A priori Convergence Theory for Reduced-Basis Approximations of Single-Parametric Elliptic Partial Differential Equations

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Yvon MADAY[*], Anthony T. PATERA[†], Gabriel TURINICI[‡]

Abstract

We consider "Lagrangian" reduced-basis methods for single-parameter symmetric coercive elliptic partial differential equations. We show that, for a logarithmic-(quasi-)uniform distribution of sample points, the reduced–basis approximation converges *exponentially* to the exact solution *uniformly* in parameter space. Furthermore, the convergence rate depends only weakly on the continuity-coercivity ratio of the operator: thus very low-dimensional approximations yield accurate solutions even for very wide parametric ranges. Numerical tests (reported elsewhere) corroborate the theoretical predictions.

1 Introduction

The development of computational methods that permit *rapid* and *reliable* evaluation of the solution of partial differential equations *in the limit of many queries* is relevant within many design, optimization, control, and characterization contexts. One particular approach is the reduced-basis method, first introduced in the late 1970s for nonlinear structural analysis [1, 9], and subsequently developed more broadly in the 1980s and 1990s [3, 5, 10, 13, 2]. The reduced-basis method recognizes that the field variable is not, in fact, some arbitrary member of the infinite-dimensional space associated with the partial differential equation; rather, it resides, or

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"evolves," on a much lower-dimensional manifold induced by the parametric dependence.

Let Y be an Hilbert space with inner product and norm $(\cdot\,,\cdot)_Y$ and $\|\cdot\|_Y = (\cdot\,,\cdot)_Y^{1/2}$, respectively. Consider a parametrized "bilinear" form $a\colon Y \times Y \times \mathcal{D} \to \mathbf{R}$, where $\mathcal{D} \equiv [0, \mu_{\max}]$, and a bounded linear form $f\colon Y \to \mathbf{R}$. We introduce the problem to be solved: Given $\mu \in \mathcal{D}$, find $u \in Y$ such that

$$a(u(\mu), v; \mu) = f(v), \quad \forall\, v \in Y \ . \tag{1}$$

Under natural conditions on the bilinear form a (e.g. continuity and coercivity) it is readily shown that this problem admits a unique solution.

We introduce an approximation index N, the parameter sample $S_N = \{\alpha_1, \ldots, \alpha_N\}$, and the solutions $u(\alpha_k)$, $k = 1, \ldots, N$, of problem (1) for this set of parameters. We next define the *reduced-basis approximation space*

$$W_N = \text{span}\,\{u(\alpha_k),\, k = 1, \ldots, N\}.$$

Our reduced-basis approximation is then: Given $\mu \in \mathcal{D}$, find $u_N(\mu) \in W_N$ such that

$$a(u_N(\mu), v; \mu) = f(v), \quad \forall\, v \in W_N \ . \tag{2}$$

This discrete problem is well posed as well under the same former continuity and coercivity conditions.

The reduced-basis approach, as earlier developped, is typically local in parameter space in both practice and theory. To wit, the α_k are chosen in the vicinity of a particular parameter point μ^* and the associated *a priori* convergence theory relies on asymptotic arguments in sufficiently small neighborhoods of μ^* [5]. In this paper we present, for single-parameter symmetric coercive elliptic partial differential equations, a first theoretical *a priori* convergence result that demonstrates exponential convergence of reduced-basis approximations *uniformly* over an extended parameter domain. The proof requires, and thus suggests, a point distribution in parameter space which does, indeed, exhibit superior convergence properties in a variety of numerical tests [15]. We refer to [16, 17] for a different analysis within the homogeneization framework.

2 Problem Formulation

Let us define the parametrized "bilinear" form $a\colon Y \times Y \times \mathcal{D} \to \mathbf{R}$ as

$$a(w, v; \mu) \equiv a_0(w, v) + \mu a_1(w, v) \ , \tag{3}$$

where the bilinear forms $a_0 : Y \times Y \rightarrow \mathbf{R}$ and $a_1 : Y \times Y \rightarrow \mathbf{R}$ are continuous, symmetric and positive semi-definite; suppose moreover that a_0 is coercive, inducing a (Y-equivalent) norm $|||\cdot|||^2 = a_0(\cdot\,,\cdot)$. It follows from our assumptions that there exists a real positive constant γ_1 such that

$$0 \leq \frac{a_1(v,v)}{a_0(v,v)} \leq \gamma_1, \quad \forall\, v \in Y\ . \tag{4}$$

For the hypotheses stated above, it is readily demonstrated that the problem (1) has a unique solution.

