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The law of iterated logarithm for additive functionals and martingale additive functionals of Harris recurrent Markov processes

Eva Löcherbach*and Dasha Loukianova†

Abstract

We use Nummelin splitting in continuous time in order to prove laws of iterated logarithm for additive functionals of a Harris recurrent Markov process, with deterministic or random renormalization.

Key words : Harris recurrence, Nummelin splitting, continuous time Markov processes, additive functionals, law of iterated logarithm.

MSC 2000 : 60 F 99, 60 J 35, 60 J 55, 60 J 60

1 Introduction

Let X be a Harris recurrent strong Markov process in continuous time, having invariant measure μ. In [9], we studied the problem of introducing Nummeling splitting in continuous time. This is a method, firstly introduced independently by Nummelin, [12], and Athreya and Ney, [1], in the discrete time case that allows to introduce renewal times for the process on an extended probability space. In [9], we translate this technique to the situation of processes in continuous time. It is a classical technique in the theory of Markov processes to translate results from discrete time to continuous time by considering what is called the R-chain in the literature. This means that we observe the continuous time process after independent exponential waiting times. We refer the reader to Meyn and Tweedie, [10] and [11], for a general survey of the subject. Hence we use the Nummelin splitting technique at random times $T_n, n \geq 1$, which we get when sampling the process after independent exponential waiting times. This allows to get a sequence of renewal times for the process in the following sense : There exists a sequence of stopping times (S_n, R_n) such that

1. For all n, $S_n < R_n < \infty$, $S_{n+1} = S_n + S_1 \circ \theta_{S_n}$, $R_n = \inf\{T_m : T_m > S_n\}$.
2. For every n, X_{R_n} is independent of $\sigma\{X_s : s \leq S_n\}$ and $\mathcal{L}(X_{R_n}) = \nu$ for some fixed probability measure ν.

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The details of this construction are recalled in section 4 of this paper.

We apply our technique to the study of the asymptotic behavior of additive functionals, for example $A_t = \int_0^t f(X_s)ds$. In [9], we have shown the existence of a deterministic equivalent for integrable additive functionals A_t. The deterministic equivalent is a deterministic function $t \mapsto v(t)$ such that $v(t) \to \infty$ as $t \to \infty$ and such that for any integrable additive functional A_t,

$$\lim_{M\to\infty} \liminf_{t\to\infty} P_\pi(1/M \leq A_t/v(t) \leq M) = 1$$

for any initial measure π. The deterministic equivalent can be defined as follows. Take any fixed positive special function g of the process having $\mu(g) > 0$ (see definition 2.3 below for the exact definition of special functions, for strong Feller processes, any bounded function having compact support is special) and define

$$v(t) := E_\eta(\int_0^t g(X_s ds),$$

where η is an arbitrary initial measure. Then the strong Chacon-Ornstein theorem implies that for any other special function g' and any other initial measure η',

$$\lim_{t\to\infty} \frac{E_\eta(\int_0^t g(X_s ds)}{E_{\eta'}(\int_0^t g'(X_s ds)} = \frac{\mu(g)}{\mu(g')}.$$

Hence the deterministic equivalent is unique up to a constant in the sense that for two choices of the deterministic equivalent, v and v', we have that $\lim_{t\to\infty} v(t)/v'(t) = c$, where c is a positive constant. In *regular* models, it can be shown that $v(t) \sim t^\alpha l(t)$, where l is a function that varies slowly at infinity. For example, for Brownian motion in dimension one, we have $\alpha = 1/2$.

In the present paper, we generalize results obtained by Chen in [3], [4] and [5] on the almost sure asymptotic behavior of integrable additive functionals to the continuous time case.

In the first case, consider $A_t = \int_0^t f(X_s)ds$ an additive functional such that $\mu(f) > 0$. In this case, it is possible to use the Chacon-Ornstein ratio limit theorem in order to compare A_t to additive functionals of the R-chain $(X_{T_n})_n$ where T_n is the sum of n independent exponential waiting times : If we write $N_t := \max\{n : T_n \leq t\}$, then

$$\frac{\int_0^t f(X_s)ds}{\sum_{k=1}^{N_t} f(X_{T_k})} \to 1 \text{ almost surely as } t \to \infty.$$

Now, it is possible to translate results obtained by Chen in [3] directly to the continuous time case, and we get the following first theorem (cf. theorem 3.1): Let $L_2(\lambda) := (\log\log\lambda) \vee 1$. Then for $v(t)$ the deterministic equivalent of the process : there exists a constant $C \in (0, \infty)$, depending only on the process but not on f, such that

$$\limsup_{t\to\infty} \frac{\int_0^t f(X_s)ds}{v(\frac{t}{L_2(v(t))})L_2(v(t))} = C\mu(f) \text{ a.s.}$$

Let $a(n)$ be the deterministic equivalent of the R-chain, i.e.

$$\lim_{M\to\infty} \liminf_{n\to\infty} P_\pi(1/M \leq \frac{\sum_{k=1}^n f(X_{T_k})}{a(n)} \leq M) = 1$$

for any initial measure π and any positive function f such that $\mu(f) > 0$. The existence of $a(n)$ has been proven in Chen [3]. Then the only difficulty in this first situation consists in the fact that we have to prove equivalence of the deterministic equivalent $a(n)$ of the R-chain and of $v(n)$. This is done in section 4.2 and in section 5 thanks to Nummelin splitting in continuous time.

Secondly, we are interested in strong limit theorems for additive functionals in the case where $A_t = \int_0^t f(X_s)ds$ having $\mu(f) = 0$. This case is much more difficult since now a direct comparison with additive functionals of the R-chain is no more possible : the ratio-limit theorem is no more valid. It is in this situation that the Nummelin spitting in continuous time shows to be very useful : Taking increments of the additive functional over life cycles, i.e. putting

$$\xi_n := \int_{R_{n-1}}^{R_n} f(X_s)ds,$$

the ξ_n are not independent, but very strongly mixing, and of the same law. (Actually, ξ_n is independent of ξ_{n+2}.) Hence it is possible to apply the classical law of iterated logarithm to $\xi_1 + \ldots + \xi_n$ under sufficient moment conditions, and we obtain the following law of iterated logarithm with random renormalization : Let f be a $\mu-$integrable function such that $\mu(f) = 0$ and such that $|f|$ is a special function (see definition 2.3 below for details), f is also called a *charge* in this case. Let $B_t = \int_0^t g(X_s)ds$ be any additive functional such that $\mu(g) > 0$. Then we have

$$\limsup_{t\to\infty} \frac{\int_0^t f(X_s)ds}{\sqrt{2B_t L_2 B_t}} = c$$

almost surely, where c is a constant that is positive if the asymptotic variance of $\xi_1+\ldots+\xi_n$ is positive.

Let us give some comments on our results. Firstly, as already pointed out, our results are a direct generalization of results obtained by Chen in the case of Markov chains, see [3]– [5]. It has not been possible to translate all results obtained by Chen to our case since in contrary to the discrete time case, the ξ_n are no longer independent (in the case of Markov chains, Nummelin splitting *does* introduce independent increments of additive functionals over life-cycles). Touati, see [18], also gives laws of iterated logarithm, also in the continuous time case, but only under the assumption of regularity of the process, i.e. R_1 is in the domain of attraction of some stable law. However, we do not need the assumption of regularity of the process. Finally, let us also point out the work by Csáki, Földes and Hu, [6], in which the authors prove a law of iterated logarithm for additive functionals of planar Brownian motion, however using deterministic renormalization.

The paper is organized as follows. In section 4, we recall the technique of Nummelin splitting in continuous time, as introduced in [9]. Moreover, we recall known results related to this construction as well as some new technical results that will be useful in the sequel. Section 5 is devoted to the proof of theorem 3.1, section 6 gives the proof of theorems 3.2 and 3.4.

2 Notation

Consider a probability space $(\Omega, \mathcal{A}, (P_x)_x)$, and on $(\Omega, \mathcal{A}, (P_x)_x)$ a process $X = (X_t)_{t\geq 0}$ which is strong Markov, taking values in a locally compact Polish space $(E, \mathcal{E})$, with càdlàg

paths, and with $X_0 = x$ P_x-almost surely, $x \in E$. We write $(P_t)_t$ for the semi group of X and we suppose that X is recurrent in the sense of Harris, with invariant measure μ, unique up to multiplication with a constant. Moreover, we shall write $(\mathcal{F}_t)_t$ for the filtration generated by the process.

We impose the following regularity condition on the transition semi-group P_t of X :

Assumption 2.1 *1. The transition semi-group P_t of the process X is strongly Feller, i.e. for every $A \in \mathcal{E}$, $x \mapsto P_t(x, A)$ is continuous.*

2. There exists a sigma-finite positive measure Λ on $(E, \mathcal{E})$ such that for every $t > 0$, $P_t(x, dy) = p_t(x, y)\Lambda(dy)$, where $(t, x, y) \mapsto p_t(x, y)$ is jointly measurable.

