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# ANALYTICAL PARAMETERIZATION OF ROTORS AND PROOF OF A GOLDBERG CONJECTURE BY OPTIMAL CONTROL THEORY\*

TÉRENCE BAYEN†

**Abstract.** Curves which can be rotated freely in an  $n$ -gon (that is, an regular polygon with  $n$  sides) so that they always remain in contact with every side of the  $n$ -gon are called rotors. Using optimal control theory, we prove that the rotor with minimal area consists of a finite union of arcs of circles. Moreover, the radii of these arcs are exactly the distances of the diagonals of the  $n$ -gon from the parallel sides. Finally, using the extension of Noether’s theorem to optimal control (as performed in [D. F. M. Torres, *WSEAS Trans. Math.*, 3 (2004), pp. 620–624]), we show that a minimizer is necessarily a regular rotor, which proves a conjecture formulated in 1957 by Goldberg (see [M. Golberg, *Amer. Math. Monthly*, 64 (1957), pp. 71–78]).

**Key words.** shape optimization, convexity, constant width bodies, rotors, support function, optimal control, Pontryagin maximum principle, switching point, bang control, Noether theorem

**AMS subject classifications.** 49J15, 49Q10, 78M50, 80M50

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**1. Introduction.** In this paper, we investigate properties of *rotors*, that is, convex curves that can be freely rotated inside a regular polygon  $P_n$  with  $n$  sides,  $n \geq 3$ , while remaining in contact with every side of  $P_n$ . When  $n = 4$ ,  $P_4$  is a square of side  $\alpha$ , and a rotor of  $P_4$  is called a curve of constant width  $\alpha$  or an orbiform. When  $n = 3$ ,  $P_3$  is an equilateral triangle, and a rotor of  $P_3$  is called a  $\Delta$ -curve. There are infinitely many such curves besides the circle (see section 2).

Orbiforms have been studied geometrically since the 19th century (see [5], [24], [26], [29], [36]). In particular, Reuleaux’s name is attached to those orbiforms obtained by intersecting a finite number of discs of equal radii  $\alpha$ . The Reuleaux triangle is the most famous of these orbiforms: it consists of the intersection of three circles of radius one and whose centers are on the vertices of an equilateral triangle of side one. Orbiforms have many interesting properties and applications in mechanics (see [5], [6], [7], [25], [26], [27], [36]). For example, Reuleaux triangles are used in boring square holes, and they are also part of the Wankel engine used by Japan’s Mazda cars. Nowadays, the study of rotors is potentially interesting in mechanics for the design of engines or propellers, for example, in the Navy.

An interesting shape optimization problem consists in determining the convex body maximizing or minimizing the area in the class of rotors. It is easy to show that the disc always has maximal area in this class. This is a consequence of the isoperimetric inequality, as all rotors have the same perimeter (see Barbier’s theorem in section 2.3). The question of finding a rotor of least area is more difficult. First, notice that the problem of minimizing the area is well posed, as rotors are convex bodies (see section 2.2). This question has been solved for  $n = 4$  (that is, in the case of orbiforms) by Blaschke using the mixed-volume (see [5]) and Lebesgue (see [24]) measures. They show that the Reuleaux triangle has the least area in the class of constant width bodies of  $\mathbb{R}^2$ . Fujiwara has given the first analytic proof of this result

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(see [12], [13]). More recently, Harrell gave a modern proof using minimization under constraints (see [20]). The study of these problems in  $\mathbb{R}^2$  is useful for extensions in  $\mathbb{R}^3$  and in the domain of spectral analysis. For example, the problem of finding a constant width body of minimal volume in  $\mathbb{R}^3$  has recently been investigated (see [4], [22]). The optimization of eigenvalues with respect to the domain  $\Omega$  is also an intense field of research (see [21] for an overview of many spectral problems involving convexity). These questions require a careful study of dimension 2.

The  $\Delta$ -curves have many similar geometrical properties to the orbiforms (see [7], [36]). Fujiwara gave an analytic proof in [12] that, among all  $\Delta$ -curves inscribed in an equilateral triangle of side one, the one of minimal area is the  $\Delta$ -biangle or lens. It consists of two circular arcs of radius  $\frac{\sqrt{3}}{2}$  and of length  $\frac{\pi}{3}$ . This result was also established by Blaschke and later by Weissbach (see [35]).

Whereas the cases  $n = 3$  and  $n = 4$  have been investigated, the question of finding the rotor of least area for  $n \geq 5$  is open. Standard geometrical proofs cannot be applied in this case (see [14]). In [17] and [18], Goldberg constructs a family of “trammel” rotors in a regular polygon,  $(O_n^{ln\pm 1})_{l \in \mathbb{N}^*}$ , that have  $2(ln \pm 1)$  symmetries, and he conjectured in [17] that the minimizer is a rotor called  $O_n^{n-1}$  obtained for  $l = 1$ . The boundary of a rotor  $O_n^{ln\pm 1}$  consists of a finite union of arcs of circles of different radii  $r_i$  and of equal sectors (see section 2.6). The values  $r_i$  are exactly the distances of the diagonals of the  $n$ -gon from the parallel sides. In this class,  $O_n^{n-1}$  has the minimum number of arcs. An analytic description of these regular rotors is given in [11] by Focke. In 1975, Klötzler made an analytic study of the minimization problem using optimal control theory (see [2], [3], [23]). He showed in [23] that a minimizer consists of a union of arcs of circles of radii  $r_i$ , but he failed to prove that a minimizer is in the class  $(O_n^{ln\pm 1})_{l \in \mathbb{N}^*}$ . His idea consists in reformulating the initial minimization problem into an optimal control problem by choosing the radius of curvature as the control variable. Unfortunately, he seems to prove that the regular rotors  $O_n^{ln\pm 1}$  are local minimizers of the area in the subclass  $\mathcal{R}_n^{ln\pm 1}$  of rotors having the same number of arcs and the same radii of curvature. This result contradicts the one of Firey (see [10]) in the case  $n = 4$ : the author shows that regular Reuleaux polygons with  $N$  sides,  $N \geq 5$ , maximize the area in the class of Reuleaux polygons with the same number of sides. Moreover, in [2], the author performs only convex perturbations of a regular rotor  $O_n^{ln\pm 1}$ . This kind of perturbation increases the area by the concavity of the functional (the Brunn–Minkowski theorem; see [8]). The main difficulty is to consider nonconvex perturbations of those rotors which are not obtained by a strictly convex combination of two rotors.

The aim of the paper is to prove the following theorem conjectured by Goldberg in 1957 (see [17]).

**THEOREM 1.1.** *Among all rotors of a regular polygon  $P_n$  ( $n \geq 3$ ), the one of minimal area is the regular rotor  $O_n^{n-1}$ .*

In section 2, we give an analytic parameterization of a rotor using the support function of a convex body (see [6] or [30] for an overview of the properties of the support function). In section 3, we formulate the minimization problem into an optimal control problem which is similar to the one obtained by Klötzler (see [23]). Indeed, the convexity constraints enable us to choose the radius of curvature of the boundary of a rotor as the control variable. Thanks to this new parameterization, the initial shape optimization is well posed. By the Pontryagin maximum principle (PMP), we show that the extremal trajectories are “bang-bang,” and we determine the corresponding number of switching points. We thus restrict the class of extremal trajectories step by step. Whereas the computation of the extremal trajectories performed by Klötzler is

incomplete (he does not show that the switching points of an extremal trajectory are equidistant), we prove, in section 4, Theorem 1.1 by using an extension of Noether's theorem to optimal control theory provided in [31]. We compute conserved quantities along an extremal trajectory, and thus we can characterize the switching points of an extremal (see section 4). This shows that the rotors corresponding to the extremal trajectories belong to the class  $(O_n^{ln\pm 1})_{l \in \mathbb{N}^*}$ . We then conclude the proof of Goldberg's conjecture by Proposition 2.11. Note that by this proposition, there is no need to examine the optimality of extremal trajectories.

## 2. Construction of a rotor.

**2.1. Support function of a convex body.** A body or a domain in  $\mathbb{R}^N$ ,  $N \geq 2$ , is a nonempty compact connected subset of  $\mathbb{R}^N$ . Let  $K$  be a convex body. The support function of  $K$  is defined as the map  $h_K : \mathbb{R}^N \setminus \{0\} \rightarrow \mathbb{R}$  with

$$h_K(\nu) := \max_{x \in K} x \cdot \nu, \quad \nu \in \mathbb{R}^N \setminus \{0\}.$$

The support function is clearly homogeneous of degree 1. A convex body is uniquely determined by its support function (see [6, p. 29] or [22]). Let  $K$  be a convex body of nonempty interior and assume that the origin is inside  $K$ . Recall that, for a convex body, a hyperplane  $H$  is a hyperplane of support for  $K$  if there exists  $x \in K \cap H$  such that  $K$  is included in one of the half-spaces defined by  $H$ . If  $\nu$  belongs to  $S^{N-1}$ ,  $h_K(\nu)$  can be interpreted as the distance from the origin to the support hyperplane of  $K$  with normal vector  $\nu$  (see Figure 1). The support function is nonnegative if and only if the origin is inside  $K$ . The next proposition characterizes the degree of regularity of the support function (see [6, p. 28] or [30]).

**PROPOSITION 2.1.** *Let  $K$  be a convex body of  $\mathbb{R}^N$  and  $h_K$  its support function. Then  $h_K$  is of class  $C^1$  if and only if  $K$  is strictly convex.*

From now on, we consider convex bodies in dimension 2. The support function of a convex body  $K$  of  $\mathbb{R}^2$  will be denoted by  $p_K(\theta) := h_K(e^{i\theta})$ ,  $\theta \in \mathbb{R}$ , or  $p(\theta)$  to simplify. The function  $p_K$  is  $2\pi$ -periodic. If  $K$  is a convex body, we denote by  $\partial K$  its boundary. Given  $(z_1, z_2) \in \mathbb{C}^2$ , their scalar product in  $\mathbb{R}^2$  will be written indifferently  $\Re(\bar{z}_1 z_2)$  or  $z_1 \cdot z_2$ .

Q2

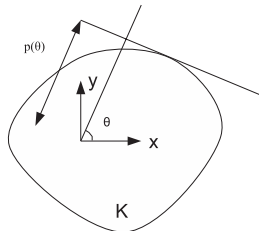


FIG. 1. *The support function of a convex body  $K$  is the distance  $p(\theta)$  between the tangent to  $K$  orthogonal to  $(\cos(\theta), \sin(\theta))$  and the origin.*

**PROPOSITION 2.2.** *Let  $K$  be a strictly convex body and  $p$  its support function. We assume that the boundary of  $K$ ,  $\partial K$ , is Lipschitz. Then  $\partial K$  can be described*

by the equations

$$(2.1) \quad \begin{cases} x(\theta) = p(\theta) \cos(\theta) - \dot{p}(\theta) \sin(\theta), \\ y(\theta) = p(\theta) \sin(\theta) + \dot{p}(\theta) \cos(\theta), \end{cases}$$

where  $\theta \in \mathbb{R}$ .

Q3 *Proof.* Let  $\theta$  be in  $[0, 2\pi]$  and  $u_\theta$  be the vector of coordinates  $(\cos(\theta), \sin(\theta))$ . The support function  $p(\theta)$  is defined by

$$p(\theta) := \max_{x \in K} x \cdot u_\theta,$$

and  $p$  is of class  $C^1$  by strict convexity. As  $K$  is compact, the maximum is reached at some point of coordinates  $(x(\theta), y(\theta))$ , and we have

$$(2.2) \quad x(\theta) \cos(\theta) + y(\theta) \sin(\theta) = p(\theta).$$

As the boundary of  $K$  is Lipschitz, the functions  $(x, y)$  are differentiable a.e. (Rademacher's theorem). Moreover, the vector  $u_\theta$  is orthogonal to the support line given by  $X \cos(\theta) + Y \sin(\theta) = 0$ ; hence, we must have

$$(\dot{x}(\theta), \dot{y}(\theta)) \cdot \vec{u}_\theta = 0.$$

Q4 By derivation of (2.2), we get

$$-x(\theta) \sin(\theta) + y(\theta) \cos(\theta) = \dot{p}(\theta),$$

which gives (2.1).  $\square$

Equation (2.1) can be rewritten as  $z(\theta) := x(\theta) + iy(\theta) = (p(\theta) + i\dot{p}(\theta))e^{i\theta}$ .

In the following, the space  $C^{1,1}$  denotes the set of maps  $p : \mathbb{R} \rightarrow \mathbb{R}$ , of class  $C^1$ , and such that  $\dot{p}$  is locally Lipschitz.

