Minimal time control of the two tanks gradostat model under a cascade inputs constraint

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Abstract

We study the minimum time control problem of a series of two interconnected chemostats under the input constraint $u_2 \leq u_1$, where u_i are the respective dilution rates in the tanks. This constraint brings controllability issues in the study of the optimal strategies. We encounter this difficulty by splitting the state domain into two sub-domains, one with no lack of controllability of the target, and its complementary is such that any optimal control fulfills $u_1 = u_2$. We explicit the complete optimal synthesis that depends on the position of the target with respect to semi-permeable curve that passes through a *steady-state singular point*.

Keywords. optimal control, minimal time problem, Pontryagin Maximum Principle, optimal synthesis, chemostat model, gradostat model, non-linear controllability, semi-permeability.

MSC. 49J15, 49K15, 49N25.

Introduction

The chemostat apparatus has been invented simultaneously by Novick & Szilard and Monod in the fifties, for the so-called "continuous culture" of micro-organisms. Its mathematical model is classically written as the system of two differential equations (see [28]):

$$\begin{cases} \dot{x} = [\mu(s) - u]x_1, \\ \dot{s} = -\mu(s)x + u_1(s_{in} - s), \end{cases}$$

where x and s stand respectively for the biomass and nutrient concentrations. The function $\mu(\cdot)$ is the specific growth rate of the micro-organisms (which is most often kept as a monotonic increasing function). The operating parameters are the input concentration of nutrient s_{in} and the dilution rate u. Notice that in the former equations the yield coefficient of the bio-conversion does not appear, because we have simply chosen it to be equal to one (that can be always done without any loss of generality).

The chemostat device is designed to be operated at steady state, that is for steady state (x_{eq}, s_{eq}) that fulfill $\mu(s_{eq}) = u$ and $x_{eq} = s_{in} - s_{eq}$. Such a model is often used as a good representation of the functioning of bioreactors in the biotechnology or waste-water industries, or for ecological investigations of the growth of micro-organisms in natural environments, such as mountain lakes.

Cascades of chemostats or bioreactors are also quite popular in microbiology (named "gradostats" [13, 20, 27, 29]) or in bio-processes (named "serial tanks" [1]), because it is a way to create a gradient of resources (see also [11, 15]). Such gradients are expected to be more realistic to mimic real environment for studying micro-organisms growth [8, 17]. In the biotechnological industry, series of bioreactors are also known to be

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more efficient for the resource conversion at steady state than single tanks [1, 16, 18, 25]. The extension of the chemostat model to a cascade of two chemostats is straightforwardly given by the following system of differential equations

$$\begin{cases} \dot{x}_1 = [\mu(s_1) - u_1]x_1, \\ \dot{s}_1 = -\mu(s_1)x_1 + u_1(s_{in} - s_1), \\ \dot{x}_2 = [\mu(s_2) - u_2]x_2, \\ \dot{s}_2 = -\mu(s_2)x_2 + u_2(s_1 - s_2), \end{cases} \tag{0.1}$$

accordingly to the picture given in Figure 1. Notice that the input variables u_1, u_2 are not independent,

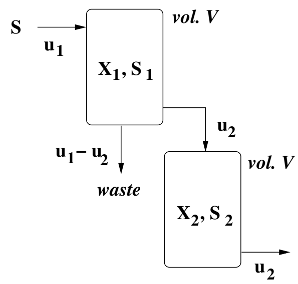

Figure 1: Representation of a cascade of two continuous stirred tank reactors. The input substrate concentration in the first reactor is equal to $s_{in} > 0$.

because the input flow of the second tank cannot be larger than the output flow of the first tank. When volumes of the tanks are identical (what we shall assume without loss of generality), this amounts to consider controls u that satisfies the *inputs cascade constraint*:

$$u \in U := \left\{(u_1, u_2) \in \mathbb{R}^2 \mid 0 \leq u_2 \leq u_1 \leq u_{\max}\right\} .$$

As for the chemostat, the gradostat device is designed to be operated at steady state. Typical practical questions that occur in laboratory or in industrial frameworks are related to changes of the operation point from the current steady state to another one. For instance, this problem has been raised in recent investigations of wine fermentation processes [9], where each tank corresponds to a precise metabolic state of the micro-organisms related to the level of substrate concentration, that one would like to adjust in each tank.

One can notice on equations (0.1) that any steady state belongs to the hyper-plane $s_i + x_i = s_{in}$ ($i = 1, 2$), that is invariant for any control law $u(\cdot)$. For our study, the dynamics (0.1) can then be reduced to the following planar one:

$$\begin{cases} \dot{s}_1 = -\nu(s_1) + u_1(s_{in} - s_1), \\ \dot{s}_2 = -\nu(s_2) + u_2(s_1 - s_2), \end{cases} \tag{0.2}$$

where we have set $\nu(s) := \mu(s)(s_{in} - s)$. The problem of change of operation point amounts then to choose a control $u(\cdot)$ that drives the system (0.2) towards a new target $\bar{s} = (\bar{s}_1, \bar{s}_2)$. It is important to mention that both controls are necessary in order to drive the system to a given equilibrium for s_1 and s_2. For this objective, one can simply adjust the control u to the value that corresponds to the new desired steady state, and wait for the asymptotic convergence of the state... Instead, a feedback strategy for the minimal time criterion appears to be a natural choice for the practitioners, that allows them to gain time while manipulating the variable u (as this is often considered for other problems in the same application fields [14, 10, 24]). Our objective in this paper is to find an optimal feedback control steering (0.2) from any initial state to a given target $\bar{s}$ in minimal time.

For minimal time control problems in the plane with dynamics linear with respect to the control variables, the theory has been mostly developed for control sets in ℓ_1 or ℓ_2-balls (see e.g. [4, 3, 19]). Notice that the inputs constraint (0.2) has a different geometry, as it is is not diffeomorphic to any ℓ_1 or ℓ_2-balls and has "corners". This unusual feature leads to new local controllability issues for the synthesis of optimal feedbacks, that we precisely investigate in the present work.

One can easily check that there exists an invariant domain $\mathcal{D}$ for (0.2) (forwardly), and we shall restrict our study for initial conditions in this domain. Given a target point $\bar{s}$ in $\mathcal{D}$, our main key is to characterize a subset $\mathcal{C}(\bar{s}) \subset \mathcal{D}$ that fulfills the following properties:

- $\bar{s} \in \mathcal{C}(\bar{s})$ is reachable from any initial condition in $\mathcal{C}(\bar{s})$, and any optimal trajectory stays in $\mathcal{C}(\bar{s})$,
- any optimal control from an initial condition outside $\mathcal{C}(\bar{s})$ is such that $u_1(t) = u_2(t)$ almost everywhere.

This feature allows us to decompose the original optimal feedback synthesis problem in $\mathcal{D}$ into two simpler sub-problems:

1. the minimal time control problem in $\mathcal{C}(\bar{s})$ with two constrained controls, without any controllability issue,
2. the minimal time control problem in $\mathcal{D}$ with $\mathcal{C}(\bar{s})$ as a target, with a scalar control, but with some controllability issues.

The paper is organized as follows. The first section gives hypotheses and introduces the set $\mathcal{C}(\bar{s})$. We also provide some qualitative properties of its boundary. In the second section, we give the optimal synthesis for initial conditions in the set $\mathcal{C}(\bar{s})$ (Proposition 2.2). We show that there exist infinitely many optimal trajectories steering a point in this set to the target. In the third section, we apply the Pontryagin Maximum Principle, and we show that for initial conditions outside the set $\mathcal{C}(\bar{s})$, we have $u_1 = u_2$. This reduction allows to use properties of affine systems in the plane with one input in order to conclude on the optimal synthesis (see e.g. [4]). We show that either the target is never reached from the set $\mathcal{D} \backslash \mathcal{C}(\bar{s})$ (case I), or it can be reached from any initial condition outside $\mathcal{C}(\bar{s})$, and the optimal strategy is singular (case II). In this section, we exhibit a partition of the domain $\mathcal{D}$ into two sub-domains A and B. We show that if the target is in A (resp. in B) then, the optimal synthesis is as in case I (resp. case II). This decomposition of $\mathcal{D}$ is based on properties of semi-permeable curves ([5, 6, 26]). Theorem 3.3 is our main result and provides a complete optimal synthesis of the problem. The last section is devoted to numerical simulations for particular choices of the growth function μ.

1 Preliminaries

In this section, we introduce the set $\mathcal{C}(\bar{s})$ for a given target $\bar{s}$, and we give some properties of this set.

1.1 Hypotheses and notations

We assume the usual hypotheses on the function $\mu(\cdot)$:

(H1) $\mu(\cdot)$ is a bounded C^2 non-negative increasing function defined $[0, +\infty[$ with $\mu(0) = 0$.

A typical example is when μ is of Monod type, see [28]: $\mu_m(s) := \frac{\bar{\mu} s}{k+s}$, where $k > 0$, $\bar{\mu} > 0$ or when $\mu(s) = \mu_m(s^\theta)$, for some $\theta > 0$. In section 4, we also consider the case where μ is linear: $\mu(s) = ms$, $m > 0$.

We consider the set $\mathcal{U}$ of admissible controls as measurable functions $u(\cdot)$ that take values in U. Classically, Hypothesis (H1) ensures the existence and uniqueness of solutions of (0.2) for any admissible control and positive time. Straightforwardly, one can check that the domain $E := \{(s_1, s_2) \in \mathbb{R}^2_+ \mid 0 \leq s_1 \leq s_2 \leq s_{in}\}$ is forwardly invariant for any $u(\cdot) \in \mathcal{U}$. Notice that the line segment

$$L := \{s_{in}\} \times [0, s_{in}]$$

that lies on the boundary of $\mathcal{D}$ is also invariant. Moreover, the minimal time problem restricted to L is a one-dimensional problem and is easy to solve. From now on, we consider targets in the interior of the domain E, and we shall consider in the following initial conditions in the set $\mathcal{D}$ defined by:

$$\mathcal{D} := E \backslash L.$$

Notice that $\mathcal{D}$ is neither open nor closed.

Next, we assume that the maximum dilution rate is large enough to compete the growth of the species on the substrate, that is $u_{\max}$ larger than $\mu(s_{in})$. This assumption is required for the controllability of the variable s_1 (otherwise s_1 is non-increasing for any control u_1). Without any loss of generality we can also take $u_{\max} = 1$ (one can always re-normalize time and $\mu(\cdot)$ of a factor $1/u_{\max}$):

(H2) $u_{\max} = 1$ and $\mu(s) < 1$ for any $s \in [0, s_{in}]$.

For any admissible control $u(\cdot)$, we shall denote, for convenience, $s(\cdot, s^0, u(\cdot))$ the forward solution of (0.2) with $s(0) = s^0$, and $s^-(\cdot, s^0, u(\cdot))$ the backward solution, that is $s^-(t, s^0, u(\cdot)) = s(-t, s^0, u(-\cdot))$ for any $t \geq 0$. We shall also denote $\gamma^+(s^0, u(\cdot))$ and $\gamma^-(s^0, u(\cdot))$ the positive and negative semi-orbits of the dynamics with control $u(\cdot)$.