Many situations may be modeled by our rather simple problem statement (1), (3). For example, if we take $Y = H_0^1(\Omega)$ where Ω is a smooth bounded subdomain of $\mathbf{R}^{d=2}$, and set $a_0(w,v) = \int_\Omega \nabla w \cdot \nabla v$, $a_1 = \int_\Omega wv$, we model conduction in thin plates; here μ represents the convective heat transfer coefficient. If we take $Y = H_0^1(\Omega)$ for $\Omega \subset \mathbf{R}^{d=1,2,\,\mathrm{or}\,3}$, with $\overline{\Omega}_1 \subset \overline{\Omega}$ ($\overline{\Omega}_1$, $\overline{\Omega}$ bounded and sufficiently regular), and set $a_0 = \int_\Omega \nabla w \cdot \nabla v$, $a_1 = \int_{\Omega_1} \nabla w \cdot \nabla v$, we model variable-property heat transfer; here $1 + \mu$ is the ratio of thermal conductivities in domains $\Omega \backslash \Omega_1$ and Ω_1. Other choices of a_0 and a_1 can model variable rectilinear geometry, variable orthotropic properties, and variable Robin boundary conditions.

The space Y is typically of infinite dimension so $u(\mu)$ is, in general, not exactly calculable. In order to construct our reduced-basis space W_N, we must therefore replace $u(\mu) \in Y$ by a "truth approximation" $u^{\mathcal{N}}(\mu) \in Y^{\mathcal{N}} \subset Y$, where $u^{\mathcal{N}}$ is the Galerkin approximation satisfying

$$a(u^{\mathcal{N}}(\mu), v; \mu) = f(v), \quad \forall\, v \in Y^{\mathcal{N}}\ .$$

Here $Y^{\mathcal{N}}$, of finite (but typically very high) dimension $\mathcal{N}$, is a sufficiently rich approximation subspace such that $|||u(\mu) - u^{\mathcal{N}}(\mu)|||$ is sufficiently small for all μ in $\mathcal{D}$; for example, for $Y = H_0^1(\Omega)$ we know that, for any desired $\varepsilon > 0$, we can indeed construct a finite-element approximation space, $Y^{\mathcal{N}(\varepsilon)}$, such that $|||u(\mu) - u^{\mathcal{N}(\varepsilon)}(\mu)||| \leq \varepsilon$.

It shall prove convenient in what follows to introduce a generalized eigenvalue problem: Find ($\varphi_i^{\mathcal{N}} \in Y^{\mathcal{N}}, \lambda_i^{\mathcal{N}} \in \mathbf{R}$), $i = 1, \ldots, \mathcal{N}$, satisfying $a_1(\varphi_i^{\mathcal{N}}, v) = \lambda_i^{\mathcal{N}}\, a_0(\varphi_i^{\mathcal{N}}, v)$, $\forall\, v \in Y^{\mathcal{N}}$. We shall order the (perforce real, non-negative) eigenvalues as $0 \leq \lambda_{\mathcal{N}}^{\mathcal{N}} \leq \lambda_{\mathcal{N}-1}^{\mathcal{N}} \leq \cdots \leq \lambda_1^{\mathcal{N}} \leq \gamma_1$, where the last inequality follows directly from (4). We may choose our eigenfunctions such that

$$a_0(\varphi_i^{\mathcal{N}}, \varphi_j^{\mathcal{N}}) = \delta_{i\,j}\ , \tag{5}$$

and hence $a_1(\varphi_i^{\mathcal{N}}, \varphi_j^{\mathcal{N}}) = \lambda_i^{\mathcal{N}}\, \delta_{i\,j}$, where $\delta_{i\,j}$ is the Kronecker-delta symbol; and such that $Y^{\mathcal{N}}$ can be expressed as span $\{\varphi_i,\, i = 1, \ldots, \mathcal{N}\}$. Note that, thanks to the *finite* dimension of our approximation space $Y^{\mathcal{N}}$, we preclude (the complications associated with) a continuous spectrum — and, as we shall see, at no loss in rigor.

