

Discrete Duality Finite Volume Method Applications to Linear Elasticity

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Abstract

In this study, we address the DDFV implementation and theory for solving linear elasticity problem in 3d. Let us recall that the principle of the DDFV discretization consists in integrating the system both over a given primal mesh and a dual mesh built from the primal one and in computing discrete operators that are in duality in a discrete sense.

Why FV for Linear Elasticity ?

The finite volume method is extensively used in CFD and the finite element method for solid mechanics problem. Howeverer there is a multitude of physical problems combining fluid and solid mechanics where finite volume methods appear to be a pertinent alternative : fluid-structure interactions, deformation of geomechanical reservoir, frost heave problems.

Primal, Dual, Diamond meshes : $\Omega = \bigcup_{\mathbf{K} \in \mathfrak{M}} \mathbf{K} = \frac{1}{2} \bigcup_{\mathbf{K}^* \in \mathfrak{M}^*} \mathbf{K}^* = \bigcup_{\mathbf{D} \in \mathfrak{D}} \mathbf{D}$.



Discrete divergence on dual and primal cells

$$\mathbf{div}^{\mathsf{K}}\xi^{\mathfrak{D}} = \frac{1}{|\mathsf{K}|} \sum_{s \in \partial \mathsf{K}} \xi^{\mathsf{D}_{\mathsf{K}s}} \mathbf{N}_{\mathsf{K}s}$$
$$\mathbf{div}^{\mathsf{K}^*}\xi^{\mathfrak{D}} = \frac{1}{|\mathsf{K}^*|} \sum_{s^* \in \partial \mathsf{K}^*} \xi^{\mathsf{D}_{\mathsf{K}s}} \mathbf{N}_{\mathsf{K}^*s^*}$$
$$\mathbf{N}_{\mathsf{K}^*s^*} = \begin{cases} \mathbf{N}_{i+1}^{\mathsf{K}s} - \mathbf{N}_{i-1}^{\mathsf{K}s} & s \in \mathfrak{D}^i \\ \mathbf{N}_{i+1}^{\mathsf{K}s} - \mathbf{N}_{i-1}^{\mathsf{K}s} + \mathbf{N}_{s,i-1,i} + \mathbf{N}_{s,i,i+1} & s \in \mathfrak{D}^{b_N} \end{cases}$$
with $x_{\mathsf{K}^*} = x_i^{\mathsf{K}s} \in \mathsf{K}$, $s \in \partial \mathsf{K}$ and $x_{\mathsf{K}^*} \in s$.
Discrete stress tensor $\sigma^{\mathfrak{D}} = (\sigma^{\mathsf{D}_{\mathsf{K}s}})_{\mathsf{K}\in\mathfrak{M},s\in\partial\mathsf{K}}$
$$\sigma^{\mathsf{D}_{\mathsf{K}s}}(\mathbf{u}^{\tau}) = \lambda_{\mathsf{K}}(\nabla^{\mathsf{D}_{\mathsf{K}s}}\mathbf{u}^{\tau} : \mathsf{Id})\mathsf{Id} + 2\mu_{\mathsf{K}}\frac{\nabla^{\mathsf{D}_{\mathsf{K}s}}\mathbf{u}^{\tau} + (\nabla^{\mathsf{D}_{\mathsf{K}s}}\mathbf{u}^{\tau})^{T}}{2}$$
DDFV f
The scheme for the numerical resolution is $\mathsf{K}(\mathcal{D}^{\mathsf{D}_{\mathsf{K}}}, \mathfrak{T}) = \mathfrak{K} \wedge \mathcal{W} = \mathfrak{M}$

Discrete gradient on half diamond cell D_{Ks}

Continuous problem Find the displacement $\mathbf{u} \in \mathbb{R}^3$

$$-\operatorname{div} \boldsymbol{\sigma}(\mathbf{u}) = \mathbf{f} \text{ on } \Omega, \ \mathbf{u} = \mathbf{g} \text{ on } \Gamma_D, \ \boldsymbol{\sigma}(\mathbf{u}) \cdot \mathbf{n} = \mathbf{h} \text{ on } \Gamma_N$$
$$Hooke relation \\ \boldsymbol{\sigma}(\mathbf{u}) = \lambda \mathbb{D}i\mathbf{v}\mathbf{u} + 2\mu \mathbb{D}\mathbf{u} \quad \text{with} \quad \mathbb{D}\mathbf{u} = \frac{\nabla \mathbf{u} + (\nabla \mathbf{u})^T}{2}$$
$$Scalar product$$