2.1 On the deterministic equivalent of additive functionals

We resume results of [9] on the deterministic equivalent of additive functionals. First, recall the definition of an additive functional:

Definition 2.2 *An additive functional of the process X is a $\bar{I\!R}_+-$valued, adapted process $A = (A_t)_{t\geq 0}$ such that*

1. Almost surely, the process is non-decreasing, right-continuous, having $A_0 = 0$.

2. For any $s, t \geq 0$, $A_{s+t} = A_t + A_s \circ \theta_t$ almost surely. Here, θ denotes the shift operator.

Examples for additive functionals are $A_t = \int_0^t f(X_s)ds$ where f is a positive measurable function. Such an additive functional is said to be *integrable*, if $\mu(f) < \infty$. In [9], we have constructed a deterministic equivalent of any integrable additive functional. This is a deterministic function $v \mapsto v(t)$ such that $v(0) = 0, v(.)$ is non-decreasing and $v(t) \to \infty$ as $t \to \infty$ satisfying that for any integrable additive functional A_t, $A_t/v(t)$ is bounded and bounded away from zero in probability (see corollary 2.8 of [9]). Recall the notion of a special function (see also [14], [2]):

Definition 2.3 *A measurable function $f : E \to I\!R_+$ is called special if for all bounded and positive measurable functions h such that $\mu(h) > 0$, the function*

$$x \mapsto E_x \int_0^\infty \exp\left[-\int_0^t h(X_s)ds\right] f(X_t)dt$$

is bounded. In the same way, an additive functional A_t is called special if

$$x \mapsto E_x \int_0^\infty \exp\left[-\int_0^t h(X_s)ds\right] dA_t$$

is bounded.

By [9], any special function g of X with $\mu(g) > 0$ defines a version of the deterministic equivalent via

$$v(t) = E_\pi \int_0^t g(X_s)ds, \tag{2.1}$$

for any arbitrary initial measure π. $v(t)$ is unique up to a constant in the following sense : For any other choice $v'(t)$ of a deterministic equivalent, we have that $\lim_{t\to\infty} v(t)/v'(t) = c$, where c is a positive constant. $v(t)$ is called deterministic equivalent due to the following result (see corollary 2.19 of [9]).

Theorem 2.4 *For any additive functional A of the process having $E_\mu(A_1) \in]0,\infty[$, we have*

$$\lim_{M\to\infty} \liminf_{t\to\infty} P_\pi\left(\frac{1}{M} \leq \frac{1}{v(t)} A_t \leq M\right) = 1.$$

3 The law of iterated logarithm for additive functionals

We put

$$L_2(\lambda) := (\log\log\lambda) \vee 1 \text{ for every } \lambda \geq 0.$$

Theorem 3.1 *Let A be a $\mu-$integrable additive functional of X. There exists a constant $0 < c < \infty$ that depends only on the process, but not on A, such that*

$$\limsup_{t\to\infty} \frac{A_t}{v(\frac{t}{L_2(v(t))})L_2(v(t))} = c\int \mu(dx)E_x(A_1) \ \ a.s. \tag{3.2}$$

Moreover, let f be a measurable $\mu-$integrable function satisfying the following assumptions:

$$\begin{array}{ll} (i) & \mu(f) = 0. \\ (ii) & |f| \text{ is a bounded special function.} \end{array}$$

Then we have the following:

Theorem 3.2 *Suppose f satisfies assumptions (i) and (ii) above. Let $A = (A_t)_t$ be any $\mu-$integrable additive functional of the process. Then there exists a constant $\lambda_f \geq 0$, such that*

$$\limsup_{t\to\infty} \frac{\int_0^t f(X_s)ds}{\sqrt{2A_t L_2(A_t)}} = (E_\mu(A_1))^{-1/2} \ \lambda_f \ \ a.s.$$

Here, $\lambda_f > 0$ if (6.20) below holds.

Remark 3.3 *Under the additional hypothesis*

$$\int_0^\infty f(x)P_t f(x)dt \text{ converges in } L^1(E,\mu),$$

we have that

$$(\lambda_f)^2 = 2\int\int_0^\infty f(x)P_t f(x)dt\mu(dx). \tag{3.3}$$

Finally, let M be a locally square integrable local $(P_x, (\mathcal{F}_t)_t)-$martingale, with $M_0 = 0$, having continuous paths, for any initial value x. We suppose moreover that

$$\forall y,\ \forall s,t:\ M_{t+s} - M_t = M_s \circ \theta_t \ P_y - \text{a.s.}, \tag{3.4}$$

and the process $< M >$ is an additive functional of X satisfying

$$E_\mu(< M >_1) < \infty. \tag{3.5}$$

Then we have the following theorem:

Theorem 3.4 *Under the assumptions (3.4) and (3.5), we have*

$$\limsup_{t\to\infty} \frac{M_t}{\sqrt{2A_t L_2(A_t)}} = (E_\mu(A_1))^{-1/2} \ \lambda_M \ a.s.$$

Moreover, we have also that

$$\limsup_{t\to\infty} \frac{M_t}{\sqrt{v(\frac{t}{L_2(v(t))})L_2(v(t))}} = c\,(E_\mu(A_1))^{-1/2} \ \lambda_M \ a.s.$$

Here, $\lambda_M^2 = E_\mu(< M >_1)$*, and* c *is a constant that depends only on the process.*

Remark 3.5 *The events* $\limsup_{t\to\infty} \frac{M_t}{\sqrt{2A_t L_2(A_t)}} = (E_\mu(A_1))^{-1/2} \ \lambda_M$ *etc of theorem 3.1 to 3.4 belong to the sigma-field* $\mathcal{J}$ *of invariant sets, and since* X *is Harris, by Revuz and Yor, [15], chapter X, proposition 3.6 and 3.10, its probability is either zero or one. Hence the convergence holds almost surely, independently of the choice of the initial distribution.*

Remark 3.6 *Touati, see [18], has shown the statement of theorem 3.4 for regular models, i.e. models where* $v(t) \sim t^\alpha l(t)$ *as* $t \to \infty$*, for some* $0 < \alpha \leq 1$*, and where* l *varies slowly at infinity. However, our result holds always.*

Example 3.7 *We consider the following statistical problem that has been studied by Höpfner and Kutoyants, see [7]. Observe the trajectory of a one-dimensional diffusion*

$$dX_t = \vartheta b(X_t)dt + \sigma dW_t, \ X_0 = 0,$$

where $\sigma > 0$ *is known, where* ϑ *is some unknown parameter and* b *continuous such that the diffusion is recurrent. The model is not necessarily regular.*

In this model, the maximum likelihood estimator is given by

$$\hat{\vartheta}_t = \int_0^t b(X_s)dX_s / \int_0^t b^2(X_s)ds.$$

Let $I_t := \int_0^t b^2(X_s)ds$ *and* $M_t := \int_0^t b(X_s)\sigma dW_s$*. Hence,*

$$I_t(\hat{\vartheta}_t - \vartheta) = M_t.$$

Now, applying theorem 3.4, we get

$$\limsup_{t\to\infty} \frac{\sqrt{I_t}}{\sqrt{L_2 I_t}}(\hat{\vartheta}_t - \vartheta) < \infty \ a.s.,$$

which means that we have to consider random rates of convergence.

4 Preliminaries on Nummelin splitting in continuous time

We use the Nummelin splitting in continuous time, as developed in [9], in order to introduce a recurrent atom for the process. We recall briefly the construction:

Introduce a sequence $(\sigma_n)_{n\geq 1}$ of i.i.d. $exp(1)$-waiting times, independent of the process X itself. Let $T_0 := 0$, $T_n := \sigma_1 + \ldots + \sigma_n$ and $\bar{X}_n := X_{T_n}$. Then the chain $\bar{X} = (\bar{X}_n)_n$ is recurrent in the sense of Harris and its one-step transition kernel $U^1(x, dy) := \int_0^\infty e^{-t} P_t(x, dy)dt$ satisfies a minorization condition:

$$U^1(x, dy) \geq \alpha 1_C(x)\nu(dy), \tag{4.6}$$

where $0 < \alpha < 1$, $\mu(C) > 0$ and ν a probability measure equivalent to $\mu(\cdot \cap C)$ (cf [14], [8], proposition 6.7). The set C can be chosen to be compact.

Then it is possible to define on an extension of the original space $(\Omega, \mathcal{A}, (P_x))$ a Markov process $Z = (Z_t)_{t\geq 0}$, taking values in $E \times [0, 1] \times E$ such that the T_n are jump times of the process and such that under P_x, $((Z_t^1)_t, (T_n)_n)$ has the same distribution as $((X_t)_t, (T_n)_n)$. We give the details as introduced in [9] :

First of all, define the following transition kernel $Q((x, u), dy)$ from $E \times [0, 1]$ to E :

$$Q((x,u),dy) = \begin{cases} \nu(dy) & \text{if } (x,u) \in C \times [0,\alpha] \\ \frac{1}{1-\alpha}\left(U^1(x,dy) - \alpha\nu(dy)\right) & \text{if } (x,u) \in C \times]\alpha, 1] \\ U^1(x,dy) & \text{if } x \notin C \end{cases}. \tag{4.7}$$

We now recall the construction of $Z_t = (Z_t^1, Z_t^2, Z_t^3)$ taking values in $E \times [0, 1] \times E$ as given in [9]. Write $u^1(x, x') := \int_0^\infty e^t p_t(x, x')dt$. Let $Z_0^1 = X_0 = x$. Choose Z_0^2 according to the uniform distribution U on $[0, 1]$. On $\{Z_0^2 = u\}$, choose $Z_0^3 \sim Q((x, u), dx')$. Then inductively in $n \geq 0$, on $Z_{T_n} = (x, u, x')$:

1. Choose a new jump time σ_{n+1} according to

$$e^{-t}\,\frac{p_t(x,x')}{u^1(x,x')}\,dt \text{ on } I\!R_+,$$

   where we define $0/0 := a/\infty := 1$, for any $a \geq 0$, and put $T_{n+1} := T_n + \sigma_{n+1}$.