The Hamiltonian $H = H(s_1, s_2, \lambda_0, \lambda_1, \lambda_2, u_1, u_2)$ associated to (0.2) is defined by:

$$H := -\lambda_1\mu(s_1)(s_{in} - s_1) - \lambda_2\mu(s_2)(s_{in} - s_2) + \lambda_0 + \lambda_1(s_{in} - s_1)u_1 + \lambda_2(s_1 - s_2)u_2. \tag{1.1}$$

Pontryagin Maximum principle can be stated as follows. Let $u := (u_1, u_2)$ an optimal control and $s := (s_1, s_2)$ the associated trajectory. There exists $t_f > 0$, $\lambda_0 \leq 0$ and $\lambda : [0, t_f] \to \mathbb{R}^2$ satisfying the adjoint equations $\dot{\lambda} = -\frac{\partial H}{\partial s}$, that is:

$$\begin{cases} \dot{\lambda}_1 = \lambda_1[\mu'(s_1)(s_{in} - s_1) - \mu(s_1) + u_1] - \lambda_2 u_2 = \lambda_1[\nu'(s_1) + u_1] - \lambda_2 u_2, \\ \dot{\lambda}_2 = \lambda_2[\mu'(s_2)(s_{in} - s_2) - \mu(s_2) + u_2] = \lambda_2[\nu'(s_2) + u_2], \end{cases} \tag{1.2}$$

Moreover, we have the maximization condition:

$$u(t) \in \arg \max_{0 \leq \omega_2 \leq \omega_1 \leq 1} H(s_1(t), s_2(t), \lambda_0, \lambda_1(t), \lambda_2(t), \omega_1, \omega_2). \tag{1.3}$$

An *extremal trajectory* is a sextuplet $(s_1(\cdot), s_2(\cdot), \lambda_0, \lambda_1(\cdot), \lambda_2(\cdot), u(\cdot))$ satisfying (0.2)-(1.2)-(1.3). We say that an extremal is *normal* whenever $\lambda_0 \neq 0$ (in that case, we can take $\lambda_0 = -1$). In the case where $\lambda_0 = 0$, we say that the extremal trajectory is *abnormal*. The Hamiltonian is zero along an extremal trajectory (as t_f is free), thus we obtain:

$$-\lambda_1\mu(s_1)(s_{in} - s_1) - \lambda_2\mu(s_2)(s_{in} - s_2) + \lambda_0 + \lambda_1(s_{in} - s_1)u_1 + \lambda_2(s_1 - s_2)u_2 = 0. \tag{1.4}$$

Given the control constraints, we introduce the the two *switching functions* that will allow to determine an extremal control:

$$\begin{cases} \phi_1 := \lambda_1(s_{in} - s_1), \\ \phi_2 := \lambda_1(s_{in} - s_1) + \lambda_2(s_1 - s_2) = \phi_1 + \lambda_2(s_1 - s_2). \end{cases} \tag{1.5}$$

We say that t_0 is a *switching point* if for any neighborhood $\mathcal{W}$ of t_0, the control u is non-constant in $\mathcal{W}$. At a switching point, we necessarily have $\phi_i(t_0) = 0$ for $i = 1$ or $i = 2$ or both.

Remark 1.1. *The control constraint set E implies the particular choice of the second switching function ϕ_2 in* (1.5)*. Notice that both controls are not independent in* (1.3)*, which justifies this choice.*

Let us now recall the definition of reachable set from a point s^0, see [7]. Given $\bar{s} \in \mathcal{D}$, $s^0 \in \mathcal{D}$, and $u \in \mathcal{U}$, we define $t_{s^0}(u) \in [0, +\infty]$ as the *first entry time* into the target, that is:

$$t_{s^0}(u) := \inf\{t \geq 0 \ \mid \ s(t) = \bar{s}\},$$

where $s(t)$ is the unique solution of (0.2) starting from s^0 at time 0. Now, let $T : \mathcal{D} \to [0, +\infty]$ denotes the minimal time function:

$$T(s^0) := \inf_{u \in \mathcal{U}} t_{s^0}(u).$$

The *reachable set* is defined as the set of starting points from which the system can reach the target in a certain time $t > 0$:

$$\mathcal{R} := \{s^0 \in \mathcal{D} \ \mid \ T(s^0) < +\infty\}.$$

1.2 Definition of the set $\mathcal{C}(\bar{s})$

We first give properties of forward and backward solutions for the extreme controls $(0,0)$ and $(1,1)$ that will be useful in the following. Recall that $\nu(s) := \mu(s)(s_{in}-s)$.

Lemma 1.1. *Take an initial state $s_0 \in (\mathcal{D} \setminus L) \setminus \{(0,0\}$.*

- *For $u=(0,0)$, one has*

$$\lim_{t\to+\infty} s_1(t) = \lim_{t\to+\infty} s_2(t) = 0 \textit{ with } \lim_{t\to+\infty} \frac{s_2(t)}{s_1(t)} = \exp\left(\int_{s_2^0}^{s_1^0} \frac{\nu'(0)}{\nu(\sigma)} d\sigma\right),$$

  and

$$\lim_{t\to-\infty} s_1(t) = \lim_{t\to-\infty} s_2(t) = s_{in} \textit{ with } \lim_{t\to-\infty} \frac{s_{in}-s_2(t)}{s_{in}-s_1(t)} = \exp\left(\int_{s_2^0}^{s_1^0} \frac{\nu'(s_{in})}{\nu(\sigma)} d\sigma\right).$$

- *For $u=(1,1)$ one has*

$$\lim_{t\to+\infty} s_1(t) = \lim_{t\to+\infty} s_2(t) = s_{in} \textit{ with } \lim_{t\to+\infty} \frac{s_2(t)}{s_1(t)} = +\infty\ .$$

  In backward time, $s(\cdot)$ leaves $\mathcal{D}$ at a finite time t_e with $s_1(-t_e)=s_2(-t_e)$ or $s_2(-t_e)=0$.

Proof. For $u=(0,0)$ the s_1 and s_2 sub-systems are decoupled with

$$\dot{s}_i = -\nu(s_i)\ , \tag{1.6}$$

$\nu(0)=\nu(s_{in})=0$ and $\nu(s)>0$ for $s\in(0,s_{in})$. Thus, $s_i^0 \notin \{0, s_{in}\}$ implies that one has necessarily

$$\lim_{t\to+\infty} s_i(t) = 0 \text{ and } \lim_{t\to-\infty} s_i(t) = s_{in},$$

for $i=1,2$. From (1.6), one obtains that the following equality is fulfilled for any t

$$\int_{s_1^0}^{s_1(t)} \frac{d\sigma}{\nu(\sigma)} = \int_{s_2^0}^{s_2(t)} \frac{d\sigma}{\nu(\sigma)} \quad \Longrightarrow \quad \int_{s_2^0}^{s_1^0} \frac{d\sigma}{\nu(\sigma)} = \int_{s_2(t)}^{s_1(t)} \frac{d\sigma}{\nu(\sigma)},$$

with

$$\lim_{t\to+\infty} \int_{s_2(t)}^{s_1(t)} \frac{d\sigma}{\nu(\sigma)} = \lim_{t\to+\infty} \int_{s_2(t)}^{s_1(t)} \frac{d\sigma}{\nu'(0)\sigma} = \frac{1}{\nu'(0)} \lim_{t\to+\infty} \log\left(\frac{s_2(t)}{s_1(t)}\right),$$

and

$$\lim_{t\to-\infty} \int_{s_2(t)}^{s_1(t)} \frac{d\sigma}{\nu(\sigma)} = \lim_{t\to-\infty} \int_{s_2(t)}^{s_1(t)} \frac{d\sigma}{\nu'(s_{in})(\sigma - s_{in})} = \frac{1}{\nu'(s_{in})} \lim_{t\to-\infty} \log\left(\frac{s_{in}-s_2(t)}{s_{in}-s_1(t)}\right),$$

that give the announced limits of the ratios.

For $u=(1,1)$ the dynamics of the s_1 sub-system is

$$\dot{s}_1 = \bar{G}(s_1), \tag{1.7}$$

where $\bar{G}(s_1) := (s_{in}-s_1)(1-\mu(s_1))$. As $\mu(s_1)<1$ (Hypothesis H2), one has then clearly $s_1(t)\to s_{in}$ when $t\to+\infty$. The s_2 sub-system can be seen as an asymptotic autonomous dynamics with the same limiting dynamics in forward time:

$$\dot{s}_2 = G(t, s_2),$$

where $G(t,s_1,s_2) := -\mu(s_2)(s_{in}-s_2) + (s_1(t)-s_2)$ satisfies $|G(t,s_2)-\bar{G}(s_2)| \to 0$ when t goes to $+\infty$. The trajectory being bounded, we deduce that $s_2(t)$ tends also to s_{in} when $t\to+\infty$, see e.g. [22]. Consider the ratio $r(t) = (s_{in}-s_2(t))/(s_{in}-s_1(t))$ whose time derivative satisfies

$$\dot{r} = (\mu(s_2(t)) - \mu(s_1(t)))r + 1$$

If $r(\cdot)$ is bounded, say by $M > 0$, there exists $T > 0$ such that $(\mu(s_1(t)) - \mu(s_2(t)))M < \alpha < 1$ for any $t > T$ as $s_1 - s_2$ is non-negative and tends towards zero. Then one should have $\dot{r}(t) > (1-\alpha) > 0$ for $t > T$, leading to a contradiction. So we conclude that r tends to $+\infty$.

In backward time, the solution $s_1(\cdot)$ of (1.7) clearly goes to negative values. So the trajectory has to leave the domain $\mathcal{D}$ in finite time with decreasing s_1. This is only possible through the parts of the boundary $s_1 = s_2$ or $s_2 = 0$. $\square$

Consider now the target point $\bar{s}$ as an initial condition. Accordingly to Lemma 1.1, there exists an exit time t_e of the domain $\mathcal{D}$ backward in time with constant controls $(1,1)$. We then define the function $p(\cdot)$ on the interval $[-t_e, +\infty)$ as

$$p(\tau) = \left| \begin{array}{ll} s^-(-\tau, \bar{s}, (1,1)), & \tau \in [-t_e, 0) \\ s^-(\tau, \bar{s}, (0,0)), & \tau \in [0, +\infty) \end{array} \right.$$

If the exit time is such that $s_1^-(t_e, \bar{s}, (1,1)) = s_2^-(t_e, \bar{s}, (1,1))$, we posit $I = [-t_e, +\infty)$. Otherwise, we extend $p(\cdot)$ on $[-t_e - s_1^-(\bar{s}, t_e, (1,1)), -t_e)$ as $p(\tau) = (\tau + t_e + s_1^-(\bar{s}, t_e, (1,1)), 0)$, and posit $I = [-t_e - s_1^-(\bar{s}, t_e, (1,1)), +\infty)$.

Finally, we define the set $\mathcal{C}(\bar{s})$ as the epigraph of the function p restricted to the domain $\mathcal{D}$:

$$\mathcal{C}(\bar{s}) := \{ s \in \mathcal{D} \,|\, s_1 = p_1(\tau),\ s_2 \geq p_2(\tau),\ \tau \in I \} \ .$$

Accordingly to Lemma 1.1, the set $\mathcal{C}(\bar{s})$ has two possible shapes, depending on which part of the boundary of $\mathcal{D}$ the semi-orbit $\gamma^-(\bar{s}, (1,1))$ leaves the domain $\mathcal{D}$ (see Figure 2).

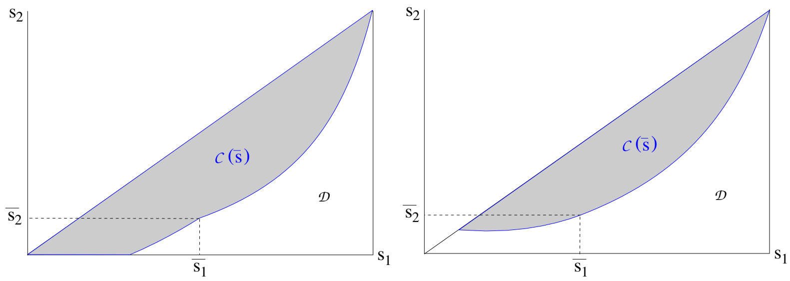

Figure 2: Picture of the set $\mathcal{C}(\bar{s})$. *On the left* : the negative semi-orbit of $\bar{s} = (1, 0.3)$ with $u_1 = u_2 = 1$ intersects the axis $s_2 = 0$. *On the right*: the negative semi-orbit of $\bar{s} = (0.9, 0.32)$ with $u_1 = u_2 = 1$ intersects the first diagonal.

