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# Uniform propagation of chaos for a class of nonlinear diffusions

Pierre Del Moral\* & Julian Tugaut<sup>†‡</sup>

## Abstract

We are interested in nonlinear diffusions in which the own law intervenes in the drift. This kind of diffusions corresponds to the hydrodynamical limit of some particle system. One also talks about propagation of chaos. It is well-known, for McKean-Vlasov diffusions, that such a propagation of chaos holds on finite-time interval. However, it has been proven that the lack of convexity of the external force implies that there is no uniform propagation of chaos if the diffusion coefficient is small enough. We here aim to establish a uniform propagation of chaos even if the external force is not convex, with a diffusion coefficient sufficiently large. The idea consists in combining the propagation of chaos on a finite-time interval with a functional inequality, already used by Bolley, Gentil and Guillin, see [BGG12a, BGG12b].

**Key words and phrases:** Nonlinear diffusions ; Propagation of chaos ; Feynman-Kac ; McKean-Vlasov models ; Functional inequality

**2000 AMS subject classifications:** Primary 60K35, 60E15 ; Secondary 82C22, 35K55, 60J60

## Introduction

We are interested in some nonlinear processes in  $\mathbb{R}^d$  defined by an equation in which the own law of the process intervenes in the drift. In this work, we consider a diffusion of the form

$$\begin{cases} X_t = X_0 + \sigma B_t - \int_0^t b(\mu_s, X_s) ds, \\ \mu_s = \mathcal{L}(X_s), \end{cases} \quad (\text{I})$$

where  $b$  is a function from  $\mathcal{P}(\mathbb{R}^d) \times \mathbb{R}^d$  to  $\mathbb{R}^d$  and  $\{B_t; t \geq 0\}$  is a  $d$ -dimensional Wiener process. The assumptions are detailed subsequently. Let us just say that

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$b(\mu, x)$  does only depend on  $x$  and on some finite number of moments of  $\mu$ . The infinitesimal generator of Diffusion (I) therefore is

$$A\mu := \frac{\sigma^2}{2} \Delta\mu - \nabla \cdot \{b(\mu, \cdot) \mu\}.$$

The associated semi-group is denoted by  $(P_t)_{t \geq 0}$ . In other words, one has  $\mu_t = \mu_0 P_t$ . We notice that  $X_t, \mu_t, P_t$  and  $A$  depend on  $\sigma$ . We do not write it for simplifying the reading.

An example of such diffusion is the McKean-Vlasov one:

$$X_t = X_0 + \sigma B_t - \int_0^t \nabla V(X_s) ds - \int_0^t \left( \nabla F * \mathcal{L}(X_s) \right) (X_s) ds, \quad (\text{II})$$

where  $V$  and  $F$  respectively are called the confinement and the interaction potentials. The notation  $*$  is used for denoting the convolution.

This equation is nonlinear in the sense of McKean, see [McK67, McK66].

It is well-known, see [McK67], that the law  $\mathcal{L}(X_t)$  is absolutely continuous with respect to the Lebesgue measure for all  $t > 0$ , provided some regularity hypotheses on  $V$  and  $F$ . Moreover, its density, which is denoted by  $u_t$ , satisfies the so-called granular media equation,

$$\frac{\partial}{\partial t} u_t = \nabla \cdot \left\{ \frac{\sigma^2}{2} \nabla u_t + (\nabla V + \nabla F * u_t) u_t \right\}.$$

The setting of this work is restricted to the McKean-Vlasov case. However, we could apply to more general hypotheses. Let us notice that we do not assume any convex properties on the confinement potential nor on the interaction one. Under easily checked assumptions, Diffusion (II) corresponds to the hydrodynamical limit of the following particle system

$$\begin{cases} X_t^1 = X_0^1 + \sigma B_t^1 - \int_0^t \left[ \nabla V(X_s^1) + \sum_{j=1}^N \frac{1}{N} \nabla F(X_s^1 - X_s^j) \right] ds, \\ \vdots \\ X_t^i = X_0^i + \sigma B_t^i - \int_0^t \left[ \nabla V(X_s^i) + \sum_{j=1}^N \frac{1}{N} \nabla F(X_s^i - X_s^j) \right] ds, \\ \vdots \\ X_t^N = X_0^N + \sigma B_t^N - \int_0^t \left[ \nabla V(X_s^N) + \sum_{j=1}^N \frac{1}{N} \nabla F(X_s^N - X_s^j) \right] ds, \end{cases} \quad (\text{III})$$

$\{B_t^i; t \geq 0\}$  being  $N$  independent  $d$ -dimensional Wiener processes. We also assume that  $\{X_0^i; i \in \mathbb{N}^*\}$  is a family of independent random variables, identically distributed with common law  $\mathcal{L}(X_0)$  (and independent from the Brownian motions). The particles therefore are exchangeable. We notice that  $X_t^1, \dots, X_t^N$  depend on  $N$  and on  $\sigma$ . We do not write it for simplifying the reading. We here focus on the first diffusion. By  $\mu_t^{1,N}$ , we denote the law at time  $t$  of the diffusion  $X^1$ .

One says, in this work, that simple propagation of chaos holds on interval  $[0; T]$  with  $T > 0$  if we have the limit

$$\lim_{N \rightarrow +\infty} \sup_{0 \leq t \leq T} \mathbb{W}_2 \left( \mu_t^{1,N}; \mu_0 P_t \right) = \lim_{N \rightarrow +\infty} \sup_{0 \leq t \leq T} \mathbb{W}_2 \left( \mu_t^{1,N}; \mu_t \right) = 0,$$

$\mathbb{W}_2$  standing for the Wasserstein distance (see Definition 1.1). This means that  $X^1$  is a good approximation of Diffusion (I) as  $N$  goes to infinity.

Such limit (with another distance) has been investigated for the special case of the McKean-Vlasov diffusion, see [BRTV98, BAZ99, Mal01, Mal03, Szn91]. Indeed, it is a consequence of the classical coupling result,

$$\sup_{t \leq T} \mathbb{E} \left\{ \|X_t - X_t^1\|^2 \right\} \leq \lambda \frac{e^{KT}}{N},$$

with  $\lambda, K > 0$ . Uniform propagation of chaos, that is

$$\lim_{N \rightarrow +\infty} \sup_{t \geq 0} \mathbb{W}_2 \left( \mu_t^{1,N}; \mu_0 P_t \right) = \lim_{N \rightarrow +\infty} \sup_{t \geq 0} \mathbb{W}_2 \left( \mu_t^{1,N}; \mu_t \right) = 0,$$

has been proven if both confinement potential  $V$  and interaction potential  $F$  are convex, see [CGM08]. The particularity in our work is that the coupling between the two diffusions is not necessarily made with the same Brownian motions.

A consequence of the uniform propagation of chaos for the nonlinear diffusion is the uniqueness of the invariant probability  $\mu^\sigma$  and the weak convergence toward this measure. However, without convex properties, it is proven in [HT10, Tug13b, Tug12] that there is non-uniqueness of the invariant probabilities under simple assumptions, provided that the diffusion coefficient  $\sigma$  is sufficiently small.

But, as pointed out in [Tug13b], if  $\sigma$  is large enough, we have a unique invariant probability. The question thus is: does uniform propagation of chaos holds if  $\sigma$  is sufficiently large? Also, can we reciprocally use the convergence toward the unique invariant probability to obtain this uniform propagation of chaos?

We positively answer to the two questions by using the simple propagation of chaos and a so-called  $WJ$ -inequality already used in [AGS08, BGG12b].

To the best of our knowledge, the first uniform propagations of chaos estimates for mean field particle models have been developed in [DMM00] and in [DMG01] in the context of Feynman-Kac interaction jump models. Further results in this direction can be found in [DMR11] as well as in the research monographs [DM13, DM04], including exponential concentration inequalities w.r.t. the time horizon, and contraction inequalities w.r.t. several classes of relative entropy criteria.

The analysis of interaction jump particle models clearly differs from the more traditional coupling analysis of the McKean-Vlasov diffusion models developed in the present article. The common feature is to enter the stability properties of

the limiting nonlinear semigroup into the estimation of the propagation of chaos properties of the finite particle systems, to deduce  $L_p$ -mean error estimates of order  $1/\sqrt{N}^\delta$ , for any  $0 < \delta < 1$  (cf. for instance theorem 2.11 in [DMM00], at the level of the empirical processes). In our context using these techniques, we obtain a variance and a  $W_2^2$ -estimate of order  $1/N^\delta$ , for some  $0 \leq \delta < 1$ . We underline that in the context of Feynman-Kac particle models, the order  $1/N$  can be obtained under stronger mixing conditions, using backward semigroup techniques. Thus, we conjecture that this decay rate is also met in our context.

The existence problem of a solution to (II) is not investigated here. We thus assume that there exists a unique strong solution  $(X_t)_{t \geq 0}$ . Let us just mention that it has been solved under simple assumptions. The method consists in applying a fixed-point theorem, see [BRTV98, HIP08].

In a first section, we introduce the framework of the  $WJ$ -inequality and we establish some functional inequalities based on the work in [BGG12a, BGG12b]. In Section 2, we provide some results on the simple propagation of chaos. Finally, in last section, we prove the main result that is the uniform propagation of chaos when the coefficient diffusion is sufficiently large. Before finishing the introduction, we give the hypotheses of the paper and the main results.

**Assumption (A):** We say that the confinement potential  $V$ , the interaction potential  $F$  and the initial law  $\mu_0$  satisfy the set of assumptions (A) if

(A-1)  $V$  is a smooth function on  $\mathbb{R}^d$ .

(A-2) for all  $\lambda > 0$ , there exists  $R_\lambda > 0$  such that  $\nabla^2 V(x) > \lambda I_d$ , for any  $\|x\| \geq R_\lambda$ .

(A-3) there exists a convex function  $V_0$  and  $\vartheta \in \mathbb{R}$  such that  $V(x) = V_0(x) - \frac{\vartheta}{2} \|x\|^2$ .

(A-4) the gradient  $\nabla V$  is slowly increasing: there exist  $m \in \mathbb{N}^*$  and  $C > 0$  such that  $\|\nabla V(x)\| \leq C \left(1 + \|x\|^{2m-1}\right)$ , for all  $x \in \mathbb{R}^d$ .