2. On $\{\sigma_{n+1} = t\}$, put $Z^2_{T_n+s} := u$, $Z^3_{T_n+s} := x'$ for all $0 \leq s < t$.

3. For every $s < t$, choose

$$Z^1_{T_n+s} \sim \frac{p_s(x,y)p_{t-s}(y,x')}{p_t(x,x')}\,\Lambda(dy).$$

   Choose $Z^1_{T_n+s} := x_0$ for some fixed point $x_0 \in E$ on $\{p_t(x, x') = 0\}$. Moreover, given $Z^1_{T_n+s} = y$, on $s + u < t$, choose

$$Z^1_{T_n+s+u} \sim \frac{p_u(y,y')p_{t-s-u}(y',x')}{p_{t-s}(y,x')}\Lambda(dy').$$

   Again, on $\{p_{t-s}(y, x') = 0\}$, choose $Z^1_{T_n+s+u} = x_0$.

4. At the jump time T_{n+1}, choose $Z^1_{T_{n+1}} := Z^3_{T_n} = x'$. Choose $Z^2_{T_{n+1}}$ independently of $Z_s, s < T_{n+1}$, according to the uniform law U. Finally, on $\{Z^2_{T_{n+1}} = u'\}$, choose $Z^3_{T_{n+1}} \sim Q((x', u'), dx'')$.

Note that by construction, given the initial value of Z at time T_n, the evolution of the process during $[T_n, T_{n+1}[$ does not depend on the chosen value of $Z^2_{T_n}$.

We will write P_π for the measure related to X, under which X starts from the initial measure $\pi(dx)$, and $\mathbb{P}_\pi$ for the measure related to Z, under which Z starts from the initial measure $\pi(dx) \otimes U(du) \otimes Q((x, u), dy)$. In the same spirit we denote E_π the expectation with respect to P_π and $\mathbb{E}_\pi$ the expectation with respect to $\mathbb{P}_\pi$. Moreover, we shall write $\mathbb{F}$ for the filtration generated by Z, $\mathbb{G}$ for the filtration generated by the first two coordinates Z^1 and Z^2 of the process, and $\mathbb{F}^X$ for the sub-filtration generated by X interpreted as first coordinate of Z.

4.1 Some auxiliary results and remarks on the construction of Z

Proposition 4.1 *The law of $(Z^1_t)_t$ under $\mathbb{P}_\pi$ equals the law of $(X_t)_t$ under P_π. Moreover, for any given fixed time t,*

$$\mathcal{L}(Z^3_t | Z^1_t)(dx') = u^1(Z^1_t, x')\Lambda(dx').$$

Proof The first assertion is proposition 2.8 c) of [9]. We show the second assertion : Let $f, g : E \to \mathbb{R}_+$ be bounded measurable positive functions . Put $\Phi(x) := \int u^1(x, x')g(x')\Lambda(dx')$. We define

$$\mathcal{A}_n := \sigma\{\mathcal{F}_{T_{n-1}}, T_n\}.$$

Note that since $Z^1_{T_n} = Z^3_{T_{n-1}}$,

$$\mathcal{L}(Z_{T_n} | \mathcal{A}_n) = \delta_{Z^1_{T_n}}(dx)U(du)Q((Z^1_{T_n}, u), dx').$$

Then

$$\begin{aligned}
\mathbb{E}_x[f(Z^1_t)g(Z^3_t)] &= \sum_n \mathbb{E}_x[f(Z^1_t)g(Z^3_t)1_{\{T_n \le t < T_{n+1}\}}] \\
&= \sum_n \mathbb{E}_x[\mathbb{E}[f(Z^1_t)g(Z^3_t)1_{\{t<T_{n+1}\}}|\mathcal{A}_n]1_{\{T_n \le t\}}] \\
&= \sum_n \mathbb{E}_x\left(\left[\int_0^1 du \int Q((Z^1_{T_n}, u), dx')g(x') \int_0^\infty e^{-s}1_{\{t-T_n \le s\}}ds \right.\right. \\
&\qquad \left.\left. \int \frac{p_{t-T_n}(Z^1_{T_n}, y)\, p_{s-(t-T_n)}(y, x')}{u^1(Z^1_{T_n}, x')} f(y)\Lambda(dy)\right] 1_{\{T_n \le t\}}\right) \\
&= \sum_n \mathbb{E}_x\left(\left[\int p_{t-T_n}(Z^1_{T_n}, y)f(y)e^{-(t-T_n)} \right.\right. \\
&\qquad \left.\left. \left(\int (\int_0^\infty e^{-s}p_s(y, x')ds)g(x')\Lambda(dx')\right)\ \Lambda(dy)\right] 1_{\{T_n \le t\}}\right) \\
&= \sum_n \mathbb{E}_x\left(\left[\int p_{t-T_n}(Z^1_{T_n}, y)f(y)\Phi(y)e^{-(t-T_n)}\Lambda(dy)\right]\right) \\
&= \sum_n \mathbb{E}_x\left[E_{Z^1_{T_n}}(f \cdot \Phi(X_{t-T_n}); t - T_n \le T_1)\ 1_{\{T_n \le t\}}\right] \\
&= E_x[f(X_t)\Phi(X_t)],
\end{aligned}$$

since $Z^1_{T_n} \sim X_{T_n}$. This yields the assertion. •

Write

$$A := C \times [0, \alpha] \times E.$$

Now we put

$$S_0 := 0,\ R_0 := 0, S_{n+1} := \inf\{T_m > R_n : Z_{T_m} \in A\}, R_{n+1} := \inf\{T_m : T_m > S_{n+1}\}.$$

Then the sequence of $I\!F-$stopping times R_n generalizes the notion of life-cycle decomposition in the case of existence of a recurrent atom in the sense which is precised in [9].

Proposition 4.2 *a) $Z_{R_n+\cdot}$ is independent of $\mathcal{F}_{R_{n-1}}$ for all $n \geq 1$.*
b) $Z_{R_n} \sim \nu(dx)U(du)Q((x,u), dx')$ for all $n \geq 1$.
c) The sequence of $(Z_{R_n})_{n\geq 1}$ is i.i.d.

Proof The assertion is true by construction, see proposition 2.13 of [9]. •

Proposition 4.3 *Let A_t be any integrable additive functional of X. Then, up to multiplication by a constant, for any initial measure π and any $n \geq 1$,*

$$I\!E_\pi(A_{R_{n+1}} - A_{R_n}) = I\!E_\nu(A_{R_1}) = E_\mu(A_1).$$

Proof This is proposition 2.20 of [9]. •

Moreover, we have the following:

Proposition 4.4 *Let f be a measurable $\mu-$integrable function. Put $R_0 := 0$,*

$$\xi_n := \int_{R_{n-1}}^{R_n} f(X_s)ds, n \geq 1.$$

Then the sequence $(\xi_n)_n$ is a stationary ergodic sequence under $I\!P_\nu$. Moreover, for $n \geq 2$, ξ_n is independent of $\mathcal{F}_{R_{n-2}}$.

Proof Fix some $n \geq 1$ and some positive measurable bounded function $\Phi : I\!R^n \to I\!R$. Then, for any $k \geq 1$, using the Markov property with respect to $\mathcal{F}_{R_{k-1}}$,

$$I\!E_\nu(\Phi(\xi_k, \ldots, \xi_{n+k})) = I\!E_\nu(I\!E_{Z_{R_{k-1}}}(\Phi(\xi_1, \ldots, \xi_{n+1}))) = I\!E_\nu(\Phi(\xi_1, \ldots, \xi_{n+1})),$$

which yields the stationarity. For the ergodicity, note that the sigma field of invariant sets of $(\xi_n)_n$ is contained in the sigma-field of invariant sets $\mathcal{J}$ of the process, which is trivial, as indicated earlier. The last assertion is an immediate consequence the construction of Z. •

Remark 4.5 *The last assertion of proposition 4.4 implies in particular that the sequence $(\xi_n)_n$ is a strongly mixing sequence with mixing coefficients $\alpha_n = 0$ for $n \geq 2; n \in I\!N$.*

In the sequel we shall also need the following technical result.