Remark 1.2. *The line segment L does not belong to $\mathcal{C}(\bar{s})$.*

2 Optimality result in $\mathcal{C}(\bar{s})$

Let us consider the set of semi-orbits that reach $\bar{s}$ with $u_1 \equiv 1$ or $u_2 \equiv 0$:

$$\mathcal{S}(\bar{s}) := \mathcal{S}^-(\bar{s}) \cup \mathcal{S}^+(\bar{s})$$

with

$$\mathcal{S}^-(\bar{s}) := \bigcup_{v(\cdot) \in \mathcal{V}} \gamma^-(\bar{s}, (1, v(\cdot))) \cap \mathcal{D} \ , \quad \mathcal{S}^-(\bar{s}) := \bigcup_{v(\cdot) \in \mathcal{V}} \gamma^-(\bar{s}, (v(\cdot), 0)) \cap \mathcal{D} \ , \tag{2.1}$$

where $\mathcal{V}$ is the set of measurable functions that take values on $[0,1]$. This set possesses the following nice property:

Proposition 2.1. *For any $\bar{s} \in \mathcal{D}$, one has $\mathcal{C}(\bar{s}) = \mathcal{S}(\bar{s})$.*

Proof. Let us first prove that $\mathcal{C}(\bar{s}) \subset \mathcal{S}(\bar{s})$.

Take $s^0 \in \mathcal{C}(\bar{s})$, and consider the trajectory $s(\cdot) := s(\cdot, s^0, (1,0))$. If there exists $t_0 \geq 0$ such that $s(t_0) = \bar{s}$, then we have $s^0 = s^-(t_0, \bar{s}, (1,0)) \in \mathcal{S}(\bar{s})$ as was to be proved. Now, we suppose that for any $t \geq 0$, we have $s(t) \neq \bar{s}$. Having $0 < s_2(0) < s_1(0)$ and $s_2(\cdot)$ decreasing, $s(\cdot)$ has to leave $\mathcal{D}$ at a finite time t_0: either it intersects at time t_0 the semi-orbit $\gamma^-(\bar{s}, (1,1))$ (case a), or it intersects at time t_0 the semi-orbit $\gamma^-(\bar{s}, (0,0))$ (case b).

Case a. Let $t_1 > 0$ be such that $\bar{s} = s(t_1, s(t_0), (1,1))$. Then, let us define a control $u : [0, t_0 + t_1] \to U$ by $u = (1,1)$ on $[0, t_1]$ and $u = (1,0)$ on $[t_1 + t_0]$. By construction, we have $s^-(t_1 + t_0, \bar{s}, u) = s^0$, and u is of the form $u = (1, u_2)$ with $u_2 \in \mathcal{V}$, and the result follows.

Case b. Let $t_1' > 0$ be such that $\bar{s} = s(t_1', s(t_0), (0,0))$. Then, let us define a control $\tilde{u} : [0, t_0 + t_1'] \to U$ by $\tilde{u} = (0,0)$ on $[0, t_1']$ and $\tilde{u} = (1,0)$ on $[t_1' + t_0]$. By construction, we have $s^-(t_1' + t_0, \bar{s}, \tilde{u}) = s^0$, and $\tilde{u}$ is of the form $\tilde{u} = (u_1, 0)$ with $u_1 \in \mathcal{V}$, and the result follows.

Let us now prove that $\mathcal{S}(\bar{s}) \subset \mathcal{C}(\bar{s})$.

Take $s^0 \in \mathcal{S}(\bar{s})$, and assume first that $s^0 = s^-(t_0, \bar{s}, (1, u_2))$ with $u_2 \in \mathcal{V}$ and $t_0 \in [0, t_e)$ where t_e is the exit time of $\mathcal{D}$. For $t \in [0, t_e)$, set $\tilde{s}(t) := s^-(t, \bar{s}, (1, u_2))$, and $\hat{s}(t) := s^-(t, \bar{s}, (1,1))$. One can straightforwardly prove that $\tilde{s}_1(t) = \hat{s}_1(t)$ and $\tilde{s}_2(t) \geq \hat{s}_2(t)$ for all $t \in [0, t_e)$.

Case a. If $s_1^-(t_e, \bar{s}, (1, u_2)) = s_2^-(t_e, \bar{s}, (1, u_2))$, one has $\tilde{s}_1(t_0) = \hat{s}_1(t_0)$, $\tilde{s}_2(t_0) \geq \hat{s}_2(t_0)$, $\hat{s}(t_0) \in \partial\mathcal{C}(\bar{s})$, and as $t_0 \leq t_e$, we get that $\tilde{s}(t_0) \in \mathcal{C}(\bar{s})$.

Case b. If $s_2^-(t_e, \bar{s}, (1,1)) = 0$, one has $\tilde{s}_1(t_0) = \hat{s}_1(t_0)$, $\tilde{s}_2(t_0) \geq \max(0, \hat{s}_2(t_0))$, $(\hat{s}_1(t_0), \max(0, \hat{s}_2(t_0))) \in \partial\mathcal{C}(\bar{s})$ so that $t_0 \leq t_e$ implies $\tilde{s}(t_0) \in \mathcal{C}(\bar{s})$.

Suppose now that $s^0 = s^-(t_0, \bar{s}, (u_1, 0))$ with $u_1 \in \mathcal{V}$ and $t_0 \in [0, t_e)$ where t_1 is the exit time of $\mathcal{D}$. For $t \in [0, t_e)$, set $\tilde{s}(t) := s^-(t, \bar{s}, (u_1, 0))$, $\check{s}(t) := s^-(t, \bar{s}, (0,0))$, and $\hat{s}(t) := s^-(t, \bar{s}, (1,0))$ Similarly as in the previous case, we obtain:

$$\tilde{s}_2(t) = \check{s}_2(t) = \hat{s}_2(t), \quad \text{and} \quad \hat{s}_1(t) \leq \tilde{s}_1(t) \leq \check{s}_1(t), \quad \forall t \in [0, t_e). \tag{2.2}$$

Notice that $\bar{s}_2 > 0$ so that at the exit time t_e, we necessarily have $\tilde{s}_2(t_1) = \tilde{s}_1(t_e)$. Moreover, combining the fact that $(\check{s}_1(t_0), \check{s}_2(t_0)) \in \partial\mathcal{C}(\bar{s})$ and that $(\hat{s}_1(t), \hat{s}_2(t)) \in \mathcal{C}(\bar{s})$ for all $t \in [0, t_e)$, we obtain that $\tilde{s}(t_0) \in \mathcal{C}(\bar{s})$ as was to be proved. □

Consider the particular semi-orbit in $\mathcal{S}(\bar{s})$:

$$\Gamma := \gamma^-(\bar{s}, (1,0)) \cap \mathcal{D},$$

and notice that it can be parametrized as a curve $s_1 \mapsto s_2(s_1)$ because we have $\dot{s}_1 > 0$ on this semi-orbit. One has the following property.

Lemma 2.1. *Take $\bar{s} \in \mathcal{D}$. The subset $\mathcal{S}^-(\bar{s})$, resp. $\mathcal{S}^+(\bar{s})$ is the set of $s \in \mathcal{S}(\bar{s})$ that is below, resp. above Γ. Furthermore $\mathcal{S}^-(\bar{s}) \cap \mathcal{S}^-(\bar{s}) = \Gamma$*

Proof. Let $\tilde{s}(\cdot) := s^-(\cdot, \bar{s}, (1,0))$, and consider a backward trajectory $s(\cdot) := s^-(\cdot, \bar{s}, (1, u_2))$ with $u_2 \in \mathcal{V}$. On has clearly $s_1(t) = \tilde{s}_1(t)$ and $s_2(t) < \tilde{s}_2(t)$ for any time t. Consequently, any trajectory in $\mathcal{C}(\bar{s})$ with $u_1 \equiv 1$ is below Γ. Similarly, any trajectory in $\mathcal{C}(\bar{s})$ with $u_2 \equiv 0$ is above Γ. Clearly, Γ is the only semi-orbit of $\mathcal{S}(\bar{s})$ that belongs to $\mathcal{S}^-(\bar{s})$ and $\mathcal{S}^+(\bar{s})$. □

Proposition 2.1 shows that for any $s^0 \in \mathcal{C}(\bar{s})$, there exists an admissible control steering s^0 to $\bar{s}$ in finite time, provided that $\bar{s} \in \mathcal{D}$. We shall now give a characterization of the optimal trajectories.

Proposition 2.2. *Let $\bar{s} \in \mathcal{D}$. For any initial state $s^0 \in \mathcal{C}(\bar{s})$, the optimal trajectories belong to $\mathcal{C}(\bar{s})$, and the optimal feedbacks $u[\cdot] : s \longmapsto u[s] \in U$ are given by: $u^\star(\cdot)$:*

- *for $s \in \mathcal{S}^-(\bar{s}) \setminus \Gamma$, $u_1[s] = 1$ and $u_2[s]$ takes any value in $[0,1]$,*
- *for $s \in \mathcal{S}^+(\bar{s}) \setminus \Gamma$, $u_2[s] = 0$ and $u_1[s]$ takes any value in $[0,1]$,*
- *for $s \in \Gamma$, $u[s] = (1,0)$*

Furthermore, the value function is:

$$T(s^0) = \max\left\{\int_{s_1^0}^{\overline{s_1}} \frac{d\sigma}{(1-\mu(\sigma))(s_{in}-\sigma)}, \int_{s_2^0}^{\overline{s_2}} \frac{d\sigma}{-\mu(\sigma)(s_{in}-\sigma)}\right\}. \tag{2.3}$$

Proof. Let $s^0 \in \mathcal{C}(\overline{s})$. We know from Proposition 2.1 that there exists a control $u \in \mathcal{U}$ of the form $(1, u_2(\cdot))$ or $(u_1(\cdot), 0)$ and a time $t_u \in [0, +\infty[$ such that

$$s(t_u, s^0, u) = \overline{s}. \tag{2.4}$$

First, suppose that u is of the form $(1, u_2)$ and set $s(\cdot) := s(\cdot, s^0, u)$. Therefore one has

$$\dot{s_1} = (1-\mu(s_1))(s_{\text{in}} - s_1), \tag{2.5}$$

hence $s_1(\cdot)$ is increasing and:

$$s_1(t) < \overline{s}_1, \quad \forall t \in [0, t_u). \tag{2.6}$$

We show that t_u is optimal. If not, there exists $\hat{u} \in \mathcal{U}$ such that $s(T(s^0), s^0, \hat{u}) = \overline{s}$ with $T(s^0) < t_u$. Set $\hat{s}(\cdot) := s(\cdot, s^0, \hat{u})$. As $\hat{u}_1 \leq 1$ and $\hat{s}_1(0) = s_1(0) = s_1^0$, we obtain by (0.2):

$$\hat{s}_1(t) \leq s_1(t), \quad \forall t \in [0, +\infty). \tag{2.7}$$

Therefore, combining (2.6) and (2.7), we get:

$$\overline{s}_1 = \hat{s}_1(T(s^0)) \leq s_1(T(s^0)) < \overline{s}_1, \tag{2.8}$$

which is a contradiction, hence $t_u = T(s^0)$. Let us now explicit $t_u = T(s^0)$. One has:

$$t_u = \int_0^{t_u} dt = \int_0^{t_u} \frac{\dot{s_1}(t)}{(1-\mu(s_1(t)))(s_{\text{in}} - s_1(t))} dt = \int_{s_1^0}^{\overline{s}_1} \frac{d\sigma}{(1-\mu(\sigma))(s_{\text{in}}-\sigma)}. \tag{2.9}$$