PROPOSITION 2.3. *Let  $K$  be a convex body and  $p$  its support function. We assume that  $p$  is of class  $C^{1,1}$ . Then the radius of curvature  $p + \ddot{p}$  of the boundary  $\partial K$  exists a.e., and, for a.e.  $\theta \in \mathbb{R}$ ,*

$$(2.3) \quad p(\theta) + \ddot{p}(\theta) \geq 0.$$

*Proof.* As  $p$  is of class  $C^{1,1}$ , the functions  $(x(\theta), y(\theta))$  are differentiable a.e., and by standard formulas, the radius of curvature  $f$  of  $\partial K$  is given by  $f = p + \ddot{p}$ . As the body  $K$  is convex,  $f$  must be nonnegative, and consequently we have  $f(\theta) = p(\theta) + \ddot{p}(\theta) \geq 0$  for a.e.  $\theta \in \mathbb{R}$ .  $\square$

If  $K$  is a convex body of support function  $p$  and if  $p$  is of class  $C^{1,1}$ , the tangent vector to  $\partial K$  is given by

$$\dot{z}(\theta) = i(p(\theta) + \dot{p}(\theta))e^{i\theta}.$$

When  $p + \dot{p} = 0$  on a set  $A$  of positive measure, then we have  $\dot{z} = 0$ . Geometrically speaking, this means that the boundary  $\partial K$  has a corner: for  $\theta \in A$ , the point  $z(\theta)$  is stationary. For a given function  $f \in L^\infty(\mathbb{R}, \mathbb{R})$  and  $2\pi$ -periodic, we denote by

$$c_1(f) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{i\theta} d\theta$$

the first Fourier coefficient of  $f$ .

**PROPOSITION 2.4.** *Let  $f \in L^\infty(\mathbb{R}, \mathbb{R})$  be a  $2\pi$ -periodic function. Then any function  $p$  that satisfies  $f = p + \ddot{p}$  is of class  $C^{1,1}$ , and  $p$  is  $2\pi$ -periodic if and only if  $c_1(f) = 0$ .*

*Proof.* Let  $f \in L^\infty(\mathbb{R}, \mathbb{R})$  be a  $2\pi$ -periodic function. A function  $p$  satisfies  $f = p + \ddot{p}$  if and only if there exists  $(a, b) \in \mathbb{R}^2$  such that, for all  $\theta \in \mathbb{R}$ ,

$$(2.4) \quad p(\theta) = \int_0^\theta f(t) \sin(\theta - t) dt + a \cos(\theta) + b \sin(\theta).$$

By (2.4), any function  $p$  that satisfies  $p + \ddot{p} = f$  is of class  $C^{1,1}$ . Moreover, any such function  $p$  is of class  $C^{1,1}$  and is  $2\pi$ -periodic if and only if its restriction on  $[0, 2\pi]$  satisfies  $p(0) = p(2\pi)$ ,  $\dot{p}(0) = \dot{p}(2\pi)$ . But we have

$$\int_0^{2\pi} (p(\theta) + \ddot{p}(\theta)) e^{i\theta} d\theta = \dot{p}(2\pi) - \dot{p}(0) - i(p(2\pi) - p(0)).$$

Hence, any function  $p$  satisfying (2.4) is  $2\pi$ -periodic if and only if  $p(2\pi) = p(0)$  and  $\dot{p}(2\pi) = \dot{p}(0)$ , that is, if and only if  $c_1(f) = 0$ .  $\square$

If we deal with  $f = p + \ddot{p}$  instead of  $p$ , we get an additional condition  $c_1(f) = 0$  which says that the boundary  $\partial K$  given by (2.1) is closed. The next theorem is a consequence of the two previous propositions.

**THEOREM 2.1.** (i) *Let  $K$  be a strictly convex body of  $\mathbb{R}^2$  and  $p$  its support function. If  $p$  is of class  $C^{1,1}$ , then  $p + \ddot{p} \geq 0$ .*

(ii) *Conversely, let  $f \in L^\infty(\mathbb{R}, \mathbb{R})$  be a  $2\pi$ -periodic function such that  $f \geq 0$  and  $c_1(f) = 0$ . If  $p$  is a function satisfying  $f = p + \ddot{p}$ , then  $p$  is of class  $C^{1,1}$ , is  $2\pi$ -periodic (in the sense of  $C^{1,1}$  maps), and is the support function of a strictly convex body.*

Let  $K$  be a strictly convex body. We denote by  $p$  its support function of class  $C^1$  and by  $\mathcal{A}(p)$  its area. By Stokes's formula and by (2.1), we have

$$(2.5) \quad \mathcal{A}(p) = \frac{1}{2} \int_0^{2\pi} (p^2(\theta) - \dot{p}^2(\theta)) d\theta.$$

By integrating by parts, the area becomes

$$(2.6) \quad \mathcal{A}(p) = \frac{1}{2} \int_0^{2\pi} p(\theta) (p(\theta) + \ddot{p}(\theta)) d\theta,$$

which has a sense because  $p + \ddot{p}$  is a positive Radon measure, and (2.6) can be interpreted as the product of a positive Radon measure and a continuous function. In the next section, we show that the support function of a rotor is of class  $C^{1,1}$ , and (2.6) is clearly defined in that case.

**2.2. Construction of a rotor by its support function.** In this section, we recall classical definitions and properties of rotors (see [6], [19], [36]). Let  $K$  be a convex domain and  $P$  be a convex polygon.  $P$  will be called a *tangential polygon* of  $K$  and  $K$  an *osculating domain* in  $P$  if  $K \subset P$  and every side of  $P$  has a nonempty intersection with  $K$  (see [19]). We say that a polygon  $P$  is *equiangular* if all of its interior angles at the vertices are equal. We say that a convex polygon  $P$  is an  *$n$ -gon* if it is a regular polygon with  $n$  sides,  $n \geq 3$ .

**DEFINITION 2.1.** *A convex domain  $K$  will be called a rotor in a polygon  $Q$  if, for every rotation  $\rho$ , there exists a translation vector  $p_\rho$  such that  $\rho K + p_\rho$  is an osculating domain in  $K$ .*

In the following, we assume that  $Q$  is a regular polygon with  $n \geq 3$  sides; that is, we consider only rotors of a regular polygon. Hence,  $K$  is a rotor in a regular  $n$ -gon  $Q$  if and only if all tangential equiangular  $n$ -gons are regular and have equal perimeters. A rotor of an  $n$ -gon  $P_n$  has the property to rotate inside  $P_n$  while remaining in contact with all sides of  $P_n$ . The disc is the most simple example of a rotor. A rotor is a strictly convex domain (see [19], [36]). Consequently, the support function of a rotor is of class  $C^1$ .

**Q5** Let  $r$  be the radius of the inscribed circle of the  $n$ -gon  $P_n$  and let  $\delta := \frac{2\pi}{n}$ . We give in the following theorem an analytic description of a rotor which will be used in the rest of the paper.

**THEOREM 2.2.** (i) *Let  $K$  be a rotor and  $p$  its support function. Then  $p$  satisfies*

$$(2.7) \quad p(\theta) - 2 \cos(\delta)p(\theta + \delta) + p(\theta + 2\delta) = 4r \sin^2\left(\frac{\delta}{2}\right) \quad \forall \theta \in [0, 2\pi].$$

Moreover,  $p$  is of class  $C^{1,1}$  and satisfies (2.3).

(ii) *Conversely, let  $p$  be a  $2\pi$ -periodic function of class  $C^{1,1}$ . Assume that  $p$  satisfies (2.3) and (2.7). Then  $p$  is the support function of a rotor  $K$ .*

The characterization of a rotor by (2.7) is well known (see [7], [11], [23]), but we show in particular that the support function of a rotor is actually of class  $C^{1,1}$ . Before doing the proof of the theorem, we set some notation:

$$(2.8) \quad S_n(p) := p(\theta) - 2 \cos(\delta)p(\theta + \delta) + p(\theta + 2\delta)$$

and

$$(2.9) \quad C_n := 4r \sin^2\left(\frac{\delta}{2}\right).$$

*Proof of (i).* We refer the reader to Chapter 8 of [36] for the following geometric property. By definition of a rotor, the tangents to  $\partial K$  at each contact point are the sides of the  $n$ -gon. Hence, the perpendiculars to these paths at their contact points meet in a point which is the instantaneous center of rotation of the body. A simple computation yields (2.7). We now prove that  $p$  is of class  $C^{1,1}$ . First, we have

$$(2.10) \quad \sum_{0 \leq k \leq n-1} p(\theta + k\delta) = nr \quad \forall \theta \in \mathbb{R}.$$

Indeed, by writing (2.7) at points  $\theta, \theta + \delta, \dots, \theta + (n-1)\delta$  and adding all of these equalities, we get (2.10). As  $K$  is strictly convex, its support function  $p$  is of class  $C^1$ . We now show that  $p$  satisfies the inequality

$$(2.11) \quad (\dot{p}(\theta') - \dot{p}(\theta)) \sin(\theta - \theta') \leq (p(\theta) + p(\theta')) (1 - \cos(\theta - \theta')) \quad \forall (\theta, \theta') \in [0, 2\pi].$$

By definition of the support function, we have, for all  $(\theta, \theta') \in [0, 2\pi]$ ,

$$(x(\theta'), y(\theta')) \cdot (\cos(\theta), \sin(\theta)) \leq p(\theta).$$

Taking into account (2.1), we get

$$\dot{p}(\theta') \sin(\theta - \theta') \leq p(\theta) - p(\theta') \cos(\theta' - \theta).$$

If we permute  $\theta$  and  $\theta'$ , we obtain

$$\dot{p}(\theta) \sin(\theta' - \theta) \leq p(\theta') - p(\theta) \cos(\theta' - \theta).$$

Adding the last two inequalities yields (2.11). We now write (2.11) at the points  $\theta + k\delta$  and  $\theta' + k\delta$ ,  $0 \leq k \leq n-1$ . We get, for all  $(\theta, \theta') \in [0, 2\pi]$  and  $0 \leq k \leq n-1$ ,

(2.12)

$$(\dot{p}(\theta' + k\delta) - \dot{p}(\theta + k\delta)) \sin(\theta - \theta') \leq (p(\theta + k\delta) + p(\theta' + k\delta))(1 - \cos(\theta - \theta')).$$

By (2.10), we obtain, for all  $(\theta, \theta') \in [0, 2\pi]$ ,

$$(2.13) \quad \sum_{1 \leq k \leq n-1} p(\theta + k\delta) = nr - p(\theta)$$

and

$$(2.14) \quad \sum_{1 \leq k \leq n-1} \dot{p}(\theta + k\delta) = -\dot{p}(\theta).$$

Combining (2.12), (2.13), and (2.14), we obtain

$$(-\dot{p}(\theta') + \dot{p}(\theta)) \sin(\theta - \theta') \leq (2nr - p(\theta) - p(\theta'))(1 - \cos(\theta - \theta')).$$

Therefore, by (2.11) and the previous inequality, we get, for all  $(\theta, \theta') \in [0, 2\pi]$ ,

$$|(\dot{p}(\theta') - \dot{p}(\theta)) \sin(\theta - \theta')| \leq 2nr \sin^2\left(\frac{\theta - \theta'}{2}\right).$$

Consequently,  $\dot{p}$  satisfies the inequality

$$|\dot{p}(\theta') - \dot{p}(\theta)| \leq 2nr \left| \tan\left(\frac{\theta - \theta'}{2}\right) \right|$$

for all  $(\theta, \theta') \in [0, 2\pi]$  such that  $|\theta - \theta'| \notin \{0, \pi, 2\pi\}$ . This inequality proves that  $\dot{p}$  is Lipschitz, and thus  $p$  is of class  $C^{1,1}$ . As  $K$  is convex and  $p$  is of class  $C^{1,1}$ , it satisfies (2.3). This concludes the proof of (i).

*Proof of (ii).* Let us assume that conditions (2.3) and (2.7) are satisfied. As  $p$  is of class  $C^{1,1}$ , is  $2\pi$ -periodic, and satisfies (2.3), it is the support function of a strictly convex body  $K$ . A straightforward computation using (2.7) shows that an osculating polygon to  $K$  is equiangular; consequently,  $K$  is a rotor.  $\square$

An example of a function  $p$  satisfying (2.7) is given by

$$(2.15) \quad p(\theta) = 1 + \frac{1}{1 - (ln - 1)^2} \cos((ln - 1)\theta),$$

where  $l \in \mathbb{N}^*$  (see Figure 2). A simple computation shows that we have  $S_n(p) = C_n$  with  $r = 1$ . Moreover, we easily have  $p(\theta) + \dot{p}(\theta) = 1 + \cos((ln - 1)\theta) \geq 0$  for all  $\theta \in \mathbb{R}$ . Hence,  $p$  is the support function of a rotor  $K$  in an  $n$ -gon. The boundary of  $K$  is of class  $C^\infty$  because  $p$  is of class  $C^\infty$ .

In the following, we denote by  $E$  the set of the functions  $p \in C^{1,1}(\mathbb{R})$  that are  $2\pi$ -periodic and that satisfy (2.3) and (2.7). The problem of finding a rotor of minimal area is now equivalent to the optimization problem

$$(2.16) \quad \min_{p \in E} \mathcal{A}(p).$$

The existence of a minimizer for problem (2.16) easily follows from standard compactness arguments (see [34], [36]).