(A-5) there exist a strictly convex function  $\Theta$  such that  $\Theta(y) > \Theta(0) = 0$  for all  $y \in \mathbb{R}^d$  and  $p \in \mathbb{N}$  such that the following limit holds for any  $y \in \mathbb{R}^d$ :

$$\lim_{r \rightarrow +\infty} \frac{V(ry)}{r^{2p}} = \Theta(y).$$

(A-6) there exist an even polynomial and strictly convex function  $G$  on  $\mathbb{R}$  and  $\alpha \in \mathbb{R}$  such that  $F(x) = F_0(x) - \frac{\alpha}{2} \|x\|^2$  with  $F_0(x) := G(\|x\|)$ . And,  $\deg(G) =: 2n \geq 2$ .

(A-7) the following inequality holds:  $p > n$ .

(A-8) the  $8q^2$ th moment of the measure  $\mu_0$  is finite with  $q := \min\{m, n\}$ .

(A-9) the measure  $\mu_0$  admits a  $C^\infty$ -continuous density  $u_0$  with respect to the Lebesgue measure. And, the entropy  $-\int_{\mathbb{R}^d} u_0(x) \log(u_0(x)) dx$  is finite.

Hypotheses (A-1)–(A-5) concern the confinement potential  $V$  and mean that it behaves like a polynomial function with degree at least four. Hypothesis (A-6) states that the interaction between the particles is a polynomial function of the distance.

The inequality  $p > n$  means that the potential  $F$  is negligible with respect to the potential  $V$  at infinity. This is a technical assumption which will be used

in order to obtain the functional inequality. This hypothesis has already been assumed in [Tug13b, Theorem 2.6] in order to characterize the exact number of invariant probabilities in the small-noise. An example of couple of potentials satisfying Hypotheses (A-1)–(A-7) is  $V(x) := \frac{x^6}{6} + \frac{x^4}{4} - \frac{x^2}{2}$  and  $F(x) = \frac{x^4}{4} - \frac{x^2}{2}$ , in the one-dimensional case.

The last two hypotheses concern the law  $\mu_0$ . Hypothesis (A-8) is required to prove the existence of a solution to the nonlinear stochastic differential equation (II), see [HIP08, BRTV98, CGM08]. And, Hypothesis (A-9) is necessary to apply the result in [AGS08] which characterizes the dissipation of the Wasserstein distance. This hypothesis was also assumed in order to get the weak convergence, see [Tug13a]. We now give the main results of the paper. We remind the reader that the Wasserstein distance is defined in Definition 1.1.

**Theorem A:** *We assume that  $V$ ,  $F$  and  $\mu_0$  satisfy the set of Hypotheses (A). Thus, there exists  $\sigma^c > 0$  such that  $\sigma > \sigma^c$  implies Diffusion (II) admits a unique invariant probability  $\mu^\sigma$ . Moreover, we have the following convergence with exponential decay if  $\sigma > \sigma^c$ :*

$$\mathbb{W}_2(\mu_t; \mu^\sigma) \leq \exp[-C(\sigma)t] \mathbb{W}_2(\mu_0; \mu^\sigma),$$

$C(\sigma)$  being a positive constant.

**Proposition B:** *We assume that  $V$ ,  $F$  and  $\mu_0$  satisfy the set of Hypotheses (A). Let  $X_0^1, \dots, X_0^N$  be  $N$  random variables with common law  $\mu_0$ . We do not assume these variables to be independent. We consider the two following particle systems:*

$$X_t^i = X_0^i + \sigma B_t^i - \int_0^t \nabla V(X_s^i) ds - \int_0^t \nabla F * \eta_s^N(X_s^i) ds,$$

where  $\eta_s^N := \left(\frac{1}{N} \sum_{j=1}^N \delta_{X_0^j}\right) P_s$  and

$$Z_t^i = X_0^i + \sigma B_t^i - \int_0^t \nabla V(Z_s^i) ds - \frac{1}{N} \sum_{j=1}^N \int_0^t \nabla F(Z_s^i - Z_s^j) ds,$$

$B^1, \dots, B^N$  being  $N$  independent Brownian motions (and independent from the initial random variables). Then, for any  $T > 0$ , we have the following inequality:

$$\sup_{t \in [0; T]} \mathbb{E} \left\{ \left\| X_t^i - Z_t^i \right\|^2 \right\} \leq \frac{C(\mu_0)}{(\vartheta + 2\alpha)^2 N} \exp[2(\vartheta + 2\alpha)T],$$

where  $C(\mu_0)$  is a positive function of  $\int_{\mathbb{R}^d} \|x\|^{8q^2} \mu_0(dx)$ .

**Theorem C:** *We assume that  $V$ ,  $F$  and  $\mu_0$  satisfy the set of Hypotheses (A). If  $\sigma \geq \sigma_c$  (where  $\sigma_c$  is defined in Theorem A), we have the uniform propagation of chaos. In other words, we have the limit*

$$\lim_{N \rightarrow +\infty} \sup_{t \geq 0} \mathbb{W}_2(\mu_t; \mu_t^{1, N}) = 0.$$

Moreover, we can compute the rate of convergence.

**First case:** The quantity  $\frac{C(\sigma)t}{\psi(t)}$  goes to  $\lambda \in \mathbb{R}_+^* \cup \{+\infty\}$  as  $t$  goes to infinity, where  $\psi$  is defined by

$$\sup_{t \in [0; T]} \mathbb{E} \left\{ \|X_t^i - Z_t^i\|^2 \right\} \leq \left( \frac{\exp[\psi(t)]}{\sqrt{N}} \right).$$

Thus, for all  $\delta > 0$ , we have:

$$\lim_{N \rightarrow +\infty} N^{\frac{1}{2(1+\lambda)} - \delta} \sup_{t \geq 0} \mathbb{W}_2 \left( \mu_t; \mu_t^{1, N} \right) = 0.$$

**Second case:** The quantity  $\frac{C(\sigma)t}{\psi(t)}$  goes to 0 as  $t$  goes to infinity. Thus, for all  $\delta > 0$ , we have:

$$\lim_{N \rightarrow +\infty} \exp \left\{ C(\sigma)\psi^{-1} \left[ \frac{1}{2}(1 - \delta) \log(N) \right] \right\} \sup_{t \geq 0} \mathbb{W}_2 \left( \mu_t; \mu_t^{1, N} \right) = 0.$$

## 1 Functional inequality

Let us give the framework (definitions and basic propositions) of the current work. We begin by introducing the Wasserstein distance.

**Definition 1.1.** For any probability measures on  $\mathbb{R}^d$ ,  $\mu$  and  $\nu$ , the Wasserstein distance between  $\mu$  and  $\nu$  is

$$\mathbb{W}_2(\mu; \nu) := \sqrt{\inf \mathbb{E} \left\{ \|X - Y\|^2 \right\}},$$

where the infimum is taken over the random variables  $X$  and  $Y$  with law  $\mu$  and  $\nu$  respectively. One can also write

$$\mathbb{W}_2(\mu; \nu) = \sqrt{\inf \iint_{\mathbb{R}^d \times \mathbb{R}^d} \|x - y\|^2 \pi(dx, dy)},$$

where the infimum runs over the probability measures on  $\mathbb{R}^d \times \mathbb{R}^d$  with marginals equal to  $\mu$  and  $\nu$ .

The Wasserstein distance can be characterized in the following way, thanks to Brenier's theorem, see [Bre91].

**Proposition 1.2.** Let  $\mu$  and  $\nu$  be two probability measures on  $\mathbb{R}^d$ . If  $\mu$  is absolutely continuous with respect to the Lebesgue measure, there exists a convex function  $\tau$  from  $\mathbb{R}^d$  to  $\mathbb{R}$  such that the following equality occurs for every bounded test function  $g$ :

$$\int_{\mathbb{R}^d} g(x) \nu(dx) = \int_{\mathbb{R}^d} g(\nabla \tau(x)) \mu(dx).$$

Then, we write

$$\nu = \nabla\tau\#\mu,$$

and we have the following equality

$$\mathbb{W}_2(\mu; \nu) = \sqrt{\int_{\mathbb{R}^d} \|x - \nabla\tau(x)\|^2 \mu(dx)}.$$

The key-idea of the paper is a so-called  $WJ_{V,F}$ -inequality. Let us present the expression that we denote by  $J_{V,F}(\nu | \mu)$  if  $\mu$  is absolutely continuous with respect to the Lebesgue measure:

$$\begin{aligned} J_{V,F}(\nu | \mu) &:= J_{V,0}(\nu | \mu) \\ &+ \frac{1}{2} \iint_{\mathbb{R}^{2d}} \langle \nabla F(Z(x, y)) - \nabla F(x - y); Z(x, y) - (x - y) \rangle \mu(dx)\mu(dy), \end{aligned} \quad (1.1)$$

with  $Z(x, y) := \nabla\tau(x) - \nabla\tau(y)$  and

$$\begin{aligned} J_{V,0}(\nu | \mu) &:= \frac{\sigma^2}{2} \int_{\mathbb{R}^d} \left( \Delta\tau(x) + \Delta\tau^*(\nabla\tau(x)) - 2d \right) \mu(dx) \\ &+ \int_{\mathbb{R}^d} \langle \nabla V(\nabla\tau(x)) - \nabla V(x); \nabla\tau(x) - x \rangle \mu(dx), \end{aligned} \quad (1.2)$$

where  $\tau^*$  denotes the Legendre transform of  $\tau$ . Here, we have  $\nu = \nabla\tau\#\mu$ . We now present the transportation inequality, already used in [AGS08, BGG12a, BGG12b], on which the article is based.

**Definition 1.3.** *Let  $\mu$  be a probability measure on  $\mathbb{R}^d$  absolutely continuous with respect to the Lebesgue measure and  $C > 0$ . We say that  $\mu$  satisfies a  $WJ_{V,F}(C)$ -inequality if the inequality*

$$C\mathbb{W}_2^2(\nu; \mu) \leq J_{V,F}(\nu | \mu) \quad (1.3)$$

holds for any probability measure  $\nu$  on  $\mathbb{R}^d$ .