Finally let us show that

$$\int_{s_2^0}^{\overline{s}_2} \frac{d\sigma}{-\mu(\sigma)(s_{\text{in}}-\sigma)} \leq \int_{s_1^0}^{\overline{s}_1} \frac{d\sigma}{(1-\mu(\sigma))(s_{\text{in}}-\sigma)}. \tag{2.10}$$

Remark that $1/(-\mu(s)(s_{\text{in}} - s)) < 0$, for any $s \in (0, s_{\text{in}})$. Therefore if $s_2^0 \leq \overline{s}_2$, one has $\int_{s_2^0}^{\overline{s}_2} \frac{d\sigma}{-\mu(\sigma)(s_{\text{in}}-\sigma)} \leq 0$ and the result is obvious. Let us now suppose that $s_2^0 > \overline{s}_2$ and set φ as the solution of the Cauchy problem:

$$\begin{cases} \dot{\varphi} = -\mu(\varphi)(s_{\text{in}} - \varphi), \\ \varphi(0) = s_2^0. \end{cases}$$

Then, φ is decreasing and converges to zero when t tends to $+\infty$. Thus, there exists $t_0 \in [0, +\infty)$ such that $\varphi(t_0) = \overline{s}_2$. Therefore, one has

$$\int_{s_2^0}^{\overline{s}_2} \frac{d\sigma}{-\mu(\sigma)(s_{\text{in}}-\sigma)} = \int_{\varphi(0)}^{\varphi(t_0)} \frac{d\sigma}{-\mu(\sigma)(s_{\text{in}}-\sigma)} = \int_0^{t_0} dt = t_0. \tag{2.11}$$

Moreover, as $u_2 \geq 0$ and $s_2(0) = \varphi(0)$ one has by (0.2) that $\varphi(t) \leq s_2(t)$ for all $t \in [0, +\infty[$. Consequently, as φ is decreasing with $\varphi(t_0) = s_2(t_u) \geq \varphi(t_u)$, one has $t_0 \leq t_u$, which by (2.9) and (2.11) gives (2.10). Thus we have proved the expression (2.3) of the value function for such initial condition.

From Proposition 2.1 and Lemma 2, any trajectory with initial condition s^0 and control $(1, u_2)$ that is below Γ is also above $\gamma^-(\bar{s}, (1, 1))$. Consequently, any such trajectory reaches $\bar{s}$ in finite time, and stays in $\mathcal{C}(\bar{s})$ until this time. As all these trajectories have exactly the same map $t \mapsto s_1(t)$, we deduce that $\bar{s}$ is reached exactly at the same time t_u. This shows the optimality of all the trajectories with control $u_1 = 1$ as long as s is below Γ.

Now, we investigate the case where u is of the form $(u_1, 0)$. We set $s(\cdot) =: s(\cdot, s^0, u)$. We have that $s_2(\cdot)$ is decreasing and therefore $s_2^0 > \bar{s}_2$. Similarly as in the previous case, we can show that $v(s^0) = t_u$ (using the fact that $s_2^0 > \bar{s}_2$), and that $t_u = \int_{s_2^0}^{\bar{s}_2} \frac{d\sigma}{-\mu(\sigma)(\sin - \sigma)}$. Finally, we show that:

$$\int_{s_1^0}^{\bar{s}_1} \frac{d\sigma}{(1-\mu(\sigma))(s_{\text{in}} - \sigma)} \leq \int_{s_2^0}^{\bar{s}_2} \frac{d\sigma}{-\mu(\sigma)(s_{\text{in}} - \sigma)}. \tag{2.12}$$

First, if $s_1^0 \geq \bar{s}_1$, then $\int_{s_1^0}^{\bar{s}_1} \frac{d\sigma}{(1-\mu(\sigma))(s_{\text{in}}-\sigma)} \leq 0$ and (2.12) is obvious. If now $s_1^0 < \bar{s}_1$, we consider φ as the solution of the following Cauchy problem:

$$\begin{cases} \dot{\varphi} = (1 - \mu(\varphi))(s_{\text{in}} - \varphi), \\ \varphi(0) = s_1^0. \end{cases}$$

As φ is increasing, there exists $t_0 > 0$ such that $\int_{s_1^0}^{\bar{s}_1} \frac{d\sigma}{(1-\mu(\sigma))(s_{\text{in}}-\sigma)} = t_0$ with $\varphi(t_0) = \bar{s}_1$. Moreover, as φ satisfies the same o.d.e. as s_1 with the constant control equal to 1 in place of u_1, we obtain $\varphi(t_0) = s_1(t_u) = \bar{s}_1 \leq \varphi(t_u)$, so that $t_0 \leq t_u$ (as φ is increasing). We conclude in a similar way than for the first case.

Finally, if $s \in \Gamma$, the optimal trajectory has to fulfill the properties of both former cases, that is $u_1[s] = 0$ and $u_2[s] = 0$. $\square$

Figure 3 depicts the infinity of optimal trajectories from an initial condition $s^0 \in \mathcal{C}(\bar{s}) \setminus \Gamma$, see Proposition 2.2.

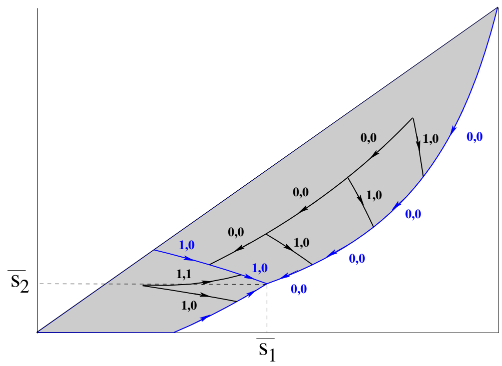

Figure 3: Picture for $\bar{s} = (1, 0.3)$. When the initial condition s^0 is below Γ, there exist infinitely many trajectories in $\mathcal{S}^-(\bar{c})$ steering s^0 to $\bar{s}$ in the same time $T(s^0)$ with $u_1 \equiv 1$. When s^0 is above Γ, there exist infinitely many trajectories in $\mathcal{S}^+(\bar{c})$ steering s^0 to $\bar{s}$ in the same time $T(s^0)$ with $u_2 \equiv 0$.

3 Optimality result outside $\mathcal{C}(\bar{s})$

In this part, we provide optimal trajectories for initial conditions in $\mathcal{D} \backslash \mathcal{C}(\bar{s})$ which allows to conclude on the optimal synthesis of the problem. Firstly, we show that for initial conditions outside the set $\mathcal{C}(\bar{s})$, an optimal control satisfies $u_1 = u_2$.

Proposition 3.1. *Let us consider an initial condition $s^0 \in \mathcal{D} \backslash \mathcal{C}(\bar{s})$, and assume that $s^0 \in \mathcal{R}$. Then, an optimal control $u \in \mathcal{U}$ steering s^0 to $\bar{s}$ satisfies $u_1 = u_2$ a.e. and $\lambda_2 > 0$.*

Proof. First, consider the case where $\lambda_2 \equiv 0$. From (1.2), λ_1 is of constant sign (either positive or negative). If $\lambda_1 > 0$, then $u_1 = 1$ a.e. and u_2 is any control taking values in $[0, 1]$. Proposition 2.1 implies that $s^0 \in \mathcal{C}(\bar{s})$ which is a contradiction. If $\lambda_1 < 0$, then we have $u_1 = 0$ a.e., thus $u_2 = 0$ a.e. implying that $s^0 \in \partial\mathcal{C}(s^0)$ which again gives a contradiction.

Now, let us investigate the case where $\lambda_2 < 0$. If there exists a time interval $[t_1, t_2]$ where $u_2 > 0$, then, we obtain a contradiction in the maximization condition (1.3) by comparing this control to the constant one equal to zero. This implies that $u_2 = 0$ a.e., and from (1.2), λ_1 is of constant sign. First, if $\lambda_1 > 0$, then $u_1 = 1$ a.e. by (1.3), and we see that $s^0 \in \Gamma \subset \mathcal{C}(\bar{s})$ implying a contradiction. Now, if $\lambda_1 < 0$, then $u_1 = 0$ a.e., and both controls are constantly equal to zero, thus $s^0 \in \partial\mathcal{C}(s^0)$ implying a contradiction. If $\lambda_1 \equiv 0$, then u_1 is any measurable control taking values within $[0, 1]$. Using Proposition 2.1 which characterizes $\mathcal{C}(\bar{s})$, we obtain that necessarily $s^0 \in \mathcal{C}(\bar{s})$ which is a contradiction.

It follows that we have $\lambda_2 > 0$. As $u_2 \leq u_1$, we obtain that

$$\lambda_1(s_{in} - s_1)u_1 + \lambda_2(s_1 - s_2)u_2 \leq \phi_2 u_1.$$

If we combine this inequality together with (1.3), we obtain the following: if $\phi_2 > 0$, then the maximum in the Hamiltonian is achieved for $u_2 = u_1 = 1$. If $\phi_2 < 0$, then it is achieved for $u_1 = u_2 = 0$. If $\phi_2 = 0$, then we have

$$\lambda_1(s_{in} - s_1)u_1 + \lambda_2(s_1 - s_2)u_2 = \lambda_2(s_1 - s_2)(u_2 - u_1),$$

and we see using $u_2 \leq u_1$ that the maximum is obtained when $u_1 = u_2$. In the three previous cases $\lambda_2 > 0$, $\lambda_2 < 0$ and $\lambda_2 = 0$, we have proved that $u_1 = u_2$. To conclude, we have proved that we always have $u_1 = u_2$ which is possible only if $\lambda_2 > 0$. This concludes the proof. □

We are now in position to study the minimal time problem outside $\mathcal{C}(\bar{s})$ using the theory of affine control systems in the plane with one input. Let us write (0.2) as:

$$\dot{s} = f(s) + ug(s), \tag{3.1}$$

where f, g are the two vector fields defined by:

$$f(s) := -\begin{pmatrix} \nu(s_1) \\ \nu(s_2) \end{pmatrix}, \; g(s) := \begin{pmatrix} s_{in} - s_1 \\ s_1 - s_2 \end{pmatrix}.$$

It is standard to introduce the *non-controllability* curve Δ_0 [3, 4, 19] by:

$$\Delta_0 := \{(s_1, s_2) \in \mathcal{D} \mid \det(f(s), g(s)) = 0\}, \tag{3.2}$$

and the singular arc Δ_{SA} by:

$$\Delta_{SA} := \{(s_1, s_2) \in \mathcal{D} \mid \det(g(s), [f, g](s)) = 0\}, \tag{3.3}$$

where $[f, g]$ denotes the Lie bracket of f and g, see e.g. [19]. Next, we define Δ_0^+ (resp. Δ_0^-) as the set of points of $\mathcal{D}$ such that $\det(f(s), g(s)) > 0$ (resp. $\det(f(s), g(s)) < 0$). Similarly, we define $\Delta_{SA}^{\pm}$. A simple computation shows that we have:

$$\det(g(s), [f, g](s)) = [s_{in} - s_1][[\mu(s_2) - \mu(s_1) - \mu'(s_1)(s_1 - s_2)](s_{in} - s_1) + \mu'(s_2)(s_{in} - s_2)(s_1 - s_2)].$$

As the point (s_{in}, s_{in}) is an equilibrium of (0.2), the singular arc is given by the implicit equation

$$[\mu(s_2) - \mu(s_1) - \mu'(s_1)(s_1 - s_2)](s_{in} - s_1) + \mu'(s_2)(s_{in} - s_2)(s_1 - s_2) = 0. \tag{3.4}$$

The next proposition gives properties of Δ_0.