$$\llbracket \mathbf{u}^{\tau}, \mathbf{v}^{\tau} \rrbracket_{\tau} = \frac{1}{3} \sum_{\mathsf{K} \in \mathfrak{M}} |\mathsf{K}| \mathbf{u}^{\mathsf{K}} \cdot \mathbf{v}^{\mathsf{K}} + \frac{1}{3} \sum_{\mathsf{K}^{*} \in \mathfrak{M}^{*}} |\mathsf{K}^{*}| \mathbf{u}^{\mathsf{K}^{*}} \cdot \mathbf{v}^{\mathsf{K}^{*}}$$
$$\left(\xi^{\mathfrak{D}} : \psi^{\mathfrak{D}} \right)_{\mathfrak{D}} = \sum_{\mathsf{D}_{\mathsf{K}s} \in \mathfrak{D}} |\mathsf{D}_{\mathsf{K}s}| \xi^{\mathsf{D}_{\mathsf{K}s}} : \psi^{\mathsf{D}_{\mathsf{K}s}}$$

Discrete Green formula

For any $\mathbf{u}^{\tau} = (\mathbf{u}_{\mathsf{K}}, \mathbf{u}_{\mathsf{K}^*})$ and any $\xi^{\mathfrak{D}} = (\xi^{\mathsf{D}_{\mathsf{K}s}})_{\mathsf{K} \in \mathfrak{M}, s \in \partial \mathsf{K}}$ /

$$\nabla^{\mathsf{D}_{\mathsf{K}s}}\mathbf{u}^{\tau} = \frac{1}{3|\mathsf{D}_{\mathsf{K}s}|} \left((\mathbf{u}_s - \mathbf{u}_{\mathsf{K}}) \otimes \mathbf{N}_{\mathsf{K}s} + \sum_{i=1}^{3} \mathbf{u}_i \otimes (\mathbf{N}_{i-1}^{\mathsf{K}s} - \mathbf{N}_{i+1}^{\mathsf{K}s}) \right)$$

 $(\mathbf{u}_i)_{i=1}^d$ are the vertices of the face s

Flux continuity for stress tensor

 \mathbf{u}_s are eliminated by writing the continuity of the flux and by boundary conditions:

$$\begin{array}{ll} \sigma^{\mathsf{D}_{\mathsf{K}s}}(\mathbf{u}^{\tau})\mathbf{N}_{\mathsf{K}s} = -\sigma^{\mathsf{D}_{\mathsf{L}s}}(\mathbf{u}^{\tau})\mathbf{N}_{\mathsf{L}s} & \forall s = \mathsf{K} \cap \mathsf{L} \in \mathfrak{D}^{i} \\ \sigma^{\mathsf{D}_{\mathsf{K}s}}(\mathbf{u}^{\tau})\mathbf{N}_{\mathsf{K}s} = \mathbf{h}(\mathbf{x}_{s}) & \forall s \in \mathfrak{D}^{b_{N}} \\ \mathbf{u}_{s} = \mathbf{g}(\mathbf{x}_{s}) & \forall s \in \mathfrak{D}^{b_{D}} \end{array}$$

formulation

on consists in finding $\mathbf{u}^{ au} \in (\mathbb{R}^3)^{ au}$ such that:

 $\mathbf{u}_{\mathsf{K}} = \mathbf{g}(\mathbf{x}_{\mathsf{K}}), \ \forall \mathsf{K} \in \partial \mathfrak{M}^{b_D} \qquad \sigma^{\mathfrak{D}}(\mathbf{u}^{\tau})\mathbf{n}_{\mathsf{K}s} = \mathbf{h}(\mathbf{x}_{\mathsf{K}}), \ \forall \mathsf{K} \in \partial \mathfrak{M}^{b_N}$ $-\operatorname{div}^{\mathsf{h}}(\sigma^{\mathfrak{V}}(\mathbf{u}^{\tau})) = \mathbf{f}^{\mathsf{h}} \quad \forall \mathsf{K} \in \mathfrak{M}$ $-\mathbf{div}^{\mathsf{K}^*}(\sigma^{\mathfrak{D}}(\mathbf{u}^{\tau})) = \mathbf{f}^{\mathsf{K}^*} \; \forall \mathsf{K}^* \in \mathfrak{M}^{*i} \cup \mathfrak{M}^{b_N}$ $\mathbf{u}_{\mathsf{K}^*} = \mathbf{g}(\mathbf{x}_{\mathsf{K}^*}), \, \forall \mathsf{K}^* \in \mathfrak{M}^{*b_D}$

