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CHARACTERIZATIONS OF PERIODS OF MULTIDIMENSIONAL SHIFTS

EMMANUEL JEANDEL AND PASCAL VANIER

ABSTRACT. We show that the sets of periods of multidimensional shifts of finite type (SFTs) are exactly the sets of integers of the complexity class **NE**. We also show that the functions counting their number are the functions of **#E**. We also give characterizations of some other notions of periodicity. We finish the paper by giving some characterizations for sofic and effective subshifts.

1. INTRODUCTION

A multidimensional shift of finite type (SFT) is a set of colorings of \mathbb{Z}^2 given by local rules. SFTs are one of the most fundamental objects in symbolic dynamics [LM95]. One important question is to determine whether two SFTs are *conjugate*. One approach to solve the problem is by the study of invariants, i.e. quantities related to subshifts that stay invariant under conjugacy. The most significant invariants for SFTs are the entropy, that measures the growth of the number of valid patterns, and the set of periodic points.

These two invariants are relatively well known in the one-dimensional case: The entropy of an SFT is the logarithm of the spectral radius of some matrix related to the SFT, and the set of periodic points relates to the cycles of the multi-graph represented by the matrix. As a consequence, the set of integers n so that there exists a periodic point of period n is a semi-linear set.

The situation becomes more complex for multi-dimensional SFTs. This is linked to the fact that the emptiness problem for SFTs becomes undecidable [Ber66; Rob71], in part due to the existence of nonempty SFTs with no periodic points. While the theory of one-dimensional SFTs indeed relates to the theory of finite automata of computer science, many properties of multidimensional SFTs are to be understood using *computability theory*.

The case of the entropy was recently solved by Hochman and Meyerovitch [HM10]: A real $\lambda \geq 0$ is the entropy of a multidimensional SFT if and only if it is right approximable, that is if we can compute a sequence of rational numbers converging to λ from above. This was one of the first articles in a series linking dynamical properties of multidimensional SFTs and computability theory [Sim09; AS09].

In this article, we give a characterization of the sets of periods of multidimensional SFTs using *complexity theory*. For a given point x in an SFT, let $\Gamma_x = \{\mathbf{v} \in \mathbb{Z}^2 \mid \forall \mathbf{z} \in \mathbb{Z}^2, x(\mathbf{z} + \mathbf{v}) = x(\mathbf{z})\}$ be the lattice of periods of x .

We will study different notions of periodic points:

- c is *strongly periodic* of period $n > 0$ if $\Gamma_c = n\mathbb{Z}^2$
- c is *1-periodic* of period $\mathbf{v} \in \mathbb{Z} \times \mathbb{N} \setminus \{(0, 0)\}$ if $\Gamma_c = \mathbf{v}\mathbb{Z}$
- c is *horizontally periodic* of period $n > 0$ if n is the least positive integer so that $n\mathbb{Z} \times \{0\} \subseteq \Gamma_c$

All these notions can readily be generalized for any dimension $d > 2$. For a given subshift X , let \mathfrak{P}_X (resp. $\mathfrak{P}_X^1, \mathfrak{P}_X^h$) denote the set of strong periods (resp. 1-periods, horizontal periods) of X . The third notion seems a bit more peculiar. It is introduced in this paper as a first, somewhat easier, result on which all other results will be built.

To give a characterization of these sets in terms of complexity classes, we will have to see these sets as languages. If $L \subseteq \mathbb{N}$, denote by $un(L) = \{a^n | n \in L\}$. If $L \subseteq \mathbb{Z} \times \mathbb{N}$, denote $un(L) = \{a^p b^q | (p, q) \in L\} \cup \{a^p c^q | (-p, q) \in L\}$.

We will prove:

Theorem 1.1. *For any $L \subseteq \mathbb{N}$, there exists an SFT X such that $L = \mathfrak{P}_X$ if and only if $un(L) \in \text{NP}$.*

Theorem 1.2. *For any $L \subseteq \mathbb{Z} \times \mathbb{N}$, there exists a two dimensional SFT X such that $L = \mathfrak{P}_X^1$ if and only if $un(L) \in \text{NSPACE}(n)$.*

Theorem 1.3. *For any $L \subseteq \mathbb{N}$, there exists a two dimensional SFT X such that $L = \mathfrak{P}_X^h$ if and only if $un(L) \in \text{NSPACE}(n)$.*

Here NP denotes as usual the class of languages computable by a nondeterministic Turing Machine in polynomial time. $\text{NSPACE}(n)$ denotes the class of languages computable by a nondeterministic Turing Machine in linear space.