Proposition 3.2. *There exist a continuous mapping $s_1 \in [0, s_{in}] \longmapsto \zeta(s_1)$ of class C^1 on $[0, s_{in})$ such that $\zeta(0) = 0$, $\zeta(s_{in}) = s_{in}$, ζ is increasing over $[0, s_{in}]$, and Δ_0 is the graph of the restriction of ζ on $(0, s_{in})$.*

Proof. Let us consider the C^1-mapping $\rho : (s_1, s_2) \in \mathcal{D} \longmapsto \rho(s_1, s_2) := -\mu(s_1)(s_1 - s_2) + \mu(s_2)(s_{in} - s_2)$ so that $\rho(s_1, s_2) = 0$ iff $s \in \Delta_0$. For $s \in \mathcal{D}\backslash\{(s_{in}, s_{in})\}$, we have:

$$\frac{\partial \rho}{\partial s_2}(s_1, s_2) = \mu(s_1) - \mu(s_2) + \mu'(s_2)(s_{in} - s_2) > 0.$$

Hence, we can apply the implicit function Theorem which provides the existence of a function $\zeta : [0, s_{in}) \to \mathcal{D}$ such that $s \in \Delta_0 \backslash \{(0,0),(s_{in}, s_{in})\}$ if and only if $s_2 = \zeta(s_1)$. Moreover, $\rho(0,0) = 0$ implies $\zeta(0) = 0$, and we have:

$$\zeta'(s_1) = \frac{\mu'(s_1)(s_1 - s_2) + \mu(s_1)}{\mu(s_1) - \mu(s_2) + \mu'(s_2)(s_{in} - s_2)} > 0, \tag{3.5}$$

where $s_2 = \zeta(s_1)$. Hence ζ is increasing over $(0, s_{in})$. As we have $\rho(s_{in}, s_{in}) = 0$, we have $\zeta(s_1) \to s_{in}$ whenever $s_1 \to s_{in}$. Hence, we can extend ζ continuously on $[0, s_{in}]$ by letting $\zeta(s_{in}) = s_{in}$.

□

Finding an optimal synthesis highly depends on the position of Δ_0 and Δ_{SA} in the invariant set. In view of numerical simulations (see section 4), we infer that the intersection in $\mathcal{D}$ of Δ_0 and Δ_{SA} is non-empty which has several consequences in term of controllability of the system and on the optimal synthesis.

In the following, we call *steady-state singular point* (see [4]) a point $s^\star \in \mathcal{D} \setminus \{(s_{in}, s_{in})\}$ such that:

$$s^\star \in \Delta_0 \cap \Delta_{SA},$$

and we assume throughout this section that there exist such points.

Remark 3.1. *(i) Such points are equilibrium of the dynamics restricted to the singular arc. In fact, along a singular extremal trajectory, the adjoint vector satisfies $\lambda(t) \cdot f(x(t)) = 1$ and $\lambda(t) \cdot g(x(t)) = 0$ which shows that it cannot reach $s^\star$ in finite time. This implies that if the singular arc is controllable, then $s^\star$ is never reached by a singular trajectory in finite time.*
(ii) Even if $s^\star$ cannot be reached in finite time, an extremal singular trajectory can be locally time minimizing. In other words, the singular arc can be hyperbolic [3].
(ii) If we consider two different trajectories steering a point s^0 to $\bar{s}$ (which intersect only at points s^0 and $\bar{s}$), then we cannot apply the clock form argument globally if the domain enclosed to the union of these curves intersects Δ_0. This tool is based on Green's Theorem [3, 21, 19] and gives a direct method to compare the cost of two trajectories which do not intersect Δ_0.

Let us recall the following result, see Lemma 13B [4] which will be useful in order to find the optimal synthesis.

Lemma 3.1. *Let us define* Tan *as the set of points of $\mathcal{D} \setminus \{(0,0)\} \setminus \{(s_{in}, s_{in})\}$ where the dynamics (3.1) is tangent to Δ_0 for any $u \in [0,1]$. Then* Tan *is non-empty, and we have:*

$$\mathrm{Tan} = \Delta_0 \cap \Delta_{SA}. \tag{3.6}$$

Proof. At any $s \in \Delta_0 \setminus \{(0,0)\} \setminus \{(s_{in}, s_{in})\}$, the vector fields f and g are non null and colinear, so that one can write $f(s) = \alpha g(s)$ where α is non null scalar. The dynamics is then tangent to Δ_0 for any control u if $f(s)$ or $g(s)$ belongs to the tangent cone of Δ_0 at s, that is defined as

$$T_{\Delta_0}(s) = \left\{\delta \in \mathcal{R}^2 \,|\, \det(f(s + h\delta), g(s + h\delta)) = O(h^2)\right\} .$$

Having $f(s) \in T_{\Delta_0}(s)$ amounts to write

$$\begin{aligned}
& \det(f(s) + h\partial_s f(s).f(s) + 0(h^2), g(s) + h\partial_s g(s).f(s) + O(h^2)) = O(h^2) \\
\Leftrightarrow \quad & \det(f(s), g(s)) + h\det(f, \partial_s g(s).f(s)) + h\det(\partial_s f(s).f(s), g(s)) = 0 \\
\Leftrightarrow \quad & \det(\alpha g(s), \partial_s g(s).f(s)) - \det(g(s), \alpha\partial_s f(s).g(s)) = 0 \\
\Leftrightarrow \quad & \det(g(s), [f, g](s)) = 0
\end{aligned}$$

that is $s \in \Delta_{SA}$. □

Throughout the rest of the paper, we make the following assumption on $s^\star$.

(H3) There exists a unique steady-state singular point $s^\star \in \mathcal{D}$.

This assumption is essential in our study. Hypothesis (H3) is satisfied whenever μ is of Monod type or linear (see section 4). For $s \in [0, s_{in}]$, we denote by $\tau(s)$ the unitary tangent vector of Δ_0 at point s.

Lemma 3.2. *Let $s \in \Delta_0$. Then, we have:*

$$s \in \Delta_{SA}^{+} \Rightarrow \det(f(s), \tau(s)) > 0, \;\; s \in \Delta_{SA}^{-} \Rightarrow \det(f(s), \tau(s)) < 0. \tag{3.7}$$

Proof. As $s \in \Delta_0$, we have $\mu(s_1)(s_1 - s_2) = \mu(s_2)(s_{in} - s_2)$. From (3.5), we can write the vector $\tau(s)$ as: $\tau(s) = \kappa \begin{pmatrix} 1 \\ \zeta'(s_1) \end{pmatrix}$, where $\kappa > 0$ is a coefficient to normalize $\tau(s)$. Using that $s_1 > s_2$, the sign of $\det(f(s), \tau(s))$ is given by the quantity defined by:

$$\alpha(s) := -\mu(s_1)(s_{in} - s_1)[\mu(s_1) + \mu'(s_1)(s_1 - s_2)] + \mu(s_2)(s_{in} - s_2)(\mu(s_1) - \mu(s_2) + \mu'(s_2)(s_{in} - s_2)).$$

Using the fact that $s \in \Delta_0$, we find that:

$$\frac{\alpha(s)}{\mu_1(s)} = (s_{in} - s_1)(\mu(s_2) - \mu(s_1) - \mu'(s_1)(s_1 - s_2)) - (s_{in} - s_1)\mu(s_2) + (s_1 - s_2)(\mu(s_1) - \mu(s_2) + \mu'(s_2)(s_{in} - s_2)).$$

Now, let us take a point $s \in \Delta_{SA}^{-}$. Then, (3.4) implies that:

$$\begin{aligned} \frac{\alpha(s)}{\mu_1(s)} &< -(s_{in} - s_1)\mu(s_2) + (s_1 - s_2)(\mu(s_1) - \mu(s_2)) \\ &= -(s_{in} - s_1)\mu(s_2) + (s_{in} - s_2)\mu(s_2) - (s_1 - s_2)\mu(s_2)) = 0. \end{aligned}$$

where the equality follows from using the fact that $s \in \Delta_0$. It follows that for $s \in \Delta_{SA}^{-}$, we have $\alpha(s) < 0$, thus $\det(f(s), \tau(s)) < 0$ which proves the result. When $s \in \Delta_{SA}^{+}$, we use the same computation and the reverse inequality. This proves the lemma. $\square$

Using Lemma 3.1, we can state the following result that is the basis of the optimality results of this section.

Proposition 3.3. *Let us take $\bar{s} \in \mathcal{D}$. Then, either $s^\star \in \mathcal{C}(\bar{s})$ or $\mathcal{C}(\bar{s}) \subset \Delta_0^{+}$.*

Proof. Recall that the boundary of $\mathcal{C}(\bar{s})$ is the union of $\gamma^{-}(\bar{s}, (0,0))$ and $\gamma^{-}(\bar{s}, (1,1))$. Let us suppose that $s^\star \notin \mathcal{C}(\bar{s})$, and assume by contradiction that there exists a point $\hat{s} \in \partial\mathcal{C}(\bar{s}) \cap \Delta_0$. We have several cases.
First case. $\hat{s} \in \Delta_{SA}^{-} \cap \gamma^{-}(\bar{s}, (0,0))$. Then, from Lemma 3.2, the boundary of $\mathcal{C}(\bar{s})$ with $u = 0$ necessarily intersects Δ_0 in Δ_{SA}^{+}. This would give $s^\star \in \mathcal{C}(\bar{s})$ which is a contradiction.
Second case. $\hat{s} \in \Delta_{SA}^{-} \cap \gamma^{-}(\bar{s}, (1,1))$. Then, from Lemma 3.2, the boundary of $\mathcal{C}(\bar{s})$ with $u = 1$ or $u = 0$ cannot intersect Δ_0 in the set Δ_{SA}^{-}. This would give $s^\star \in \mathcal{C}(\bar{s})$ which is a contradiction.
It remains to study the case where the intersection point $\hat{s}$ is in Δ_{SA}^{+}. The same argument as above allow to conclude. $\square$

Given a target point $\bar{s} \in \mathcal{D}$, one should determine whenever we have $s^\star \in \mathcal{C}(\bar{s})$. To do so, we introduce a curve Λ passing through $s^\star$ and that provides a partition of the set $\mathcal{D}$:

$$\Lambda := \gamma^{+}(\bar{s}, (0,0)) \cup \gamma^{+}(\bar{s}, (1,1)).$$

This curve is depicted on Figure 6 (green curve) and satisfies the following properties.

Lemma 3.3. *(i) The curve Λ is such that $\Lambda \cap \Delta_0 = \{s^\star\}$*
(ii) There exists a mapping $q : \mathbb{R} \to \mathcal{D}$ which satisfies $\dot{q}_1 > 0$ such that Λ coincides with the graph of q.

Proof. The proof of (i) follows by using Lemma 3.2 and a similar reasoning as in the proof of Proposition 3.3. As Λ is defined via the controls $(0,0)$ and $(1,1)$, the parametrization follows from Proposition 1.1. $\square$

We now show a semi-permeability property on the curve Λ [5, 6, 26]. To do so, let us write the dynamics (0.2) as $\dot{s} = F(s,u)$, where $F : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ is given by (0.2).

Definition 3.1. *A closed set $C \subseteq \mathbb{R}^2$ has a semipermeable boundary for the inclusion $\dot{s} = F(s,u)$ in a neighborhood of $\bar{s} \in \partial C$, if there exists a neighborhood V of $\bar{s}$ such that*

$$\forall s \in \partial C \cap V, \ \forall p \in N_C(s), \inf_{u \in U} \langle F(s,u), p \rangle = 0,$$

where $N_C(s)$ is the normal cone to C at point s.

Proposition 3.4. *The curve Λ is semi-permeable for the system* (0.2).