We conclude this section by noting that $u^{\mathcal{N}}(\mu)$ can be expressed as

$$u^{\mathcal{N}}(\mu) = \sum_{i=1}^{\mathcal{N}} \frac{f_i^{\mathcal{N}}\, \varphi_i^{\mathcal{N}}}{1+\mu\lambda_i^{\mathcal{N}}}\ , \tag{6}$$

where $f_i^{\mathcal{N}} = f(\varphi_i^{\mathcal{N}})$.

3 A Priori Convergence Theory

We propose here to choose the sample points $\alpha_k,\ k = 1,\dots,N$, log-equidistributed in $\mathcal{D}$, in the sense that, if we set $\delta_N = \ln(\gamma\mu_{\max}+1)/N$, and γ is any finite upper bound for γ_1 [1], then

$$\alpha_k = \exp\{-\ln\gamma + \sum_{\ell=1}^{k} \tilde{\delta}_{\ell N}\} - \gamma^{-1},$$

where we assume that there exists a constant c^* such that

$$\frac{\tilde{\delta}_{kN}}{\delta_N} \le c^*, \quad \forall k, k = 1,\dots,N$$

and also that $\sum_{\ell=1}^{N} \tilde{\delta}_{\ell N} = \ln(\gamma\mu_{\max}+1)$.

Denote the reduced-basis approximation space as $W_N^{\mathcal{N}} = \mathrm{span}\,\{u^{\mathcal{N}}(\alpha_k),\ k = 1,\dots,N\}$. Although in general $\dim(W_N^{\mathcal{N}}) \le N$, we can suppose that $\dim(W_N^{\mathcal{N}}) = N$ (otherwise we eliminate elements from $W_N^{\mathcal{N}}$ until it contains only linearly independent vectors). Then, the (reduced basis) problem is : Given $\mu \in \mathcal{D}$, find $u_N^{\mathcal{N}}(\mu) \in W_N^{\mathcal{N}}$ such that

$$a(u_N^{\mathcal{N}}(\mu), v; \mu) = f(v), \quad \forall\, v \in W_N^{\mathcal{N}}\ . \tag{7}$$

This problem admits a unique solution.

Our goal is to (sharply) bound $|||u^{\mathcal{N}}(\mu) - u_N^{\mathcal{N}}(\mu)|||$, for all $\mu \in \mathcal{D}$, as a function of N (and ultimately $\mathcal{N}$ as well). This error bound in the energy norm can be readily translated into error bounds on continuous-linear-functional outputs [12]; we do not consider this extension further here.

We shall need two standard results from the theory of Galerkin approximation of symmetric coercive problems [14]:

$$a(u^{\mathcal{N}} - u_N^{\mathcal{N}}, u^{\mathcal{N}} - u_N^{\mathcal{N}}; \mu) = \inf_{w_N^{\mathcal{N}} \in W_N^{\mathcal{N}}} a(u^{\mathcal{N}} - w_N^{\mathcal{N}}, u^{\mathcal{N}} - w_N^{\mathcal{N}}; \mu)\ ; \tag{8}$$

[1] Note that γ_1, γ, and hence S_N, are *independent* of $\mathcal{N}$.

and

$$a(u^{\mathcal{N}}, u^{\mathcal{N}}; \mu) \leq a(u, u; \mu) \ . \tag{9}$$

From the definition of the $|||\cdot|||$ norm, the positive semidefiniteness of a_1, (3), (4) and (8) we can write

$$\begin{aligned}
&|||u^{\mathcal{N}}(\mu) - u_N^{\mathcal{N}}(\mu)|||^2 \leq a(u^{\mathcal{N}}(\mu) - u_N^{\mathcal{N}}(\mu), u^{\mathcal{N}}(\mu) - u_N^{\mathcal{N}}(\mu), \mu) \\
&\leq \inf_{w_N^{\mathcal{N}} \in W_N^{\mathcal{N}}} a(u^{\mathcal{N}}(\mu) - w_N^{\mathcal{N}}, u^{\mathcal{N}}(\mu) - w_N^{\mathcal{N}}, \mu) \\
&\leq (1 + \mu_{\max}\gamma_1) \inf_{w_N^{\mathcal{N}} \in W_N^{\mathcal{N}}} |||u^{\mathcal{N}}(\mu) - w_N^{\mathcal{N}}|||^2, \qquad \forall\, \mu \in \mathcal{D}.
\end{aligned} \tag{10}$$