Please note the slight difference between theorem 1.1 and the others: the two other theorems are valid for a fixed dimension $d = 2$. Theorem 1.1 needs however to be formulated for all dimensions at once: given a language $L \in \text{NP}$ the dimension of the SFT for which L is a set of strong periods depends on L . It is in fact hard to provide a statement valid exactly in dimension $d = 2$. Intuitively, the reason is that SFTs can be seen as a model of computation. For most models of computation, the space complexity of a problem L is generally the same. However the time complexity of a problem depends on the exact definition of the model: the problem to decide if a word is a palindrome is provably quadratic in a model of a Turing Machine with one tape, but becomes linear if the Turing Machine has two tapes. The two-dimensional SFTs being assimilated as a new model of computation, there is no reason for it to behave like a specific, already known, model of Turing Machine. That's why we have to use a more robust class, NP, which coincides for all reasonable models of computation (See e.g. the *Invariance Thesis* [EB]). The problem does not appear for space classes, as they are already robust.

Rather than the set of periodic points, another interesting quantity is the *number* of periodic points. This quantity makes only sense for strongly periodic points. If X is an SFT, denote by \mathfrak{N}_X the map from $\{a\}^*$ to \mathbb{N} that maps a^n to the number of points of strong period n . Then we will prove:

Theorem 1.4. *Let $f : \{a\}^* \rightarrow \mathbb{N}$, there exists an SFT X such that $f = \mathfrak{N}_X$ if and only if $f \in \#P$*

Here $\#P$ denotes as usual the class of functions corresponding to the number of accepting paths of a nondeterministic Turing Machine working in polynomial time. This theorem gives a first insight on the behavior of the Zeta function of a multidimensional SFT [Lin96].

These characterizations in terms of complexity classes lead to some closure properties on the set of periods. NP and $\text{NSPACE}(n)$ are closed under intersection and union, so the sets of periods are also closed under intersection and union. The

closure by union could of course be proven directly, by taking the disjoint union of the two SFTs. Note that the closure by intersection is not trivial, as the classical construction by Cartesian product does not work as usefully as it would seem: It is not true that the set of strong periods of $X \times Y$ is the intersection of the sets of strong periods of X and Y : If $\{2\}$ (resp. $\{3\}$) is the set of strong periods of X (resp. Y), then the set of strong periods of $X \times Y$ will be $\{6\}$ and not \emptyset as intended. Note in particular that due to the peculiar form of theorem 1.1, the SFT that realizes the intersection of the strong periods of X and Y can be of higher dimension than X and Y . Finally, since nondeterministic space is closed under complementation [Imm88; BDG88b], 1-periods are closed under complementation. The question whether the sets of strong periods are closed under complementation is of course related to the P vs NP problem (More accurately, it is related to the question $\text{NE} = \text{coNE}$, and to Asser's Problem [JS74; DJMM]).

This paper is organized as follows: The first section gives the necessary background both in multidimensional symbolic dynamics and complexity theory. We then proceed to prove all four theorems. We first prove Theorem 1.3 on horizontal periods. The techniques used for this proof are the core of this article. The other three theorems then build on this first proof, adding more and more complex structures in the various constructions. We end this paper with a discussion on similar results for multidimensional sofic and effective shifts rather than SFTs.

Some of the results of this paper were announced at the DLT conference in the extended abstract [JV10].

2. PRELIMINARIES

2.1. Symbolic Dynamics. We give here a primer on Multidimensional Symbolic Dynamics. See [LM95; Lin04] for more information.

2.1.1. Subshifts. Let Σ be a finite alphabet and $d > 0$ an integer. A *configuration* over Σ is a coloring of \mathbb{Z}^d by Σ , that is a map $c : \mathbb{Z}^d \rightarrow \Sigma$. We denote by $\Sigma^{\mathbb{Z}^d}$ the set of all configurations over Σ , $\Sigma^{\mathbb{Z}^d}$ is also called the *full shift* on Σ . A *pattern* P is a coloring of a subset $D \subset \mathbb{Z}^d$. D is the *support* of the pattern. A pattern is finite if D is finite.