Proof. Recall that $\Lambda = \gamma^+(\bar{s},(0,0)) \cup \gamma^+(\bar{s},(1,1))$ and that $\Lambda \subset \Delta_0^+ \cup \{s^\star\}$. First, we consider a point $s = (s_1, s_2) \in \gamma^+(\bar{s},(0,0))$. The outward normals of Λ at point s are of the form $\lambda \cdot n(s)$ with $\lambda > 0$ and

$$n(s) = \begin{pmatrix} \mu(s_2)(s_{\text{in}} - s_2) \\ -\mu(s_1)(s_{\text{in}} - s_1) \end{pmatrix}.$$

We show that $\inf_{(u_1,u_2)\in U} \langle F(s,(u_1,u_2)), n(s) \rangle = 0$. It is clear that $\langle F(s,(0,0)), n(s) \rangle = 0$, so we have to show that $\langle F(s,(u_1,u_2)), n(s) \rangle \geq 0$, for all $(u_1,u_2) \in U$. Let $(u_1,u_2) \in U$. One has:

$$\langle F(s,(u_1,u_2)), n(s) \rangle = (s_{\text{in}} - s_1)[u_1 \mu(s_2)(s_{\text{in}} - s_2) - u_2 \mu(s_1)(s_1 - s_2)].$$

As $(s_1,s_2) \in \Delta_0^+ \cup \Delta_0$, we have $\mu(s_2)(s_{\text{in}} - s_2) \geq \mu(s_1)(s_1 - s_2)$, implying that $u_1 \mu(s_2)(s_{\text{in}} - s_2) \geq u_2 \mu(s_1)(s_1 - s_2)$, that is, $\langle F(s,(u_1,u_2)), n(s) \rangle \geq 0$ as wanted.

Suppose now that $s \in \gamma^+(\bar{s},(0,0))$. Then, the outward normals of Λ at point s are of the form $\lambda \cdot n(s)$ with $\lambda > 0$ and

$$n(s) = \begin{pmatrix} -\mu(s_2)(s_{\text{in}} - s_2) + (s_1 - s_2) \\ (\mu(s_1) - 1)(s_{\text{in}} - s_1) \end{pmatrix}.$$

Notice that we have $\langle F(s,(1,1)), n(c(t)) \rangle = 0$. Let us now take $(u_1,u_2) \in U$. One has:

$$\langle F(s,(u_1,u_2)), n(s) \rangle = (s_{\text{in}} - s_1)[(1-u_1)(\mu(s_2)(s_{\text{in}} - s_2) - \mu(s_1)(s_1 - s_2)) + (u_1 - u_2)(s_1 - s_2)(1 - \mu(s_1))].$$

Now, as $(s_1,s_2) \in \Delta_0^+ \cup \Delta_0$, we have $\mu(s_1)(s_1 - s_2) \leq \mu(s_2)(s_{\text{in}} - s_2)$. Using that $u_2 \leq u_1$ together with $\mu(s_1) < 1$, we obtain

$$\langle F(s,(u_1,u_2)), n(s) \rangle \geq 0,$$

which concludes the proof. □

This property will lead to consider two cases for $\bar{s}$, either $\bar{s}$ is above or below Λ. Let u define a subset $A \subset \mathcal{D}$ as the epigraph of q in $\mathcal{D}$:

$$A := \{s = (s_1,s_2) \in \mathcal{D} \backslash \{(0,0),(s_{in},s_{in})\} \ | \ s_2 \geq q(s_1) \ \text{ and } \ s_2 < s_1\}.$$

Proposition 3.3 implies the following result.

Corollary 3.1. *If $\bar{s} \in \mathrm{Int}A$ then $\mathcal{C}(\bar{s}) \subset \Delta_0^+$; if $\bar{s} \in \Lambda$ then $s^\star \in \partial \mathcal{C}(\bar{s})$; and if $\bar{s} \notin A$, then $s^\star \in \mathcal{C}(\bar{s})$.*

We now provide an optimal control in the two cases of the previous corollary.

3.1 Study of $\bar{s} \in \mathcal{D} \backslash A$

In this part, we show that the optimal strategy outside $\mathcal{C}(\bar{s})$ is singular (see [2, 24, 14]). Roughly speaking, the optimal strategy consists in choosing the control that steers the system to the singular arc in minimal time. If the singular arc is reached, then optimal trajectories are singular until reaching $\partial \mathcal{C}(\bar{s})$. Let $s \in \mathcal{D} \backslash \mathcal{C}(\bar{s})$. The *singular arc strategy* (SAS) is defined by:

$$u_1[s] = u_2[s] = \left| \begin{array}{cl} 1 & s \in \Delta_{SA}^-, \\ u_s[s] & s \in \Delta_{SA}, \\ 0 & s \in \Delta_{SA}^+. \end{array} \right.$$

where $u_s[\cdot]$ is a *singular* control such that the solution of (0.2) with $u = u_s$ belongs to Δ_{SA}. The control $u_s[\cdot]$ can be computed in feedback form:

$$u_s = -\frac{\langle \lambda, [f,[f,g]] \rangle}{\langle \lambda, [g,[f,g]] \rangle},$$

where the adjoint vector is given by $\lambda(\cdot) = -\frac{-g(s)^{\perp}}{\mathrm{get}(f(s),g(s))}$, see e.g. [4, 12]. Let us recall the definition of *turnpike* and *anti-turnpike* (see [4] p.45) adapted to our setting.

Definition 3.2. *A turnpike (resp. anti-turnpike) is a singular arc S that satisfies:*
(i) For every $s \in S$, the vectors $f(s) + g(s)$ and $f(s)$ are not tangent to Δ_{SA} and point to opposite sides of Δ_{SA}.
(ii) For every $s \in S$, we have $\det(g(s), [f,g](s)) = 0$ and $\det(f(s), g(s)) \neq 0$.
(iii) The mapping $s \longmapsto \theta(s) := -\frac{\det(g(s),[f,g](s))}{\det(f(s),g(s))}$ satisfies $\theta > 0$ (resp. $\theta < 0$) on Δ_{SA}^+ and $\theta < 0$ (resp. $\theta > 0$) on Δ_{SA}^-.

From hypothesis (H3), we know that Δ_{SA} can be written:

$$\Delta_{SA} = \Delta_{SA}^1 \cup \Delta_{SA}^2 \cup \{s^\star\},$$

where Δ_{SA}^i, $i = 1, 2$ is either a turnpike or an anti-turnpike and $\Delta_{SA}^1 \subset \Delta_0^-$ and $\Delta_{SA}^2 \subset \Delta_0^+$. Next, we assume that the singular arc is controllable.

(H4) The singular arc is controllable i.e. $u_s \in [0, 1]$.

We can now prove the following property on the singular arc.

Lemma 3.4. *The set Δ_{SA}^1 (resp. Δ_{SA}^2) is a turnpike (resp. anti-turnpike).*

Proof. First, notice by letting $s_1, s_2 \to s_{in}$ that the point (s_{in}, s_{in}) belongs to the closure of Δ_{SA} in E. Now, for $s \in \Delta_{SA}$ with $s_1 \neq s_2$, one can write:

$$\frac{s_{in} - s_2}{s_{in} - s_1} = \frac{1}{\mu'(s_2)} \left[-\int_0^1 \mu'(s_2 + r(s_1 - s_2))dr + \mu'(s_1) \right].$$

This shows that the limit of $\frac{s_{in}-s_2}{s_{in}-s_1}$ is finite whenever s goes to (s_{in}, s_{in}). Moreover, Proposition 3.2 implies that $\zeta'(s_1) \to +\infty$ when s_1 goes to s_{in}. We deduce that in a neighborhood of (s_{in}, s_{in}), Δ_{SA} is above Δ_0.

As the singular arc is controllable, the two vectors $f(s)$ and $f(s) + g(s)$ for $s \in \Delta_{SA}$ are pointing into opposite directions w.r.t. Δ_{SA}. Moreover, one can check that the point $(s_{in}, 0)$ belongs to Δ_{SA}^+ and that $f(s)$ is pointing in the direction of the point $(0, 0)$ (both components of f are negative). As the singular arc is the graph of an increasing function, $f(s) + g(s)$ is pointing into Δ_{SA}^+ and $f(s)$ into Δ_{SA}^- as was to be proved. The result follows by verifying the sign of the function θ in $\Delta_{SA}^{\pm}$. is pointing into Δ_{SA}^+ and f is pointing in the direction of Δ_{SA}^-. $\square$

Equivalently, $\Delta_{SA} \cap \Delta_0^-$ is hyperbolic or time-minimizing and $\Delta_{SA} \cap \Delta_0^+$ is elliptic or time-maximizing (see [12]).

Theorem 3.1. *The optimal strategy steering any point $s^0 \in \mathcal{D} \backslash \mathcal{C}(\bar{s})$ to $\bar{s}$ is the strategy SAS.*

Proof. Let us take a point $s^0 \in \Delta_{SA}+$. Assume that an optimal trajectory contains an arc $u = 1$ on some time interval $[t_1, t_2]$. Then, the trajectory necessarily contains a switching point (otherwise it would not reach the target). So, we can assume that $\phi_2(t_2) = 0$, and as $s^0 \in \Delta_{SA}^+$ we have $s(t_2) \in \Delta_{SA}^+$, and the trajectory switches to $u = 0$. A simple computation shows that we have:

$$\dot{\phi}_2 = \mu'(s_1)(s_{in} - s_1)\phi_2 + \lambda_2 \frac{\det(g(s), [f,g](s))}{s_{in} - s_1}. \tag{3.8}$$

Therefore, $\lambda_2 > 0$ implies that $\dot{\phi}_2(t_2) > 0$. On the other hand, as we have $u = 1$ over $[t_1, t_2]$, we have $\phi_2 > 0$ over $[t_1, t_2)$, hence $\dot{\phi}_2(t_2) \leq 0$ which gives a contradiction. Hence, the optimal trajectory necessarily satisfies $u = 0$ until reaching either the singular arc or $\partial\mathcal{C}(\bar{s})$. In the same way, we can show that an optimal trajectory starting at a point $s^0 \in \Delta_{SA}^-$ necessarily satisfies $u = 1$ until reaching either the singular arc or $\partial\mathcal{C}(\bar{s})$. The same argument as above show that it is not optimal for a trajectory to leave the singular before reaching $\bar{s}$. $\square$

As mentioned in the proof of the previous theorem, optimal trajectories may not necessarily reach the singular arc. This can happen if $\Delta_{SA} \subset \mathcal{C}(\bar{s})$ (see Fig. 5, picture right). Nevertheless, the singular strategy remains optimal. Notice that the optimal trajectory steering s^0 to $\bar{s}$ is unique in this case.

We conclude this part by a characterization of abnormal trajectories that are optimal.

Proposition 3.5. *Consider an abnormal optimal trajectory steering a point s^0 to the target $\bar{s} \in \mathcal{D}\backslash A$. Then, $s^0 \in \gamma^+(\bar{s}, (1,0)) \cup \gamma^+(\bar{s}, (0,0)) \cup \gamma^+(\bar{s}, (1,1))$.*

Proof. Consider an abnormal extremal trajectory. From (1.2), λ_2 is of constant sign. From $H = 0$, we cannot have $\lambda_2 \equiv 0$. Assume now that $\lambda_2 < 0$. From (1.3), we obtain that $u_2 = 0$ a.e. and so λ_1 is of constant sign. From $H = 0$, the only possible case is $\lambda_1 > 0$ which gives $u_1 = 1$ a.e. This case leads to $s^0 \in \gamma^+(\bar{s}, (1,0))$.

