|_{L^p(\mathbb{S}^d)}$ . Since  $p$  and  $\frac{q}{2}$  are Hölder conjugate exponents, it follows from Hölder's inequality that

$$\int_{\mathfrak{M}} |\nabla u|^2 dv_g - \int_{\mathfrak{M}} V |u|^2 dv_g + \alpha(\mu) \int_{\mathfrak{M}} |u|^2 dv_g \geq \|\nabla u\|_{L^2(\mathbb{S}^d)}^2 - \mu \|u\|_{L^q(\mathbb{S}^d)}^2 + \alpha(\mu) \|u\|_{L^2(\mathbb{S}^d)}^2$$

with equality if  $V^{p-1}$  and  $|u|^2$  are proportional. The right-hand side is nonnegative according to Proposition 3. By taking the infimum of the left-hand side, we can deduce an estimate of the lowest, nonpositive eigenvalue  $\lambda_1(-\Delta_g - V)$  of  $-\Delta_g - V$ , which provides us with our first main result.

**Theorem 4** Let  $d \geq 1$ ,  $p \in (1, +\infty)$  if  $d = 1$  and  $p \in (\frac{d}{2}, +\infty)$  if  $d \geq 2$  and assume that  $\Lambda_* > 0$ . With the above notations and definitions, for any nonnegative  $V \in L^p(\mathfrak{M})$ , we have

$$|\lambda_1(-\Delta_g - V)| \leq \alpha(\|V\|_{L^p(\mathfrak{M})}). \quad (4)$$

Moreover, we have  $\alpha(\mu)^{p-\frac{d}{2}} = L_{p-\frac{d}{2},d}^1 \mu^p (1 + o(1))$  as  $\mu \rightarrow +\infty$ .

The estimate (4) is optimal in the sense that for any  $\mu \in (0, +\infty)$ , there exists a nonnegative function  $V$  such that  $\mu = \|V\|_{L^p(\mathfrak{M})}$  and  $|\lambda_1(-\Delta_g - V)| = \alpha(\mu)$ . Moreover, if  $\mu < \frac{\kappa \Delta_*}{q-2}$ ,  $\alpha(\mu) = \frac{\mu}{\kappa}$  and equality in (4) is achieved by constant potentials.

In the case of operators  $-\Delta_g + W$  on  $\mathfrak{M}$ , where  $W$  is a nonnegative potential, following again the same arguments as in [4] in the case of the sphere, and the rigidity result of Theorem 1, we obtain our second main result.

**Theorem 5** *Let  $d \geq 1$ ,  $p \in (0, +\infty)$ . There exists an increasing concave function  $\nu : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ , satisfying  $\nu(\beta) = \beta/\kappa$ , for any  $\beta \in (0, \frac{p+1}{2} \kappa \Lambda)$  if  $p > 1$ , such that for any positive potential  $W$  we have*

$$\lambda_1(-\Delta + W) \geq \nu(\beta) \quad \text{with} \quad \beta = \left(\int_{\mathfrak{M}} W^{-p} dv_g\right)^{1/p}.$$

Moreover, for large values of  $\beta$ , we have  $\nu(\beta)^{-(p+\frac{d}{2})} = L_{-(p+\frac{d}{2}),d}^1 \beta^{-p} (1 + o(1))$  as  $\beta \rightarrow +\infty$ .

With  $p = \frac{q}{2-q}$ , the spectral estimate of Theorem 5 is derived from the interpolation inequality

$$\|\nabla u\|_{L^2(\mathfrak{M})}^2 + \beta \left(\int_{\mathfrak{M}} |u|^q dv_g\right)^{2/q} \geq \nu(\beta) \|u\|_{L^2(\mathfrak{M})}^2 \quad \forall u \in H^1(\mathfrak{M}).$$

The concentration phenomena leading to the asymptotics for large norms of  $W$  can be studied as in Proposition 3: see [4] for the proof in the case of a sphere. We omit the details of the proof of Theorem 5.

## References

- [1] D. BAKRY AND M. LEDOUX, *Sobolev inequalities and Myers's diameter theorem for an abstract Markov generator*, Duke Math. J., 85 (1996), pp. 253–270.
- [2] M.-F. BIDAUT-VÉRON AND L. VÉRON, *Nonlinear elliptic equations on compact Riemannian manifolds and asymptotics of Emden equations*, Invent. Math., 106 (1991), pp. 489–539.
- [3] J. DOLBEAULT, M. J. ESTEBAN, M. KOWALCZYK, AND M. LOSS, *Sharp Interpolation Inequalities on the Sphere: New Methods and Consequences*, Chin. Ann. Math. Ser. B, 34 (2013), pp. 99–112.
- [4] J. DOLBEAULT, M. J. ESTEBAN, AND A. LAPTEV, *Spectral estimates on the sphere*. Preprint hal-00770755, to appear in Analysis and PDE.
- [5] J. DOLBEAULT, M. J. ESTEBAN, AND M. LOSS, *Nonlinear flows and rigidity results on compact manifolds*. Preprint hal-00784887.
- [6] B. GIDAS AND J. SPRUCK, *Global and local behavior of positive solutions of nonlinear elliptic equations*, Comm. Pure Appl. Math., 34 (1981), pp. 525–598.
- [7] E. HEBEY, *Nonlinear analysis on manifolds: Sobolev spaces and inequalities*, vol. 5 of Courant Lecture Notes in Mathematics, New York University Courant Institute of Mathematical Sciences, New York, 1999.
- [8] J. R. LICOIS AND L. VÉRON, *Un théorème d'annulation pour des équations elliptiques non linéaires sur des variétés riemanniennes compactes*, C. R. Acad. Sci. Paris Sér. I Math., 320 (1995), pp. 1337–1342.
- [9] J. R. LICOIS AND L. VÉRON, *A class of nonlinear conservative elliptic equations in cylinders*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 26 (1998), pp. 249–283.

**Acknowledgments.** J.D. and M.J.E. have been partially supported by ANR grants *CBDif* and *NoNAP*. They thank the Mittag-Leffler Institute, where part of this research was carried out, for hospitality. M.L. was supported in part by NSF grant DMS-0901304.

© 2013 by the authors. This paper may be reproduced, in its entirety, for non-commercial purposes.

July 23, 2013