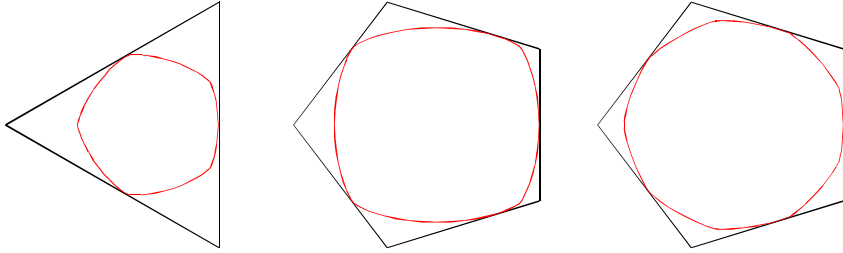


FIG. 2. Example of rotors whose support function is given by (2.15) for  $n = 3$ ,  $l = 2$  and  $n = 5$ ,  $l = 1, 2$ .

**2.3. Basic properties of rotors.** This section is devoted to well-known results about rotors which can be found in the case  $n = 3$  or  $n = 4$  in [5], [7], and [36]. Let us first recall Barbier's theorem, which is a simple consequence of (2.7).

**THEOREM 2.3.** *Let  $r$  be the radius of the inscribed circle in  $P_n$ . Then the perimeter of every rotor  $\mathcal{R}$  of  $P_n$  is equal to  $2\pi r$ .*

*Proof.* Let  $\mathcal{R}$  be a rotor and  $p$  be its support function. The perimeter  $L$  of  $\mathcal{R}$  is given by the integral of the radius of curvature:

$$L = \int_0^{2\pi} (p(\theta) + \ddot{p}(\theta)) d\theta,$$

which is well defined, as  $p$  is of class  $C^{1,1}$ . As  $\dot{p}$  is  $2\pi$ -periodic, the perimeter becomes  $L = \int_0^{2\pi} p(\theta) d\theta$ . Now integrating (2.7) on the interval  $[0, 2\pi]$  and using the  $2\pi$ -periodicity of  $p$ , we get  $L = 2\pi r$ .  $\square$

**PROPOSITION 2.5.** *Among all rotors of a regular polygon  $P_n$ , the one of maximal area is the disc of radius  $r$ .*

*Proof.* By the isoperimetric inequality, the body of maximal area among all closed curves having the same perimeter is the disc, and the disc is a rotor of  $P_n$ .  $\square$

When  $n = 4$ , a rotor is called a constant width body.

**DEFINITION 2.2.** *The width of a convex curve in a given direction is the distance between a pair of supporting lines of the curve perpendicular to this direction. If the width is constant in every direction, the curve is a curve of constant width.*

Equivalently, a constant width body has the property to rotate inside a square while remaining tangent to the four sides of the square. The relation (2.7) can be simplified in the case  $n = 4$ , which corresponds to the constant width bodies. The support function of  $K$  in this case satisfies

$$(2.17) \quad p(\theta) + p(\theta + \pi) = 2r \quad \forall \theta \in \mathbb{R},$$

which is exactly saying that any pair of parallel support lines to  $K$  is separated by the distance  $2r$  (see [15]).

**2.4. Formulation of the constraints on the interval  $[0, 2\delta]$ .** In this section, we derive consequences of (2.7) which will be useful in formulating the optimal control problem associated with the minimization problem. Let us define the reals  $s_k$  and  $t_k$  for  $k = 0, \dots, n-1$  by

$$(2.18) \quad s_k := \frac{\sin(k\delta)}{\sin(\delta)}, \quad t_k := 2 \frac{\sin(\frac{k\delta}{2}) \sin(\frac{(k-1)\delta}{2})}{\cos(\frac{\delta}{2})} r.$$

LEMMA 2.1. *Let  $p$  be a  $2\pi$ -periodic map in  $C^{1,1}(\mathbb{R})$  satisfying (2.7). Then we have*

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$$(2.19) \quad p(\theta + k\delta) = s_k p(\theta + \delta) - s_{k-1} p(\theta) + t_k \quad \forall \theta \in [0, 2\pi].$$

*Proof.* Let  $\theta \in [0, 2\pi]$  and  $v_k := p(\theta + k\delta)$ . We have by (2.7)

$$(2.20) \quad v_k - 2 \cos(\delta) v_{k+1} + v_{k+2} = 4r \sin^2\left(\frac{\delta}{2}\right).$$

We solve this linear recurrent sequence and get

$$v_k = a\omega^k + \overline{a\omega^k} + r,$$

where  $\omega := e^{i\delta}$  and  $v_0 = p(\theta)$ ,  $v_1 = p(\theta + \delta)$ . This gives (2.19).  $\square$

COROLLARY 2.1. *If  $n$  is even, a rotor  $K$  in an  $n$ -gon is a constant width body.*

*Proof.* Let  $K$  be a rotor and  $p$  be its support function which satisfies (2.7). We assume that  $n = 2m$ ,  $m \in \mathbb{N}^*$ . Using (2.19) with  $k = m$ , we get  $s_m = 0$ ,  $s_{m-1} = 1$ , and  $t_m = 2r$ . Consequently,  $p$  satisfies

$$p(\theta + m\delta) = -p(\theta) + 2r,$$

which is exactly saying that  $K$  is of constant width as  $m\delta = \pi$ .  $\square$

We now reformulate the area of a rotor on the interval  $[0, 2\delta]$ . Let  $r$  be the radius of the inscribed circle to the  $n$ -gon and  $P \in C^{1,1}(\mathbb{R}, \mathbb{R})$ ,  $F \in L^\infty(\mathbb{R}, \mathbb{R})$  be the maps defined by

$$(2.21) \quad \begin{cases} P(\theta) := p(\theta) - r, \\ F(\theta) := p(\theta) + \ddot{p}(\theta) - r = P(\theta) + \ddot{P}(\theta). \end{cases}$$

LEMMA 2.2. *Let  $p$  be the support function of a rotor and  $f$  its radius of curvature. The area of a rotor is given by*

$$\mathcal{A}(p) = \frac{n}{4 \sin^2\left(\frac{\delta}{2}\right)} \tilde{\mathcal{A}}(P) + \pi r^2,$$

where

$$(2.22) \quad \begin{aligned} \tilde{\mathcal{A}}(P) = \int_0^\delta & \left( P(\theta)F(\theta) + P(\theta + \delta)F(\theta + \delta) \right. \\ & \left. - \cos(\delta)(F(\theta)P(\theta + \delta) + F(\theta + \delta)P(\theta)) \right) d\theta. \end{aligned}$$

*Proof.* We have by (2.6)

$$\begin{aligned} \mathcal{A}(f) &= \frac{1}{2} \int_0^{2\pi} p(\theta) f(\theta) d\theta = \frac{1}{2} \sum_{0 \leq k \leq n-1} \int_{k\delta}^{(k+1)\delta} p(\theta) f(\theta) d\theta \\ &= \frac{1}{2} \sum_{0 \leq k \leq n-1} \int_0^\delta p(\theta + k\delta) f(\theta + k\delta) d\theta. \end{aligned}$$

Replacing  $p(\theta + k\delta)$  and  $f(\theta + k\delta)$  using (2.19), we get the result by the equalities

$$\sum_{0 \leq k \leq n-1} s_k^2 = \sum_{0 \leq k \leq n-1} s_{k-1}^2 = \frac{2}{2 \sin^2(\delta)}$$

and

$$\sum_{0 \leq k \leq n-1} s_k t_k = -\frac{n}{4 \cos^2(\frac{\delta}{2})}, \quad \sum_{0 \leq k \leq n-1} s_k s_{k-1} = \frac{n \cos(\delta)}{2 \sin^2(\delta)}. \quad \square$$

Note that in the special case of sets of constant width ( $n = 4$ ), one finds the usual functional (see [15]):

$$(2.23) \quad \mathcal{A}(p) = \pi r^2 - \int_0^\pi p(\theta)(1 - f(\theta))d\theta,$$

which can be easily obtained by (2.6) and (2.17).

**2.5. Simplification of the functional.** Before going into details for solving the minimization problem (2.16), we diagonalize the functional (2.22) (see [23] for the same parameterization). In particular, we establish the equivalence between the parameterization of a rotor by its support function and the new parameterization. The following parameterization will be useful in defining an optimal control problem equivalent to (2.16). We set

$$\gamma := \cos(\delta), \quad \sigma := \sin(\delta), \quad \omega^{\frac{1}{2}} := e^{\frac{i\delta}{2}}, \quad \omega^{-\frac{1}{2}} := e^{-\frac{i\delta}{2}};$$

that is, we denote by  $\omega^{\frac{1}{2}}$  and  $\omega^{-\frac{1}{2}}$  a square root of  $\omega$  and  $\bar{\omega}$ .

Recall that given a rotor  $K$  of support function  $p$ , the functions  $P$  and  $F$  are defined by (2.21), and by (2.8) and (2.9) we have  $S_n(f) = C_n$  if and only if  $S_n(F) = 0$ . We now define the functions  $W \in C^{1,1}(\mathbb{R}, \mathbb{C})$  and  $Z \in L^\infty(\mathbb{R}, \mathbb{C})$  by

$$(2.24) \quad \begin{cases} W(\theta) := P(\theta) - \bar{\omega}P(\theta + \delta), \\ Z(\theta) := F(\theta) - \bar{\omega}F(\theta + \delta), \end{cases}$$

where  $\theta \in \mathbb{R}$ , so that

$$(2.25) \quad W + \ddot{W} = Z.$$

The functions  $W$  and  $Z$  can be interpreted as the *complex support function* and the *complex radius of curvature* associated with a rotor. We denote by  $X_1, X_3, U, V$  the real and imaginary parts of  $W$  and  $Z$ :

$$\begin{cases} W = X_1 + iX_3, \\ Z = U + iV, \end{cases}$$

so that we have

$$(2.26) \quad \begin{cases} X_1(\theta) = P(\theta) - \gamma P(\theta + \delta), \\ X_3(\theta) = \sigma P(\theta + \delta), \\ U(\theta) = F(\theta) - \gamma F(\theta + \delta), \\ V(\theta) = \sigma F(\theta + \delta). \end{cases}$$

We have, equivalently,

$$(2.27) \quad \begin{cases} P(\theta) = X_1(\theta) + \frac{\gamma}{\sigma} X_3(\theta), \\ P(\theta + \delta) = \frac{1}{\sigma} X_3(\theta), \\ F(\theta) = U(\theta) + \frac{\gamma}{\sigma} V(\theta), \\ F(\theta + \delta) = \frac{1}{\sigma} V(\theta + \delta). \end{cases}$$

PROPOSITION 2.6. *The functions  $W$  and  $Z$  satisfy the relations*

$$(2.28) \quad \begin{cases} W(\theta + \delta) = \bar{\omega}W(\theta) \quad \forall \theta \in \mathbb{R}, \\ Z(\theta + \delta) = \bar{\omega}Z(\theta) \quad \text{a.e. } \theta \in \mathbb{R}. \end{cases}$$

*Proof.* Let  $p$  be the support function of a rotor. We have by (2.7)  $S_n(p) = C_n$ , where  $C_n$  is given by (2.9). Thus,  $S_n(P) = 0$ , that is,

$$(2.29) \quad \forall \theta \in \mathbb{R}, \quad P(\theta) - 2\gamma P(\theta + \delta) + P(\theta + 2\delta) = 0.$$

Eliminating  $P(\theta + 2\delta)$  in the equation above, we get

$$\forall \theta \in \mathbb{R}, \quad W(\theta + \delta) = P(\theta + \delta) - \bar{\omega}(2\gamma P(\theta + \delta) - P(\theta)),$$

which gives  $W(\theta + \delta) = \bar{\omega}W(\theta)$  for all  $\theta \in \mathbb{R}$ . By derivation of the previous equation, we get  $Z(\theta + \delta) = \bar{\omega}Z(\theta)$  for all  $\theta \in \mathbb{R}$ .  $\square$  Q7

In the following,  $\mathcal{P}_n$  denotes the regular polygon whose center is the origin and whose vertices are the points of coordinates  $(r^* \omega^k e^{i\alpha})_{0 \leq k \leq n-1}$ , where  $r^* := 2r \sin(\frac{\delta}{2})$  and  $\alpha := -\frac{\pi}{2} - \frac{\delta}{2}$ .