Assume now that $\lambda_2 > 0$. From Proposition 3.1, we obtain that $u_1 = u_2$. Clearly, the constant controls $(0,0)$ and $(1,1)$ are possible and conduct to $s^0 \in \gamma^+(\bar{s}, (0,0)) \cup \gamma^+(\bar{s}, (1,1))$. Moreover, if an abnormal optimal trajectory has a switching point at some time t_0, then we must have $s(t_0) \in \Delta_0$. From Theorem 3.1, we see that such a trajectory is not optimal. □

3.2 Study of $\bar{s} \in A$

Theorem 3.2. *Consider a target point $\bar{s} \in A$. Then, $\mathcal{C}(\bar{s})$ is not reachable from $\mathcal{D}\backslash\mathcal{C}(\bar{s})$.*

Proof. First, let us assume that $\bar{s} \in \mathrm{Int}A$ and suppose that there exists a trajectory steering (0.2) from $s^0 \in \mathcal{D}\backslash\mathcal{C}(\bar{s})$ to the target. If follows that $\mathcal{C}(\bar{s})$ is reachable from s^0, hence optimal trajectories starting from the point s^0 satisfy $u = u_1 = u_2$. From Proposition 3.3, we obtain that $\mathcal{C}(\bar{s}) \subset \Delta_0^+$. An optimal trajectory necessarily necessarily satisfies $u = 0$, $u = 1$ or $u = u_s$. As $\mathcal{C}(\bar{s}) \subset \Delta_0^+$, the trajectory cannot reach $\mathcal{C}(\bar{s})$ with the singular arc. Now, the trajectory necessarily has a switching point in Δ_0^+ at some time t_0, otherwise it would reach $\bar{s}$ with $u = 0$ or $u = 1$ which means that $s^0 \in \partial\mathcal{C}(s^0)$ in contradiction with the choice of s^0. Using the expression of the Hamiltonian and $\phi_2(t_0) = 0$, we find that:

$$\lambda_1(t_0) = \frac{s_1 - s_2}{s_{in} - s_1}\frac{1}{\rho(s_1, s_2)}, \ \lambda_2(t_0) = -\frac{1}{\rho(s_1, s_2)}, \tag{3.9}$$

where $\rho(s_1, s_2) = \mu(s_2)(s_{in} - s_2) - \mu(s_1)(s_1 - s_2)$. But, we have $\rho > 0$ in Δ_0^+, hence we obtain that $\lambda(t_2) < 0$ which contradicts the fact that $\lambda_2 > 0$ for initial conditions outside of $\mathcal{C}(\bar{s})$. This means that $\bar{s}$ is not reachable from s^0 which ends the proof.

Now, we have to investigate the case where $\bar{s} \in \Lambda$. Let $s^0 \in \mathcal{D}\backslash\mathcal{C}(\bar{s})$. If $\bar{s} = s^\star$, then from Lemma 3.1, we know that $\mathcal{C}(\bar{s})\backslash s^\star \subset \Delta_0^+$. As we cannot reach $s^\star$ with the singular arc, we can proceed as in the previous case. Assume now that $\bar{s} \in \gamma^+(\bar{s}, (0,0))\backslash s^\star$. Notice that the part of the boundary of $\mathcal{C}(\bar{s})$ defined with the controls $u_1 = u_2 = 0$ coincides with Λ in the set Δ_{SA}^-. Lemma 3.1 implies that the part of the boundary of $\mathcal{C}(\bar{s})$ defined with the controls $u_1 = u_2 = 1$ does not intersect Δ_0. Therefore, $\mathcal{C}(\bar{s})\backslash s^\star \subset \Delta_0^+$ and we can use the same argument as in the case $\bar{s} = s^\star$ to show that $\mathcal{C}(\bar{s})$ cannot be reached from s^0. In the case where $\bar{s} \in \gamma^+(\bar{s}, (1,1))\backslash s^\star$, we can use a similar reasoning to show that $\mathcal{C}(\bar{s})$ cannot be reached from s^0. This concludes the proof. □

The next theorem is our main result and is a rephrasing of Proposition 2.2, Theorem 3.1, Theorem 3.2.

Theorem 3.3. *Let us take a target point $\bar{s} \in \mathcal{D}$. Then, the we have the two following cases.*
(i) If $\bar{s} \in A$, then $\bar{s}$ is reachable from initial conditions in $\mathcal{C}(\bar{s})$ only.
(ii) If $\bar{s} \notin A$, then $\bar{s}$ is reachable from any initial conditions in $\mathcal{D}$.
(iii) Moreover, the optimal synthesis is as follows:

- *For $s^0 \in \mathcal{C}(\bar{s})$, there exist infinitely many controls of the form $(u_1, 0)$ or $(1, u_2)$ steering s^0 to $\bar{s}$.*
- *For $s^0 \in \mathcal{D}\backslash\mathcal{C}(\bar{s})$ (case ii only), the optimal control is given by the singular arc strategy.*

In the first case of Theorem 3.3, we can write the value function as:

$$T(s^0) := \begin{cases} \max\left\{\int_{s_1^0}^{\bar{s}_1} \frac{d\sigma}{(1-\mu(\sigma))(s_{\mathrm{in}}-\sigma)}, \int_{s_2^0}^{\bar{s}_2} \frac{d\sigma}{-\mu(\sigma)(s_{\mathrm{in}}-\sigma)}\right\}, \ s \in \mathcal{C}(\bar{s}), \\ +\infty, \ s \in \mathcal{D}\backslash\mathcal{C}(\bar{s}). \end{cases}$$

In the second case of Theorem 3.3, we denote by $T_1(s^0)$ the time to drive (0.2) from s^0 to $\bar{s}$ by the singular arc strategy. We can write the value function as:

$$T(s^0) := \begin{cases} \max\left\{\int_{s_1^0}^{\bar{s}_1} \frac{d\sigma}{(1-\mu(\sigma))(s_{\rm in}-\sigma)}, \int_{s_2^0}^{\bar{s}_2} \frac{d\sigma}{-\mu(\sigma)(s_{\rm in}-\sigma)}\right\}, \ s \in \mathcal{C}(\bar{s}), \\ T_1(s^0) \ s \in \mathcal{D}\backslash\mathcal{C}(\bar{s}). \end{cases}$$

4 Numerical Simulations

4.1 Singular arc when μ is linear

In this part, we suppose that $\mu(s) = \alpha s$ with $0 < \alpha < \frac{1}{s_{in}}$ (see (H2)). We can prove the following properties of the singular arc.

Proposition 4.1. *(i) The set Δ_{SA} is given by*

$$\Delta_{SA} = \{(s_1, s_2) \in \mathcal{D} \mid s_2 = 2s_1 - s_{in}\}.$$

(ii) There exists exactly one steady-state singular point $s^\star := (\frac{2s_{in}}{3}, \frac{s_{in}}{3})$.
(iii) The singular control is given by $u_s := 2\alpha(s_{in} - s_1)$, and $u_s \in [0,1]$.
(iv) The steady-state singular point is attractive and:

$$\dot{s}_1 > 0 \text{ iff } s_1 \in [\frac{s_{in}}{2}, \frac{2s_{in}}{3}) \text{ and } \dot{s}_1 < 0 \text{ iff } s_1 \in (\frac{2s_{in}}{3}, s_{in}].$$

(v) The adjoint vector is given by

$$\lambda_1 = -\lambda_2 = -\frac{1}{\alpha(s_{in} - s_1)(-3s_1 + 2s_{in})}, \ s_1 \neq \frac{2s_{in}}{3}.$$

Proof. The proof of (i) and (ii) is straightforward. Notice that we have $s_{in} - s_1 = s_1 - s_2$ along the singular arc. The expression of u_s follows from (0.2) using $\dot{s}_2 = 2\dot{s}_1$ which proves (iii). Now replacing u_s into (0.2) gives the closed-loop system:

$$\begin{cases} \dot{s}_1 = \alpha(s_{in} - s_1)(2s_{in} - 3s_1), \\ \dot{s}_2 = \alpha(s_{in} - s_1)(s_{in} - 3s_2), \end{cases} \tag{4.1}$$

hence we obtain (iv). The proof of (v) follows by solving $\phi_2 = 0$ together with $H = 0$ along a singular arc. □

This case is illustrated on Fig. 4. The arrows indicate that $s^\star$ is attractive. The singular arc in Δ_0^- (part below the green curve Δ_0) is turnpike (time minimizing) whereas the singular arc in Δ_0^+ (part above Δ_0) is anti-turnpike (time maximizing). The figures 5 and 6 are an illustration of Theorem 3.3. Several examples of optimal trajectories for initial conditions outside the set $\mathcal{C}(\bar{s})$ are depicted.

4.2 Singular arc when μ is Monod

In this part, we suppose that the growth function is of Monod type (see [27]): $\mu_m(s) = \frac{\bar{\mu}s}{k+s}$, where $k > 0$, $\bar{\mu} > 0$ and $\bar{\mu}s_{in} < 1$ see (H2). The situation is quite similar to the linear case, but the expression of Δ_0, Δ_{SA} and u_s are more delicate to obtain. We have used a symbolic software in order to verify the next proposition.

Proposition 4.2. *(i) There exists $\check{s}_1 \in (0, s_{in})$ and a C^1-mapping $\xi : [\check{s}_1, s_{in}) \to [0, s_{in})$ which is increasing and such that $(s_1, s_2) \in \Delta_{SA}$ if and only if $s_2 = \xi(s_1)$, where :*

$$\xi(s_1) := \frac{1}{2(s_{in} - s)}\left[-s_{in}s_1 - k^2 + ks_1 - 3ks_{in} + \sqrt{(k+s_1)^2(s_{in}+k)(5s_{in}+k-4s_1)}\right],$$

and we have $\check{s}_1 := \frac{-2k - s_{in} + k\sqrt{4k^2 + 8ks_{in} + 5s_{in}^2}}{2(s_{in}+k)}$.
(ii) The function $\zeta : [0, s_{in}] \to [0, s_{in}]$ is given by:

$$\zeta(s_1) = -\frac{1}{2k}\left[s_1^2 - ks_1 - ks_{in} - s_{in}s_1 + \sqrt{(s_{in} - s_1)(ks_{in} + s_{in}s_1 - s_1^2 + 3ks_1)}\right], \ s_1 \in (0, s_{in}).$$

By using the expression of ξ and ζ, we can check numerically that there exists exactly one singular point $s^\star$, see Fig. 4.

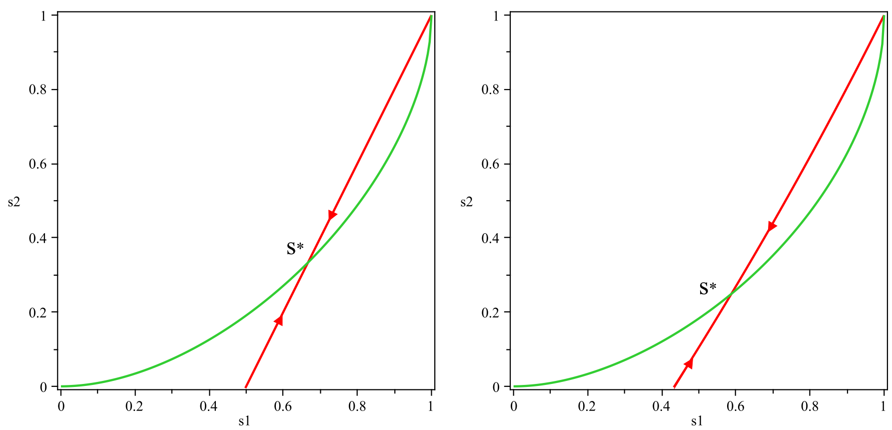

Figure 4: *Picture left*: plot of the singular arc Δ_{SA} (in red) and of Δ_0 (in green) in the linear case: $\mu(s) = s$. *Picture right*: plot of the singular arc Δ_{SA} (in red) and of Δ_0 (in green) in the Monod case: $\mu_m(s) = \frac{s}{5+s}$.