A pattern P of support D *appears* in a pattern P' of support D' if there exists a position $\mathbf{v} \in D'$ so that $\mathbf{v} + D \subseteq D'$ and $P(\mathbf{v} + \mathbf{z}) = P'(\mathbf{z})$ for all $\mathbf{z} \in D$. We will write $P \in P'$ to say that P appears in P' .

Let \mathcal{F} be a set of *finite* patterns. A pattern is *admissible* for \mathcal{F} if it contains no patterns of \mathcal{F} . The *subshift* $X_{\mathcal{F}}$ defined by \mathcal{F} is the set of all configurations where no pattern of \mathcal{F} appears:

$$X_{\mathcal{F}} = \{c \in \Sigma^{\mathbb{Z}^d} \mid \forall P \in \mathcal{F}, P \notin c\}$$

A set X is a subshift if there exists a set \mathcal{F} so that $X = X_{\mathcal{F}}$. Subshifts can also be characterized by a topological property, but we will not need it here.

A *subshift of finite type* (or shortly SFT) is a subshift $X_{\mathcal{F}}$ where \mathcal{F} is finite. In this case, we can assume that all patterns of \mathcal{F} are over the same finite support D . In this setting, a configuration c is a *valid* if all patterns of support D appearing in c are not in \mathcal{F} , such a configuration is called a point of X . The *radius* of D is the smallest r so that $D \subseteq [-r, r]^d$. The radius of $X_{\mathcal{F}}$ is the radius of D .

An *effective subshift* is a subshift $X_{\mathcal{F}}$ where \mathcal{F} is recursively enumerable.

Let X and Y be two d -dimensional subshifts, a *block code* is a map $F : X \rightarrow Y$ such that there exists a map $f : \Sigma_X^V \rightarrow \Sigma_Y$, with $V = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ a finite subset of \mathbb{Z}^2 such that for any $\mathbf{z} \in \mathbb{Z}$:

$$F(x)_{\mathbf{z}} = f(x_{\mathbf{z}+\mathbf{v}_1}, \dots, x_{\mathbf{z}+\mathbf{v}_k})$$

A map $F : X \rightarrow Y$ is a factor map if it is surjective block code, Y is then called a *factor* of X . A subshift is called *sofic* if it is a factor of some SFT.

2.1.2. Periodic points. Let $c \in \Sigma^{\mathbb{Z}^d}$ be a configuration. A vector $\mathbf{v} \in \mathbb{Z}^d$ is a *vector of periodicity* for c if $c(\mathbf{z}) = c(\mathbf{z} + \mathbf{v})$ for all $\mathbf{z} \in \mathbb{Z}^d$. We write $\Gamma_c = \{\mathbf{v} \in \mathbb{Z}^d \mid \forall \mathbf{z} \in \mathbb{Z}^d, c(\mathbf{z}) = c(\mathbf{z} + \mathbf{v})\}$ for the set of vectors of periodicity of c . Γ_c is of course a lattice (i.e. a (discrete) subgroup of \mathbb{Z}^d). There are three cases for Γ_c :

- $\Gamma_c = \{0\}$: c has no vector of periodicity.
- Γ_c has rank d , then c is *periodic*. This corresponds to the notion of periodicity in dimension 1: a finite orbit. In particular, in this case, one can prove that there exists n such that $n\mathbb{Z}^d \subseteq \Gamma_c$. If $n\mathbb{Z}^d = \Gamma_c$ we will say that c is *strongly periodic* of strong period n .
- Γ_c has an intermediate rank $1 \leq k < d$. We will say that c is *k-periodic*.

If c is 1-periodic, then $\gamma_c = v\mathbb{Z}$ for a unique vector v (upto opposite direction) which is called the *1-period* of c .