Also from the definition of the $|||\cdot|||$ norm and the positive semidefiniteness of a_1, (3), (4) and (9), we obtain

$$|||u^{\mathcal{N}}(\mu)||| \leq (1 + \mu_{\max}\gamma_1)^{1/2}\, |||u(\mu)|||, \qquad \forall\, \mu \in \mathcal{D} \tag{11}$$

We begin with a preparatory result in

Lemma 3.1 Let

$$g(z, \lambda) = \frac{1}{1 - \frac{\lambda}{\gamma} + \lambda e^z} \tag{12}$$

for $z \in Z \equiv [\ln(\gamma^{-1}), \infty]$ and $\lambda \in \Lambda \equiv [0, \gamma]$ (recall γ is our strictly positive upper bound for γ_1). Then, for any $q \geq 0$

$$|D_1^q\, g(z, \lambda)| \leq 2^q\, q! \ , \quad \forall\, z \in Z,\ \forall\, \lambda \in \Lambda \ ,$$

where $D_1^q g$ denotes the q^{th}-derivative of g with respect to the first argument.

Proof. We first remark that for any $p \geq 0$

$$0 \leq g^p(z, \lambda) \leq 1, \quad \forall\, z \in Z,\ \forall\, \lambda \in \Lambda, \ , \tag{13}$$

where g^p is the p^{th}-power of g. This follows since $\forall\, z \in Z$, $e^z \geq \gamma^{-1}$, and hence, $\forall\, z \in Z$, $\forall\, \lambda \in \Lambda$, $1 - \lambda/\gamma + \lambda e^z \geq 1$.

We next claim that, for $m \geq 2$,

$$D_1^{m-1}\, g(z, \lambda) = \sum_{n=1}^{m} a_n^m\, g^n(z, \lambda) \ , \tag{14}$$

where, for $m \geq 1$ (and $a_1^1 \equiv 1$)

$$\begin{aligned}
a_1^{m+1} &= -a_1^m \\
a_n^{m+1} &= -n\, a_n^m + (n-1)\left(1 - \tfrac{\lambda}{\gamma}\right) a_{n-1}^m, \quad 2 \leq n \leq m \\
a_{m+1}^{m+1} &= m\left(1 - \tfrac{\lambda}{\gamma}\right) a_m^m.
\end{aligned} \tag{15}$$

We prove this result by induction. We first differentiate g to obtain

$$D_1^1\, g(z,\lambda) = \frac{-\lambda e^z}{\left(1-\frac{\lambda}{\gamma}+\lambda e^z\right)^2} = \frac{-\left(1-\frac{\lambda}{\gamma}+\lambda e^z\right)+1-\frac{\lambda}{\gamma}}{\left(1-\frac{\lambda}{\gamma}+\lambda e^z\right)^2}$$

$$= -g(z,\lambda) + \left(1-\frac{\lambda}{\gamma}\right) g^2(z,\lambda)\ ; \tag{16}$$

(14) and (15) for $m = 2$ directly follows. We now differentiate (14) for $m = \overline{m}$, and exploit (16), to obtain

$$\begin{aligned}
&D_1^{(\overline{m}+1)-1}(z,\lambda) = \sum_{n=1}^{\overline{m}} a_n^{\overline{m}}\, n\, g^{n-1}(z,\lambda)\, D_1^1 g(z,\lambda)\\
&= \sum_{n=1}^{\overline{m}} a_n^{\overline{m}}\, n\, g^{n-1}(z,\lambda) \left[-g(z,\lambda) + \left(1-\frac{\lambda}{\gamma}\right) g^2(z,\lambda)\right]\\
&= \sum_{n=1}^{\overline{m}} -n\, a_n^{\overline{m}}\, g^n(z,\lambda) + \sum_{n=2}^{\overline{m}+1} (n-1)\left(1-\frac{\lambda}{\gamma}\right) a_{n-1}^{\overline{m}}\, g^n(z,\lambda)\ ;
\end{aligned}$$

(14) and (15) for $m = \overline{m}+1$ directly follows. Thus, (14), (15) are valid for $m \geq 2$, as required.