5 Conclusion

For this minimal time problem with a control set of a triangular shape, that is due to the constraint $u_2 \leq u_1$, we have shown the benefit of considering a particular subset $\mathcal{C}(\bar{s})$ of the state space, that is target dependent and presents the following features:

- Outside this set, either the target is non reachable, or the optimal synthesis fulfills $u_1 = u_2$ with a possible singular arc,
- in $\mathcal{C}(\bar{s})$, the extra controllability over controls $u_1 = u_2$ leads to an infinity of optimal trajectories, all of them with $u_2 \neq u_1$ (excepted on part of the boundary of the set).

Furthermore, when the the target is reachable from the exterior of $\mathcal{C}(\bar{s})$, the particular cascade structure of the problem leads to a non intuitive feature of some of the optimal trajectories: it consists in rolling far away from the target until reaching the set $\mathcal{C}(\bar{s})$ or a singular arc to be followed until eventually reaching $\mathcal{C}(\bar{s})$, and then come back along the set $\mathcal{C}(\bar{s})$.

The geometric analysis has revealed the role of two particular curves $p(\cdot)$ and $q(\cdot)$, that can be easily computed numerically and that indicate to the practioners if the target is reachable and the nature of the optimal feedback depending on the position of the initial condition with respect to these curves.

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References

[1] W. Bakker, H. Beeftink, C. de Gooijer and J. Tramper, *Bioreactors in series: An overview of design procedures and practical applications*, Enzyme and Microbial Technology, 18 (1996), 202–219.

[2] T. Bayen, F. Mairet, P. Gajardo, *Optimal synthesis for the minimum time control problems of fed-batch bioprocesses for growth functions with two maxima*, Journal of Optim. Theory and Applications, vol. 158, 2, pp 521-553, 2013.

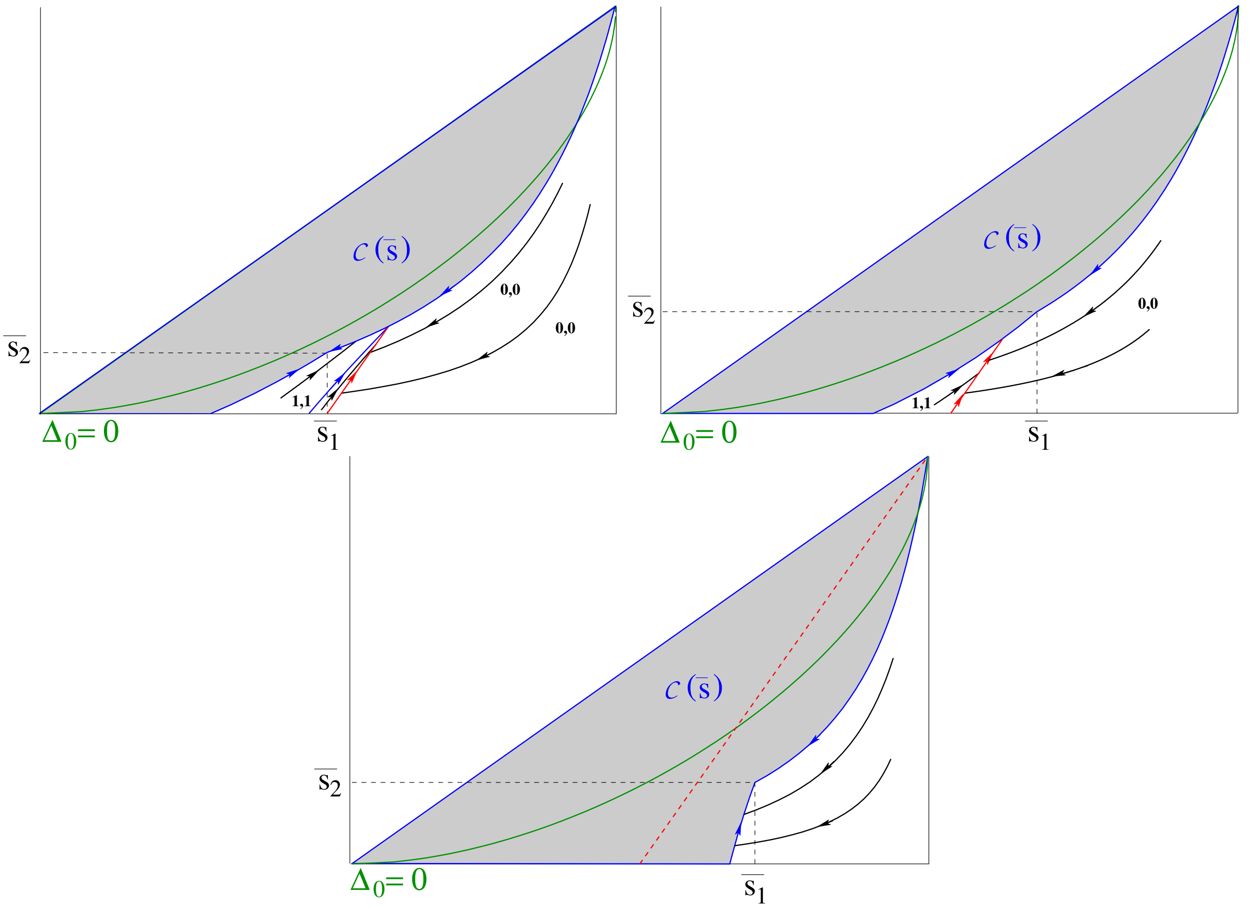

Figure 5: Examples of optimal trajectories when $s^\star \in \mathcal{C}(\bar{s})$. *Picture left*: target in Δ_{SA}^-. *Picture in the middle*: target in Δ_{SA}^+. *Picture right*: the singular arc satisfies $\Delta_{SA} \subset \mathcal{C}(\bar{s})$ and the singular strategy reduces to $u = 0$ until reaching $\partial\mathcal{C}(\bar{s})$

[3] B. Bonnard and M. Chyba, *Singular Trajectories and their role in Control Theorey*, Springer, SMAI, vol. 40, 2002.

[4] U. Boscain and B. Piccoli, *Optimal Syntheses for Control Systems on 2-D Manifolds*, Springer SMAI, vol. 43, 2004.

[5] P. Cadarliaguet, *On the regularity of semipermeable surfaces in control theory with application to the optimal exit-time problem (part I)*, SIAM J. Control Optim, Vol. 35, 5, pp. 1638–1652, 1997.

[6] P. Cadarliaguet, *On the regularity of semipermeable surfaces in control theory with application to the optimal exit-time problem (part II)*, SIAM J. Control Optim, Vol. 35, 5, pp. 1653–1671, 1997.

[7] M. Bardi, I. Capuzzo-Dolcetta, *Optimal Control and Visosity Solutions of Hamilton-Jacobi-Bellman Equations, Birkhauser*, 1997.

[8] I. Cacciari, S. Grego and E. Di Mattia, *Eco-physiological characterization of soil bacterial populations in different states of growth*, Microb. Ecol. 43(1) (2002), 34–43.

[9] C. Camarasa, T. Clement, M. Perez, J.R. Mouret and J.M. Sablayrolles, *Use of a continuous multistage bioreactor to mimic winemaking fermentation*, International Journal of Food Microbiology, 150(1):42?49, 2011.

[10] D. Dochain and A. Rapaport, *Minimal time control of fed-batch processes for growth functions with several maxima*, IEEE Transactions on Automatic Control, vol. 56, 11, pp. 2671-2676, 2011.

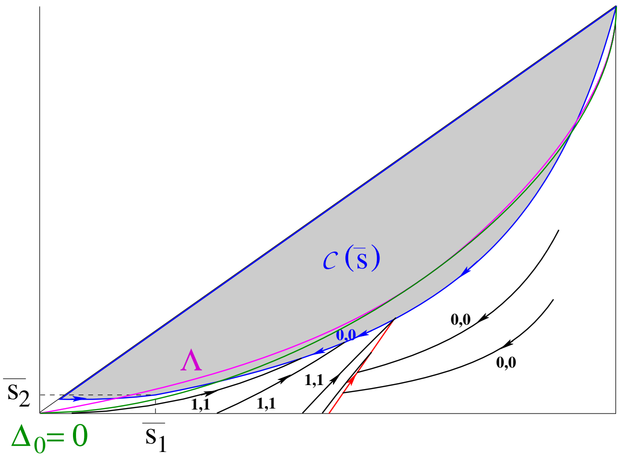

Figure 6: In green, the semi-permeable curve Λ, in red the curve Δ_0, in blue, the boundary of $\mathcal{C}(\bar{s})$, in purple, the singular arc Δ_{SA}. In black, we have plotted different optimal trajectories for different choices of the initial state outside of $\mathcal{C}(\bar{s})$.

[11] A. Dramé, J. Harmand, C. Lobry and A. Rapaport, *Multiple steady state profiles in interconnected biological systems*, Mathematical and Computer Modelling of Dynamical Systems, 12 (2006), 379–393.

[12] B. Bonnard, J.-B. Caillau, E. Trélat, *Second order optimality conditions in the smooth case and applications in optimal control*, ESAIM Control Optim. Calc. Var. 13, 2, pp. 207–236, 2007.

[13] H. El-Owaidy and O. El-Leithy, *Theoretical studies on extinction in the gradostat*, Mathematical Biosciences, 101(1) (1990), 1–26.

[14] P. Gajardo, H. Ramirez, A. Rapaport, *Minimal time sequential batch reactors with bounded and impulse controls for one or more species*, SIAM J. Control Optim., Vol. 47, 6, pp. 2827–2856, 2008.

[15] F. Gérard, I. Haidar and A. Rapaport, *Effects of spatial structure and diffusion on the performances of the chemostat*, Mathematical Biosciences and Engineering, 8(4) (2011), 953–971.

[16] J. Harmand, A. Rapaport and A. Trofino, *Optimal design of two interconnected bioreactors–some new results*, American Institute of Chemical Engineering Journal, 49 (1999), 1433–1450.

[17] Y. Higashi, N. Ytow, H. Saida and H. Seki, *In situ gradostat for the study of natural phytoplankton community with an experimental nutrient gradient* Environmental Pollution, 99 (1998), 395–404.

[18] G. Hill and C. Robinson, *Minimum tank volumes for CFST bioreactors in series*, The Canadian Journal of Chemical Engineering, 67 (1989), 818–824.

[19] U. Ledzewicz and H. Schattler, *Geometric Optimal Control*, Springer 2012.

[20] R. Lovitt and J. Wimpenny, *The gradostat: A bidirectional compound chemostat and its applications in microbial research*, Journal of General Microbiology, 127 (1981), 261–268.

[21] A. Miele, *Application of Green's Theorem to the extremization of linear integrals*, Symp. on Vehicle Systems Optimization, Garden City, L. I. New York, pp. 26–35, 1961.

[22] K. Mischaikow, H.L. Smith, H. Thieme *Asymptotically autonomous semiflows: chain recurrence and Liapunov functions*, Trans. Amer. Math. Soc., vol. 347, 1995, pp.1669–1685, 1995.

[23] J. Monod, *Recherches sur la Croissance des Cultures Bactériennes*, Hermann, Paris 1942.

[24] J. A. Moreno, *Optimal time control of bioreactors for the wastewater treatment*, Optim. Control Appl. Meth., 20, pp. 145–164, 1999.

[25] M. Nelson and H. Sidhu, *Evaluating the performance of a cascade of two bioreactors*, Chemical Engineering Science, 61 (2006), 3159–3166.

[26] M. Quincampoix, *Differential inclusions and target problems*, SIAM J. Control Optim., Vol 30, 2, pp. 324–335, 1992.

[27] H.L. Smith and P. Waltman, *The gradostat: a model of competition along a nutrient gradient*, J. Microb. Ecol. 22 (1991) 207–226.

[28] H.L. Smith and P. Waltman, *The theory of the chemostat*, Dynamics of microbial competition, Cambridge University Press, 1995.

[29] B. Tang, *Mathematical investigations of growth of microorganisms in the gradostat*,J. Math. Biol. Vol 23 (1986) 319–339.