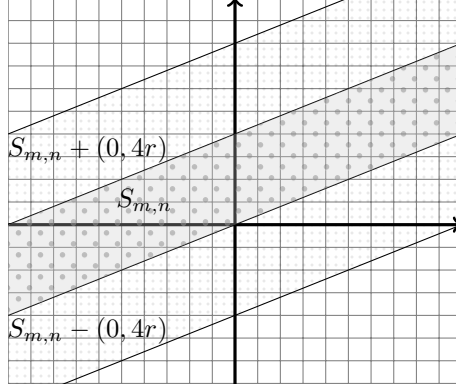


FIGURE 1. Representation of $S_{m,n}$ for $m = 5, n = 2$ and $r = 1$. Note that for every point x in $S_{m,n}$, the square of size $2r$ and of bottom right corner x is entirely contained in $S_{m,n} \cup (S_{m,n} + (0, 4r)) = D_{m,n}$.

A configuration which is 1-periodic in a subshift of dimension d can be seen as a configuration in a subshift of dimension $d - 1$. To make this statement exact, we need some definitions. Let X be an SFT of radius r . Let (m, n) be integers with $m \geq n \geq 0$. Consider $S_{m,n} = \{(x, y) \in \mathbb{Z}^2 \mid 0 \leq -nx + my < 4mr\}$. and $D_{m,n} = \{(x, y) \in \mathbb{Z}^2 \mid 0 \leq -nx + my < 8mr\} = S_{m,n} \cup (S_{m,n} + (0, 4r))$ represented in figure 1.

Now the number of patterns of support $S_{m,n}$ that are (m, n) -periodic is finite, as any such pattern is entirely determined by $S \cap ((0, m-1) \times \mathbb{Z})$ which contains at most $4rm$ points. So there are at most $|\Sigma|^{4rm}$ such patterns.

Now consider the following directed graph $G_{m,n}(X)$:

- The vertices of $G_{m,n}(X)$ are all patterns of support $S_{m,n}$ that are (m, n) periodic.
- There is a edge from P to P' if the pattern $P \ominus P'$ of support $D_{m,n}$ defined by $P \ominus P'(z) = P(z)$ if $z \in S_{m,n}$ and $P \ominus P'(z) = P'(z - (0, 4r))$ otherwise is valid for X .

Now it is clear that there exists a bijection between configurations of X with (m, n) as a period and bi-infinite paths in $G_{m,n}(X)$. It is due to the fact that $\mathbb{Z}^2 = \uplus_i (S_{m,n} + (0, 4r)i)$ and that any square of size $2r$ whose bottom right corner is in $S_{m,n}$ is in $D_{m,n}$ (this is where the hypothesis $m \geq n$ is used) so that any square of size $2r$ in \mathbb{Z}^2 is contained in $D_{m,n} + (0, 4r)i$ for some i .

Thus the entire information is contained in this graph. It is then easy to obtain the following consequences:

Lemma 2.1. *Let X be a two-dimensional SFT of radius r .*

- *If X contains a configuration which is periodic of period (m, n) , then it contains a configuration which is fully periodic. (If the finite graph $G_{m,n}(X)$ contains an infinite path, it contains a cycle)*
- *More precisely, if X contains a configuration with (m, n) as a period, $m \geq n > 0$, then it contains a fully periodic configuration with (m, n) and $(0, p)$ as periods, for some $0 < p \leq |\Sigma|^{4rm}$.*

We could also obtain a characterization of 1-periods with the graph:

Lemma 2.2. *X admits (m, n) as a 1-period if and only if the graph $G_{m,n}(X)$ contains a path $u_0 \dots u_k$ so that*

- $u_i = u_0$ for some $i < k$.
- $u_{i+1} \neq u_1$
- $u_j = u_k$ for some $i \leq j < k$
- *For each (m', n') so that $\exists d \in \mathbb{N}, (m, n) = d \cdot (m', n')$, there exists some pattern u_l which is not (m', n') periodic.*

We can of course choose $k \leq 3m|\Sigma|^{4rm}$. The first three conditions ensure that there exists a configuration c which is 1-periodic and admits (m, n) as a period. The last condition ensures that (m, n) is indeed the least period of c .