PROPOSITION 2.7. *Let  $K$  be a rotor,  $p$  its support function, and  $f = p + \ddot{p}$  its radius of curvature. We denote by  $Z$  its complex radius of curvature. Then we have  $f \geq 0$  if and only if  $Z(\theta) \in \mathcal{P}_n$  for a.e.  $\theta \in [0, \delta]$ .*

*Proof.* Let us consider for  $0 \leq k \leq n-1$  the map defined by

$$u_k(x, y) = s_k y - s_{k-1} x + t_k.$$

By Lemma 2.1, we have, for  $\theta \in [0, \delta]$  and for  $0 \leq k \leq n-1$ ,

$$f(\theta + k\delta) = u_k(f(\theta), f(\theta + \delta)).$$

Therefore, we have, for  $\theta \in [0, \delta]$ ,

$$\begin{aligned} f \geq 0 &\iff u_k(f(\theta), f(\theta + \delta)) \geq 0, \quad k = 0, \dots, n-1 \\ &\iff s_k(f(\theta + \delta) - r) - s_{k-1}(f(\theta) - r) + t_k + r(s_k - s_{k-1}) \geq 0 \\ &\iff \sin(k\delta)F(\theta + \delta) - \sin((k-1)\delta)F(\theta) \geq -\sigma r \\ &\iff \Im(\sin(k\delta)Z(\theta) - \sin((k-1)\delta)Z(\theta - \delta)) \geq -\sigma^2 r \\ &\iff \Im(\sin(k\delta)Z(\theta) - \sin((k-1)\delta)\omega Z(\theta)) \geq -\sigma^2 r \\ &\iff \Im(\omega^{k-1}Z(\theta)) \geq -\sigma r. \end{aligned}$$

Let  $z = x + iy$  be a complex number,  $D_k$  the hyperplane of equation  $\Im(\omega^{k-1}z) = -\sigma r$ , and  $H_k$  the half-plane defined for  $z \in \mathbb{C}$  by  $\Im(\omega^{k-1}z) \geq -\sigma r$ . We easily have that  $z \in D_{k+1}$  if and only if  $\omega z \in D_k$ . Hence, for  $\theta \in [0, \delta]$ ,  $Z(\theta)$  satisfies  $\Im(\omega^{k-1}Z(\theta)) \geq -\sigma r$ ,  $0 \leq k \leq n-1$ , if and only if  $Z(\theta)$  belongs to the intersection of the half-spaces  $H_k$ . This intersection is nonempty, as 0 belongs to  $H_k$  for all  $0 \leq k \leq n-1$  and is convex as all  $H_k$  are convex; hence it is a nonempty convex polygon. Moreover, a simple computation yields that the vertices of  $\mathcal{P}_n$  are given by the intersection  $D_k \cap D_{k+1}$  and are of coordinates  $-2ir \sin(\frac{\delta}{2})e^{i(k-\frac{1}{2})\delta}$  for  $0 \leq k \leq n-1$ .  $\square$

Q8

It is convenient to work with  $\mathcal{P}_n$  because we will see in the next section that the optimal control takes its values at the vertices of  $\mathcal{P}_n$  (the extremal points of  $\mathcal{P}_n$ ).

PROPOSITION 2.8. *Let  $p$  be the support function of a rotor  $K$ . Then the area of  $K$  is given by*

$$(2.30) \quad \mathcal{A}(p) = \pi r^2 + \frac{n}{4\sigma^2} \int_0^\delta UX_1 + VX_3 = \pi r^2 + \frac{n}{4\sigma^2} \int_0^\delta \Re(Z\bar{W}).$$

*Proof.* The area of the rotor  $K$  described by  $p \in E$  is given by (2.22). Replacing  $P(\theta)$ ,  $P(\theta + \delta)$ ,  $F(\theta)$ , and  $F(\theta + \delta)$  by  $W(\theta)$ ,  $W(\theta + \delta)$ ,  $Z(\theta)$ , and  $Z(\theta + \delta)$ , we get (2.30) by using (2.28).  $\square$

Notice the similarity between (2.6) and (2.30).

DEFINITION 2.3. *Let  $\Gamma$  be the set of the complex functions  $W$  in  $C^{1,1}([0, \delta])$  that satisfy*

$$(2.31) \quad \begin{cases} W(\delta) = \bar{\omega}W(0), \\ \dot{W}(\delta) = \bar{\omega}\dot{W}(0) \end{cases}$$

and such that the function  $Z = W + \dot{W}$  takes its values in the polygon  $\mathcal{P}_n$ .

DEFINITION 2.4. *We denote by  $\mathcal{Z}$  the set of the complex valued functions  $Z \in L^\infty(\mathbb{R}, \mathbb{C})$  satisfying*

$$Z(\theta + \delta) = \bar{\omega}Z(\theta) \quad \forall \theta \in \mathbb{R}$$

and

$$Z(\theta) \in \mathcal{P}_n \quad \forall \theta \in \mathbb{R}.$$

We can now prove the equivalence between the parameterization of a rotor  $K$  by its support function  $p$  and its complex support function  $W$ .

THEOREM 2.4. (i) *Let  $W = X_1 + iX_3$  be a function in  $\Gamma$ . Let us define the function  $\tilde{p}$  on  $[0, 2\delta]$  by  $\tilde{p} = P + r$ , where  $P$  is given by (2.27). Then, if we extend  $\tilde{p}$  on the interval  $[0, 2\pi]$  by (2.19) and if we denote by  $p$  this extension, then  $p$  is the support function of a rotor.*

(ii) *Conversely, if  $p$  is the support function of a rotor  $K$  and  $P := p - r$ , then the function  $W|_{[0, \delta]}$  defined by (2.24) belongs to  $\Gamma$ .*

*Proof of (i).* First, let us take  $W = X_1 + iX_3 \in \Gamma$ . We have by (2.31)

$$(2.32) \quad \begin{cases} \frac{1}{\sigma}X_3(0) = X_1(\delta) + \frac{\gamma}{\sigma}X_3(\delta), \\ \sigma X_1(0) - \gamma X_3(0) = -X_3(\delta) \end{cases}$$

and

$$(2.33) \quad \begin{cases} \frac{1}{\sigma}\dot{X}_3(0) = \dot{X}_1(\delta) + \frac{\gamma}{\sigma}\dot{X}_3(\delta), \\ \sigma\dot{X}_1(0) - \gamma\dot{X}_3(0) = -\dot{X}_3(\delta). \end{cases}$$

We now define a function  $P$  on the interval  $[0, 2\delta]$  by

$$P(\theta) = X_1(\theta) + \frac{\gamma}{\sigma} X_3(\theta), \quad P(\theta + \delta) = \frac{1}{\sigma} X_3(\theta)$$

for  $\theta \in [0, \delta]$ . By (2.32), we have

$$P(\delta^-) = P(\delta^+),$$

and by (2.33) we have

$$\dot{P}(\delta^-) = \dot{P}(\delta^+).$$

Consequently, the function  $P$  is of class  $C^1$  on  $[0, 2\delta]$ . By (2.32) we also get

$$S_n(P)(0) = 0,$$

and by (2.33) we get

$$S_n(\dot{P})(0) = 0.$$

Hence, the functions  $P$  and  $\dot{P}$  satisfy  $S_n(P) = 0$  and  $S_n(\dot{P}) = 0$  for  $\theta = 0$ . If we extend  $p = P + r$  to the interval  $[0, 2\pi]$  by (2.19) and to  $\mathbb{R}$  by  $2\pi$ -periodicity, it satisfies, by construction,  $S_n(p) = C_n$ . We also have  $p(0) = p(2\pi)$  and  $\dot{p}(0) = \dot{p}(2\pi)$  by (2.19) so that the function  $p$  is of class  $C^1$ . Finally, we have  $p + \ddot{p} \geq 0$  because  $Z \in \mathcal{P}_n$ . We conclude that  $p$  is the support function of a rotor.

*Proof of (ii).* Let us now consider the support function  $p$  of a rotor. We define a function  $W$  by (2.24). First, the condition (2.3) satisfied by  $p$  implies that  $Z = W + \ddot{W}$  takes its value in  $\mathcal{P}_n$ . Let us show that  $W$  satisfies (2.31). By (2.26), we have

$$\frac{1}{\sigma} X_3(0) = X_1(\delta) + \frac{\gamma}{\sigma} X_3(\delta),$$

and by using  $S_n(P)(0) = 0$ , we get

$$\sigma X_1(0) - \gamma X_3(0) = -X_3(\delta).$$

These two real conditions imply  $W(\delta) = \bar{\omega}W(0)$ . By using (2.27) and the equality  $S_n(\dot{P})(0) = 0$ , we get  $\dot{W}(\delta) = \bar{\omega}\dot{W}(0)$ . Hence,  $W$  belongs to  $\Gamma$ .  $\square$

*Remark 2.1.* Let us make two remarks. First, any function  $W \in \Gamma$  such that  $Z = W + \ddot{W}$  satisfies, by (2.31), the condition

$$(2.34) \quad \int_0^\delta Z(\theta) e^{i\theta} d\theta = 0.$$

Second, (2.30) remains unchanged if we replace  $W$  by  $W e^{i\alpha}$  and  $Z$  by  $Z e^{i\alpha}$ , where  $\alpha \in \mathbb{R}$ .

From now on, we will mainly deal with the set  $\Gamma$  instead of the set  $E$ , as there is a one-to-one correspondence between these two sets. For  $W \in \Gamma$  such that  $W = X_1 + iX_3$  and  $Z = W + \ddot{W} = U + iV$ , we denote by  $J(W)$  the functional

$$(2.35) \quad J(W) = \int_0^\delta U X_1 + V X_3 = \int_0^\delta \Re(Z \bar{W})$$

and by  $\mathcal{A}(W)$  the area of a rotor. An integration by parts shows that we have

$$J(W) = \int_0^\delta Z\overline{W} = \int_0^\delta |W|^2 - |\dot{W}|^2,$$

and as  $J(W) \in \mathbb{R}$ , we have

$$\int_0^\delta \Im(Z\overline{W}) = 0.$$

The area of a rotor becomes

$$\mathcal{A}(W) = \pi r^2 + \frac{n}{4\sigma^2} J(W).$$

The initial problem, finding the rotor of least area (problem (2.16)), is now equivalent to

$$(2.36) \quad \min_{W \in \Gamma} J(W).$$

In sections 3 and 4, we will solve problem (2.36) using the optimal control theory.

**2.6. Fourier series of regular rotors.** Before going further into the analysis of (2.36), we describe by Fourier series the two families of regular rotors  $O_n^{ln \pm 1}$  introduced in section 1. An analogous description is given by Focke (see [11]), but here we use the new parameterization  $(W, Z)$ , which simplifies the computations.

We consider the subset  $J \subset \mathbb{Z}$  defined for  $n \geq 3$  by

$$J = (n\mathbb{Z} + 1) \cup (n\mathbb{Z} - 1) \setminus \{\pm 1\}$$

and let  $p$  be the support function of a rotor. Then  $p$  is given by

$$(2.37) \quad p(\theta) = r + c_1 e^{i\theta} + c_{-1} e^{-i\theta} + \sum_{j \in J} c_j e^{ij\theta},$$

where  $c_j$  are the Fourier coefficients of  $p$ . In the case of constant width bodies, the support function becomes

$$p(\theta) = r + c_1 e^{i\theta} + c_{-1} e^{-i\theta} + \sum_{l \in \mathbb{Z}^*} \left( c_{4l-1} e^{i(4l-1)\theta} + c_{4l+1} e^{i(4l+1)\theta} \right).$$

By the Parseval equality, the area of a rotor  $K$  becomes

$$(2.38) \quad \mathcal{A}(p) = \pi \left( r^2 - \sum_{j \in J} \frac{|c_j|^2}{j^2 - 1} \right).$$

Let  $m \in \mathbb{N}^*$ ,  $\varepsilon = \pm 1$ ,  $L = mn - \varepsilon$ ,  $\tau = \frac{\delta}{L}$ , and  $s = L - 1$ . We can easily check that the complex function defined by

$$(2.39) \quad Z(\theta) = \sum_{0 \leq j \leq s} \omega^{\varepsilon j} \mathbb{1}_{[j\tau, (j+1)\tau[}$$

is an element of  $\mathcal{Z}$ . We will define the regular rotors by (2.39).

DEFINITION 2.5. We call regular rotor any element  $W$  of  $\Gamma$  such that  $W + \ddot{W}$  is of the form (2.39). The first series consists of the rotors obtained for  $\varepsilon = 1$ , and the second series is obtained for  $\varepsilon = -1$ .