In the same way, we say that  $\mu$  satisfies a  $WJ_{V,0}(C)$ -inequality if we have

$$C\mathbb{W}_2^2(\nu; \mu) \leq J_{V,0}(\nu | \mu),$$

for any probability measure  $\nu$  on  $\mathbb{R}^d$ . In the following, we aim to establish a  $WJ_{V,F}$ -inequality for an invariant probability  $\mu^\sigma$  of Diffusion (II). However, it is well known that  $\mu^\sigma$  is absolutely continuous with respect to the Lebesgue measure. Consequently, we can apply Brenier's theorem. So, the  $WJ_{V,F}$ -inequality consists in obtaining an inequality on the convex function  $\tau$  from  $\mathbb{R}^d$  to  $\mathbb{R}$ .

We now give a classical result which explains why a  $WJ_{V,F}$ -inequality has consequences on the long-time behavior of McKean-Vlasov diffusions (II). It is similar to [BGG12b, Proposition 1.1]. See also [AGS08].



**Proposition 1.4.** *Let  $V$  and  $F$  be two functions satisfying Hypotheses (A-1)–(A-7). Let  $\mu_0$  and  $\nu_0$  be two probability measures on  $\mathbb{R}^d$  absolutely continuous with respect to the Lebesgue measure. Set  $(X_t)_{t \in \mathbb{R}_+}$  and  $(Y_t)_{t \in \mathbb{R}_+}$  two McKean-Vlasov diffusions (II) starting with law  $\mu_0$  and  $\nu_0$ . By  $\mu_t$  (respectively  $\nu_t$ ), we denote the law of  $X_t$  (respectively  $Y_t$ ). Therefore, we have the inequality*

$$\frac{1}{2} \frac{d}{dt} \mathbb{W}_2^2(\mu_t; \nu_t) \leq -J_{V,F}(\nu_t | \mu_t). \quad (1.4)$$

Consequently, if  $\mu^\sigma$  is an invariant probability of Diffusion (II) and if  $\mu^\sigma$  satisfies a  $WJ_{V,F}(C)$ -inequality, by combining Ineq. (1.3) and Ineq. (1.4), we obtain

$$\frac{1}{2} \frac{d}{dt} \mathbb{W}_2^2(\mu_t; \mu^\sigma) \leq -J_{V,F}(\mu_t | \mu^\sigma) \leq -C \mathbb{W}_2^2(\mu_t; \mu^\sigma),$$

for any  $\mu_0$  absolutely continuous with respect to the Lebesgue measure. This leads to

$$\mathbb{W}_2(\mu_t; \mu^\sigma) \leq e^{-Ct} \mathbb{W}_2(\mu_0; \mu^\sigma).$$

This also establishes the uniqueness of the invariant probability.

Bolley, Gentil and Guillin suggested a method to obtain a  $WJ_{V,F}$ -inequality in the non-convex case. But, we proceed in a slightly different way. We first use their result which provides a  $WJ_{V_0,0}(C^\sigma)$ -inequality. Then, we prove that  $C^\sigma$  goes to infinity as  $\sigma$  goes to infinity. Finally, we remark that  $J_{V,F}(\mu | \mu^\sigma) \geq J_{V_0,0}(\mu | \mu^\sigma) - (\max\{\alpha; 0\} + \vartheta) \mathbb{W}_2^2(\mu; \mu^\sigma)$  for any measure  $\mu$ .

In the following,  $\mu^\sigma$  denotes an invariant probability of Diffusion (II). We know that such a measure exists, see [Tug12, Proposition 2.1]. Moreover, the measure satisfies the following implicit equation

$$\mu^\sigma(dx) := \frac{\exp\left\{-\frac{2}{\sigma^2} W^\sigma(x)\right\}}{\int_{\mathbb{R}^d} \exp\left\{-\frac{2}{\sigma^2} W^\sigma(y)\right\} dy} dx$$

with  $W^\sigma(x) := V(x) + F * \mu^\sigma(x)$ . Let us now give a  $WJ_{V_0,0}$ -inequality on the measure  $\mu^\sigma$ .

**Proposition 1.5.** *We assume that  $V$ ,  $F$  and  $\mu_0$  satisfy the set of Hypotheses (A). Thus, the measure  $\mu^\sigma$  satisfies a  $WJ_{V_0,0}(C^\sigma)$ -inequality where the constant  $C^\sigma$  is defined by*

$$C^\sigma := \max_{R>0} C^\sigma(R) > 0$$

$$\text{with } C^\sigma(R) := \min \left\{ \frac{K(R)}{3}; \frac{\sigma^2}{72R^2} e^{-\frac{2}{\sigma^2} S(R)}; \frac{K(R)}{3} \frac{3^d - 2^d}{2^d} e^{\frac{2}{\sigma^2} (I(R) - S(R))} \right\},$$

$$K(R) := \inf_{\|x\| \geq R} \nabla^2 V_0(x), \quad I(R) := \inf_{\|x\| \leq 2R} W^\sigma(x)$$

$$\text{and } S(R) := \sup_{\|x\| \leq 3R} W^\sigma(x).$$

The proof is left to the reader and consists in a simple adaptation of the proof of [BGG12a, Proposition 3.4] that is to say [BGG12a, Section 5]. Let us mention that we do not need to apply the whole set of assumptions. Indeed, to prove this result, we simply use Hypotheses (A-1)-(A-2)-(A-6). More precisely, we need the potential  $W^\sigma$  to be  $\mathcal{C}^1$ -continuous (which is an immediate consequence of (A-1) and (A-6)). We also need the function  $V_0$  to be convex at infinity, which is proven by (A-2).

From Proposition 1.5, we also deduce the following corollary which is central in the section.

**Corollary 1.6.** *We assume that  $V$ ,  $F$  and  $\mu_0$  satisfy the set of Hypotheses (A). Therefore, we have the following inequality:*

$$(\mathcal{C}^\sigma - \max\{\alpha; 0\} - \vartheta) \mathbb{W}_2^2(\mu_t; \mu^\sigma) \leq J_{V,F}(\mu_t | \mu^\sigma). \quad (1.5)$$

Particularly, if  $\mathcal{C}^\sigma - \max\{\alpha; 0\} - \vartheta > 0$ , Diffusion (II) admits a unique invariant probability  $\mu^\sigma$  and we have

$$\mathbb{W}_2(\mu_t; \mu^\sigma) \leq \exp[-(\mathcal{C}^\sigma - \max\{\alpha; 0\} - \vartheta)t] \mathbb{W}_2(\mu_0; \mu^\sigma), \quad (1.6)$$

for any  $t \geq 0$ .

Like with Proposition 1.5, we do not need the whole set of assumptions. We assume  $V$  and  $F$  to verify (A-1)-(A-2)-(A-6). And, in order to apply Proposition 1.4, we assume that the initial law  $\mu_0$  satisfy (A-8)-(A-9).

*Proof.* By Proposition 1.5, we have

$$\mathcal{C}^\sigma \mathbb{W}_2^2(\mu_t; \mu^\sigma) \leq J_{V_0,0}(\mu_t | \mu^\sigma). \quad (1.7)$$

However, by definition, the quantity  $J_{V,F}(\mu_t | \mu^\sigma)$  is equal to

$$\begin{aligned} J_{V,F}(\mu_t | \mu^\sigma) &= J_{V_0,0}(\mu_t | \mu^\sigma) - \vartheta \int_{\mathbb{R}^d} \|\nabla\varphi_t(x) - x\|^2 \mu^\sigma(dx) \\ &\quad - \frac{\alpha}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \|(\nabla\varphi_t(x) - \nabla\varphi_t(y)) - (x - y)\|^2 \mu^\sigma(dx) \mu^\sigma(dy) \\ &\quad + \frac{1}{2} \iint_{\mathbb{R}^{2d}} \langle \nabla F_0(Z(x,y)) - \nabla F_0(x-y); Z(x,y) - (x-y) \rangle \mu(dx) \mu(dy), \end{aligned}$$

with  $Z(x,y) := \nabla\tau(x) - \nabla\tau(y)$ . However,  $F_0$  is a convex function. Consequently, we have

$$\begin{aligned} J_{V,F}(\mu_t | \mu^\sigma) &\geq J_{V_0,0}(\mu_t | \mu^\sigma) - \vartheta \int_{\mathbb{R}^d} \|\nabla\varphi_t(x) - x\|^2 \mu^\sigma(dx) \\ &\quad - \frac{\alpha}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \|(\nabla\varphi_t(x) - \nabla\varphi_t(y)) - (x - y)\|^2 \mu^\sigma(dx) \mu^\sigma(dy) \\ &\geq J_{V_0,0}(\mu_t | \mu^\sigma) - (\max\{\alpha; 0\} + \vartheta) \int_{\mathbb{R}^d} \|\nabla\varphi_t(x) - x\|^2 \mu^\sigma(dx). \end{aligned}$$

By Brenier's theorem, we obtain

$$J_{V,F}(\mu_t | \mu^\sigma) \geq J_{V_0,0}(\mu_t | \mu^\sigma) - (\max\{\alpha; 0\} + \vartheta) \mathbb{W}_2^2(\mu_t; \mu^\sigma),$$

which with (1.7) gives (1.5). Here, the convex function  $\varphi_t$  is defined by  $\mu_t =: \nabla\varphi_t\#\mu^\sigma$ . The uniqueness of the stationary measure if  $\mathcal{C}^\sigma - \alpha - \vartheta > 0$  and the exponential decay in (1.6) are consequences of Proposition 1.4.  $\square$

Let us note that the inequality

$$\sup_{\mathbb{R}^d} -\nabla^2 V < \inf_{\mathbb{R}^d} \nabla^2 F < 0$$

implies the uniqueness of the stationary measure  $\mu^\sigma$  and the exponential convergence toward  $\mu^\sigma$  for any  $\sigma > 0$ . Such a result has already been proven in [CMV03].

Reciprocally, the inequality

$$\inf_{\mathbb{R}^d} \nabla^2 F < \inf_{\mathbb{R}^d} \nabla^2 V < 0$$

is not sufficient to ensure the uniqueness and the convergence result. We need also the center of mass to be fixed, see [Tug13a, CMV03].