It now follows from (13) and (14) that, for any $q \geq 1$, $|D_1^q\, g(z,\lambda)| \leq \mathcal{S}^q$, where $\mathcal{S}^q = \sum\limits_{n=1}^{q+1} |a_n^{q+1}|$. We now invoke (15), and observe that $\forall\, \lambda \in \Lambda$, $|1-\frac{\lambda}{\gamma}| \leq 1$, to obtain $\mathcal{S}^q \leq 2q\, \mathcal{S}^{q-1}$; since $\mathcal{S}^1 \leq 2$, we conclude that $\mathcal{S}^q \leq 2^q\, q!$, and the lemma is thus proven. $\square$

We now prove a bound for the best approximation result in

Lemma 3.2 For $N \geq N_{\rm crit}$

$$\inf_{w_N^{\mathcal{N}} \in W_N^{\mathcal{N}}} |||u^{\mathcal{N}}(\mu) - w_N^{\mathcal{N}}||| \leq |||u^{\mathcal{N}}(0)|||\, \exp\left\{\frac{-N}{N_{\rm crit}}\right\}, \qquad \forall\, \mu \in \mathcal{D}\ ,$$

where $N_{\rm crit} \equiv c^* e\, \ln(\gamma\, \mu_{\max} + 1)$.

Proof. To facilitate the proof, we shall effect a change of coordinates in parameter space. To wit, we let $\widetilde{\mathcal{D}} \equiv [\ln \gamma^{-1}, \ln(\mu_{\max}+\gamma^{-1})]$, and introduce $\tau\colon \widetilde{\mathcal{D}} \to \mathcal{D}$, as $\tau(\tilde\mu) = e^{\tilde\mu} - \gamma^{-1}$ so that $\tau^{-1}(\mu) = \ln(\mu+\gamma^{-1})$. We then set $\tilde u(\tilde\mu) = u(\tau(\tilde\mu))$, $\tilde u^{\mathcal{N}}(\tilde\mu) = u^{\mathcal{N}}(\tau(\tilde\mu))$, and $\tilde u_N^{\mathcal{N}}(\tilde\mu) = u_N^{\mathcal{N}}(\tau(\tilde\mu))$. We note that

$$\tilde u^{\mathcal{N}}(\tilde\mu) = \sum_{i=1}^{\mathcal{N}} \frac{f_i^{\mathcal{N}}\, \varphi_i^{\mathcal{N}}}{1-\frac{\lambda_i^{\mathcal{N}}}{\gamma}+\lambda_i^{\mathcal{N}}\, e^{\tilde\mu}} = \sum_{i=1}^{\mathcal{N}} f_i^{\mathcal{N}}\, \varphi_i^{\mathcal{N}}\, g(\tilde\mu, \lambda_i^{\mathcal{N}}), \tag{17}$$

from (6), our change of variable, and the definition (12).

We now observe that in our mapped coordinate, the sample points $\tilde{\alpha}_k \equiv \tau^{-1}(\alpha_k)$, $k = 1, \ldots, N$, are equi-distributed with separation $\tilde{\alpha}_{k+1} - \tilde{\alpha}_k \simeq \ln(\gamma\mu_{\max} + 1)/N$. It thus follows that, given any $\tilde{\mu} \in \widetilde{\mathcal{D}}$, we can construct a closed interval $\widetilde{I}^{\tilde{\mu}}_{\tilde{\Delta}}$ of length $\tilde{\Delta}$, that includes $\tilde{\mu}$ and $M^{\tilde{\mu}}(\tilde{\Delta}, \delta_N)$ distinct points $\tilde{\alpha}_{P^{\tilde{\mu}}_n}$, $n = 1, \ldots, M$. Here $M^{\tilde{\mu}}(\tilde{\Delta}, \delta_N)$ is of the order of $\frac{\tilde{\Delta}}{\delta_N}$; more precisely,

$$M^{\tilde{\mu}}(\tilde{\Delta}, \delta_N) \geq \frac{\tilde{\Delta}}{c^* \delta_N}. \tag{18}$$

In what follows, we shall often abbreviate $M^{\tilde{\mu}}(\tilde{\Delta}, \delta_N)$ as M.