Proposition 4.6 *Let f satisfy the assumptions (i) and (ii) of theorem 3.2. Let ξ_n be as in proposition 4.4. Then $\mathbb{E}_\pi(|\xi_k|^3) < \infty$.*

Proof We interpret X as first coordinate of Z. Note that

$$\begin{aligned}
\xi_k^3 &= \int_{R_{k-1}}^{R_k}\int_{R_{k-1}}^{R_k}\int_{R_{k-1}}^{R_k} f(X_s)f(X_u)f(X_v)dsdudv \\
&= 6\int_{R_{k-1}}^{R_k}(\int_s^{R_k}(\int_u^{R_k} f(X_v)dv)f(X_u)du)f(X_s)ds \\
&\leq 6\int_{R_{k-1}}^{R_k}(\int_s^{R_k}(\int_u^{R_k} |f|(X_v)dv)|f|(X_u)du)|f|(X_s)ds.
\end{aligned}$$

Now, using Markov's property with respect to $\mathcal{F}_u$, we get

$$\begin{aligned}
\mathbb{E}_\pi(|\xi_k|^3) &\leq 6\mathbb{E}_\nu\int_0^{R_1}\left(\int_s^{R_1}|f|(X_u)\mathbb{E}_{Z_u}(\int_0^{R_1}|f|(X_t)dt)du\right)|f|(X_s)ds \\
&\leq C\mathbb{E}_\nu\int_0^{R_1}|f|(X_s)ds\left(\int_s^{R_1}|f|(X_u)\mathbb{E}_{Z_u}(T_1)du\right) \\
&\quad +C\mathbb{E}_\nu\int_0^{R_1}\left(\int_s^{R_1}|f|(X_u)du\right)|f|(X_s)ds. \qquad (4.8)
\end{aligned}$$

In this last step, we have used that

$$\mathbb{E}_{Z_u}(\int_{T_1}^{R_1}|f|(X_t)dt) \leq \mathbb{E}_{Z_u^3}(\int_0^{R_1}|f|(X_t)dt) \leq C$$

by proposition 2.16 of [9]. Recall that $\mathbb{E}_{Z_u^3}$ designs expectation when the first component starts from $x = Z_u^3$, the second is chosen according to the uniform distribution on $[0,1]$ and the third component is chosen according to Q.

Have a look at the second expression in (4.8): Using Markov's property with respect to $\mathcal{F}_s$, we get that

$$\begin{aligned}
&\mathbb{E}_\nu\int_0^{R_1}\left(\int_s^{R_1}|f|(X_u)du\right)|f|(X_s)ds = \mathbb{E}_\nu\int_0^{R_1}\left(|f|(X_s)\mathbb{E}_{Z_s}\int_0^{R_1}|f|(X_u)du\right)ds \\
&\leq C\mathbb{E}_\nu\int_0^{R_1}|f|(X_s)\mathbb{E}_{Z_s}(T_1)ds + \mathbb{E}_\nu\int_0^{R_1}\left(|f|(X_s)\mathbb{E}_{Z_s}\int_{T_1}^{R_1}|f|(X_u)du\right)ds \\
&\leq C\mathbb{E}_\nu\int_0^{R_1}|f|(X_s)\mathbb{E}_{Z_s}(T_1)ds + \mathbb{E}_\nu\int_0^{R_1}\left(|f|(X_s)\mathbb{E}_{Z_s^3}\int_0^{R_1}|f|(X_u)du\right)ds \\
&\leq C\mathbb{E}_\nu\int_0^{R_1}|f|(X_s)\mathbb{E}_{Z_s}(T_1)ds + C\mathbb{E}_\nu\int_0^{R_1}|f|(X_s)ds \\
&= C\mathbb{E}_\nu\int_0^{R_1}|f|(X_s)\mathbb{E}_{Z_s}(T_1)ds + C\mu(|f|).
\end{aligned}$$

Here we have used proposition 5.16 of [9]:

$$\mathbb{E}_{Z_s^3}\int_0^{R_1}|f|(X_u)du \leq C.$$

Hence, writing $\Psi(Z_s) := \mathbb{E}_{Z_s}(T_1)|f|(X_s)$, we have to study

$$\mathbb{E}_\nu \int_0^{R_1} \Psi(Z_s)ds.$$

But

$$\mathbb{E}_\nu \int_0^{R_1} \Psi(Z_s)ds = \mathbb{E}_\nu \int_0^{T_2} \Psi(Z_s)ds + \sum_{n\geq 1} \mathbb{E}_\nu \left[1_{\{Z_{T_k} \notin A \ \forall 1\leq k\leq n\}} \int_{T_{n+1}}^{T_{n+2}} \Psi(Z_s)ds \right]. \quad (4.9)$$

Have a look at the first term:

$$\begin{aligned}
\mathbb{E}_\nu \textstyle\int_0^{T_2} \Psi(Z_s)ds &= \mathbb{E}_\nu \int_0^{T_1} \Psi(Z_s)ds + \mathbb{E}_\nu \int_{T_1}^{T_2} \Psi(Z_s)ds \\
&= \mathbb{E}_\nu \int_0^{T_1} \Psi(Z_s)ds + \mathbb{E}_\nu \mathbb{E}_{Z_0^3} \int_0^{T_1} \Psi(Z_s)ds \\
&= \mathbb{E}_\nu \int_0^{T_1} \Psi(Z_s)ds \\
&\qquad + \int \nu(dx) \int u^1(x,x')\Lambda(dx')\mathbb{E}_{x'} \int_0^{T_1} \Psi(Z_s)ds. \quad (4.10)
\end{aligned}$$

But

$$\mathbb{E}_\nu \int_0^{T_1} \Psi(Z_s)ds = \int_0^\infty \mathbb{E}_\nu \left(1_{\{s\leq T_1\}} \Psi(Z_s) \right) ds,$$

and since $\Psi(Z_s)$ does not depend on Z_s^2,

$$\begin{aligned}
&\mathbb{E}_\nu \left(1_{\{s\leq T_1\}} \Psi(Z_s) \right) = \\
&= \int \nu(dx) \int u^1(x,x')\Lambda(dx') \int_s^\infty e^{-t} \frac{p_t(x,x')}{u^1(x,x')} dt \int \frac{p_s(x,y)p_{t-s}(y,x')}{p_t(x,x')} \Psi(y,x')\Lambda(dy) \\
&= \int \nu(dx) \int e^{-s} p_s(x,y)\Lambda(dy) \int u^1(y,x')\Psi(y,x')\Lambda(dx').
\end{aligned}$$

A simple calculus shows that

$$\int u^1(y,x')\Psi(y,x')\Lambda(dx') = |f|(y).$$

Hence,

$$\mathbb{E}_\nu \left(1_{\{s\leq T_1\}} \Psi(Z_s) \right) = \mathbb{E}_\nu \left(1_{\{s\leq T_1\}} |f|(X_s) \right).$$

The same argument applies to the second expression in (4.10), and we get

$$\mathbb{E}_{x'} \int_0^{T_1} \Psi(Z_s)ds = \mathbb{E}_{x'} \int_0^{T_1} |f|(X_s)ds.$$

Finally, for all the other terms in (4.9), we first use Markov's property with respect to $\mathcal{F}_{T_n}$, noticing that $\{Z_{T_k} \notin A \ \forall 1 \leq k \leq n\} \in \mathcal{F}_{T_n}$. Hence we have to investigate the following expression

$$\mathbb{E}_{Z_{T_n}} \int_{T_1}^{T_2} \Psi(Z_s)ds = \mathbb{E}_{Z^3_{T_n}} \int_0^{T_1} \Psi(Z_s)ds,$$

and the same argument as above shows that this equals

$$\mathbb{E}_{Z^3_{T_n}} \int_0^{T_1} |f|(X_s)ds.$$

We deduce that

$$\mathbb{E}_\nu \int_0^{R_1} \Psi(Z_s)ds = \mathbb{E}_\nu \int_0^{R_1} |f|(X_s)ds = \mu(|f|).$$

We are now going to treat the first term in (4.8): Using once again Markov's property, we get

$$\begin{aligned}
&\mathbb{E}_\nu \int_0^{R_1} |f|(X_s)ds \left(\int_s^{R_1} |f|(X_u)\mathbb{E}_{Z_u}(T_1)du \right) \\
&= \mathbb{E}_\nu \int_0^{R_1} |f|(X_s)ds \left(\mathbb{E}_{Z_s} \int_0^{R_1} |f|(X_u)\mathbb{E}_{Z_u}(T_1)du \right) \\
&\leq C\mathbb{E}_\nu \int_0^{R_1} |f|(X_s)ds \left(\mathbb{E}_{Z_s} \int_0^{T_1} \mathbb{E}_{Z_u}(T_1)du \right) \\
&\qquad + \mathbb{E}_\nu \int_0^{R_1} |f|(X_s)ds \left(\mathbb{E}_{Z_s} \int_{T_1}^{R_1} |f|(X_u)\mathbb{E}_{Z_u}(T_1)du \right)
\end{aligned} \tag{4.11}$$

Have a look at the second term in (4.11):

$$\mathbb{E}_{Z_s} \int_{T_1}^{R_1} |f|(X_u)\mathbb{E}_{Z_u}(T_1)du = \mathbb{E}_{Z^3_s} \int_0^{R_1} |f|(X_u)\mathbb{E}_{Z_u}(T_1)du,$$

and an argument similar to that in (4.9) leads to

$$\mathbb{E}_{Z^3_s} \int_0^{R_1} |f|(X_u)\mathbb{E}_{Z_u}(T_1)du \leq C.$$

Hence by proposition 5.16 of [9], the second term in (4.11) is bounded.
Write $\Phi(Z_s) := \mathbb{E}_{Z_s} \int_0^{T_1} \mathbb{E}_{Z_u}(T_1)du$, hence it remains to investigate

$$\begin{aligned}
\mathbb{E}_\nu \int_0^{R_1} |f|(X_s)\Phi(Z_s)ds &= \mathbb{E}_\nu \int_0^{T_2} |f|(X_s)\Phi(Z_s)ds \\
&\quad + \sum_{n\geq 1} \mathbb{E}_\nu \left[1_{\{Z_{T_k} \notin A \ \forall 1\leq k\leq n\}} \int_{T_{n+1}}^{T_{n+2}} |f|(X_s)\Phi(Z_s)ds \right].
\end{aligned}$$