**Theorem 1.7.** *We assume that  $V$ ,  $F$  and  $\mu_0$  satisfy the set of Hypotheses (A). Thus, there exists  $\sigma_c$  such that  $\mathcal{C}^\sigma > \alpha + \vartheta$  for any  $\sigma \geq \sigma_c$ . Consequently, if  $\sigma$  is large enough, Diffusion (II) admits a unique invariant probability  $\mu^\sigma$ . Moreover, it satisfies the exponential decay (1.6).*

*Proof.* In order to prove it, we first admit the following limit

$$\lim_{\sigma \rightarrow +\infty} \frac{1}{\sigma^2} \int_{\mathbb{R}^d} \|x\|^{2n} \mu^\sigma(dx) = 0, \quad (1.8)$$

for any family  $\{\mu^\sigma; \sigma \geq 1\}$  of invariant probabilities of Diffusion (II). In a first step, we prove that Limit (1.8) implies the statement of Theorem 1.7. In a second step, we prove (1.8).

**Step 1.** We admit the limit (1.8). We remind the reader the following equality

$$W^\sigma(x) = V(x) + F * \mu^\sigma(x).$$

Moreover, Hypothesis (A-6) on  $F$  implies

$$|F * \mu^\sigma(x)| \leq C \left(1 + \|x\|^{2n}\right) \left(1 + \int_{\mathbb{R}^d} \|y\|^{2n} \mu^\sigma(dy)\right)$$

so that, for any  $R > 0$ , we have the limit

$$\lim_{\sigma \rightarrow +\infty} \frac{1}{\sigma^2} \sup_{\|x\| \leq 3R} \|W^\sigma(x)\| = 0,$$

thanks to Limit (1.8). Therefore, for any  $R > 0$ , the quantities  $\exp[-\frac{2}{\sigma^2}S(R)]$  and  $\exp[\frac{2}{\sigma^2}(I(R) - S(R))]$  go to 1 as  $\sigma$  goes to infinity. We remind the reader that  $I(R)$  and  $S(R)$  are defined in Proposition 1.5 by

$$I(R) := \inf_{\|x\| \leq 2R} W^\sigma(x) \quad \text{and} \quad S(R) := \sup_{\|x\| \leq 3R} W^\sigma(x).$$

We obtain the following limit for any  $R > 0$ :

$$\lim_{\sigma \rightarrow \infty} \mathcal{C}^\sigma(R) = \frac{K(R)}{3} \min \left\{ 1; \frac{3^d - 2^d}{2^d} \right\},$$

where  $K(R) := \inf_{\|x\| \geq R} \nabla^2 V_0(x)$ . By Assumption (A-2), the quantity  $K(R)$  goes to infinity as  $R$  goes to infinity. We take  $R_0$  such that

$$\frac{K(R_0)}{3} \min \left\{ 1; \frac{3^d - 2^d}{2^d} \right\} > 2(\alpha + \vartheta).$$

Then, we take  $\sigma_c$  large enough such that  $\mathcal{C}^\sigma(R_0) > \frac{1}{2} \lim_{\xi \rightarrow \infty} \mathcal{C}^\xi(R_0)$  for any  $\sigma \geq \sigma_c$ . Thus, we have the inequality

$$\mathcal{C}^\sigma \geq \mathcal{C}^\sigma(R_0) > (\alpha + \vartheta)$$

for any  $\sigma \geq \sigma_c$ . Consequently, if Limit (1.8) is satisfied, the statement of the theorem is proven.

**Step 2.** We now achieve the proof by establishing Limit (1.8). *It is in this step that we use the hypothesis  $p > n$ .* We proceed a *reductio ad absurdum*. Let us assume the existence of a positive constant  $C$  and an increasing sequence  $(\sigma_k)_{k \in \mathbb{N}}$  which goes to infinity such that for any  $k \in \mathbb{N}$ , Diffusion (II) admits an invariant probability  $\mu^{\sigma_k}$  satisfying

$$\eta_{2n}(k) := \int_{\mathbb{R}^d} \|x\|^{2n} \mu^{\sigma_k}(dx) \geq C\sigma_k^2.$$

In particular, we deduce that the sequence  $(\eta_{2n}(k))_{k \in \mathbb{N}}$  goes to infinity as  $k$  goes to infinity. Since  $\mu^{\sigma_k}$  is an invariant probability, we have

$$\eta_{2n}(k) = \frac{\int_{\mathbb{R}^d} \|x\|^{2n} \exp \left\{ -\frac{2}{\sigma_k^2} [V(x) + F * \mu^{\sigma_k}(x)] \right\} dx}{\int_{\mathbb{R}^d} \exp \left\{ -\frac{2}{\sigma_k^2} [V(x) + F * \mu^{\sigma_k}(x)] \right\} dx}.$$

By making the transformation  $x := (\eta_{2n}(k))^{\frac{1}{2n}} y$ , we obtain

$$1 = \frac{\int_{\mathbb{R}^d} \|y\|^{2n} \exp \left\{ -\frac{2}{\sigma_k^2} \left[ \frac{V((\eta_{2n}(k))^{\frac{1}{2n}} y)}{(\eta_{2n}(k))^{\frac{p}{n}}} + \frac{F * \mu^{\sigma_k}((\eta_{2n}(k))^{\frac{1}{2n}} y)}{(\eta_{2n}(k))^{\frac{p}{n}}} \right] \right\} dy}{\int_{\mathbb{R}^d} \exp \left\{ -\frac{2}{\sigma_k^2} \left[ \frac{V((\eta_{2n}(k))^{\frac{1}{2n}} y)}{(\eta_{2n}(k))^{\frac{p}{n}}} + \frac{F * \mu^{\sigma_k}((\eta_{2n}(k))^{\frac{1}{2n}} y)}{(\eta_{2n}(k))^{\frac{p}{n}}} \right] \right\} dy}, \quad (1.9)$$

with  $\widehat{\sigma}_k := \frac{\sigma_k}{\sqrt{\eta_{2n}(k)}} (\eta_{2n}(k))^{-\frac{p-n}{2n}} \leq \frac{1}{\sqrt{C}} (\eta_{2n}(k))^{-\frac{p-n}{2n}} \rightarrow 0$  as  $k$  goes to infinity.

For any  $y \in \mathbb{R}^d$ , Hypothesis (A-7) implies

$$\lim_{k \rightarrow +\infty} \frac{F * \mu^{\sigma_k} \left( (\eta_{2n}(k))^{\frac{1}{2n}} y \right)}{(\eta_{2n}(k))^{\frac{p}{n}}} = 0.$$

And, Assumption (A-5) yields

$$\lim_{k \rightarrow +\infty} \frac{V \left( (\eta_{2n}(k))^{\frac{1}{2n}} y \right)}{(\eta_{2n}(k))^{\frac{p}{n}}} = \Theta(y),$$

the function  $\Theta$  being strictly convex and such that  $\Theta(y) > \Theta(0) = 0$  for any  $y \neq 0$ . Consequently, by applying [Tug12, Lemma A.2], the right hand term in (1.9) goes to 0 as  $k$  goes to infinity. Nevertheless, the left hand term is equal to 1. The initial assumption of Step 2 is absurd. This achieves the proof.  $\square$

Let us remark that Theorem 1.7 goes further than the results in [Tug13b] concerning the uniqueness of the invariant probability for sufficiently large  $\sigma$ . Moreover, it provides, with Corollary 1.6 a method for simulating a lower-bound of the critical value above which there is a unique invariant probability. Nevertheless, this method needs more computation than those described in [Tug13b]. Let us mention that the difference with the results obtained in [BGG12b] is that the confinement potential  $V$  is not assumed to be convex.

## 2 Propagation of chaos

The other keystone of the uniform propagation of chaos is the simple propagation of chaos. We here consider it with the Wasserstein distance. Let us define it precisely.

**Definition 2.1.** *One says that simple propagation of chaos holds if the inequality*

$$\lim_{N \rightarrow +\infty} \sup_{t \in [0; T]} \mathbb{W}_2 \left( \mu_t ; \mu_t^{1, N} \right) = 0$$

*is true for any  $T > 0$ , where  $\mu_t$  (resp.  $\mu_t^{1, N}$ ) is the law of Diffusion (II) (resp. the law of the first particle in the system (III)).*

*Uniform propagation of chaos holds if we have the limit*

$$\lim_{N \rightarrow +\infty} \sup_{t \in \mathbb{R}_+} \mathbb{W}_2 \left( \mu_t ; \mu_t^{1, N} \right) = 0. \quad (2.1)$$

In order to obtain the uniform propagation of chaos, we first need to prove the simple propagation of chaos. It is a classical result. However, we will provide the proof because we will need to establish a coupling result starting from non i.i.d. random variables.

**Proposition 2.2.** *We assume that  $V$ ,  $F$  and  $\mu_0$  satisfy the set of Hypotheses (A). Let  $X_0^1, \dots, X_0^N$  be  $N$  random variables with common law  $\mu_0$ . We do not assume these variables to be independent. We consider the two following particle systems:*

$$X_t^i = X_0^i + \sigma B_t^i - \int_0^t \nabla V(X_s^i) ds - \int_0^t \nabla F * \eta_s^N(X_s^i) ds, \quad (2.2)$$

where  $\eta_s^N := \left(\frac{1}{N} \sum_{j=1}^N \delta_{X_s^j}\right) P_s$ ,  $P_s$  being defined as the semi-group associated to Diffusion (II) and

$$Z_t^i = X_0^i + \sigma B_t^i - \int_0^t \nabla V(Z_s^i) ds - \frac{1}{N} \sum_{j=1}^N \int_0^t \nabla F(Z_s^i - Z_s^j) ds, \quad (2.3)$$

$B^1, \dots, B^N$  being  $N$  independent Brownian motions (and independent from the initial random variables). Then, for any  $T > 0$ , we have the following inequality:

$$\sup_{t \in [0; T]} \mathbb{E} \left\{ \|X_t^i - Z_t^i\|^2 \right\} \leq \frac{C(\mu_0)}{(\vartheta + 2\alpha)^2 N} \exp[2(\vartheta + 2\alpha)T], \quad (2.4)$$

where  $C(\mu_0)$  is a positive function of  $\int_{\mathbb{R}^d} \|x\|^{8q^2} \mu_0(dx)$ .