The integer  $L = s + 1$  denotes the number of intervals of the subdivision  $[0, \delta]$ . We now consider the set

$$J_\varepsilon = \left\{ k \in \mathbb{Z}, k \equiv \varepsilon[n] \right\}.$$

PROPOSITION 2.9. The Fourier series of a regular rotor is given by

$$(2.40) \quad Z(\theta) = \frac{n}{\pi} e^{-\frac{i\varepsilon\delta}{2}} \sin\left(\frac{\varepsilon\delta}{2}\right) \sum_{k \in J_\varepsilon} \frac{e^{ikL\theta}}{k}.$$

*Proof.* The function  $\theta \mapsto e^{i\theta} Z(\theta)$  is  $\delta$ -periodic, as we have  $Z(\theta + \delta) = \overline{\omega} Z(\theta)$ . Thus, one has, for a.e.  $\theta \in \mathbb{R}$ ,

$$Z(\theta) e^{i\theta} = \sum_{k \in \mathbb{Z}} c_k e^{ikn\theta},$$

where the Fourier coefficients are given by

$$c_k = \frac{n}{2\pi} \int_0^\delta e^{-i(kn-1)\theta} Z(\theta) d\theta.$$

Using (2.39), we get, for  $k \in \mathbb{Z}$ ,

$$c_k = \frac{i}{kn-1} (e^{-i(kn-1)\tau} - 1) \sum_{0 \leq j \leq s} \omega^{\varepsilon j} e^{-i(kn-1)j\tau}.$$

The previous sum can be easily computed, and we get  $c_0 = 0$  and

$$c_k \neq 0 \iff \omega^\varepsilon e^{-i(kn-1)\tau} = 1,$$

because  $\tau = \frac{\delta}{L}$ . For  $\varepsilon = 1$ , one has

$$\omega^\varepsilon e^{-i(kn-1)\tau} = 1 \iff \exists j \in \mathbb{Z}, \quad kn - 1 = (jn + 1)L.$$

For  $\varepsilon = +1$ , we finally obtain

$$c_k = \frac{n}{\pi(jn+1)} e^{-i\frac{\delta}{2}} \sin\left(\frac{\delta}{2}\right).$$

For  $\varepsilon = -1$ , a similar computation yields

$$c_k = -\frac{n}{\pi(jn-1)} e^{i\frac{\delta}{2}} \sin\left(\frac{\delta}{2}\right).$$

This gives (2.40).  $\square$

The Fourier series of  $Z$  can also be written as

$$Z(\theta) = \frac{n}{\pi} e^{-i\varepsilon\frac{\delta}{2}} \sin\left(\frac{\delta}{2}\right) \sum_{j \in \mathbb{Z}} \frac{e^{i((mnj - \varepsilon j + \varepsilon m)n - 1)\theta}}{jn + \varepsilon}.$$



The first series of rotors obtained for  $\varepsilon = +1$  will be called  $O_n^{mn-1}$ , and the second series obtained for  $\varepsilon = -1$  will be called  $O_n^{mn+1}$  (see [11], [23]). For  $n = 4$ , the two families  $O_4^{4m-1}$  and  $O_4^{4m+1}$  describe the odd Reuleaux polygons (see [10]). A Reuleaux polygon consists of the intersection of  $N$  circles of radii 1 ( $N$  is odd) and whose centers are the vertices of an  $N$ -gon of side 1. An analogous geometrical description of  $O_n^{ln\pm 1}$  can be found in [18].

Q9

PROPOSITION 2.10. *Let  $K$  be a rotor and  $Z$  its complex radius of curvature. If  $Z$  is given by (2.39), then the area of  $K$  becomes*

$$(2.41) \quad \mathcal{A}(K) = \pi r^2 - \frac{r^2 n^2}{2\pi} \tan^2 \left( \frac{\delta}{2} \right) \sum_{j \in \mathbb{Z}} \frac{1}{(jn+1)^2 ((mn-\varepsilon)^2 (jn+1)^2 - 1)}.$$

*Proof.* By (2.30), we have

$$\mathcal{A}(K) = \pi r^2 + \frac{n}{4\sigma^2} \int_0^\delta \overline{Z}(\theta) W(\theta) d\theta,$$

where  $W$  is in  $\Gamma$  and satisfies  $W + \ddot{W} = Z$ . By (2.40), the function  $W$  is given by

$$W(\theta) = -\frac{n}{\pi} e^{-i\varepsilon \frac{\delta}{2}} \sum_{k \in J_\varepsilon} \frac{e^{ikL\theta}}{k(k^2 L^2 - 1)}.$$

Applying the Parseval equality yields (2.41).  $\square$

The following proposition has been proved in [11]. It will be useful for proving Goldberg's conjecture (see section 4). We give a short proof using the expression of the area of a rotor given by (2.41).

PROPOSITION 2.11. *In the class of the regular rotors  $O_n^{mn\pm 1}$ , the one of minimal area is  $O_n^{n-1}$  obtained for  $m = 1$  and  $\varepsilon = +1$ . Its Fourier series is given by*

$$(2.42) \quad Z(\theta) = \frac{n}{\pi} e^{-i\varepsilon \frac{\delta}{2}} \sin \left( \frac{\delta}{2} \right) \sum_{j \in \mathbb{Z}} \frac{e^{i((n-1)j+1)n-1)\theta}}{jn+1}.$$

*Proof.* The area of a rotor  $K$  described by  $Z \in \mathcal{Z}$  is an increasing function of  $m \in \mathbb{N}^*$  by (2.41). Thus the minimum in the class of regular rotors is obtained for  $m = 1$ . The minimum between  $O_n^{n-1}$  and  $O_n^{n+1}$  is clearly  $O_n^{n-1}$ .  $\square$

It is easy to see that  $O_n^{n-1}$  is invariant with respect to the action of the dihedral group of order  $2(n-1)$ ,  $D_{n-1}$ . For example, the Reuleaux triangle is invariant with respect to the group  $D_3$  and is invariant with the  $\Delta$ -biangle with respect to the group  $D_2$ . Anyway, it seems difficult to prove that a minimizer of problem (2.36) has these symmetries.

### 3. The minimization problem as an optimal control problem.

**3.1. First consequences of the PMP.** In the case of the sets of constant width ( $n = 4$ ), one can deal with one control on the interval  $[0, \pi]$  because the functional to minimize is given by (2.23) (see [15]). The optimal control problem in the general case ( $n \geq 3$ ) requires a sharper analysis here because we have to deal with a control  $(U, V) \in \mathbb{R}^2$  on  $[0, \delta]$  as  $\gamma \neq 0$ .

Let us consider the polygon  $\mathcal{P}'_n$  which corresponds to the initial polygon  $\mathcal{P}_n$  by a homotheticity of ratio  $\lambda = \frac{1}{2 \sin(\frac{\delta}{2})}$  and a rotation of angle  $\alpha = \frac{\pi}{2} + \frac{\delta}{2}$ . Hence, the vertices of the polygon  $\mathcal{P}'_n$  are the points of coordinates  $(\omega^j)_{0 \leq j \leq n-1}$ . We consider

the differential system (harmonic oscillator) on the interval  $[0, \delta]$  described by the equations

$$(3.1) \quad \begin{cases} \dot{X}_1 = X_2, \\ \dot{X}_2 = -X_1 + U, \\ \dot{X}_3 = X_4, \\ \dot{X}_4 = -X_3 + V, \end{cases}$$

where the control  $(U, V)$  takes its values within the polygon  $\mathcal{P}'_n$ . As the vector  $(X_1, X_3)$  satisfies the boundary conditions given by (2.31), the PMP will lead to transversality conditions. Notice that the initial and final states are not fixed, but they are linked by (2.31).

By the linearity of (3.1), the problem (2.36) is clearly equivalent to minimizing (2.30), where  $(X_1, X_2, X_3, X_4)$  satisfies (2.31) and (3.1) and the control  $(U, V)$  takes its values within the polygon  $\mathcal{P}'_n$ . We have thus reformulated the initial shape optimization problem into an optimal control problem:

$$(3.2) \quad \min \left\{ \int_0^\delta U X_1 + V X_3, (U, V) \in \mathcal{P}'_n, (X_1, X_2, X_3, X_4) \text{ satisfies (2.31) and (3.1)} \right\}.$$

DEFINITION 3.1. *We denote by  $X = (X_1, X_2, X_3, X_4) \in \mathbb{R}^4$  the state variable and  $q = (q_1, q_2, q_3, q_4) \in \mathbb{R}^4$  the dual variable. The Hamiltonian of the system  $H := H(X, q, U, V, p_0)$  is given by*

$$(3.3) \quad H = q_1 X_2 + q_2(-X_1 + U) + q_3 X_4 + q_4(-X_3 + V) + p_0(U X_1 + V X_3),$$

where  $p_0 \in \mathbb{R}$ .

We first prove the existence of an optimal control of (3.2).

THEOREM 3.1. *There exists an optimal control for problem (3.2).*

*Proof.* There exists an admissible trajectory of (3.2) corresponding to  $Z = 0$ ; hence, the set of admissible trajectories is nonempty. The existence of an optimal control will follow from an application of Filipov's theorem (see [1] or [33, p. 98]). First, we check that the trajectories are uniformly bounded. Indeed, the set of admissible controls is compact, and by linearity of (3.1), we obtain a uniform bound by Gronwall's lemma. Second, given  $(X_1, X_2, X_3, X_4) \in \mathbb{R}^4$ , the set defined by

$$\left\{ (X_1 U + X_3 V, X_2, -X_1 + U, X_4, -X_3 + V), (U, V) \in \mathcal{P}'_n \right\}$$

is clearly convex. By Filipov's theorem (see [33]), we get the result.  $\square$

By the PMP, there exists a map  $X : [0, \delta] \rightarrow \mathbb{R}^4$  absolutely continuous, a map  $q : [0, \delta] \rightarrow \mathbb{R}^4$  absolutely continuous, a constant  $p_0 \leq 0$ , and an optimal control  $Z(\theta) = (U(\theta), V(\theta))$  satisfying the equations

$$(3.4a) \quad \dot{X} = \frac{\partial H}{\partial q},$$

$$(3.4b) \quad \dot{q} = -\frac{\partial H}{\partial X},$$

and

$$(3.5) \quad \max_{(\tilde{U}, \tilde{V}) \in \mathcal{P}'_n} H(X(\theta), q(\theta), \tilde{U}, \tilde{V}, p_0) = H(X(\theta), q(\theta), U(\theta), V(\theta), p_0).$$

Moreover, the pair  $(p_0, q)$  is nontrivial, and  $q$  satisfies transversality conditions that we will make explicit in the paragraph below.

DEFINITION 3.2. *We will call an extremal trajectory a quadruplet  $(X, q, p_0, Z)$  satisfying (3.4a), (3.4b), (3.5) and such that the pair  $(X, q)$  is absolutely continuous on  $[0, \delta]$ ,  $p_0 \leq 0$ , and  $(p_0, q)$  is nonzero. The control  $Z = (U, V)$  corresponding to an extremal trajectory will be called an extremal control.*

As the system is autonomous, the Hamiltonian of the system is conserved along the extremal trajectories of the system. By (3.4b), the variable  $q$  satisfies the dual system:

$$(3.6) \quad \begin{cases} \dot{q}_1 = q_2 - p_0 U, \\ \dot{q}_2 = -q_1, \\ \dot{q}_3 = q_4 - p_0 V, \\ \dot{q}_4 = -q_3. \end{cases}$$

The system (3.1) can also be written as

$$(3.7) \quad \ddot{W} + W = Z,$$

where

$$W = X_1 + iX_3, \quad Z = U + iV,$$

and from now on, for convenience, we will mainly deal with complex variables. We write the dual variable  $q = (q_1, q_2, q_3, q_4)$  in the following way:

$$(3.8) \quad \Pi = q_2 + iq_4,$$

so that we have

$$(3.9) \quad \dot{\Pi} = -q_1 - iq_3.$$

We get from (3.6)

$$(3.10) \quad \ddot{\Pi} + \Pi = p_0 Z.$$

It follows that  $W$  and  $\Pi$  are of class  $C^{1,1}$  on the interval  $[0, \delta]$ , as the control  $Z$  is bounded. Let us now compute the transversality conditions by using the variables  $(W, \Pi)$ . The vector of  $\mathbb{C}^4$ ,

$$(W(0), \dot{W}(0), W(\delta), \dot{W}(\delta)),$$

takes its values in the subspace  $M$  of  $\mathbb{C}^4$  defined by

$$M := \{(A, B, \bar{\omega}A, \bar{\omega}B), (A, B) \in \mathbb{C}^2\}.$$

The orthogonal of  $M$  in  $\mathbb{C}^4$  (with respect to the canonical scalar product in  $\mathbb{C}^4$ ) is simply

$$M^\perp = \{(A', B', -\bar{\omega}A', -\bar{\omega}B'), (A', B') \in \mathbb{C}^2\}.$$

By the PMP, the vector  $(-q(0), q(\delta)) = (-\Pi(0), -\dot{\Pi}(0), \Pi(\delta), \dot{\Pi}(\delta))$  is in  $M^\perp$  (see [28], [33] for transversality conditions in the periodic case). Hence, we have  $\Pi(\delta) = \bar{\omega}\Pi(0)$  and  $\dot{\Pi}(\delta) = \bar{\omega}\dot{\Pi}(0)$ ; consequently,  $\Pi$  satisfies (2.31), that is, the same boundary conditions as  $W$ . Note that the Hamiltonian can be expressed as follows:

$$(3.11) \quad H = -\Re(W\bar{\Pi}) - \Re(\dot{W}\bar{\dot{\Pi}}) + \Re((p_0W + \Pi)\bar{Z}).$$

By (3.8) and (3.9), the scalar product in  $\mathbb{C}^2$  between  $W$  and  $\Pi$  is given by

$$(3.12) \quad \langle W, \Pi \rangle := \sum_{1 \leq i \leq 4} q_i X_i = -\Re(W\bar{\Pi}) + \Re(\dot{W}\bar{\dot{\Pi}}).$$

We now simplify the system (3.4a)–(3.4b) by expressing the dual variable  $\Pi$  as a function of the state variable  $W$ . This corresponds to a reduction of the number of degrees of freedom of the system (3.4a)–(3.4b).