In each of these terms, we take conditional expectation of $\Phi(Z_s)$ with respect to $X_s = Z^1_s$, in the same way as in (4.9). On $\{X_s = x\}$, we thus have to calculate

$$\begin{aligned}
\mathbb{E}(\Phi(Z_s)|X_s = x) &= \int u^1(x,x')\Lambda(dx') \int_0^\infty e^{-t} \frac{p_t(x,x')}{u^1(x,x')} dt \int_0^t du \\
&\qquad \int \frac{p_u(x,y)p_{t-u}(y,x')}{p_t(x,x')} \Lambda(dy) \int_0^\infty e^{-s} \frac{p_s(y,x')}{u^1(y,x')} sds \\
&= \int \Lambda(dx') \int_0^\infty e^{-t} dt \int_0^t du \\
&\qquad \int p_u(x,y)p_{t-u}(y,x')\Lambda(dy) \int_0^\infty e^{-s} \frac{p_s(y,x')}{u^1(y,x')} sds
\end{aligned}$$

$$
\begin{aligned}
&= \int \Lambda(dx') \int_0^\infty e^{-u} du \int p_u(x,y)\Lambda(dy) \int_0^\infty e^{-t} p_t(y,x') dt \\
&\qquad \int_0^\infty e^{-s} \frac{p_s(y,x')}{u^1(y,x')} s ds \\
&= \int \Lambda(dx') \int_0^\infty e^{-u} du \int p_u(x,y)\Lambda(dy) u^1(y,x') \\
&\qquad \int_0^\infty e^{-s} \frac{p_s(y,x')}{u^1(y,x')} s ds \\
&= \int_0^\infty e^{-u} du \int p_u(x,y)\Lambda(dy) \\
&\qquad \int_0^\infty e^{-s} \left(\int \Lambda(dx') p_s(y,x') \right) s ds = 1.
\end{aligned}
$$

This concludes our proof. •

4.2 Some results related to the deterministic equivalent

Proposition 4.7 *Let f be a bounded, positive special function of X, let C be a constant such that $\sup_x \mathbb{E}_x \int_0^{R_1} f(X_s) ds \leq C$ (see proposition 2.16 of [9]). Put*

$$
v^*(t) := C + E_\nu \int_0^t f(X_s) ds.
$$

Then we have for any $c \geq 1$,

$$
v^*(ct) \leq (c+1) v^*(t).
$$

Proof Using Markov's property for X we have:

$$
v^*(2t) - v^*(t) = E_\nu (E_{X_t} \int_0^t f(X_s) ds).
$$

Now we identify X with the first coordinate of Z:

$$
\begin{aligned}
E_x \int_0^t f(X_s) ds &= \mathbb{E}_x \int_0^t f(Z_s^1) ds \leq \\
&\leq \mathbb{E}_x (\int_0^{R_1} f(Z_s^1) ds + \int_{R_1}^{t+R_1} f(Z_s^1) ds) \\
&\leq C + \mathbb{E}_x (\mathbb{E}_{Z_{R_1}} (\int_0^t f(X_s) ds)) = C + \mathbb{E}_\nu \int_0^t f(Z_s^1) ds.
\end{aligned}
$$

Hence, $v^*(nt) \leq n v^*(t)$ for all n. Then, for any $c \geq 1$,

$$
v^*(ct) \leq v^*(([c]+1)t) \leq ([c]+1) v^*(t) \leq (c+1) v^*(t).
$$

•

5 Proof of theorem 3.1

In this section we give the proof of the law of iterated logarithm for additive functionals as stated in theorem 3.1.

Proof Let f be some special function of X. We suppose $f \geq 0$, bounded, with $\mu(f) > 0$. It is enough to prove that

$$\limsup_{t\to\infty} \frac{\int_0^t f(X_s)ds}{v(\frac{t}{L_2(v(t))})L_2(v(t))} = c\mu(f) \text{ a.s.} \tag{5.12}$$

with $v(t) = E_\nu \int_0^t f(X_s)ds$.

Let σ_n, $n \in I\!N$, be i.i.d. exp(1) random variables independent of the process X and $T_0 := 0$, $T_n = \sigma_1 + \ldots + \sigma_n$, as introduced in the section 5 for splitting construction. Define $\bar{X}_n := X_{T_n}$. Then the chain $(\bar{X}_n)$ is recurrent in the sense of Harris, with the same invariant measure μ. The special functions of the chain $(\bar{X}_n)$ and those of the process X are the same.

Now, let

$$a(n) := I\!E_\nu \sum_{k=1}^{n} f(\bar{X}_k).$$

This is a version of deterministic equivalent of the chain $(\bar{X}_n)$, see Chen ([3]).

In the sequel, c denotes a positive constant that depends only on the process, not on f. Chen's law of iterated logarithm for discrete Harris chains, see [3], theorem 2.2, applied to the chain $(\bar{X}_n)$ gives :

$$\limsup_{n\to\infty} \frac{\sum_{k=0}^{n} f(\bar{X}_k)}{a(\frac{n}{L_2(a(n))})L_2(a(n))} = L\mu(f) \text{ a.s.} \tag{5.13}$$

Here the constant L depends only on the recurrence behavior of the chain $(\bar{X}_n)$: Chen shows (5.13) first for a particular additive functional which is the number of visits to an atom before time n. The constant L is the limit obtained for this chain. Then, (5.13) is obtained by passing to a general additive functional, using Chacon-Ornstein theorem.

Let $N_t = \sum_{n\geq 1} 1_{\{T_n \leq t\}}$. N_t is a Poisson process with intensity 1, $\lim_{t\to\infty} N_t = \infty$ a.s., and the substitution of N_t in (5.13) gives:

$$\limsup_{t\to\infty} \frac{\sum_{k=0}^{N_t} f(\bar{X}_k)}{a(\frac{N_t}{L_2(a(N_t))})L_2(a(N_t))} = L\mu(f) \text{ a.s.} \tag{5.14}$$

Using the Chacon-Ornstein limit-quotient theorem for additive functionals or the law of large numbers for Poisson process we get:

$$\lim_{t\to\infty} \frac{\sum_{k=0}^{N_t} f(\bar{X}_k)}{\int_0^t f(X_s)ds} = 1 \quad \text{a.s.,} \tag{5.15}$$

and thus

$$\limsup_{t\to\infty} \frac{\int_0^t f(X_s)ds}{a(\frac{N_t}{L_2(a(N_t))})L_2(a(N_t))} = L\mu(f) \text{ a.s.} \tag{5.16}$$

We now want to compare the normalization in (5.12) with that of (5.16).

Remark that

$$M_t = \sum_{k=0}^{N_t} f(\bar{X}_k) - \int_0^t f(X_s)ds = \int_0^t f(X_s)dN_s - \int_0^t f(X_s)ds$$

is a martingale. For all $k > 0$, $T_n \wedge k$ is a bounded stopping time. Using the stopping-rule,

$$\mathbb{E}_\nu \sum_{k=0}^{N_{T_n \wedge k}} f(\bar{X}_k) = \mathbb{E}_\nu \int_0^{T_n \wedge k} f(X_s)ds. \tag{5.17}$$

By monotone convergence, we can replace in the previous equation $T_n \wedge k$ by T_n and obtain

$$a(n) = \mathbb{E}_\nu \int_0^{T_n} f(X_s)ds.$$

But under $\mathbb{P}_\nu$, $(T_n)_n$ and X are independent and T_n is the sum of n independent random variables that are exponentially $exp(1)$ distributed. So if we firstly integrate with respect to X, we obtain

$$a(n) = E(v(T_n)), \tag{5.18}$$

where expectation is taken only with respect to T_n which is the sum of n independent $exp(1)-$variables.

Now denote as previously $v^*(t) = C + v(t)$, where $C > 0$ is a constant such that $\sup_x \mathbb{E}_x \int_0^{R_1} f(X_s)ds \leq C$.

We write

$$v^*(T_n) \leq v^*(n)1_{\{T_n \leq n\}} + v^*(\frac{T_n}{n}n)1_{\{T_n \geq n\}}.$$

But

$$v^*(\frac{T_n}{n}n)1_{\{T_n \geq n\}} \leq (\frac{T_n}{n} + 1)v^*(n)1_{\{T_n \geq n\}}.$$

Hence we get

$$v^*(T_n) \leq v^*(n) + \frac{T_n}{n}v^*(n)1_{\{T_n \geq n\}} \leq v^*(n)(1 + \frac{T_n}{n}),$$

and taking expectations with respect to T_n yields

$$a^*(n) \leq 2v^*(n),$$

where $a^*(n) = C + a(n)$.