2.2. Aperiodic SFTs and determinism. Let X be a \mathbb{Z}^d SFT, X is *aperiodic* when no point of X admits a periodicity vector. There are several well known such SFTs for \mathbb{Z}^2 , most of them come from the study of tilings, see e.g. Berger [Ber66; Ber64], Robinson [Rob71], Kari [Kar96; Kar92]. We will need two dimensional aperiodic SFTs with a particular property in this paper : determinism. A two dimensional SFT is *north-west deterministic* if for any two symbols a, b at positions (i, j) and $(i+1, j+1)$ there is at most one symbol c allowed at position $(i+1, j)$. Such an SFT was constructed by Kari [Kar92], his particular SFT will be used in section 5. *East-determinism* can be determined in the same way: for any two symbols a, b at positions (i, j) and $(i, j+1)$ there is at most one symbol c allowed at position $(i+1, j)$.

2.3. Computational Complexity. In this section we provide some background on computational complexity and its links with subshifts of finite type. More information about computational complexity and computability can be found in [BDG88a; AB09; Rog87].

Usually to model computation, Turing machines are used. Despite its power, this model is quite simple to describe : shortly, a Turing machine is a device with finite memory but that can read/write on an infinite tape at the position of its “head”. Formally, it is a tuple $(Q, \Gamma, B, \Sigma, \delta, q_0, H)$ where:

- Q is a finite set of states,
- Γ is the tape alphabet, the finite set of symbols that can appear on the tape,
- $B \in \Gamma$ is the “Blank” symbol,
- $\Sigma \subseteq \Gamma \setminus \{B\}$ is the input alphabet,
- q_0 is the initial state,
- H is the set of halting states,
- $\delta : Q \setminus F \times \Gamma \rightarrow Q \times \Gamma \times \{\leftarrow, \cdot, \rightarrow\}$ is the transition function. The symbols \rightarrow, \leftarrow and \cdot stand for moving the head to the right, left and to let it where it is respectively.

If a problem can be answered by a Turing machine, then it is called *decidable* and otherwise *undecidable*. The most famous undecidable problem is the Halting Problem: deciding whether a given Turing machine halts with itself as an input. Another well known undecidable problem is the Domino Problem: given a set of Wang tiles, does it tile the plane?

A problem is called *recursively enumerable* (r.e.) if there exists a Turing machine enumerating its elements and *co-recursively enumerable* (co-r.e.) when its complement is r.e.

A complexity class is a class of problems decidable by a Turing machine such that some resource is bounded. The usual restrictions of the resources are on time or space :

- the classes $\text{TIME}(f(n))$ are the classes of problems decidable in time $f(n)$, where n is the size of the input and f a fonction from \mathbb{N} to \mathbb{N} .
- the classes $\text{SPACE}(f(n))$ are the classes of problems decidable in space $f(n)$.

Turing machines may be nondeterministic, which means that the transition function is multivalued. In this case, the input is accepted if there exists a sequence of transitions leading to an accepting state. Time and space complexity classes are also defined in the case of nondeterministic Turing machines :

- the classes $\text{NTIME}(f(n))$ are the classes of problems nondeterministically decidable in time $f(n)$, where n is the size of the input and f a fonction from \mathbb{N} to \mathbb{N} .
- the classes $\text{NSPACE}(f(n))$ are the classes of problems nondeterministically decidable in space $f(n)$.

As said earlier, tilings and recursivity are intimately linked. In fact, it is quite easy to encode Turing machines in tilings. Such encodings can be found e.g. in [Kar94; Cha08]. Given a Turing machine M , we can build a tiling system τ_M in figure 2. The tiling system is given by *Wang tiles*, i.e., we can only glue two tiles together if they coincide on their common edge. This tiling system τ_M has the following property: there is an accepting path for the word u in time (less than) t using space (less than) w if and only if we can tile a rectangle of size $(w+2) \times t$ with white borders, the first row containing the input. Note that this method works for both deterministic and nondeterministic machines.