Now, for any $\tilde{\mu} \in \widetilde{\mathcal{D}}$, we introduce $\hat{u}^{\tilde{\mu}} \in W^{\mathcal{N}}_N$ given by

$$\begin{aligned} \hat{u}^{\tilde{\mu}} &\equiv \sum_{n=1}^{M} \widetilde{Q}^{\tilde{\mu}}_n(\tilde{\mu})\, u^{\mathcal{N}}(\tau(\tilde{\alpha}_{P^{\tilde{\mu}}_n})) = \sum_{n=1}^{M} \widetilde{Q}^{\tilde{\mu}}_n(\tilde{\mu})\, \tilde{u}^{\mathcal{N}}(\tilde{\alpha}_{P^{\tilde{\mu}}_n}) \\ &= \sum_{n=1}^{M} \widetilde{Q}^{\tilde{\mu}}_n(\tilde{\mu}) \sum_{i=1}^{\mathcal{N}} f^{\mathcal{N}}_i\, \varphi^{\mathcal{N}}_i\, g(\tilde{\alpha}_{P^{\tilde{\mu}}_n}, \lambda^{\mathcal{N}}_i)\ , \end{aligned}$$

where the characteristic functions $\widetilde{Q}^{\tilde{\mu}}_n$ are uniquely determined by $\widetilde{Q}^{\tilde{\mu}}_n \in \mathbf{P}_{M-1}(\widetilde{I}^{\tilde{\mu}}_{\tilde{\Delta}})$, $n = 1, \ldots, M$, and $\widetilde{Q}^{\tilde{\mu}}_n(\tilde{\alpha}_{P^{\tilde{\mu}}_{n'}}) = \delta_{nn'}$, $1 \leq n,\ n' \leq M$; here $\mathbf{P}_{M-1}(\widetilde{I}^{\tilde{\mu}}_{\tilde{\Delta}})$ refers to the space of polynomials of degree $\leq M-1$ over $\widetilde{I}^{\tilde{\mu}}_{\tilde{\Delta}}$. We thus obtain

$$\hat{u}^{\tilde{\mu}} = \sum_{i=1}^{\mathcal{N}} f^{\mathcal{N}}_i\, \varphi^{\mathcal{N}}_i\, [\widetilde{\mathcal{I}}^{\tilde{\mu}}_{M-1}\, g(\cdot, \lambda^{\mathcal{N}}_i)]\, (\tilde{\mu})\ , \tag{19}$$

where, for given λ, $\widetilde{\mathcal{I}}^{\tilde{\mu}}_{M-1}\, g(\cdot, \lambda)$ is the $(M-1)^{\text{th}}$-order polynomial interpolant of $g(\cdot, \lambda)$ through the $\tilde{\alpha}_{P^{\tilde{\mu}}_n}$, $n = 1, \ldots, M$; more precisely, $\widetilde{\mathcal{I}}^{\tilde{\mu}}_{M-1}\, g(\cdot, \lambda) \in \mathbf{P}_{M-1}(\widetilde{I}^{\tilde{\mu}}_{\tilde{\Delta}})$, and $(\widetilde{\mathcal{I}}^{\tilde{\mu}}_{M-1}\, g(\cdot, \lambda))(\tilde{\alpha}_{P^{\tilde{\mu}}_n}) = g(\tilde{\alpha}_{P^{\tilde{\mu}}_n}, \lambda)$, $n = 1, \ldots, M$. Note that $[\widetilde{\mathcal{I}}^{\tilde{\mu}}_{M-1}\, g(\cdot, \lambda)](\tau^{-1}(\mu))$ is *not* a polynomial in μ.

It now follows from (5), (6), (17) and (19) that

$$\begin{aligned} |||\tilde{u}^{\mathcal{N}}(\tilde{\mu}) - \hat{u}^{\tilde{\mu}}||| &\leq \left|\left|\left| \textstyle\sum_{i=1}^{\mathcal{N}} f^{\mathcal{N}}_i\, \varphi^{\mathcal{N}}_i \left(g(\tilde{\mu}, \lambda^{\mathcal{N}}_i) - [\widetilde{\mathcal{I}}^{\tilde{\mu}}_{M-1}\, g(\cdot, \lambda^{\mathcal{N}}_i)]\, (\tilde{\mu}) \right) \right|\right|\right| \\ &\leq \sup\nolimits_{\lambda \in \Lambda} |g(\tilde{\mu}, \lambda) - [\widetilde{\mathcal{I}}^{\tilde{\mu}}_{M-1}\, g(\cdot, \lambda)]\, (\tilde{\mu})|\ |||u^{\mathcal{N}}(0)|||\ . \end{aligned} \tag{20}$$