LEMMA 3.1. *Let  $W$  be an extremal trajectory of the system and  $\Pi = q_2 + iq_4$  its dual variable. Then there exists  $A \in \mathbb{C}$  such that the function  $\Pi - p_0W$  is of the form*

$$\Pi(\theta) - p_0W(\theta) = Ae^{-i\theta}, \quad \theta \in [0, \delta].$$

*Proof.* We have by (3.7) and (3.10)

$$\ddot{\Pi} + \Pi = p_0(U + iV) = p_0Z = p_0(\ddot{W} + W),$$

and, consequently, the function  $y = \Pi - p_0W$  satisfies  $\ddot{y} + y = 0$ . There exist two constants  $(A, B) \in \mathbb{C}^2$  such that, for all  $\theta \in [0, \delta]$ , we have

$$(3.13) \quad \Pi(\theta) - p_0W(\theta) = Ae^{-i\theta} + Be^{i\theta}.$$

Let us prove that  $B = 0$ . For  $\theta = 0$  and  $\theta = \delta$ , we get

$$\Pi(0) - p_0W(0) = A + B, \quad \Pi(\delta) - p_0W(\delta) = A\bar{\omega} + B\omega.$$

But, as  $(W, \Pi)$  belong to  $\Gamma$ , we have by the transversality conditions

$$\Pi(\delta) - p_0W(\delta) = \bar{\omega}\Pi(0) - p_0\bar{\omega}W(0) = A\bar{\omega} + B\bar{\omega}.$$

Thus, we conclude that  $B = 0$ .  $\square$

We now show that an extremal trajectory is not abnormal.

LEMMA 3.2. *Let  $(X, q, p_0, Z)$  be an extremal trajectory. Then the constant  $p_0$  is strictly negative.*

*Proof.* Let us assume that  $p_0 = 0$ . As the point  $(0, 0)$  belongs to  $\mathcal{P}'_n$ , we get by the PMP the following: for almost  $\theta \in [0, \delta]$ ,

$$q_2(\theta)U(\theta) + q_4(\theta)V(\theta) \geq 0.$$

Consequently,

$$\int_0^\delta (q_2(\theta)U(\theta) + q_4(\theta)V(\theta))d\theta \geq 0.$$

But, we have

$$\int_0^\delta (q_2(\theta)U(\theta) + q_4(\theta)V(\theta))dt = \int_0^\delta \operatorname{Re}(\bar{\Pi}(\theta)Z(\theta))d\theta,$$

and by the previous lemma and (2.34), we have

$$\int_0^\delta \bar{\Pi}Z = \int_0^\delta \bar{A}e^{i\theta}Z(\theta)d\theta = 0.$$

Hence, the function  $\Re(\Pi Z)$  must be zero on the interval  $[0, \delta]$ . If  $\Pi$  is not zero, then the extremal control associated with this trajectory is orthogonal to  $\Pi$ . This contradicts (3.5) by choosing a control  $\tilde{Z} \in \mathcal{P}'_n$  such that  $\Re(\Pi\tilde{Z}) > 0$ . Hence,  $\Pi$  must be 0 everywhere. This is not possible because by the PMP, the pair  $(\Pi, p_0)$  is not zero.  $\square$

In the following, we take  $p_0 = -1$  for any extremal trajectory of the system. Let  $(W, \Pi, Z)$  be an extremal trajectory defined by  $\frac{\partial H}{\partial U} = \frac{\partial H}{\partial V} = 0$ ; that is, we have  $\Pi = W$ . As  $p_0 = -1$ , we get by Lemma 3.1

$$W(\theta) = \frac{A}{2}e^{-i\theta}, \quad \theta \in [0, \delta].$$

Such an extremal trajectory represents the disc which maximizes the area, and this case can be excluded.

LEMMA 3.3. *Let  $W$  be an extremal trajectory of the system and  $\Pi$  its dual variable. Then there exists an extremal trajectory of the system,  $W_1$ , with dual variable  $\Pi_1$ , such that*

$$\Pi_1 = -W_1$$

and such that the functional of both extremals is identical.

*Proof.* For  $\lambda \in \mathbb{C}$ , we consider the functions  $(W_1, \Pi_1)$  defined on  $[0, \delta]$  by

$$\begin{cases} W_1(\theta) = W(\theta) + \lambda e^{-i\theta}, \\ \Pi_1(\theta) = \Pi(\theta) + \lambda e^{-i\theta}. \end{cases}$$

We have

$$\ddot{W}_1 + W_1 = Z, \quad \ddot{\Pi}_1 + \Pi_1 = -Z.$$

We can easily check that  $W_1$  and  $\Pi_1$  satisfy (2.31). By (2.34), the functional remains unchanged:

$$\int_0^\delta \Re(Z\bar{W}_1) = \int_0^\delta \Re(Z\bar{W}).$$

Hence,  $(W_1, \Pi_1)$  is also an optimal trajectory. Recall that the Hamiltonian along this trajectory is defined by

$$H_1 = -\Re(W_1\bar{\Pi}_1) - \Re(W'_1\bar{\Pi}'_1) + \Re((\Pi_1 - W_1)\bar{Z}).$$

Using Lemma 3.1, we have  $\Pi = -W + Ae^{-i\theta}$ , and by a computation, we get

$$H_1 = H + 2\Re(A\bar{\lambda}) + 2|\lambda|^2,$$

where  $H$  is given by (3.11). This shows that the PMP (3.5) gives the same extremal control for  $(W, \Pi)$  and for  $(W_1, \Pi_1)$ , as both Hamiltonian are equal up to a constant. Finally, we have

$$\Pi_1 + W_1 = (A + 2\lambda)e^{-i\theta},$$

and by taking  $\lambda$  such that  $A = -2\lambda$ , we get the lemma.  $\square$

From now on, we consider extremal solutions  $(W, Z)$  of the system such that the dual variable  $\Pi$  satisfies  $\Pi = -W$  (by Lemma 3.3). To simplify, we will say that  $W$  is an extremal trajectory of the system if  $\Pi = -W$  and if it satisfies the PMP. The Hamiltonian of the system is constant along such an extremal and can be written using (3.11):

$$(3.14) \quad H = |W|^2 + |\dot{W}|^2 - 2\Re(W \cdot \bar{Z}) = |W - Z|^2 + |\dot{W}|^2 - |Z|^2.$$

*Remark 3.1.* By (3.14), and by using (3.5), we get  $H \geq 0$  along an extremal trajectory.

**3.2. Computation of the extremal control.** We now examine in more detail the consequences of the PMP to describe the extremal trajectories. Let us recall the definition of a switching point.

**DEFINITION 3.3.** *Let  $Z = (U, V)$  be an extremal control of problem (3.2). A point  $\tau \in ]0, \delta[$  is called a switching point if, for every  $\varepsilon > 0$  such that  $[\tau - \varepsilon, \tau + \varepsilon] \subset ]0, \delta[$ , the control  $Z$  is nonconstant on  $[\tau - \varepsilon, \tau + \varepsilon]$ .*

To restrict the class of extremal trajectories, we prove step by step the following:

- An extremal is bang-bang, and the associated control takes its values on the vertices of  $\mathcal{P}'_n$  (Lemma 3.4).
- An extremal control takes its values regularly on the vertices of  $\mathcal{P}'_n$  (Theorem 3.2).
- The number of switching points of an extremal control is finite (Theorem 3.3).
- The number of switching points of an extremal control is prescribed (Theorem 3.4).
- The distance between two consecutive switching points is constant (Proposition 4.1).

We first prove two lemmas which will be useful in proving Theorems 3.2 and 3.3.

**LEMMA 3.4.** *Let  $W$  be an extremal trajectory of the system. Then the extremal control takes its values on the vertices of  $\mathcal{P}'_n$ .*

*Proof.* First, we show that the extremal control takes its values on the vertices of  $\mathcal{P}'_n$ . By (3.5) and (3.14), the extremal control is a solution of the maximization problem

$$(3.15) \quad \max_{z \in \mathcal{P}'_n} \phi(z),$$

where  $\phi$  is defined on  $\mathcal{P}'_n$  by  $\phi(z) := -2\Re(\bar{z}W(\theta))$  and  $\theta \in [0, \delta]$  is fixed. Let  $z_0$  be a point where the maximum in (3.15) is obtained.

If  $W(\theta) = 0$ , then the maximum in (3.15) can be taken arbitrarily in  $\mathcal{P}'_n$  and, in particular, on a vertex of  $\mathcal{P}'_n$ . Let us now assume that  $W(\theta) \neq 0$ . The maximum of  $\phi$  is necessarily on the boundary of  $\mathcal{P}'_n$  because  $\nabla\phi(z_0) \neq 0$ . Hence,  $z_0$  is of the form  $z_0 = t_0\omega^j + (1 - t_0)\omega^{j+1}$ , where  $t_0 \in [0, 1]$  and  $0 \leq j \leq n - 1$ . If  $W(\theta)$  is orthogonal to  $\omega^{j+1} - \omega_j$ , then we can take  $z_0 = \omega^j$  or  $z_0 = \omega^{j+1}$ . If this is not the case, let us define the function  $\psi$  on  $[0, 1]$  by

$$\psi(t) = -2\Re((t\bar{\omega}^j + (1 - t)\bar{\omega}^{j+1})W(\theta)).$$

As we have  $\dot{\psi}(t_0) \neq 0$ , the maximum in (3.15) cannot be reached at  $t_0$ . Hence, the maximum in (3.15) is reached on a vertex of  $\mathcal{P}'_n$ , and this proves the lemma.  $\square$

LEMMA 3.5. *Let  $W$  be an extremal trajectory of the system and  $\tau_j, j \in \mathbb{N}$ , a switching point of the extremal control  $Z$  such that  $Z(\tau_j^-) = \omega^{k_j}$  and  $Z(\tau_j^+) = \omega^{k_{j+1}}$  with  $(k_j, k_{j+1}) \in \mathbb{N}^2$ . Then there exists  $t_j \in \mathbb{R}$  such that*

$$(3.16) \quad W(\tau_j) = t_j \omega^{\frac{k_j + k_{j+1}}{2}}.$$

*Proof.* The Hamiltonian is constant along an extremal trajectory, and the functions  $\theta \mapsto |W(\theta)|^2$  and  $\theta \mapsto |W'(\theta)|^2$  are continuous. Hence, the function  $\theta \mapsto \Re(W(\theta)\overline{Z(\theta)})$  is continuous, and at a switching point  $\tau_j$ , we get

$$\Re(W(\tau_j)\overline{\omega^{k_j}}) = \Re(W(\tau_j)\overline{\omega^{k_{j+1}}}).$$

Geometrically speaking, the vector  $W(\tau_j)$  is orthogonal to the segment  $[\omega^{k_j}, \omega^{k_{j+1}}]$ ; hence it takes the form given by (3.16).  $\square$

By Lemma 3.4, an extremal trajectory is bang-bang: the extremal control associated with this trajectory takes the extremal values of the convex polygon  $\mathcal{P}'_n$ . We now show that the extremal control goes all over the vertices  $\omega^j$  clockwise or counterclockwise.

THEOREM 3.2. *Let  $W$  be an extremal trajectory of the system. There exists  $\varepsilon \in \{\pm 1\}$  such that if  $\tau_j$  is a switching point with  $Z(\tau_j^-) = \omega^{k_j}$  and  $Z(\tau_j^+) = \omega^{k_{j+1}}$ , then*

$$k_{j+1} - k_j = \varepsilon.$$

*Proof.* By Lemma 3.5, we have at a switching point  $\tau_j$

$$W(\tau_j) = t_j \omega^{\frac{k_j + k_{j+1}}{2}},$$

where  $t_j \in \mathbb{R}$ . Geometrically speaking, the vector  $W(\tau_j)$  is parallel to the median of the segment  $[\omega^{k_j}, \omega^{k_{j+1}}]$ , which is a side or a diagonal of the polygon  $\mathcal{P}'_n$ . The line  $\Delta$  directed by  $W(\tau_j)$  contains 0, 1, or 2 vertices of  $\mathcal{P}'_n$ .

First, assume that  $\Delta$  does not contain any vertex of  $\mathcal{P}'_n$ . If  $|k_j - k_{j+1}| \neq 1$ , there exists another vertex  $\omega^s := (U_s, V_s)$  of  $\mathcal{P}'_n$ , which is different from  $\omega^j$  and  $\omega^{j+1}$ , and such that

$$-2\Re(\overline{W(\tau_j)}\omega^s) > -2\Re(\overline{W(\tau_j)}\omega^j)$$

or

$$-2\Re(\overline{W(\tau_j)}\omega^s) > -2\Re(\overline{W(\tau_j)}\omega^{j+1}).$$

This means that the scalar product between  $W(\tau_j)$  and  $\omega^s$  is less than the scalar product between  $W(\tau_j)$  and  $\omega^{k_j}$  or  $\omega^{k_{j+1}}$ . Assume, for example, that the first inequality is satisfied by  $\omega^s$ . We obtain by (3.14)

$$H(W(\tau_j), \Pi(\tau_j), U_s, V_s, p_0) > H(W(\tau_j), \Pi(\tau_j), U(\tau_j^-), V(\tau_j^-), p_0).$$

This contradicts (3.5), that is, the maximality of the Hamiltonian along an extremal.