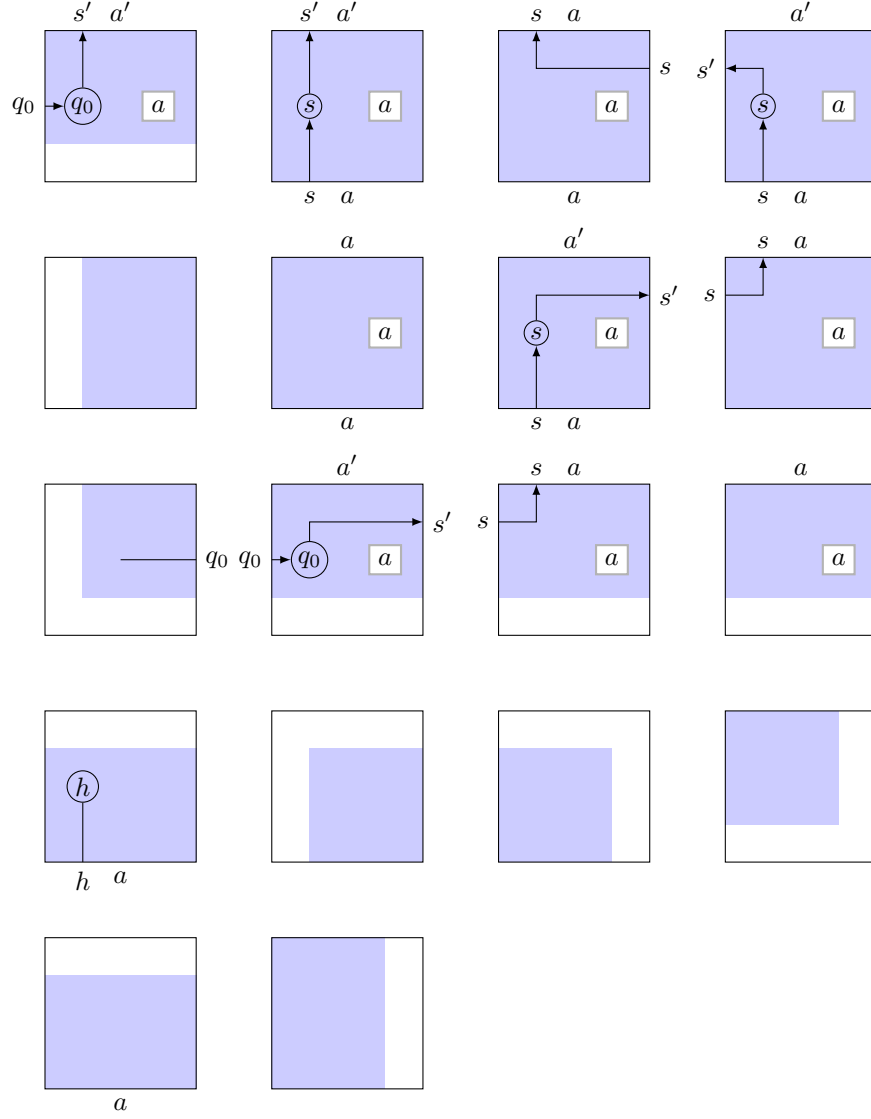


FIGURE 2. A tiling system, given by Wang tiles, simulating a Turing machine. The meaning of the labels are the following:

- label s_0 represents the initial state of the Turing machine.
- The top-left tile corresponds to the case where the Turing machine, given the state s and the letter a on the tape, writes a' , moves the head to the left and to change from state s to s' . The two other tiles are similar.
- h represents a halting state. Note that the only states that can appear in the last step of a computation (before a border appears) are halting states.

3. HORIZONTAL PERIODICITY IN SFTs OF DIMENSION 2 AND SPACE COMPLEXITY

In this section, we give a proof for theorem 1.3. We first prove that the unary language corresponding to horizontal periods can be recognized in linear space by a nondeterministic Turing machine (lemma 3.1) and then the reciprocal (lemma 3.2).

Lemma 3.1. *Let $L \subseteq \mathbb{N}^*$ be the set of horizontal periods of a two-dimensional SFT X , then $un(L) \in \text{NSPACE}(n)$.*

Proof. Let $X = X_{\mathcal{F}}$ be a 2-dimensional SFT on the alphabet Σ . We will construct a nondeterministic Turing machine accepting 1^n if and only if $n + 1$ is a horizontal period of X . The machine has to work in space $\mathcal{O}(n)$, the input being given in unary.