We next invoke the standard polynomial interpolation remainder formula [4] and Lemma 3.1 to obtain

$$\begin{aligned} \sup_{\lambda \in \Lambda} |g(\tilde{\mu}, \lambda) - [\widetilde{\mathcal{I}}^{\hat{u}}_{M-1}\, g(\cdot, \lambda)]\, (\tilde{\mu})| &\leq \sup\nolimits_{\lambda \in \Lambda} \sup\nolimits_{z \in Z} \tfrac{1}{M!} |D^M_1\, g(z, \lambda)|\, \tilde{\Delta}^M \\ &\leq (2\tilde{\Delta})^{M^{\tilde{\mu}}(\tilde{\Delta}, \delta_N)}. \end{aligned} \tag{21}$$

We now assume that $\frac{c^*\delta_N}{2} \leq \tilde{\Delta}$ and $\tilde{\Delta} \leq \frac{1}{2}$; under these conditions (recall (18)) we obtain $(2\tilde{\Delta})^{M^{\tilde{\mu}}(\tilde{\Delta},\delta_N)} \leq (2\tilde{\Delta})^{\tilde{\Delta}/c^*\delta_N}$, and hence, from (20) and (21), we can write

$$|||\tilde{u}^{\mathcal{N}}(\tilde{\mu}) - \hat{u}^{\tilde{\mu}}||| \leq |||u^{\mathcal{N}}(0)|||(2\tilde{\Delta})^{\tilde{\Delta}/c^*\delta_N} \ . \tag{22}$$

It remains to select a best $\tilde{\Delta}$ satisfying $\frac{c^*\delta_N}{2} \leq \tilde{\Delta} \leq \frac{1}{2}$.

To provide the sharpest possible bound, we choose $\tilde{\Delta} = \tilde{\Delta}^* \equiv \frac{1}{2e}$, the minimizer (over all positive $\tilde{\Delta}$) of $(2\tilde{\Delta})^{\tilde{\Delta}/\delta_N}$. Our conditions on $\tilde{\Delta}$ are readily verified: $\frac{c^*\delta_N}{2} \leq \tilde{\Delta}^*$ follows directly from the hypothesis of our lemma, $N \geq N_{\mathrm{crit}}$; and $\tilde{\Delta}^* \leq \frac{1}{2}$ follows from inspection. We now insert $\tilde{\Delta} = \tilde{\Delta}^*$ into (22) to obtain

$$|||\tilde{u}^{\mathcal{N}}(\tilde{\mu}) - \hat{u}^{\tilde{\mu}}||| \leq |||u^{\mathcal{N}}(0)|||\, e^{-N/N_{\mathrm{crit}}}, \qquad \forall\, \tilde{\mu} \in \widetilde{\mathcal{D}} \ .$$

It immediately follows that, for any $\mu \in \mathcal{D}$,

$$\begin{aligned} &\inf_{w_N^{\mathcal{N}} \in W_N^{\mathcal{N}}} |||u^{\mathcal{N}}(\mu) - w_N^{\mathcal{N}}||| = \inf_{w_N^{\mathcal{N}} \in W_N^{\mathcal{N}}} |||\tilde{u}^{\mathcal{N}}(\tau^{-1}(\mu)) - w_N^{\mathcal{N}}||| \\ &\leq |||\tilde{u}^{\mathcal{N}}(\tau^{-1}(\mu)) - \hat{u}^{\tau^{-1}(\mu)}||| \leq |||u^{\mathcal{N}}(0)|||\, e^{-N/N_{\mathrm{crit}}} \ , \end{aligned}$$

since $\hat{u}^{\cdot} \in W_N^{\mathcal{N}}$ and, for $\mu \in \mathcal{D}$, $\tau^{-1}(\mu) \in \widetilde{\mathcal{D}}$. This concludes the proof. $\square$

Then, from (10),(11), Lemma 3.1, and Lemma 3.2, we obtain

Theorem 3.3 For $N \geq N_{\mathrm{crit}} \equiv c^* e \, \ln(\gamma \mu_{\max} + 1)$,

$$|||u^{\mathcal{N}}(\mu) - u_N^{\mathcal{N}}(\mu)||| \leq (1 + \mu_{\max}\gamma_1)^{1/2}\, |||u^{\mathcal{N}}(0)|||\, e^{-N/N_{\mathrm{crit}}},\ \forall\, \mu \in \mathcal{D};$$

furthermore,

$$|||u(\mu) - u_N^{\mathcal{N}(\varepsilon)}(\mu)||| \leq \varepsilon + (1 + \mu_{\max}\gamma_1)\, |||u(0)|||\, e^{-N/N_{\mathrm{crit}}},\ \forall\, \mu \in \mathcal{D},$$

for $\mathcal{N}(\varepsilon)$ such that $|||u(\mu) - u^{\mathcal{N}(\varepsilon)}(\mu)||| \leq \varepsilon$.