Now assume that  $\Delta$  contains only one vertex of  $\mathcal{P}'_n$  (in this case  $n$  is necessarily even) and  $|k_j - k_{j+1}| \neq 1$ . The segment  $[\omega^{k_j}, \omega^{k_{j+1}}]$  is parallel to a side  $[\omega^r, \omega^{r+1}]$  of  $\mathcal{P}'_n$ . Let us call  $\omega^l$  the vertex of  $\mathcal{P}'_n$  opposite to  $[\omega^r, \omega^{r+1}]$ . As in the previous case, we get a contradiction in (3.5). Indeed, one has

$$H(W(\tau_j), \Pi(\tau_j), U_s, V_s, p_0) > H(W(\tau_j), \Pi(\tau_j), U(\tau_j^-), V(\tau_j^-), p_0),$$

with  $s$  equal to  $r, r+1$ , or  $l$  and with  $\omega^s := (U_s, V_s)$ .

If  $\Delta$  contains two vertices  $\omega^s$  and  $\omega^l$  of  $\mathcal{P}_n$  and if  $|k_j - k_{j+1}| \neq 1$ , we get a similar contradiction in (3.5) by considering the vertex  $\omega^s$  or  $\omega^l$ .

We have thus proved that  $|k_{j+1} - k_j| = 1$  for any switching point  $\tau_j$ . To conclude the proof of the theorem, we have to show that the extremal control does not contain a subsequence of the form  $\{\omega^p, \omega^{p+1}, \omega^p, \dots\}$ , where  $p \in \mathbb{N}$ . Let us assume that an extremal control  $Z$  takes the form

$$Z(\theta) = \mathbb{1}_{[\tau_1, \tau_2[} + \omega \mathbb{1}_{[\tau_2, \tau_3[} + \mathbb{1}_{[\tau_3, \tau_4[} + \tilde{Z}(\theta), \quad \theta \in [0, \delta],$$

where  $\tau_1 < \tau_2 < \tau_3 < \tau_4$  and  $(\tau_2, \tau_3)$  are two consecutive switching points, and  $\tilde{Z}$  is the restriction of  $Z$  on  $[0, \delta] \setminus [\tau_1, \tau_4]$ :

$$\tilde{Z} = Z|_{[0, \delta] \setminus [\tau_1, \tau_4]}.$$

It is always possible to consider this case by multiplying  $Z$  by  $\bar{\omega}^p$ , since it does not change the extremality of  $(W, Z)$ . As  $Z$  is switching from 1 to  $\omega$  for  $\theta = \tau_2$ , we have by Lemma 3.5  $W(\tau_2) = t_2 \omega^{\frac{1}{2}}$ ,  $t_2 \in \mathbb{R}$ . Notice that by (3.14), we necessarily have  $t_1 < 0$ . Indeed, by the maximality condition, the value of the Hamiltonian on the extremal is greater than the value of the Hamiltonian obtained with  $(\tilde{U}, \tilde{V}) = (0, 0)$ . At the switching point  $\tau_3$ , we similarly have  $W(\tau_3) = t_3 \omega^{\frac{1}{2}}$ , where  $t_3 < 0$ . Hence, the vectors  $W(\tau_2)$  and  $W(\tau_3)$  are parallel. For  $\theta \in [\tau_2, \tau_3]$ , the function  $\theta \mapsto W(\theta)$  describes an arc of an ellipse whose center is the point  $\omega$ . Indeed, by (3.7), we have

$$W(\theta) = \omega + A_2 e^{i\theta} + B_2 e^{-i\theta}, \quad (A_2, B_2) \in \mathbb{C}^2.$$

Hence, the vectors  $W(\tau_2)$  and  $W(\tau_3)$  are equal or opposite because the line directed by  $W(\tau_2)$  crosses the ellipse in at most two points. But, as we have  $W(\tau_2) \cdot W(\tau_3) = t_2 t_3 > 0$ , we must have

$$W(\tau_2) = W(\tau_3).$$

This condition will bring a contradiction. Let  $\mathcal{E}$  be the ellipse of center  $\omega$  on which the function  $W$  takes its values for  $\theta \in [\tau_1, \tau_2]$ .

*First case.*  $\mathcal{E}$  is not degenerated. The function  $W$  satisfies  $W(\tau_2) = W(\tau_3)$ . As  $W$  is of class  $C^1$ , it must go all over the ellipse, and this is possible only if  $\tau_2 = \tau_1 + 2k\pi$ ,  $k \in \mathbb{N}^*$ . As  $(\tau_2, \tau_3)$  belong to the interval  $[0, \delta]$ , we get a contradiction.

*Second case.*  $\mathcal{E}$  is a segment which contains  $W(\tau_2)$  and  $\omega$ . For  $\theta \in [\tau_2, \tau_3]$ ,  $W(\theta)$  takes its values within this segment. For  $\theta \in [\tau_1, \tau_2]$ , the function  $\theta \mapsto W(\theta)$  takes its values within an ellipse  $\mathcal{E}'$  whose center is the point  $(1, 0)$ . By Lemma 3.5,  $W$  satisfies, for  $\theta = \tau_2$ ,  $W(\tau_2) = t_2 \omega^{\frac{1}{2}}$ . Hence, the function  $W$  cannot be of class  $C^1$  at the point  $\theta = \tau_2$ , since  $W(\theta)$  is parallel to  $W(\tau_2)$  for  $\theta \in [\tau_2, \tau_3]$ . We thus get a contradiction.

We have thus proved that for any switching point  $\tau_j$ ,  $k_{j+1} - k_j = \varepsilon$ , where  $\varepsilon = \pm 1$  is fixed by the rotation of  $Z$  clockwise or counterclockwise. This concludes the proof of the theorem.  $\square$

We now show that an extremal control switches a finite number of times on the interval  $[0, \delta]$ .

**THEOREM 3.3.** *Let  $W$  be an extremal trajectory of the system. Then there exists a subdivision  $(\tau_j)_{0 \leq j \leq r}$  of  $[0, \delta]$  with  $r \in \mathbb{N}^*$  such that  $\tau_0 = 0$  and  $\tau_{r+1} = \delta$  and such that on each  $[\tau_j, \tau_{j+1}[$  the extremal control  $(U, V)$  satisfies  $Z = \omega^{\varepsilon_j + h}$ , where  $h \in \mathbb{N}$ ,  $\varepsilon = \pm 1$ .*



*Proof.* Let us prove that the number of switching points is finite on the interval  $[0, \delta]$ . Assume that there exists a sequence  $(\tau_j)$  of switching points in  $[0, \delta]$  that converges to a point  $\tau \in [0, \delta]$ . We will show that

$$(3.17) \quad W(\tau) = 0, \quad \dot{W}(\tau) = 0.$$

Assume that  $Z$  rotates clockwise, that is,  $\varepsilon = \pm 1$ . We have by Lemma 3.5

$$W(\tau_j) = t_j \omega^{j+\frac{1}{2}}.$$

As  $W$  is of class  $C^{1,1}$  on  $[0, \pi]$ , the sequence  $(t_j)$  is bounded. Consequently (up to a subsequence), we can assume that the sequence  $(t_j)$  converges to a real  $t \in \mathbb{R}$ . Assume that  $t \neq 0$ ; then there exists  $j_0 \in \mathbb{N}$  such that, for  $j \geq j_0$ , we have  $t_j \neq 0$ . Hence,  $\frac{W(\tau_j)}{W(\tau_{j+1})}$  converges to 1 and

$$\frac{W(\tau_j)}{W(\tau_{j+1})} = \frac{t_j}{t_{j+1}} \bar{\omega},$$

which converges to  $\bar{\omega}$ . Thus  $t = 0$  and  $W(\tau) = 0$ . Again, we get a contradiction if we assume that  $\dot{W}(\tau) \neq 0$ . This shows (3.17). The Hamiltonian  $H$  along this extremal is 0. By (3.5) and by (3.14), the value of  $H$  is greater than the value of  $H$  for  $(\tilde{U}, \tilde{V}) = (0, 0)$ . It follows that  $W \equiv 0$  and  $Z \equiv 0$ . This extremal represents the disc, which is not a minimizer. An extremal trajectory then has a finite number of switching points. Finally, if we consider  $\omega^h$ ,  $h \in \mathbb{N}$ , the initial value of the control, and  $\varepsilon = \pm 1$ , the rotation clockwise or counterclockwise of the control, then we get the theorem.  $\square$

We now compute the exact number of switching points of an extremal. We prove the following result.

**THEOREM 3.4.** *Let  $W$  be an extremal trajectory and  $Z$  the extremal control. Then we have*

$$(3.18) \quad Z = \sum_{0 \leq j \leq s} \omega^{\varepsilon j+h} \mathbb{1}_{[\tau_j, \tau_{j+1}[},$$

where  $\varepsilon \in \{\pm 1\}$ ,  $h \in \mathbb{N}$ , and  $\tau_0 = 0 < \tau_1 < \dots < \tau_s < \tau_{s+1} = \delta$ . Moreover, the number  $L$  of switching points of  $Z$  in the interval  $[0, \delta]$  is given by

$$(3.19) \quad L = s + 1 = ln - \varepsilon, \quad l \in \mathbb{N}^*.$$

*Proof.* By Theorem 3.3, an extremal control  $Z$  takes the values  $(\omega^{\varepsilon j+h})_{0 \leq j \leq n-1}$  with  $h \in \mathbb{N}$  and  $\varepsilon = \pm 1$  on a finite subdivision of  $[0, \delta]$  denoted by  $(\tau_j)_{0 \leq j \leq s+1}$  with  $\tau_0 = 0$  and  $\tau_{s+1} = \delta$ . Without loss of generality, we can assume that  $\varepsilon = +1$ . If  $Z = \omega^h$  for  $\theta = 0^+$ , by performing a rotation of the control, that is, by changing  $Z$  into  $Z\bar{\omega}^h$ , we can always assume that  $Z(0^+) = 1$ . By extending the function  $W$  to  $\mathbb{R}$  by the relation  $W(\theta + \delta) = \bar{\omega}W(\theta)$  (recall that  $W$  is in  $\Gamma$ ), we can assume that 0 is a switching point. The function  $Z$  is now given by

$$Z = \sum_{0 \leq j \leq s} \omega^j \mathbb{1}_{[\tau_j, \tau_{j+1}[},$$

with  $\tau_0 = 0 < \tau_1 < \dots < \tau_s < \tau_{s+1} = \delta$ . As  $Z$  is in  $\mathcal{Z}$ , we must have  $Z(\delta^+) = \bar{\omega}Z(0^+) = \bar{\omega}$ . On the interval  $[\tau_s, \delta]$ , we have  $Z = \omega^s$ . Consequently, the point  $\delta$  is

a switching point, and we must have  $\omega^{s+1} = \bar{\omega}$ . Thus,  $s+1$  is of the form  $s+1 = -1+ln$ ,  $l \in \mathbb{N}^*$ . The number of switching points in the interval  $[0, \delta]$  is  $s+1$ , as  $\delta$  is not considered as a switching point of this interval. We have proved the theorem in the case where  $\varepsilon = +1$ . When the control  $Z$  satisfies  $Z = \bar{\omega}^j$ , the proof is the same, and we must have  $\bar{\omega}^{s+1} = \bar{\omega}$ . Consequently,  $s$  is given by  $s = ln$ ,  $l \in \mathbb{N}^*$ . In this case the number of switching points is  $s+1 = ln+1$ . This ends the proof of the theorem.  $\square$

In the case of regular rotors  $O_n^{ln \pm 1}$ , the switching points are of the form  $j\tau$ ,  $j = 1, \dots, s = ln \pm 1 - 1$  with  $\tau = \frac{\delta}{s+1}$ , and the associated control is given by (2.39). In the next section, we show that the distance between two consecutive switching points  $\tau_j$  and  $\tau_{j+1}$  of an extremal is constant. This will prove that a minimizer is necessarily a regular rotor.

An extremal  $(W, Z)$  given by (3.18) satisfies on each interval  $[\tau_j, \tau_{j+1}]$

$$(3.20) \quad W(\theta) = A_j e^{i\theta} + B_j e^{-i\theta} + \omega^{\varepsilon j + h}.$$

A simple computation using (3.14) shows that the Hamiltonian along this trajectory is

$$(3.21) \quad H = 2|A_j|^2 + 2|B_j|^2 - 1 \quad \forall 0 \leq j \leq s,$$

and, as  $H$  is constant, we have

$$|A_j|^2 + |B_j|^2 \equiv cst \quad \forall 0 \leq j \leq s.$$

**4. Conserved quantities along the extremal trajectories.** In this section we prove by an extension of Noether's theorem in optimal control theory that the angular momentum is conserved along an extremal trajectory. Combining the two conserved quantities (Hamiltonian and angular momentum) we will show that extremal trajectories describe regular rotors. We use the results of Torres (see [31], [32]) in order to prove the conservation of the angular momentum.