*Proof.* By  $\mu_t$ , we denote the law  $\mathcal{L}(X_t^1) = \dots = \mathcal{L}(X_t^N)$ . By definition, for any  $1 \leq i \leq N$ , we have

$$\begin{aligned} Z_t^i - X_t^i &= - \int_0^t \{ \nabla V(Z_s^i) - \nabla V(X_s^i) \} ds \\ &\quad - \int_0^t \left\{ \frac{1}{N} \sum_{j=1}^N \nabla F(Z_s^i - Z_s^j) - \nabla F * \eta_s^N(X_s^i) \right\} ds. \end{aligned}$$

We apply Itô formula to  $Z_t^i - X_t^i$  with the function  $x \mapsto \|x\|^2$ . By introducing the notation  $\xi_i(t) := \|Z_t^i - X_t^i\|^2$ , we obtain

$$\begin{aligned} d\xi_i(t) &= -2\Delta_1(i, t)dt - \frac{2}{N}\Delta_2(i, t)dt \\ \text{with } \Delta_1(i, t) &:= \langle Z_t^i - X_t^i; \nabla V(Z_t^i) - \nabla V(X_t^i) \rangle \\ \text{and } \Delta_2(i, t) &:= \left\langle Z_t^i - X_t^i; \sum_{j=1}^N \left[ \nabla F(Z_t^i - Z_t^j) - \nabla F * \eta_t^N(X_t^i) \right] \right\rangle. \end{aligned}$$

By taking the sum on the integer  $i$  running between 1 and  $N$ , we get

$$d \sum_{i=1}^N \xi_i(t) = -2\Delta_1(t)dt - \frac{2}{N} \sum_{i=1}^N \sum_{j=1}^N \left( \Delta_2(i, j, t) + \Delta_3(i, j, t) \right) dt$$

$$\text{with } \Delta_1(t) := \sum_{i=1}^N \Delta_1(i, t),$$

$$\Delta_2(i, j, t) := \left\langle \nabla F(Z_t^i - Z_t^j) - \nabla F(X_t^i - X_t^j); Z_t^i - X_t^i \right\rangle$$

$$\text{and } \Delta_3(i, j, t) := \left\langle \nabla F(X_t^i - X_t^j) - \nabla F * \eta_t^N(X_t^i); Z_t^i - X_t^i \right\rangle.$$

According to the definition of the function  $F_0$  in Hypothesis (A-6), it is convex. This implies  $\langle x - y; \nabla F_0(x - y) \rangle \geq 0$  for any  $x, y \in \mathbb{R}^d$ . This inequality yields

$$\frac{1}{N} \sum_{i=1}^N \sum_{j=1}^N (\Delta_2(i, j, t) + \Delta_2(j, i, t)) \geq -4\alpha \sum_{i=1}^N \|Z_t^i - X_t^i\|^2.$$

Indeed, we have

$$\begin{aligned} & \Delta_2(i, j, t) + \Delta_2(j, i, t) \\ &= \left\langle \nabla F(Z_t^i - Z_t^j) - \nabla F(X_t^i - X_t^j); (Z_t^i - X_t^i) - (Z_t^j - X_t^j) \right\rangle \\ &= \left\langle \nabla F_0(Z_t^i - Z_t^j) - \nabla F_0(X_t^i - X_t^j); (Z_t^i - X_t^i) - (Z_t^j - X_t^j) \right\rangle \\ &\quad - \alpha \left\langle \nabla(Z_t^i - Z_t^j) - (X_t^i - X_t^j); (Z_t^i - X_t^i) - (Z_t^j - X_t^j) \right\rangle \\ &= \left\langle \nabla F_0(Z_t^i - Z_t^j) - \nabla F_0(X_t^i - X_t^j); (Z_t^i - Z_t^j) - (X_t^i - X_t^j) \right\rangle \\ &\quad - \alpha \left\| (Z_t^i - Z_t^j) - (X_t^i - X_t^j) \right\|^2 \\ &\geq -\alpha \left\| (Z_t^i - Z_t^j) - (X_t^i - X_t^j) \right\|^2 \\ &\geq -2\alpha \left\{ \|Z_t^i - X_t^i\|^2 + \|Z_t^j - X_t^j\|^2 \right\}. \end{aligned}$$

Consequently, we have

$$\begin{aligned} \mathbb{E} \left\{ -\frac{2}{N} \sum_{i=1}^N \sum_{j=1}^N \Delta_2(i, j, t) \right\} &= \frac{1}{2} \mathbb{E} \left\{ -\frac{2}{N} \sum_{1 \leq i, j \leq N} (\Delta_2(i, j, t) + \Delta_2(j, i, t)) \right\} \\ &\leq 4\alpha \sum_{i=1}^N \|Z_t^i - X_t^i\|^2. \end{aligned} \quad (2.5)$$

By definition of  $\vartheta$  in Assumption (A-3), for any  $x, y \in \mathbb{R}^d$  we have the inequality  $\langle \nabla V(x) - \nabla V(y); x - y \rangle \geq -\vartheta \|x - y\|^2$ . This implies

$$-2 \sum_{i=1}^N \Delta_1(i, t) \leq 2\vartheta \sum_{i=1}^N \xi_i(t). \quad (2.6)$$

We now deal with the sum containing  $\Delta_3(i, j, t)$ . We apply Cauchy-Schwarz inequality:

$$-\mathbb{E} \left[ \sum_{j=1}^N \Delta_3(i, j, t) \right] \leq \left\{ \mathbb{E} \left[ \|Z_t^i - X_t^i\|^2 \right] \right\}^{\frac{1}{2}} \left\{ \sum_{j=1}^N \sum_{k=1}^N \mathbb{E} [\langle \rho_j^i(t); \rho_k^i(t) \rangle] \right\}^{\frac{1}{2}}$$

with  $\rho_j^i(t) := \nabla F(X_t^i - X_t^j) - \nabla F * \eta_t^N (X_t^i)$ .

The idea now is to prove an inequality of the form

$$\sum_{j=1}^N \sum_{k=1}^N \mathbb{E} [\langle \rho_j^i(t); \rho_k^i(t) \rangle] \leq CN,$$

where  $C$  is a positive constant. We use the following conditioning:

$$\mathbb{E} [\langle \rho_j^i(t); \rho_k^i(t) \rangle] = \mathbb{E} \left\{ \mathbb{E} [\langle \rho_j^i(t); \rho_k^i(t) \rangle \mid X_0^1, \dots, X_0^N] \right\}.$$

The particles  $X^r$ ,  $1 \leq r \leq N$ , are not independent but they are independent conditionally to the knowlege of the initial random variables  $X_0^1, \dots, X_0^N$ . Therefore, we have the equality

$$\mathbb{E} [\langle \rho_j^i(t); \rho_k^i(t) \rangle] = \mathbb{E} \left\{ \langle \mathbb{E} [\rho_j^i(t) \mid X_0^1, \dots, X_0^N]; \mathbb{E} [\rho_k^i(t) \mid X_0^1, \dots, X_0^N] \rangle \right\},$$

if  $j \neq k$ . Consequently, for any  $1 \leq j \leq N$ , we have

$$\begin{aligned} & \sum_{k=1}^N \mathbb{E} [\langle \rho_j^i(t); \rho_k^i(t) \rangle] \\ &= \mathbb{E} \left\{ \left\langle \mathbb{E} [\rho_j^i(t) \mid X_0^1, \dots, X_0^N]; \mathbb{E} \left[ \sum_{k=1}^N \rho_k^i(t) \mid X_0^1, \dots, X_0^N \right] \right\rangle \right\} \\ &+ \mathbb{E} \left[ \|\rho_j^i(t)\|^2 \right] - \mathbb{E} \left\{ \|\mathbb{E} [\rho_j^i(t) \mid X_0^1, \dots, X_0^N]\|^2 \right\} \\ &\leq \mathbb{E} \left\{ \left\langle \mathbb{E} [\rho_j^i(t) \mid X_0^1, \dots, X_0^N]; \mathbb{E} \left[ \sum_{k=1}^N \rho_k^i(t) \mid X_0^1, \dots, X_0^N \right] \right\rangle \right\} \\ &+ \mathbb{E} \left[ \|\rho_j^i(t)\|^2 \right]. \end{aligned}$$

Now, we will prove that

$$\mathbb{E} \left[ \sum_{k=1}^N \rho_k^i(t) \mid X_0^1, \dots, X_0^N \right]$$

is equal to 0. Indeed, for any  $1 \leq k \leq N$ , we have

$$\mathbb{E} [\nabla F(X_t^i - X_t^k) \mid X_0^1, \dots, X_0^N, X_t^i] = \nabla F * \nu_t^{X_0^k, \frac{1}{N} \sum_{l=1}^N \delta_{X_0^l}} (X_t^i),$$



where  $\nu_t^{x_0, \mu_0}$  is the law of the diffusion

$$Y_t := x_0 + \sigma B_t - \int_0^t \nabla V(Y_s) ds - \int_0^t (\nabla F * \mu_0 P_s)(Y_s) ds.$$

We remark that

$$\sum_{k=1}^N \nu_t^{X_0^k, \frac{1}{N} \sum_{i=1}^N \delta_{X_0^i}} = \left( \frac{1}{N} \sum_{l=1}^N \delta_{X_0^l} \right) P_t.$$

Consequently, we get

$$\mathbb{E} \left[ \sum_{k=1}^N \nabla F(X_t^i - X_t^k) \mid X_0^1, \dots, X_0^N \right] = NF * \eta_t^N(X_t^i).$$

This yields

$$\mathbb{E} \left[ \sum_{k=1}^N \rho_k^i(t) \mid X_0^1, \dots, X_0^N \right] = 0.$$

We obtain immediately:

$$\sum_{j=1}^N \sum_{k=1}^N \mathbb{E} [\langle \rho_j^i(t); \rho_k^i(t) \rangle] \leq \sum_{j=1}^N \mathbb{E} [|\rho_j^i(t)|^2].$$