Let r be the radius of X , a point is in X if and only if all its $r \times r$ blocks have no sub-pattern contained in \mathcal{F} .

Furthermore, we can prove that if there exists a point of horizontal period n , then there also exists such a point, with vertical period at most $|\Sigma|^{r^n}$. Here is an algorithm, starting from n as an input that checks whether n is a horizontal period of some point of X :

- Initialize an array P of size n so that $P[i] = 1$ for all i .
- First choose nondeterministically $p \leq |\Sigma|^{r^n}$
- Choose r bi-infinite rows $(c_i)_{0 \leq i \leq r-1}$ of period n (that is, choose $r \times n$ symbols).
- For each $r + 1 \leq i \leq p$, choose a bi-infinite row c_i of period n (that is, choose n symbols), and verify that all $r \times r$ blocks in the rows $c_i \dots c_{i-r+1}$ do not contain a forbidden pattern. At each time, keep only the last r rows in memory (we never forget the r first rows though).
- (Verification of the *least* period) If at any of the previous steps, the row c_i is not periodic of period $k < n$, then $P[k] \leftarrow 0$
- For $i \leq r$, verify that all $r \times r$ blocks in the rows $c_{p-i} \dots c_p c_0 \dots c_{i-r-1}$ do not contain a forbidden pattern.
- If there is some k such that $P[k] = 1$, reject. Otherwise accept.

This algorithm needs to keep in memory only $2r$ rows and the array P at each time, hence is in space $\mathcal{O}(n)$. \square

Lemma 3.2. *Let $L \subseteq \mathbb{N}$ be a language such that $un(L) \in \text{NSPACE}(n)$, then there exists a two-dimensional SFT X such that $n \in L$ if and only if there exists a point $c \in X$ with horizontal period n .*

Proof. Take a nondeterministic Turing machine M accepting $un(L)$ in linear space. Using traditional tricks from complexity theory, we can suppose that on input 1^n the Turing machine uses exactly $n + 1$ cells of the tape (i.e. the input, with one additional cell on the right) and works in time exactly c^n for some constant c (depending only on the Turing machine M).

We are going to construct an SFT X' such that $1^n \in L$ if and only if $n + 4$ is a horizontal period of some point of X' . The modification to obtain $n + 1$ rather than $n + 4$, and thus prove the lemma, is left to the reader (basically “fatten” the vertical lines of \mathbf{I} presented in lemma 3.3 below so that they absorb 3 adjacent tiles), and serves no interest other than technical.

The proof may basically be split into two parts:

- First produce an SFT Y_c such that every point of horizontal period n looks like a grid of rectangles of size n by c^n delimited by horizontal and vertical markers (see fig. 3b) and whose horizontal periods are $\mathbb{N} \setminus \{0, 1\}$. Lemma 3.3 shows how to construct such an SFT.
- The Turing machine M is then encoded inside these rectangles on a layer C : the nondeterministic transitions are *synchronized* on every line to ensure the computations inside the rectangles are the same.

The main difficulty lies in the first part, while the second part is straightforward and does not need further explanation.

Now we prove that $1^n \in L$ if and only if $n + 4$ is a horizontal period of the the SFT X' .

- For the component C to be valid, the input 1^{p-4} ($4 = 1$ (■ symbol of Y_c) + 1 (left border for the TM) + 1 (right border for the TM) + 1 (blank marker on the end of the tape)) must be accepted by the Turing machine, hence $1^{p-4} \in L$
- Finally, due to the synchronization of the nondeterministic transitions, the C component is also p -periodic. As a consequence, our tiling is p -periodic, hence $n = p - 4$. Therefore $1^{n+4} \in L$

Conversely, suppose $1^n \in L$. Consider the coloring of period $n + 4$ obtained as follows (only a period is described):

- The component A consists of $n + 3$ correctly tiled columns of our aperiodic East-deterministic SFT, with an additional column of ■. Note that aperiodic points exist.
- The component C corresponds to a successful computation path of the Turing machine on the input 1^n , that exists by hypothesis. As the computation lasts less than c^n steps, the computation fits exactly inside the $n \times c^n$ rectangle.
- We then add all other layers according to the rules to obtain a valid configuration, thus obtaining a point of X' of period exactly $n + 4$.