4 Conclusions

We make several observations about the results of Theorem 3.3. First, we obtain *exponential convergence* with respect to N. Second, our convergence result applies *uniformly* for all $\mu \in \mathcal{D}$. Third, our convergence parameter N_{crit} depends only very *weakly* — logarithmically — on γ_1, related to the form of the operator, and on $\mu_{\max}$, related to the range of the parameter (note we may also view the product $\gamma_1\mu_{\max}$ as the continuity-coercivity ratio). As a result, N_{crit} will, in general, be small, and we will thus achieve convergence "soon" ($N \geq N_{\text{crit}}$) with a "large" ($\frac{1}{N_{\text{crit}}}$) exponential decay rate. Fourth, we obtain convergence with respect to *both* N and $\mathcal{N}$: $u_N^{\mathcal{N}}(\mu) \to u(\mu)$ as $N, \mathcal{N} \to \infty$.

Let us now make several remarks concerning the point distribution. First, the logarithmic point distribution is intimately related to our particular abstract problem, (1), (3), and, relatedly, the parametric dependence of the solution, (6). In brief, for larger values of μ, the derivatives of $(1 + \mu\lambda)^{-1}$ will be smaller, thus permitting a larger interval in which to recruit the points required for accurate interpolation. Second, it should be clear from the proof of Lemma 3.2 that the requirement on the point distribution is, in fact, rather weak: the location of the M points within $\widetilde{I}_{\bar{\Delta}}^{\bar{\mu}}$ is (save for conditioning issues) irrelevant. This permits, for example, log-*random* distributions — particularly attractive in higher parameter dimensions in which tensor-product grids are prohibitively costly.

Third, the logarithmic point distribution is not an artifact of our (*interpolant*-based) proof: in numerical tests [15] the error in the actual *Galerkin approximation* is also "minimized" by a logarithmic point distribution; even point distributions that enjoy general optimality properties, such as Chebyshev, do not perform as well as the logarithmic distribution for our particular problem. (Indeed, for Chebyshev interpolation over the interval $\mathcal{D}$, it may be shown that N_{crit} scales as $\sqrt{\mu_{\max}}$ — much worse than our $\ln \mu_{\max}$.)

Fourth, we note that numerical tests [15] roughly confirm the dependence of $|||u^{\mathcal{N}}(\mu) - u_N^{\mathcal{N}}(\mu)|||$ on N, γ, and $\mu_{\max}$. However, in general, our theoretical bound can be quite pessimistic, as might be expected given that our proof is based on (albeit, tailored) interpolation arguments: Galerkin optimality can always do better, for example, choosing to "illuminate" only an appropriate subsample of S_N so as to construct the best "subapproximation" (or sub-interpolant) amongst all $O(N!)$ possibilities. This property is no doubt crucial in higher parameter dimensions, in which effective scattered-data higher-order interpolants are very difficult to construct; it is here that the superiority of the reduced-basis approach over simple parameter-space interpolation is most evident. We do not yet have any (uniform in μ) theory for higher parameter dimensions, although numerical results again suggest extremely rapid convergence.

Finally, we note that we address in this paper only one aspect — rapid uniform convergence — of successful reduced-basis approaches. In other papers [6, 7, 8, 12] we focus on (i) off-line/on-line computational decompositions that permit real-time response, and (ii) a posteriori error estimators that ensure both efficiency and certainty.

Acknowledgements. We would like to thank Christophe Prud'homme, Dimitrios Rovas, and Karen Veroy of MIT for sharing their numerical results prior to publication. This work was performed while ATP was an Invited Professor at the University of Paris VI in February, 2001. This work was supported by the

Singapore-MIT Alliance, by DARPA and ONR under Grant F49620-01-1-0458, by DARPA and AFOSR under Grant N00014-01-1-0523 (Subcontract 340-6218-3), and by NASA under Grant NAG-1-1978.

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