Let us now compute  $\mathbb{E} [|\rho_j^i(t)|^2]$ . The diffusions  $X^i$  and  $X^j$  are not independent but they are independent conditionally to the initial random variables. However, according to Hypothesis (A-6), we have  $F(x) = G(\|x\|)$  where  $G$  is a polynomial function of degree  $2n$ , we have the following inequality:

$$\begin{aligned} & \mathbb{E} \left[ \left| \nabla F(X_t^i - X_t^j) - \nabla F * \eta_t^N(X_t^i) \mid X_0^1, \dots, X_0^N \right|^2 \right] \\ & \leq C \left( 1 + \mathbb{E} [|\|X_t^i\|^{4n-2} \mid X_0^1, \dots, X_0^N] + \mathbb{E} [|\|X_t^j\|^{4n-2} \mid X_0^1, \dots, X_0^N] \right). \end{aligned}$$

Then, we use the control of the moments obtained in [HIP08, Theorem 2.13] and we obtain the following majoration:

$$\sup_{t \geq 0} \mathbb{E} \left[ \left| \nabla F(X_t^i - X_t^j) - \nabla F * \eta_t^N(X_t^i) \right|^2 \mid X_0^1, \dots, X_0^N \right] \leq C \left( \frac{1}{N} \sum_{k=1}^N \delta_{X_0^k} \right),$$

$C(\nu)$  being a function of the  $8q^2$  moment of the law  $\nu$  for any measure  $\nu$ . We deduce

$$\begin{aligned} & \sup_{t \geq 0} \mathbb{E} \left[ \left| \nabla F(X_t^i - X_t^j) - \nabla F * \nu_t^N(X_t^i) \right|^2 \mid X_0^1, \dots, X_0^N \right] \\ & \leq K \left( 1 + \frac{1}{N} \sum_{k=1}^N \|X_0^k\|^{8q^2} \right), \end{aligned}$$

$K$  being a positive constant. Consequently, we have

$$\sup_{t \geq 0} \mathbb{E} \left[ \left\| \nabla F(X_t^i - X_t^j) - \nabla F * \eta_t^N(X_t^i) \right\|^2 \right] \leq K \left( 1 + \int_{\mathbb{R}^d} \|x\|^{8q^2} \mu_0(dx) \right).$$

Therefore, we deduce the following inequality:

$$-\mathbb{E} \left[ \sum_{j=1}^N \Delta_3(i, j, t) \right] \leq \sqrt{C(\mu_0)} \sqrt{N \mathbb{E} [\xi_i(t)]}. \quad (2.7)$$

By combining (2.5), (2.6) and (2.7), we obtain

$$\frac{d}{dt} \sum_{i=1}^N \mathbb{E} [\xi_i(t)] \leq 2 \sum_{i=1}^N \left\{ (\vartheta + 2\alpha) \mathbb{E} [\xi_i(t)] + \frac{\sqrt{C(\mu_0)}}{\sqrt{N}} \sqrt{\mathbb{E} [\xi_i(t)]} \right\}. \quad (2.8)$$

However, the particles are exchangeable. Consequently, for any  $1 \leq i \leq N$ , we have

$$\frac{d}{dt} \mathbb{E} \{\xi_i(t)\} \leq 2(\vartheta + 2\alpha) \mathbb{E} \{\xi_i(t)\} + \frac{2\sqrt{C(\mu_0)}}{\sqrt{N}} \sqrt{\mathbb{E} \{\xi_i(t)\}}.$$

By introducing  $\tau_i(t) := \sqrt{\mathbb{E} \{\xi_i(t)\}}$ , we obtain

$$\tau_i'(t) \leq (\vartheta + 2\alpha) \left\{ \tau_i(t) + \frac{\sqrt{C(\mu_0)}}{(\vartheta + 2\alpha) \sqrt{N}} \right\}$$

The application of Grönwall lemma yields

$$\mathbb{E} \left\{ \|Z_t^i - X_t^i\|^2 \right\} \leq \frac{C(\mu_0)}{N(\vartheta + 2\alpha)^2} \exp [2(\vartheta + 2\alpha)t].$$

We obtain (2.4) by taking the supremum for  $t$  running between 0 and  $T$ .  $\square$

This result is not the classical propagation of chaos because the initial random variables are not supposed to be independent. However, we have the same inequality and this is one of the main tools of the proof of the main theorem. Before giving the proof of the main theorem, we give the following result to control the moments.

**Proposition 2.3.** *We assume that the potentials  $V$  and  $F$  and the probability measure  $\mu_0$  satisfy the set of Assumptions (A). Let  $Z_0^1, \dots, Z_0^N$  be  $N$  i.i.d. random variables with common law  $\mu_0$ . We consider the following particle system:*

$$Z_t^i = Z_0^i + \sigma B_t^i - \int_0^t \nabla V(Z_s^i) ds - \frac{1}{N} \sum_{j=1}^N \int_0^t \nabla F(Z_s^i - Z_s^j) ds, \quad (2.9)$$

$B^1, \dots, B^N$  being  $N$  independent Brownian motions (and independent from the initial random variables). Then, there exists a constant  $M(\mu_0)$  such that

$$\max_{1 \leq k \leq 8q^2} \sup_{t \geq 0} \mathbb{E} \left\{ \|Z_t^i\|^k \right\} \leq M(\mu_0), \quad (2.10)$$

for any  $N \in \mathbb{N}$ .

The proof is classical and can be adapted from [CGM08, Section 2.1] so it is left to the reader. Let us just mention that the only hypotheses that we need on the potentials  $V$  and  $F$  are (A-1), (A-2), (A-4) and (A-6). Indeed, these hypotheses are sufficient to ensure the convexity at infinity of the drift  $V + F * \mu_t$ . About the law  $\mu_0$ , we do need the initial entropy to be finite.

### 3 Main results

In this paragraph, we prove the main results: there is uniform (with respect to the time) propagation of chaos with sufficiently large  $\sigma$ . In all this section, we assume the inequality (of simple propagation of chaos) (3.1).

Let  $X_0^1, \dots, X_0^N$  be  $N$  random variables with common law  $\mu_0$ . We do not assume these variables to be independent. We consider the two following particle systems:

$$X_t^i = X_0^i + \sigma B_t^i - \int_0^t \nabla V(X_s^i) ds - \int_0^t \nabla F * \eta_s^N(X_s^i) ds,$$

where  $\eta_s^N := \left(\frac{1}{N} \sum_{j=1}^N \delta_{X_0^j}\right) P_s$  and

$$Z_t^i = X_0^i + \sigma B_t^i - \int_0^t \nabla V(Z_s^i) ds - \frac{1}{N} \sum_{j=1}^N \int_0^t \nabla F(Z_s^i - Z_s^j) ds,$$

$B^1, \dots, B^N$  being  $N$  independent Brownian motions (and independent from the initial random variables). Then, for any  $T > 0$ , we have the following inequality:

$$\sup_{t \in [0; T]} \mathbb{E} \left\{ \|X_t^i - Z_t^i\|^2 \right\} \leq \left( \frac{\exp[\psi(T)]}{\sqrt{N}} \right)^2, \quad (3.1)$$

for any  $T > 0$ . Here, the function  $\psi$  is nondecreasing. Let us remark that under the set of assumptions (A), this inequality is satisfied thanks to the coupling result obtained in Proposition 2.2,

$$\sup_{t \in [0; T]} \mathbb{E} \left\{ \|X_t^i - Z_t^i\|^2 \right\} \leq \frac{C(\mu_0)}{(\vartheta + 2\alpha)^2 N} \exp[2(\vartheta + 2\alpha)T].$$

In this case, we can write  $\psi(T) = (\vartheta + 2\alpha)T + \log\left(\frac{\sqrt{C(\mu_0)}}{\vartheta + 2\alpha}\right)$ .

**Theorem 3.1.** *We assume that the potentials  $V$  and  $F$  and the probability measure  $\mu_0$  satisfy the set of Assumptions (A). If  $\sigma \geq \sigma_c$  ( $\sigma_c$  being defined in Theorem 1.7), we have the uniform propagation of chaos:*

$$\lim_{N \rightarrow +\infty} \sup_{t \geq 0} \mathbb{W}_2(\mu_t; \mu_t^{1, N}) = 0. \quad (3.2)$$

Moreover, we can compute the rate of convergence. By  $C(\sigma)$ , we denote the expression  $C^\sigma - \max\{\alpha; 0\} - \vartheta > 0$ .

**First case:** The quantity  $\frac{C(\sigma)t}{\psi(t)}$  goes to  $\lambda \in \mathbb{R}_+^* \cup \{+\infty\}$  as  $t$  goes to infinity. Therefore, for all  $\delta > 0$ , we have:

$$\lim_{N \rightarrow +\infty} N^{\frac{1}{2(1+\lambda)} - \delta} \sup_{t \geq 0} \mathbb{W}_2(\mu_t; \mu_t^{1,N}) = 0. \quad (3.3)$$

**Second case:** The quantity  $\frac{C(\sigma)t}{\psi(t)}$  goes to 0 as  $t$  goes to infinity. Thus, for all  $\delta > 0$ , we have:

$$\lim_{N \rightarrow +\infty} \exp\left\{C(\sigma)\psi^{-1}\left[\frac{1}{2}(1-\delta)\log(N)\right]\right\} \sup_{t \geq 0} \mathbb{W}_2(\mu_t; \mu_t^{1,N}) = 0. \quad (3.4)$$

*Proof. Step 1.* Let  $t$  be a positive real. The idea is to consider the nonlinear diffusion (2.3). Let  $T$  be a positive real.

**Step 2.** The triangular inequality implies

$$\begin{aligned} \mathbb{W}_2(\mu_{T+t}; \mu_{T+t}^{1,N}) &\leq \mathbb{W}_2(\mu_{T+t}; \mu^\sigma) + \mathbb{W}_2(\mu^\sigma; \eta_{T,t}^N) \\ &\quad + \mathbb{W}_2(\eta_{T,t}^N; \mu_{T+t}^{1,N}), \end{aligned}$$

where  $\eta_{T,t}^N$  is the law at time  $T$  of diffusion

$$Y_s = X_t^{1,N} + \sigma(B_{t+s}^1 - B_t^1) - \int_0^s [\nabla V + \nabla F * \nu_u^N](Y_u) du,$$

where  $\nu_u^N := \left(\frac{1}{N} \sum_{i=1}^N \delta_{X_t^{i,N}}\right) P_u$ .