In the same way,

$$\frac{v^*(T_n)}{v^*(n)} \geq \frac{v^*(\frac{T_n}{n}n)1_{\{\frac{T_n}{n} > 1-\varepsilon\}}}{v^*(n(1-\varepsilon)\frac{1}{1-\varepsilon\}})} \geq \frac{v^*((1-\varepsilon)n)1_{\{\frac{T_n}{n} > 1-\varepsilon\}}}{(1 + \frac{1}{1-\varepsilon})v^*(n(1-\varepsilon))}$$

hence,

$$\frac{v^*(T_n)}{v^*(n)} \geq \frac{1_{\{\frac{T_n}{n} > 1-\varepsilon\}}}{(1 + \frac{1}{1-\varepsilon})},$$

and taking expectation yields

$$a^*(n) \geq v^*(n)\left(\frac{1}{2+\varepsilon'}\right)p_n,$$

where $p_n = P(T_n \geq n(1-\varepsilon)) \to 1$ as $n \to \infty$. Hence we have shown:

$$\frac{1}{2} = \liminf \frac{a^*(n)}{v^*(n)} \leq \limsup \frac{a^*(n)}{v^*(n)} \leq 2.$$

And thus almost surely

$$\frac{1}{2} \leq \liminf \frac{a^*(N_t)}{v^*(N_t)} \leq \limsup \frac{a^*(N_t)}{v^*(N_t)} = 2.$$

In the same way, using $\lim_{t\to\infty} \frac{N_t}{t} = 1$, one shows that almost surely

$$\frac{1}{2} \leq \liminf \frac{v^*(N_t)}{v^*(t)} \leq \limsup \frac{v^*(N_t)}{v^*(t)} = 2,$$

and finally, since $v(t)/v^*(t) \to 1$,

$$\frac{1}{2} \leq \liminf \frac{a(N_t)}{v(t)} \leq \limsup \frac{a(N_t)}{v(t)} = 2.$$

In the same way we show that with some constants $C_1 > 0$ and $C_2 < \infty$

$$C_1 = \liminf \frac{a(\frac{N_t}{L_2 a(N_t)})L_2 a(N_t)}{v(\frac{t}{L_2 v(t)})L_2 v(t)} \leq \limsup \frac{a(\frac{N_t}{L_2 a(N_t)})L_2 a(N_t)}{v(\frac{t}{L_2 v(t)})L_2 v(t)} = C_2. \tag{5.19}$$

Hence,

$$\begin{aligned} \limsup \frac{\int_0^t f(X_s)ds}{v(\frac{t}{L_2 v(t)})L_2 v(t)} &\leq \limsup \frac{\int_0^t f(X_s)ds}{a(\frac{N_t}{L_2 a(N_t)})L_2 a(N_t)} \limsup \frac{a(\frac{N_t}{L_2 a(N_t)})L_2 a(N_t)}{v(\frac{t}{L_2 v(t)})L_2 v(t)} \\ &= LC_2\mu(f), \end{aligned}$$

and this yields the desired result. •

Corollary 5.1 *As a consequence of the above proof, in particular of (5.18), it is evident that in the regular case, i.e. if*

$$v(t) \sim t^\alpha l(t) \text{ as } t \to \infty$$

for some $0 < \alpha \leq 1$ *and for some function* l *that varies slowly at infinity,*

$$a(n) = v(n).$$

6 Proof of theorem 3.2 and theorem 3.4

Let f be a measurable $\mu-$integrable function such that $\mu(f) = 0$ and such that $|f|$ is a special function. Put $\xi_n := \int_{R_{n-1}}^{R_n} f(X_s)ds$. Then the sequence of $\xi_1, \xi_3, \xi_5, \ldots$ is a sequence of i.i.d. variables having $\mathbb{E}(\xi_i) = 0$ and $\mathbb{E}(|\xi_i|^3) < \infty$, see proposition 4.6. The same is true of the sequence $\xi_2, \xi_4, \ldots$. As a consequence, $(\xi_n)_n$ is a strictly stationary sequence

that is uniformly strong mixing in a very strong sense, since for any n, $\sigma\{\xi_k, k \leq n\}$ is independent of $\sigma\{\xi_{n+l}, l \geq 2\}$. Put

$$\Sigma_n = \xi_1 + \ldots + \xi_n,$$

then

$$I\!E_\nu(\Sigma_n^2) = nI\!E_\nu(\xi_1^2) + 2(n-1)I\!E_\nu(\xi_1\xi_2) \sim n\sigma^2$$

where

$$\sigma^2 = I\!E_\nu(\xi_1^2) + 2I\!E_\nu(\xi_1\xi_2).$$

We suppose that

$$\sigma^2 > 0. \tag{6.20}$$

We apply the Hartman-Wintner law of iterated logarithm for strongly mixing variables which is a consequence of theorem 2 of Rio, [16]. Note that the condition (1.5) of [16] on exponential mixing rates is satisfied due to proposition 4.6. We thus get, putting

$$\lambda_f := \sigma, \tag{6.21}$$

$$\limsup \frac{\Sigma_n}{\sqrt{2nL_2n}} = \limsup \frac{\int_0^{R_n} f(X_s)ds}{\sqrt{2nL_2n}} = \lambda_f \text{ almost surely as } n \to \infty. \tag{6.22}$$

We are now able to give the proof of theorem 3.2 :

Proof of theorem 3.2

Let

$$N_t := \sum_{n\geq 1} 1_{\{R_n \leq t\}}.$$

Using (6.22), we get immediately that

$$\limsup \frac{\int_0^{R_{N_t}} f(X_s)ds}{\sqrt{2N_tL_2(N_t)}} = \lambda_f. \tag{6.23}$$

Here λ_f depends only on the function f and on the transition of the process, but not on the specific choice of R_n. Now, note that, using Markov's property,

$$\begin{aligned} I\!P_\nu\left(\sup_{R_n\leq t\leq R_{n+1}} |\int_{R_n}^t f(X_s)ds| \geq \varepsilon\sqrt{nL_2(n)}\right) &= I\!P_\nu\left(\sup_{0\leq t\leq R_1} |\int_0^t f(X_s)ds| \geq \varepsilon\sqrt{nL_2(n)}\right) \\ &\leq I\!P_\nu\left(\frac{1}{\varepsilon^2}(\sup_{0\leq t\leq R_1} |\int_0^t f(X_s)ds|)^2 \geq n\right). \end{aligned}$$

Summing over n yields

$$\begin{aligned} \textstyle\sum_n I\!P_\nu\left(\sup_{R_n\leq t\leq R_{n+1}} |\int_{R_n}^t f(X_s)ds| \geq \varepsilon\sqrt{nL_2(n)}\right) &\leq \frac{1}{\varepsilon^2} I\!E_\nu\left((\sup_{0\leq t\leq R_1} |\int_0^t f(X_s)ds|)^2\right) \\ &\leq \frac{1}{\varepsilon^2} I\!E_\nu\left(\int_0^{R_1} |f|(X_s)ds\right)^2 < \infty \end{aligned}$$

by proposition 4.6. Using Borel-Cantelli's lemma, we thus conclude that

$$\limsup \frac{\int_0^t f(X_s)ds}{\sqrt{2N_tL_2(N_t)}} = \lambda_f,$$

since

$$\limsup \frac{\sup_{R_{N_t}\le s\le t}|\int_{R_{N_t}}^s f(X_u)du|}{\sqrt{2N_tL_2(N_t)}}=0.$$

Note that the additive functional N_t can be replaced by any $\mu-$integrable additive functional A, using the ratio limit theorem. This concludes the proof. •

Proof of remark 3.3 The proof is given in several steps and follows the ideas of the proof of lemma 2.3 of Chen [4].

1. We start by showing the following result. Let $N_t := \sum_{n\ge 1} 1_{\{S_n\le t\}}$. Let $v(t) = I\!E_\nu(N_t+1)$. Put $R_t = \inf\{R_n : R_n > S_{N_t+1}\}$. Then we have

$$\lim_{t\to\infty}\frac{1}{v(t)}I\!E_\nu\left((\int_0^{R_t} f(X_s)ds)^2\right)=\lambda_f^2 \tag{6.24}$$

and in particular also

$$\lim_{n\to\infty}\frac{1}{\int_0^n v(t)dt}\int_0^n\left(I\!E_\nu\left((\int_0^{R_t} f(X_s)ds)^2\right)\right)dt=\lambda_f^2. \tag{6.25}$$

Equation (6.24) is shown as follows. As in the proof of proposition 3.8 of [9], writing $S_0 := R_0 = 0$, we have

$$\begin{aligned}\left(\int_0^{R_t} f(X_s)ds\right)^2 &= \left(\sum_{n\ge 0}1_{\{S_n\le t\}}\int_{R_n}^{R_{n+1}} f(X_s)ds\right)^2\\ &= \sum_{n\ge 0}\sum_{m\ge 0}1_{\{S_n\le t\}}1_{\{S_m\le t\}}(\int_{R_n}^{R_{n+1}} f(X_s)ds)(\int_{R_m}^{R_{m+1}} f(X_s)ds)\\ &= \sum_{n\ge 0}\sum_{m\ge 0}1_{\{S_n\le t\}}1_{\{S_m\le t\}}\xi_{n+1}\xi_{m+1}.\end{aligned}$$