**4.1. Conservation of the angular momentum.** Let  $M$  be the function defined on the interval  $[0, \delta]$  by

$$M(\theta) = \Im((\bar{W}(\theta) - \bar{Z}(\theta))\dot{W}(\theta)), \quad \theta \in [0, \delta],$$

where  $(W(\theta), Z(\theta))$  is an admissible trajectory of problem (3.2). This quantity is usually called the *angular momentum* in mechanics (cross product between the position and the velocity). If  $(W(\theta), Z(\theta))$  is an extremal trajectory of (3.2) given by (3.18), we have, for  $0 \leq j \leq s$ , and  $\theta \in [\tau_j, \tau_{j+1}[$ ,

$$M(\theta) = \Im((\bar{W}(\theta) - \bar{\omega}^{\varepsilon j + h})\dot{W}(\theta)).$$

By differentiating, we get

$$\dot{M}(\theta) = 0 \quad \forall \theta \in [\tau_j, \tau_{j+1}[.$$

This proves that the function  $M(\theta)$  is piecewise constant on each  $[\tau_j, \tau_{j+1}]$ . We now show a stronger result.

**THEOREM 4.1.** *Along an extremal trajectory of (3.2), the quantity  $M(\theta)$  is constant:*

$$\forall \theta \in [0, \delta], \quad \dot{M}(\theta) = 0.$$

*Proof.* Let us consider the  $C^1$  transformation  $h^\alpha : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ ,  $\alpha \in \mathbb{R}_+$ , defined by

$$(4.1) \quad h^\alpha(W, Z) = e^{i\alpha}(W - Z) + Z.$$

Q11 Geometrically speaking,  $h^\alpha(W, Z)$  is the image of  $W - Z$  by the rotation of angle  $\alpha$  and whose center is  $Z$ . For any  $(W, Z) \in \mathbb{C}^2$ , we have  $h^0(W, Z) = W$ . Now, given an extremal trajectory  $(W(\theta), Z(\theta))$  of (3.2), we denote by  $W^\alpha$  the image of  $(W(\theta), Z(\theta))$  by  $h^\alpha$ . We then have on  $[0, \delta]$

$$\ddot{W}^\alpha + W^\alpha = Z.$$

Consequently,  $W^\alpha$  satisfies the same equation as  $W$ , and the extremal control associated with  $W^\alpha$  is  $Z$ . Let  $L : \mathbb{C} \times \mathbb{C}$  be the  $C^1$  map defined by

$$L(W, Z) = \Re(W\bar{Z}).$$

If  $(W(\theta), Z(\theta))$  is an extremal trajectory, we have

$$L(W^\alpha, Z) = \cos(\alpha)L(W, Z) - \sin(\alpha)\Im(W\bar{Z}) + 1 - \cos(\alpha).$$

Considering the  $C^1$  map  $F : \mathbb{C} \times \mathbb{C} \times \mathbb{R}_+ \rightarrow \mathbb{R}$  defined by

$$F(W, \dot{W}, \alpha) = -\sin(\alpha)\Im(W\bar{\dot{W}}),$$

we then have along an extremal trajectory  $(W(\theta), Z(\theta))$

$$L(W^\alpha(\theta), Z(\theta)) = \cos(\alpha)L(W(\theta), Z(\theta)) + \frac{d}{d\theta}F(W(\theta), \dot{W}(\theta), \alpha) + 1 - \cos(\alpha) \quad \forall \theta \in [0, \delta].$$

By (3.12), the scalar product between the state variable  $W$  and the dual variable  $\Pi$  is

$$\langle W, \Pi \rangle = -\Re(W\bar{\Pi}) + \Re(\dot{W}\bar{\Pi}).$$

Now we are in position to derive consequences of the invariance theorem (see [31]). Let  $(W(\theta), Z(\theta))$  be an extremal trajectory of (3.2),  $H$  the Hamiltonian along this trajectory, and  $\Pi(\theta)$  the dual variable. We then have

$$(4.2) \quad p_0 \frac{\partial F(W(\theta), Z(\theta), \alpha)}{\partial \alpha} \Big|_{\alpha=0} + \left\langle \frac{\partial W^\alpha(\theta)}{\partial \alpha} \Big|_{\alpha=0}, \Pi(\theta) \right\rangle - H \equiv cst$$

for all  $\theta \in [0, \delta]$ . But, we have

$$p_0 \frac{\partial F(W(\theta), Z(\theta), \alpha)}{\partial \alpha} \Big|_{\alpha=0} = -\Im(\bar{W}(\theta)\dot{W}(\theta)) \quad \forall \theta \in [0, \delta],$$

and by Lemma 3.3, we can take  $\Pi = -W$  so that

$$\left\langle \frac{\partial W^\alpha(\theta)}{\partial \alpha} \Big|_{\alpha=0}, \Pi(\theta) \right\rangle = \Im(\bar{W}(\theta)Z(\theta)) + 2\Im(\dot{W}(\theta)\bar{W}(\theta)) \quad \forall \theta \in [0, \delta].$$

As the Hamiltonian is constant along an extremal trajectory, we get by (4.2)

$$\Im((\bar{W} - \bar{Z})\dot{W}) \equiv cst.$$

This ends the proof of the theorem.  $\square$

#### 4.2. Conserved quantities and equidistance of the switching points.

Thanks to the two conserved quantities along a Pontryagin extremal, we are now in position to prove the equidistance of the switching points. We first show the following proposition.

**PROPOSITION 4.1.** *For an extremal trajectory given by (3.18), we have for  $0 \leq j \leq s$ ,  $\tau_{j+1} - \tau_j = \tau_1 - \tau_0$ .*

*Proof.* A simple computation shows that for an extremal given by (3.18) we have on each  $[\tau_j, \tau_{j+1}[$ ,  $0 \leq j \leq s$ ,

$$(4.3) \quad M(\theta) = |A_j|^2 - |B_j|^2, \quad \theta \in [\tau_j, \tau_{j+1}[.$$

Thus, by (3.21) and Theorem 4.1, we get, for  $0 \leq j \leq s$ ,

$$(4.4) \quad \begin{cases} |A_j| = |A_0|, \\ |B_j| = |B_0|. \end{cases}$$

Since  $W$  is of class  $C^1$  at each switching point  $\tau_j$ , the coefficients  $A_j$  and  $B_j$ ,  $1 \leq j \leq s$ , are given by

$$(4.5) \quad \begin{cases} A_j = A_{j-1} + \frac{1}{2}(\omega^{j-1} - \omega^j)e^{-i\tau_j}, \\ B_j = B_{j-1} + \frac{1}{2}(\omega^{j-1} - \omega^j)e^{i\tau_j}. \end{cases}$$

Combining (4.4) and (4.5), we get

$$(4.6) \quad \Re(A_j \overline{A_{j-1}}) \equiv cst, \quad 1 \leq j \leq s.$$

Geometrically speaking, the complex  $(A_j)_{0 \leq j \leq s}$  lie on a circle whose center is the origin and whose radius is  $|A_0|$ , and  $A_{j+1}$  is the image of  $A_j$  by a rotation of a fixed angle by (4.6). In terms of the switching point  $(\tau_j)_{1 \leq j \leq s}$ , the phase between  $A_{j+1} - A_j$  and  $A_j - A_{j-1}$  is  $\delta - (\tau_{j+1} - \tau_j)$  by (4.5). But using (4.4) and (4.6), the phase between these two complex numbers is the same as the phase between  $A_j$  and  $A_{j-1}$ . By (4.6), the phase between  $A_{j+1} - A_j$  and  $A_j - A_{j-1}$  is constant, which ends the proof of the proposition.  $\square$

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**COROLLARY 4.1.** *Let  $W$  be an extremal trajectory of the system and  $Z$  the extremal control. Then the corresponding rotor is in the class  $(O_n^{ln\pm 1})_{l \in \mathbb{N}^*}$ , and the extremal control  $Z$  is given by (2.39).*

*Proof.* This is a consequence of the previous proposition, as two consecutive switching points of an extremal are equidistant. The corresponding rotor given by (3.18) satisfies  $\tau_{j+1} - \tau_j \equiv cst$ , and it is necessarily an element of  $(O_n^{ln\pm 1})_{l \in \mathbb{N}^*}$ .  $\square$

By Proposition 2.11, the rotor of minimal area in the class  $O_n^{ln\pm 1}$  is  $O_n^{n-1}$  (with the least number of arcs). As the rotor of minimal area necessarily belongs to this class (by the PMP), it is  $O_n^{n-1}$ . By (2.39) the optimal control  $Z_{min}$  corresponding to  $O_n^{n-1}$  is obtained for  $s+1 = n-1$  and is given by

$$(4.7) \quad Z_{min} = \sum_{0 \leq j \leq n-2} \omega^j \mathbb{1}_{[j \frac{\delta}{n-1}, (j+1) \frac{\delta}{n-1}[}$$

This proves Goldberg's conjecture (Theorem 1.1). Note that there is no necessity of verifying the optimality of the extremal trajectories corresponding to  $(O_n^{ln\pm 1})_{l \in \mathbb{N}^*} \setminus \{O_n^{n-1}\}$ , as  $O_n^{n-1}$  is of minimal area in this class.

TABLE 1  
 Values of the  $r_j$  for  $n = 3, 4, 5, 6$ .

$n$	$r_0$	$r_1$	$r_2$	$r_3$	$r_4$	$r_5$
3	0	$3r$	0			
4	0	$2r$	$2r$	0		
5	0	$r_1$	$r_2$	$r_1$	0	
6	0	$r$	$2r$	$2r$	$r$	0

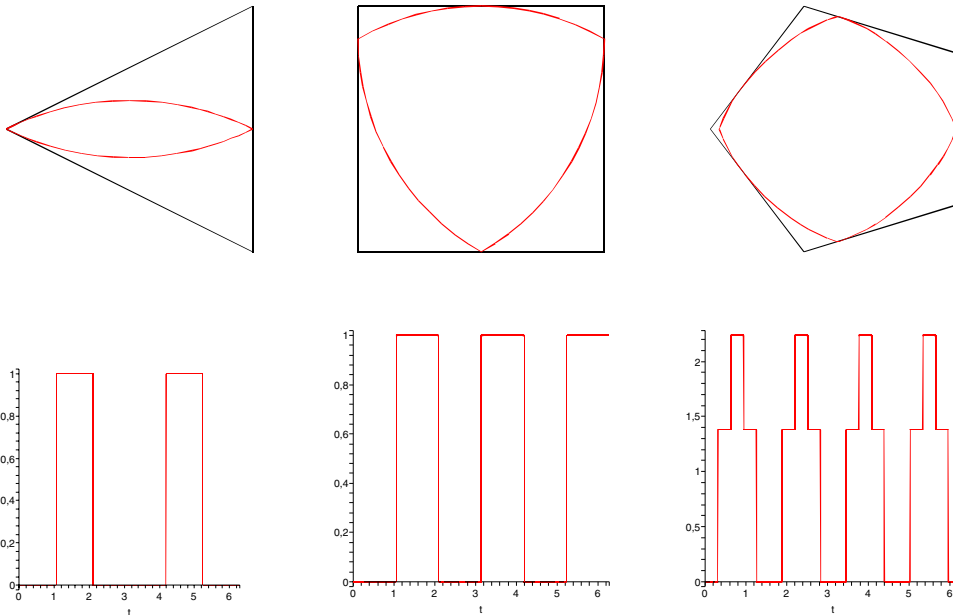


FIG. 3. The minimizers in the cases  $n = 3$  ( $\Delta$ -biangle),  $n = 4$  (Reuleaux triangle), and  $n = 5$  ( $O_5^{\frac{\delta}{2}}$ ) and their respective radii of curvature on the interval  $[0, 2\pi]$ .

Geometrically speaking, if we come back to the initial parameterization of a rotor by its support function  $p$ , the rotor  $O_n^{n-1}$  is the union of arcs of circles of radii  $r_j$ :

$$\begin{aligned}
 r_j &= \frac{r}{\cos\left(\frac{\delta}{2}\right)} \left( \cos\left(\frac{\delta}{2}\right) - \cos\left(\left(j + \frac{1}{2}\right)\delta\right) \right) \\
 &= \frac{r}{\cos\left(\frac{\delta}{2}\right)} \Re(\omega^{1/2} - \omega^{j+1/2}), \quad j = 0, \dots, n-1.
 \end{aligned}$$

These values of the radii of curvature are precisely equal to the distances of the diagonals of the  $n$ -gon from the parallel sides (see [17], [18]), and the sectors are all equal to  $\frac{2\pi}{n(n-1)}$ , as the switching points are equidistant. In Table 1, we give the different values of the radii  $r_j$  for  $n = 3, 4, 5, 6$ . For  $n = 5$ , there are two different radii  $r_1 < r_2$ , and  $r$  denotes the radius of the inscribed circle. If  $n$  is even, there are exactly  $\frac{n-2}{2}$  values of the  $r_j$ , and if  $n$  is odd, there are exactly  $\frac{n-1}{2}$  values of the  $r_j$ . By (2.39), the radius of curvature of the boundary of  $O_n^{n-1}$  is  $\frac{2\pi}{n-1}$ -periodic (see [11]). We have represented in Figure 3 the minimizers of the area for  $n = 3$ ,  $n = 4$ , and  $n = 5$  and their respective radii of curvature.

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