**Step 3.** Let us bound each of the three terms.

**Step 3.1.** We can bound easily the last term. By definition and by assumption, we have:

$$\mathbb{W}_2(\eta_{T,t}^N; \mu_{T+t}^{1,N}) \leq \frac{\exp[\psi(T)]}{\sqrt{N}}.$$

**Step 3.2.** The first term can be bounded like so:

$$\mathbb{W}_2(\mu_{T+t}; \mu^\sigma) \leq e^{-C(\sigma)(T+t)} \mathbb{W}_2(\mu_0; \mu^\sigma).$$

**Step 3.3.** We proceed in a similar way with the second term. We introduce the McKean-Vlasov diffusion  $Y^{x_0^1, \dots, x_0^N}$  starting from the law  $\frac{1}{N} \sum_{j=1}^N \delta_{x_0^j}$ . We stress that the law  $\eta_{T,t}^N$  is different from the law of diffusion  $Y^{X_t^{1,N}, \dots, X_t^{N,N}}$ . However, we have

$$\mathbb{W}_2(\mu^\sigma; \eta_{T,t}^N)^2 = \inf \|X_1 - X_2\|^2$$

where the infimum runs for  $X_1$  which follows the law  $\mu^\sigma$  and for  $X_2$  which has the same law as the diffusion  $Z_{T+t}^{1,N}$  defined by

$$Z_{s+t}^{i,N} = X_t^{i,N} + \sigma(B_{t+s}^i - B_t^i) - \int_0^s [\nabla V + \nabla F * \nu_u^N](Z_{u+t}^{i,N}) du.$$

Consequently, we can write

$$\mathbb{W}_2(\mu^\sigma; \eta_{T,t}^N)^2 = \mathbb{E} \left\{ \mathbb{W}_2 \left( \mu^\sigma; \left( \frac{1}{N} \sum_{j=1}^N \delta_{X_t^{j,N}} \right) P_T \right) \right\}.$$

Indeed, the infimum of  $\|X_1 - Z_T^{i,N}\|^2$  over the law of  $Z_T^{i,N}$  and over  $1 \leq i \leq N$  is exactly the infimum of  $\|X_1 - X_2\|^2$ , where  $\mathcal{L}(X_2) = \left( \frac{1}{N} \sum_{j=1}^N \delta_{X_t^{j,N}} \right) P_T$ . We deduce immediately:

$$\begin{aligned} \mathbb{W}_2(\mu^\sigma; \eta_{T,t}^N) &\leq \mathbb{E} \left\{ e^{-C(\sigma)T} \mathbb{W}_2 \left( \mu^\sigma; \frac{1}{N} \sum_{j=1}^N \delta_{X_t^{j,N}} \right) \right\} \\ &\leq e^{-C(\sigma)T} \mathbb{W}_2(\mu^\sigma; \mu_t^{1,N}) \\ &\leq e^{-C(\sigma)T} \left( \mathbb{W}_2(\mu^\sigma; \mu_t) + \mathbb{W}_2(\mu_t; \mu_t^{1,N}) \right) \\ &\leq e^{-C(\sigma)(T+t)} \mathbb{W}_2(\mu^\sigma; \mu_0) + e^{-C(\sigma)T} \mathbb{W}_2(\mu_t; \mu_t^{1,N}). \end{aligned}$$

**Step 3.4.** Let us briefly justify why we can extend the WJ-inequality by starting from a sum of Dirac measures,  $\eta_0$ . To do so, we consider a sequence of probability measures with finite entropy  $(\eta_0^{(n)})_{n \geq 1}$  which converges for the Wasserstein distance to  $\eta_0$ . By  $\eta_t$  (respectively  $\eta_t^{(n)}$ ), we denote the law at time  $t$  of the McKean-Vlasov diffusion starting from the law  $\eta_0$  (respectively the law  $\eta_0^{(n)}$ ). Then, we have :

$$\mathbb{W}_2(\eta_t; \mu^\sigma) \leq \mathbb{W}_2(\eta_t; \eta_t^{(n)}) + \mathbb{W}_2(\eta_t^{(n)}; \mu^\sigma).$$

By applying the inequality to  $\eta_t^{(n)}$ , we get

$$\mathbb{W}_2(\eta_t; \mu^\sigma) \leq \mathbb{W}_2(\eta_t; \eta_t^{(n)}) + e^{-C(\sigma)t} \mathbb{W}_2(\eta_0^{(n)}; \mu^\sigma).$$

By making a coupling, one can easily show that the quantity  $\mathbb{W}_2(\eta_t; \eta_t^{(n)})$  converges to 0. Finally, since  $\mathbb{W}_2(\eta_0^{(n)}; \eta_0)$  goes to 0 as  $n$  tends to infinity, we obtain the formula

$$\mathbb{W}_2(\eta_t; \mu^\sigma) \leq e^{-C(\sigma)t} \mathbb{W}_2(\eta_0; \mu^\sigma).$$

**Step 4.** Consequently, we have:

$$\begin{aligned} \mathbb{W}_2(\mu_{T+t}; \mu_{T+t}^{1,N}) &\leq e^{-C(\sigma)T} \mathbb{W}_2(\mu_t; \mu_t^{1,N}) + 2e^{-C(\sigma)(T+t)} \mathbb{W}_2(\mu_0; \mu^\sigma) \\ &\quad + \frac{\exp[\psi(T)]}{\sqrt{N}}. \end{aligned}$$

We now take the supremum for  $t$  running between  $(k-1)T$  and  $kT$  and we obtain

$$\begin{aligned} \sup_{kT \leq t \leq (k+1)T} \mathbb{W}_2 \left( \mu_t; \mu_t^{1,N} \right) &\leq e^{-C(\sigma)T} \sup_{(k-1)T \leq t \leq kT} \mathbb{W}_2 \left( \mu_t; \mu_t^{1,N} \right) \\ &+ 2e^{-C(\sigma)kT} \mathbb{W}_2(\mu_0; \mu^\sigma) + \frac{\exp[\psi(T)]}{\sqrt{N}}. \end{aligned} \quad (3.5)$$

We denote  $\lambda_k(T) := \sup_{kT \leq t \leq (k+1)T} \mathbb{W}_2 \left( \mu_t; \mu_t^{1,N} \right)$  and  $\gamma := 2\mathbb{W}_2(\mu_0; \mu^\sigma)$ . The inequality (3.5) can be written in the following way:

$$\lambda_k(T) \leq e^{-C(\sigma)T} \lambda_{k-1}(T) + \gamma e^{-C(\sigma)kT} + \frac{\exp[\psi(T)]}{\sqrt{N}}.$$

**Step 5.** By elementary computations, we have:

$$\lambda_k(T) \leq \frac{1}{1 - e^{-C(\sigma)T}} \frac{\exp[\psi(T)]}{\sqrt{N}} + k\gamma e^{-C(\sigma)kT}.$$

By taking  $T > \frac{1}{C(\sigma)}$ , the sequence  $(ke^{-C(\sigma)kT})_{k \geq 1}$  is decreasing. Therefore, we have  $\sup_{k \geq 0} ke^{-C(\sigma)kT} = e^{-C(\sigma)T}$ . Consequently, we deduce

$$\sup_{t \geq 0} \mathbb{W}_2 \left( \mu_t; \mu_t^{1,N} \right) = \sup_{k \geq 0} \lambda_k(T) \leq \frac{\exp[\psi(T)]}{1 - e^{-C(\sigma)T}} \frac{1}{\sqrt{N}} + \gamma e^{-C(\sigma)T}. \quad (3.6)$$

Let  $\epsilon > 0$  be arbitrarily small. We take  $T > \frac{1}{C} \log \left( \frac{2\gamma}{\epsilon} \right)$  so that  $\gamma e^{-C(\sigma)T} < \frac{\epsilon}{2}$ . Then, by taking  $N$  large enough, we have  $\frac{\exp[\psi(T)]}{1 - e^{-C(\sigma)T}} \frac{1}{\sqrt{N}} < \frac{\epsilon}{2}$ . This implies  $\sup_{t \geq 0} \mathbb{W}_2 \left( \mu_t; \mu_t^{1,N} \right) < \epsilon$  if  $N$  is large enough. This proves Limit (3.2).

**Step 6.** We now prove the rate of convergence result. Let  $\delta > 0$  be arbitrarily small.

**Step 6.1.** We look at the first case. Inequality (3.6) holds for any  $T > 0$ . We take  $T_N := \frac{1}{C(\sigma)} \left( \frac{1}{2(1+1/\lambda)} \log(N) \right)$ . We immediately deduce  $\frac{1}{1 - e^{-C(\sigma)T_N}} \leq 2$  for  $N$  large enough. We thus have:

$$\frac{\exp[\psi(T_N)]}{\sqrt{N}} \frac{1}{1 - e^{-C(\sigma)T_N}} \leq 2 \frac{\exp \left[ \frac{1}{2} \frac{1}{1+1/\lambda} \log(N) \frac{\psi(T_N)}{C(\sigma)T_N} \right]}{\sqrt{N}}.$$

For  $N$  large enough, the quantity  $\frac{\psi(T_N)}{C(\sigma)T_N}$  is less than  $\frac{1}{\lambda} + \delta \left( 1 + \frac{1}{\lambda} \right)$  so that the quantity  $\frac{\exp[\psi(T_N)]}{\sqrt{N}}$  is less than  $N^{-\left( \frac{1}{2(1+1/\lambda)} - \frac{\delta}{2} \right)}$ . We deduce

$$N^{\frac{1}{2(1+1/\lambda)} - \delta} \frac{\exp[\psi(T_N)]}{1 - e^{-\varphi(T_N)}} \frac{1}{\sqrt{N}} \leq N^{-\frac{\delta}{2}} \rightarrow 0,$$

as  $N$  goes to infinity. The second term,  $\gamma e^{-C(\sigma)T_N}$ , is equal to  $\gamma N^{-\frac{1}{2(1+1/\lambda)}}$  so

$$N^{\frac{1}{2(1+1/\lambda)}-\delta} \gamma e^{-C(\sigma)T_N} = \gamma N^{-\delta} \longrightarrow 0,$$

as  $N$  goes to infinity. This achieves the proof of Limit (3.3).