Now, using the independence of ξ_{n+1} of $\{S_n\le t\}$ and the independence properties of the sequence $(\xi_n)_n$ itself, we obtain immediately, as in [9],

$$I\!E_\nu\left((\int_0^{R_t} f(X_s)ds)^2\right)=\sum_{n\ge 0}I\!E_\nu(1_{\{S_n\le t\}}\xi_{n+1}^2)+2\sum_{n\ge 0}I\!E_\nu(1_{\{S_{n+1}\le t\}}\xi_{n+1}\xi_{n+2}). \tag{6.26}$$

The first term in (6.26) equals

$$\sum_{n\ge 0}I\!E_\nu(1_{\{S_n\le t\}}\xi_{n+1}^2)=I\!E_\nu(N_t+1)I\!E_\nu(\xi_1^2), \tag{6.27}$$

and hence

$$\lim_{t\to\infty}\frac{\sum_{n\ge 0}I\!E_\nu(1_{\{S_n\le t\}}\xi_{n+1}^2)}{v(t)}=I\!E_\nu(\xi_1^2).$$

Concerning the second term in (6.26), note that

$$\begin{aligned}&\sum_{n\ge 0}I\!E_\nu(1_{\{S_{n+1}\le t\}}\xi_{n+1}\xi_{n+2})\\ &=\sum_{n\ge 0}I\!E_\nu(1_{\{S_n\le t\}}\xi_{n+1}\xi_{n+2})-\sum_{n\ge 0}I\!E_\nu(1_{\{S_n\le t<S_{n+1}\}}\xi_{n+1}\xi_{n+2})\\ &=I\!E_\nu(N_t+1)I\!E_\nu(\xi_1\xi_2)-\sum_{n\ge 0}I\!E_\nu(1_{\{S_n\le t<S_{n+1}\}}\xi_{n+1}\xi_{n+2}).\end{aligned} \tag{6.28}$$

The first term converges to $\mathbb{E}_\nu(\xi_1\xi_2)$, once divided by $v(t)$. In order to treat the second term in (6.28), note that

$$\begin{aligned}
&\sum_{n\geq 0} \mathbb{E}_\nu(1_{\{S_n\leq t<S_{n+1}\}}\xi_{n+1}\xi_{n+2}) \\
&= \sum_{n\geq 0} \mathbb{E}_\nu(1_{\{S_n\leq t<S_{n+1}\}}(\int_{R_n}^{S_{n+1}} f(X_s)ds)(\int_{R_{n+1}}^{R_{n+2}} f(X_s)ds)) \\
&\qquad + \sum_{n\geq 0} \mathbb{E}_\nu(1_{\{S_n\leq t<S_{n+1}\}}(\int_{S_{n+1}}^{R_{n+1}} f(X_s)ds)(\int_{R_{n+1}}^{R_{n+2}} f(X_s)ds)). \qquad (6.29)
\end{aligned}$$

Recall that $(\mathcal{G}_t)_t$ is the filtration generated by the first two coordinates of Z. In the first term of (6.29), we are going to use conditional expectation with respect to $\mathcal{G}_{S_{n+1}}$:

$$\begin{aligned}
&\mathbb{E}_\nu(1_{\{S_n\leq t<S_{n+1}\}}(\int_{R_n}^{S_{n+1}} f(X_s)ds)(\int_{R_{n+1}}^{R_{n+2}} f(X_s)ds)) \\
&\quad = \mathbb{E}_\nu(1_{\{S_n\leq t<S_{n+1}\}}(\int_{R_n}^{S_{n+1}} f(X_s)ds)\mathbb{E}_\nu[\int_{R_{n+1}}^{R_{n+2}} f(X_s)ds|\mathcal{G}_{S_{n+1}}]).
\end{aligned}$$

But, since $\mathcal{L}(Z^3_{S_{n+1}}|\mathcal{G}_{S_{n+1}}) = \nu(dx')$, and then using Markov's property firstly with respect to $\mathcal{F}_{S_{n+1}}$ and then with respect to $\mathcal{F}_{T_1}$,

$$\begin{aligned}
\mathbb{E}_\nu[\int_{R_{n+1}}^{R_{n+2}} f(X_s)ds|\mathcal{G}_{S_{n+1}}] &= \int \nu(dx')\mathbb{E}_{(Z^1_{S_{n+1}},Z^2_{S_{n+1}},x')}\left(\int_{T_1}^{R_1} f(X_s)ds\right) \\
&= \int \nu(dx')\mathbb{E}_{x'}\left(\int_0^{R_1} f(X_s)ds\right) = \mathbb{E}_\nu\xi_1 = c\mu(f) = 0,
\end{aligned}$$

where the last equality follows from proposition 4.3 and from $\mu(f) = 0$. Hence the first term in (6.29) vanishes. In order to treat the second term, we use that $|f|$ is bounded by a constant. Hence, writing $\mathcal{A}_n := \sigma\{\mathcal{F}_{S_{n+1}}, R_{n+1}\}$,

$$\begin{aligned}
&\mathbb{E}_\nu(1_{\{S_n\leq t<S_{n+1}\}}(\int_{S_{n+1}}^{R_{n+1}} |f|(X_s)ds)(\int_{R_{n+1}}^{R_{n+2}} |f|(X_s)ds)) \\
&\quad \leq C\,\mathbb{E}_\nu\left(1_{\{S_n\leq t<S_{n+1}\}}(R_{n+1}-S_{n+1})\mathbb{E}_\nu\left[\int_{R_{n+1}}^{R_{n+2}} |f|(X_s)ds\,|\,\mathcal{A}_n\right]\right).
\end{aligned}$$

But $\mathbb{E}_\nu[\int_{R_{n+1}}^{R_{n+2}} |f|(X_s)ds\,|\,\mathcal{A}_n]$ does not depend on R_{n+1}, and since $Z^1_{R_{n+1}} = Z^3_{S_{n+1}}$, we get

$$\mathbb{E}_\nu\left[\int_{R_{n+1}}^{R_{n+2}} |f|(X_s)ds\,|\,\mathcal{A}_n\right] = \mathbb{E}_{Z^3_{S_{n+1}}}\left[\int_0^{R_1} |f|(X_s)ds\right] \leq C,$$

since $|f|$ is a special function, see proposition 2.19 of [9]. Thus,

$$\begin{aligned}
&\mathbb{E}_\nu\left(1_{\{S_n\leq t<S_{n+1}\}}(R_{n+1}-S_{n+1})\mathbb{E}_\nu\left[\int_{R_{n+1}}^{R_{n+2}} |f|(X_s)ds\,|\,\mathcal{A}_n\right]\right) \\
&\quad \leq C\mathbb{E}_\nu\left(1_{\{S_n\leq t<S_{n+1}\}}(R_{n+1}-S_{n+1})\right) \\
&\quad = C\mathbb{E}_\nu\left(1_{\{S_n\leq t<S_{n+1}\}}\mathbb{E}_\nu[R_{n+1}-S_{n+1}|\mathcal{G}_{S_{n+1}}]\right) \\
&\quad \leq \frac{C}{\alpha}\mathbb{E}_\nu\left(1_{\{S_n\leq t<S_{n+1}\}}\right),
\end{aligned}$$

since

$$\begin{aligned}
\mathbb{E}_\nu[R_{n+1} - S_{n+1}|\mathcal{G}_{S_{n+1}}] &= \int \nu(dx') \int_0^\infty t\, e^{-t} \frac{p_t(Z^1_{S_{n+1}}, x')}{u^1(Z^1_{S_{n+1}}, x')} dt \\
&\leq \int_0^\infty t\, e^{-t} dt \int \frac{1}{\alpha} p_t(Z^1_{S_{n+1}}, x') \Lambda(dx') \\
&= \frac{1}{\alpha},
\end{aligned}$$

since by construction, $u^1(Z^1_{S_{n+1}}, x') \geq \alpha\nu(x')$. Here $\nu(dx') = \nu(x')\Lambda(dx')$.

Thus, after summation in n, the second term in (6.29) is bounded by

$$C' \sum_n \mathbb{E}_\nu \left(1_{\{S_n \leq t < S_{n+1}\}}\right) = C',$$

since $\sum_n 1_{\{S_n \leq t < S_{n+1}\}} = 1$.

Thus, the second term in (6.28), once divided by $v(t)$, converges to 0, as $t \to \infty$. Finally, (6.26), (6.27) and (6.28) give the desired (6.24).