Example 1: 3D cube

We impose an embedding condition on left, front and bottom faces (Γ) and a given traction otherwise such that

$$\mathbf{f} = 10^{-2} \begin{cases} y + 2z \\ x - z \\ 2x - y \end{cases} \quad \mathbf{u} = 10^{-2} \begin{cases} xyz \\ -2xyz \\ -xyz \end{cases}$$

Example 2 : 3D two materials structure

Left embedding $(\mathbf{\Gamma}_1)$ and imposed displacement on the right (Γ_2) : $\mathbf{u} = (1, 0, 0)$. Free traction otherwise.



$$\begin{split} \xi^{\mathsf{D}_{\mathsf{K}s}}\mathbf{N}_{\mathsf{K}s} &= -\xi^{\mathsf{D}_{\mathsf{L}s}}\mathbf{N}_{\mathsf{L}s} \\ \llbracket \mathsf{div}^{\tau}\xi^{\mathfrak{D}}, \mathbf{u}^{\tau} \rrbracket_{\tau} &= -\left(\xi^{\mathfrak{D}}: \nabla^{\mathfrak{D}}\mathbf{u}^{\tau}\right)_{\mathfrak{D}} + <\xi^{\mathfrak{D}}\cdot\vec{\mathbf{n}}, \gamma^{\tau}(\mathbf{u}^{\tau}) >_{\Gamma} \\ \text{where the trace operator } \gamma^{\tau} \text{ is defined by:} \end{split}$$

$$\gamma_s(\mathbf{u}^{\tau}) = \frac{1}{3}\mathbf{u}_s + \frac{2}{9}\sum_{i=1}^3 \mathbf{u}_i \qquad \forall s \in \partial \mathfrak{M}$$

Discrete Korn inequality

For any $\mathbf{u}^{\tau} \in (\mathbb{R}^3)^{\tau}$ such that $\mathbf{u}_{\mathsf{K}} = 0, \ \forall \mathsf{K} \in \partial \mathfrak{M}$ and $\mathbf{u}_{\mathsf{K}^*} = 0, \ \forall \mathsf{K}^* \in \partial \mathfrak{M}^*$, there is a constant $C(\Omega)$ such that

 $\|\nabla^{\mathfrak{D}}\mathbf{u}^{\tau}\|_{2} \leqslant C(\Omega) \|\mathbb{D}^{\mathfrak{D}}\mathbf{u}^{\tau}\|_{2}$

Theoretical results

- Under mesh regularity assumptions
- Existence :
- The DDFV scheme yields to a sym. pos. def. system of linear equations and admits exactly one solution.

• Stability :

Let $\mathbf{u}^{\tau} \in (\mathbb{R}^3)^{\tau}$ be the solution of the discrete pb (1), then there exists a constant $C(\Omega)$ such that

 $\frac{\mu}{2} \mid\mid \nabla^{\mathfrak{D}} \mathbf{u}^{\tau} \mid\mid_{2}^{2} + \frac{\lambda}{3} \mid\mid \mathbb{D}\mathsf{i} \mathbf{v}^{\mathfrak{D}} \mathbf{u}^{\tau} \mid\mid_{2}^{2} \leqslant C(\Omega) \mid\mid \mathbf{f}^{\tau} \mid\mid_{2}^{2}$









O DDFV Linf 0.9

DDFV L1 1.38

DDFV L2 1.43





Initial mesh



DDFV solution



FE solution



Convergence : There exists a constant $C(\Omega)$ such that $|| \mathbf{u} - \mathbf{u}^{\tau} ||_2 + || \nabla \mathbf{u} - \nabla^{\mathfrak{D}} \mathbf{u}^{\tau} ||_2 \leq C(\Omega) \operatorname{size}(\mathfrak{M})$

Why DDFV ?

We have to construct a discrete stress tensor which satisfies transmission conditions on each interface. One idea is to compute it on diamond cells [2]. However, in case of non orthogonal meshes, vertices unknowns are eliminated using a suitable interpolation on cell-center unknowns, breaking symmetry. DDFV scheme introduces a dual mesh which leads to consider vertex values as unknowns, so that symmetry is guaranteed.



10⁻²

Conclusion / Open questions / References

The DDFV discretization method has been introduced for linear elasticity using both cell and vertex centers values as unknowns. This method satisfies compatibility with discontinuities at the interfaces of the mesh : discrete gradients are definied on half diamond cells and stress flux continuities are imposed to eliminate auxiliary unknowns on faces. However, there are still open problems : proof of discrete Korn inequality in case of mixed boundary conditions, proof of convergence in case of Lamé coefficients constant on half diamond cells.

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