**Step 6.2.** We now look at the second case. Here, we obtain

$$\sup_{t \geq 0} \mathbb{W}_2 \left( \mu_t; \mu_t^{1,N} \right) \leq \frac{N^{-\frac{\delta}{2}}}{1 - e^{-C(\sigma)T_N}} + \gamma e^{-C(\sigma)\psi^{-1}(\frac{1}{2}(1-\delta)\log(N))},$$

by taking  $T_N := \psi^{-1}(\frac{1}{2}(1-\delta)\log(N))$ . This implies Limit (3.4).  $\square$

The second case, that is to say when  $\lim_{t \rightarrow +\infty} \frac{\psi(t)}{t} = 0$ , does not hold with McKean-Vlasov diffusion. However, the current work aims to be applied for more general diffusions.

This uniform propagation of chaos has a consequence on the interacting particle system. Let us denote by  $(\mathcal{P}_t^N)_{t \geq 0}$  the semi-group associated to Diffusion (III) on  $(\mathbb{R}^d)^N$ . Let us precise the norm that we use. For any  $\mathcal{X} = (X_1, \dots, X_N) \in (\mathbb{R}^d)^N$ , we consider

$$\|\mathcal{X}\|^2 := \frac{1}{N} \sum_{i=1}^N \|X_i\|^2.$$

With this norm, we obtain the following result.

**Corollary 3.2.** *We assume that the triples  $(V, F, \mu_0)$  and  $(V, F, \nu_0)$  satisfy the set of Assumptions (A). Then, if  $\sigma \geq \sigma_c$  ( $\sigma_c$  being defined in Theorem 1.7) for any  $N \in \mathbb{N}^*$  and  $t \geq 0$ , we have the inequality:*

$$\mathbb{W}_2(\mu_0^{\otimes N} \mathcal{P}_t^N; \nu_0^{\otimes N} \mathcal{P}_t^N) \leq \gamma e^{-C(\sigma)t} + 2\eta(N),$$

with  $\lim_{N \rightarrow +\infty} \eta(N) = 0$  and  $\gamma := \mathbb{W}_2(\mu_0; \mu^\sigma) + \mathbb{W}_2(\nu_0; \mu^\sigma)$ .

*Proof.* The triangular inequality gives us

$$\begin{aligned} \mathbb{W}_2(\mu_0^{\otimes N} \mathcal{P}_t^N; \nu_0^{\otimes N} \mathcal{P}_t^N) &\leq \mathbb{W}_2(\mu_0^{\otimes N} \mathcal{P}_t^N; \mu_0^{\otimes N} P_t^{\otimes N}) \\ &+ \mathbb{W}_2(\mu_0^{\otimes N} P_t^{\otimes N}; \nu_0^{\otimes N} P_t^{\otimes N}) + \mathbb{W}_2(\nu_0^{\otimes N} P_t^{\otimes N}; \nu_0^{\otimes N} \mathcal{P}_t^N). \end{aligned}$$

According to the definition of the norm that we use in  $(\mathbb{R}^d)^N$ , for any probability measure on  $\mathbb{R}^d$ ,  $\mu$  and  $\nu$ , we have

$$\mathbb{W}_2^2(\mu^{\otimes N}; \nu^{\otimes N}) = \inf \frac{1}{N} \sum_{i=1}^N \mathbb{E} \left\{ \|X_i - Y_i\|^2 \right\},$$

the infimum running over the random variables  $X_1, \dots, X_N$  and  $Y_1, \dots, Y_N$  such that  $\mathcal{L}(X_i) = \mu$ ,  $\mathcal{L}(Y_i) = \nu$  and  $X_i$  (respectively  $Y_i$ ) is independent from  $X_j$  (respectively  $Y_j$ ) for any  $1 \leq i \neq j \leq N$ . We deduce

$$\mathbb{W}_2^2(\mu^{\otimes N}; \nu^{\otimes N}) = \mathbb{W}_2^2(\mu; \nu).$$

According to the definition of the semi-group  $(P_t)_{t \geq 0}$ , we have

$$\mathbb{W}_2(\mu_0^{\otimes N} P_t^{\otimes N}; \nu_0^{\otimes N} P_t^{\otimes N}) = \mathbb{W}_2(\mu_t^{\otimes N}; \nu_t^{\otimes N}) = \mathbb{W}_2(\mu_t; \nu_t).$$

Then, we apply Theorem 1.7 and we obtain

$$\begin{aligned} \mathbb{W}_2(\mu_0^{\otimes N} P_t^{\otimes N}; \nu_0^{\otimes N} P_t^{\otimes N}) &\leq \mathbb{W}_2(\mu_t; \mu^\sigma) + \mathbb{W}_2(\nu_t; \mu^\sigma) \\ &\leq e^{-C(\sigma)t} \{ \mathbb{W}_2(\mu_0; \mu^\sigma) + \mathbb{W}_2(\nu_0; \mu^\sigma) \}. \end{aligned}$$

Finally, we apply Theorem 3.1 to achieve the proof.  $\square$

We remind the reader that in [CGM08, Theorem 3.2], the authors obtain a uniform propagation of chaos of the form

$$\sup_{t \geq 0} \mathbb{E} \left\{ \|X_t - X_t^1\|^2 \right\} \leq \frac{K}{N^{-(1-\rho)}},$$

with  $0 < \rho < 1$ . However, by using a method similar to the one of the proof of Theorem 3.1, we obtain a better inequality with the Wasserstein distance.

**Corollary 3.3.** *Let us assume that  $V$ ,  $F$  and  $\mu_0$  satisfy the set of Assumptions (A) and that  $\max\{\alpha; \vartheta\} \leq 0$ . For any  $\sigma > 0$ , we have the following uniform propagation of chaos result:*

$$\lim_{N \rightarrow \infty} N^{1-\delta} \sup_{t \geq 0} \mathbb{W}_2^2(\mu_t; \mu_t^{1,N}) = 0$$

for any  $0 < \delta < 1$ .

*Proof.* By proceeding exactly like in [BRTV98, Lemma 5.4], there exists  $K > 0$  such that the following inequality holds:

$$\sup_{0 \leq t \leq T} \mathbb{E} \left\{ \|X_t - X_t^1\|^2 \right\} \leq \frac{KT^2}{N} \quad (3.7)$$

for any  $T > 0$ . Here, there are two differences with the proof in [BRTV98]. First, here, there is the presence of a confinement potential but since this potential is convex, we can proceed similarly. And, in [BRTV98], the initial random variables are assumed to be independent. However, we need here to relax this independence hypothesis (like in the proof of Theorem 3.1). We use the same technic than the one in Proposition 2.2 by conditioning with respect to the initial random variables and we have the result. Inequality (3.7) implies

$$\sup_{0 \leq t \leq T} \mathbb{W}_2(\mu_t; \nu_t^N) \leq \frac{\exp[\psi(T)]}{\sqrt{N}}$$



with  $\psi(T) := \frac{1}{2} \log(K) + \log(T)$ .

Now, since  $\alpha \leq 0$  and  $\vartheta \leq 0$ , any invariant probability  $\mu^\sigma$  satisfies a  $WJ_{V,F}$ -inequality with a constant  $C(\sigma) > 0$ . Consequently, we have

$$\lim_{t \rightarrow +\infty} \frac{C(\sigma)t}{\psi(t)} = +\infty.$$

We apply Theorem 3.1 and we obtain the statement for any  $\delta > 0$ .  $\square$

The rate that we obtain is better. However, the coupling that we proceed is not necessarily with the same Brownian motions. Indeed, we do not obtain a coupling between  $X$  and  $X^1$ .

In Theorem 3.1, we remark that the stationary measure plays a particular role in the WJ-inequality. Consequently, we can obtain a better result if we start from  $\mu_0 = \mu^\sigma$ , the unique invariant probability.

**Corollary 3.4.** *We assume that the potentials  $V$  and  $F$  satisfy the hypotheses (A-1)–(A-7). If  $\sigma \geq \sigma_c$  ( $\sigma_c$  being defined in Theorem 1.7), we have the following uniform propagation of chaos result if we start from  $\mu_0 = \mu^\sigma$ :*

$$\sup_{t \geq 0} \mathbb{W}_2 \left( \mu_t ; \mu_t^{1,N} \right) = \sup_{t \geq 0} \mathbb{W}_2 \left( \mu^\sigma ; \mu_t^{1,N} \right) \leq \frac{K}{N},$$

$K$  being a positive constant.

*Proof.* Since the invariant probability  $\mu^\sigma$  satisfies Hypotheses (A-8) and (A-9), we can apply Theorem 1.7. However, in the proof of Theorem 1.7, we obtained Inequality (3.6):

$$\sup_{t \geq 0} \mathbb{W}_2 \left( \mu_t ; \mu_t^{1,N} \right) \leq \frac{\exp[\psi(T)]}{1 - e^{-CT}} \frac{1}{N} + \gamma e^{-CT},$$

with  $\gamma = 2\mathbb{W}_2(\mu_0 ; \mu^\sigma) = 0$ . Therefore, we immediately obtain

$$\sup_{t \geq 0} \mathbb{W}_2 \left( \mu^\sigma ; \mu_t^{1,N} \right) \leq \frac{\exp[\psi(T)]}{1 - e^{-CT}} \frac{1}{N},$$

for any  $T > 0$ .  $\square$

We stress that the contraction which holds in Theorem 1.7 links the measure  $\mu_t$  with the invariant probability  $\mu^\sigma$ . But we do not have any contraction between  $\mu_0 P_t$  and  $\nu_0 P_t$ . Let us note that if we had the contraction inequality

$$\mathbb{W}_2(\mu_t ; \nu_t) \leq e^{-Ct} \mathbb{W}_2(\mu_0 ; \nu_0),$$

we would have the following uniform propagation of chaos

$$\sup_{t \geq 0} \mathbb{W}_2 \left( \mu_t ; \mu_t^{1,N} \right) \leq \frac{K}{N},$$

$K$  being a positive constant. However, to obtain such an inequality requires the strictly uniform convexity of  $V$ , see [SvR05]. And, in this case, we directly have the uniform propagation of chaos without using the convergence.

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