2. Let

$$h(x) := f(x) \int_0^\infty P_t f(x) dt.$$

Note that

$$\begin{aligned}
E_\nu(\int_0^t f(X_s)ds)^2 &= 2E_\nu \int_0^t \int_s^t f(X_s) f(X_r) dr ds \\
&= 2E_\nu \int_0^t \int_0^{t-s} f(X_s) P_r f(X_s) dr ds \\
&= 2E_\nu \int_0^t h(X_s) ds - 2E_\nu \int_0^t f(X_s) \int_{t-s}^\infty P_r f(X_s) dr ds. \qquad (6.30)
\end{aligned}$$

By the Chacon-Ornstein ratio limit theorem, we have

$$\lim_{t\to\infty} \frac{E_\nu \int_0^t h(X_s)ds}{v(t)} = \mu(h) = \int \int_0^\infty f(x) P_t f(x) dt \mu(dx). \qquad (6.31)$$

Moreover, write

$$h_t(x) := f(x) \int_t^\infty P_r f(x) dr.$$

The second term in (6.30) can be handled as follows.

$$|E_\nu \int_0^t f(X_s) \int_{t-s}^\infty P_r f(X_s) dr ds| \leq E_\nu \int_0^t |h_{t-s}(X_s)| ds. \qquad (6.32)$$

Unfortunately, the term on the right hand side of (6.32) is not an additive functional of the process, since the function h_{t-s} depends also on s. That's why we have to integrate once more. Integrating with respect to t over the interval $[0, n]$ and using Fubini gives

$$\begin{aligned}
\int_0^n |E_\nu \int_0^t f(X_s) \int_{t-s}^\infty P_r f(X_s) dr ds| dt &\leq E_\nu \int_0^n ds (\int_0^{n-s} |h_t|(X_s) dt) ds \\
&\leq E_\nu \int_0^n ds (\int_0^n |h_t|(X_s) dt) ds. \qquad (6.33)
\end{aligned}$$

Put

$$H_n(x) := \int_0^n |h_t|(x)dt,$$

then as in the proof of proposition 3.8 of [9], the right hand side of (6.33) can be bounded by

$$E_\nu \int_0^n ds(\int_0^n |h_t|(X_s)dt)ds \leq I\!E_\nu(N_n+1) \cdot \int H_n(x)\mu(dx) = I\!E_\nu(N_n+1)\int_0^n \mu(|h_t|)dt. \tag{6.34}$$

As a consequence,

$$\frac{E_\nu \int_0^n ds(\int_0^n |h_t|(X_s)dt)ds}{\int_0^n v(t)dt} \leq \frac{I\!E_\nu(N_n+1)\int_0^n \mu(|h_t|)dt}{\int_0^n v(t)dt}. \tag{6.35}$$

Since $t \mapsto v(t)$ is non-decreasing, we have $\int_0^n v(t)dt \geq v(n/2)\cdot n/2$. Recall the definition of $v^*(t)$ of proposition 4.7. Then

$$\limsup \frac{I\!E_\nu(N_n+1)}{v(n/2)} = \limsup \frac{v(n)}{v(n/2)} = \limsup \left[\frac{v(n)}{v^*(n)} \cdot \frac{v^*(n)}{v^*(n/2)} \cdot \frac{v^*(n/2)}{v(n/2)}\right] < \infty$$

by proposition 4.7 and by the Chacon-Ornstein ratio limit theorem. Moreover, $\mu(|h_t|) \to 0$ as $t \to \infty$ by our assumption. Hence,

$$\frac{1}{n}\int_0^n \mu(|h_t|)dt \to 0.$$

As a consequence of (6.35) and (6.30), (6.31), we obtain finally

$$\lim_{n\to\infty} \frac{\int_0^n \left(E_\nu(\int_0^t f(X_s)ds)^2\right)dt}{\int_0^n v(t)dt} = 2\mu(h). \tag{6.36}$$

3. Finally,

$$\lim_{n\to\infty} \frac{1}{\int_0^n v(t)dt} \int_0^n \left[I\!E_\nu\left((\int_0^{R_t} f(X_s)ds)^2\right) - I\!E_\nu\left((\int_0^t f(X_s)ds)^2\right)\right]dt = 0. \tag{6.37}$$

This is seen as follows. Write $\Delta_t := \int_t^{R_t} f(X_s)ds$. Then

$$\begin{aligned} &I\!E_\nu\left((\int_0^{R_t} f(X_s)ds)^2\right) - I\!E_\nu\left((\int_0^t f(X_s)ds)^2\right) \\ &\quad = I\!E_\nu\left(\Delta_t^2\right) - 2I\!E_\nu\left((\int_0^t f(X_s ds)\Delta_t\right). \end{aligned}$$

Similar arguments to that of proof of proposition 4.6 give $I\!E_\nu \Delta_t^2 \leq C$ for some constant independent of t, and thus

$$\lim_{n\to\infty} \frac{\int_0^n I\!E_\nu\left(\Delta_t^2\right)dt}{\int_0^n v(t)dt} = 0,$$

since $n/(\int_0^n v(t)dt) \to 0$. Moreover,

$$\int_0^n I\!E_\nu\left((\int_0^t f(X_s ds)\Delta_t\right)dt \leq \left(\int_0^n I\!E_\nu(\int_0^t f(X_s ds))^2 dt\right)^{1/2}\left(\int_0^n I\!E_\nu \Delta_t^2 dt\right)^{1/2},$$

by Hölder's inequality, but

$$\limsup \frac{\left(\int_0^n I\!E_\nu(\int_0^t f(X_s ds))^2 dt\right)^{1/2}}{\left(\int_0^n v(t)dt\right)^{1/2}} < \infty$$

due to (6.36), and

$$\limsup \frac{\left(\int_0^n I\!E_\nu \Delta_t^2 dt\right)^{1/2}}{\left(\int_0^n v(t)dt\right)^{1/2}} = 0,$$

which gives (6.37).

(6.25), (6.36) and (6.37) finally yield the desired result

$$\lambda_f^2 = 2\mu(h).$$

•

Proof of theorem 3.4 Put $\xi_n := M_{R_n} - M_{R_{n-1}}$. Since M has continuous paths, it is well-known that M_{R_n} is measurable with respect to $\mathcal{F}_{R_n-}$. Hence we have

$$I\!E(\xi_n|\sigma\{\xi_1,\ldots,\xi_{n-1}\}) = I\!E\left(I\!E(\xi_n|\mathcal{F}_{R_{n-1}-})|\sigma\{\xi_1,\ldots,\xi_{n-1}\}\right).$$

Since M is an additive functional, we have that $\xi_n = \xi_1 \circ \theta_{R_{n-1}}$ Using Markov's property, we get

$$I\!E(\xi_n|\mathcal{F}_{R_{n-1}-}) = I\!E(I\!E_{Z_{R_{n-1}}}(\xi_1)|\mathcal{F}_{R_{n-1}-}) = I\!E_{Z^1_{R_{n-1}}}(\xi_1).$$

Now, as in (4.9), we write

$$\xi_1 = M_{R_1} = M_{T_2} + \sum_{n\geq 1} 1_{\{Z_{T_k}\notin A \forall 1\leq k\leq n\}}(M_{T_{n+2}} - M_{T_{n+1}})$$

in order to conclude that for all x,

$$I\!E_x(M_{R_1}) = 0.$$

Hence the sequence $(\xi_n)_n$ is a $\sigma\{\xi_1,\ldots,\xi_n\}-$martingale difference sequence, which is stationary and ergodic due to proposition 4.4. The same argument as above applies and allows us to get that

$$I\!E_\nu(\xi_1^2) = I\!E_\nu(< M >_{R_1}) = E_\mu(< M >_1) < \infty.$$

Then the Hartmann-Wintner law of iterated logarithm, see [17], yields

$$\limsup \frac{M_{R_n}}{\sqrt{2n\log\log n}} = E_\mu(< M >_1)^{1/2} =: \lambda_M$$

almost surely. Now, the proof of the first part of theorem 3.4 is exactly as the proof of theorem 3.2.

In order to prove the second assertion of theorem 3.4, note moreover that $\bar{M}_n := M_{T_n}$ is a $\mathcal{F}_{T_n}-$martingale such that

$$\bar{M}_{n+k} - \bar{M}_n = \bar{M}_k \circ \theta_{T_n} = \bar{M}_k \circ \bar{\theta}_n,$$

where $\bar{\theta}_n$ is the shift operator of the discrete time chain $\bar{X}_n = X_{T_n}$. Let $a(n)$ be the deterministic equivalent of the chain $\bar{X}$, then by theorem 6.1 of Chen, [5],

$$\limsup \frac{M_{T_n}}{\sqrt{a(\frac{n}{L_2(a(n))})L_2(a(n))}} = \lambda_M$$

almost surely. Write for short $b(n) := \sqrt{a(\frac{n}{L_2(a(n))})L_2(a(n))}$. Hence we have

$$\limsup \frac{M_{\bar{N}_t}}{b(\bar{N}_t)} = \lambda_M \text{ almost surely,}$$

where $\bar{N}_t := \sum_n 1_{\{T_n \leq t\}}$. The same argument as in the proof of theorem 3.1 shows that for $c(t) := \sqrt{v(\frac{t}{L_2(v(t))})L_2(v(t))}$,

$$0 < \liminf \frac{b(n)}{c(n)} \leq \limsup \frac{b(n)}{c(n)} < \infty,$$

and thus

$$\limsup \frac{M_{\bar{N}_t}}{\sqrt{v(\frac{\bar{N}_t}{L_2(v(\bar{N}_t))})L_2(v(\bar{N}_t))}} = c\lambda_M$$

almost surely, where the constant c depends only on the process. Hence we get the lower bound

$$\limsup \frac{M_t}{\sqrt{v(\frac{t}{L_2(v(t))})L_2(v(t))}} \geq c\lambda_M.$$

On the other hand, by the first assertion of theorem 3.4,

$$\limsup \frac{M_t}{\sqrt{2N_t L_2(N_t)}} \leq c,$$

and then the upper bound follows since

$$\frac{\limsup \sqrt{2N_t L_2(N_t)}}{\limsup \sqrt{v(\frac{t}{L_2(v(t))})L_2(v(t))}} < \infty \text{ almost surely,}$$

which is a consequence of theorem 3.1. •

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