□

Lemma 3.3. *There exists an SFT Y_k such that for any point $y \in Y_k$ of horizontal period p , y is composed of rectangles of size $p \times k^{p-1}$ with marked boundaries. Furthermore, all integers $n \geq 2$ are horizontal periods.*

We will construct the SFT by superimposing several components (or layers) each of them addressing a specific issue $Y_k = A \times C_k \times T$:

- A will allow us to force periodic tilings to have columns separated by vertical lines,
- C_k will make the horizontal lines and at the same time force the regularity of the vertical ones,
- T will force that within a horizontal period, only one vertical line can appear.

The components and their rules are as follows:

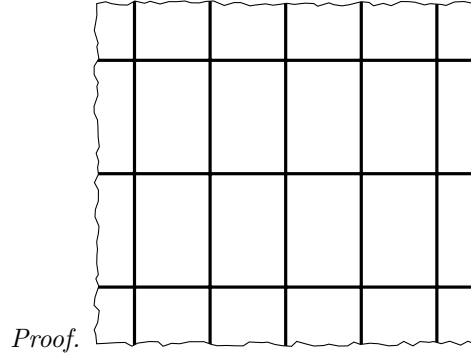


FIGURE 3. The shape of the base SFT Y_k : whenever a point of Y_k is periodic, it must have the above shape where the width of the rectangles is exactly the period p and their height k^{p-1} .

$w_{0,9} w_{1,9} w_{2,9}$	$w_{3,9} w_{4,9} w_{5,9}$	$w_{6,9} w_{7,9} w_{8,9} w_{9,9} w_{10,9} w_{11,9}$	$w_{0,9} w_{1,9} w_{2,9}$	$w_{3,9} w_{4,9} w_{5,9}$	$w_{6,9} w_{7,9} w_{8,9} w_{9,9} w_{10,9} w_{11,9}$
$w_{0,8} w_{1,8} w_{2,8}$	$w_{3,8} w_{4,8} w_{5,8}$	$w_{6,8} w_{7,8} w_{8,8} w_{9,8} w_{10,8} w_{11,8}$	$w_{0,8} w_{1,8} w_{2,8}$	$w_{3,8} w_{4,8} w_{5,8}$	$w_{6,8} w_{7,8} w_{8,8} w_{9,8} w_{10,8} w_{11,8}$
$w_{0,7} w_{1,7} w_{2,7}$	$w_{3,7} w_{4,7} w_{5,7}$	$w_{6,7} w_{7,7} w_{8,7} w_{9,7} w_{10,7} w_{11,7}$	$w_{0,7} w_{1,7} w_{2,7}$	$w_{3,7} w_{4,7} w_{5,7}$	$w_{6,7} w_{7,7} w_{8,7} w_{9,7} w_{10,7} w_{11,7}$
$w_{0,6} w_{1,6} w_{2,6}$	$w_{3,6} w_{4,6} w_{5,6}$	$w_{6,6} w_{7,6} w_{8,6} w_{9,6} w_{10,6} w_{11,6}$	$w_{0,6} w_{1,6} w_{2,6}$	$w_{3,6} w_{4,6} w_{5,6}$	$w_{6,6} w_{7,6} w_{8,6} w_{9,6} w_{10,6} w_{11,6}$
$w_{0,5} w_{1,5} w_{2,5}$	$w_{3,5} w_{4,5} w_{5,5}$	$w_{6,5} w_{7,5} w_{8,5} w_{9,5} w_{10,5} w_{11,5}$	$w_{0,5} w_{1,5} w_{2,5}$	$w_{3,5} w_{4,5} w_{5,5}$	$w_{6,5} w_{7,5} w_{8,5} w_{9,5} w_{10,5} w_{11,5}$
$w_{0,4} w_{1,4} w_{2,4}$	$w_{3,4} w_{4,4} w_{5,4}$	$w_{6,4} w_{7,4} w_{8,4} w_{9,4} w_{10,4} w_{11,4}$	$w_{0,4} w_{1,4} w_{2,4}$	$w_{3,4} w_{4,4} w_{5,4}$	$w_{6,4} w_{7,4} w_{8,4} w_{9,4} w_{10,4} w_{11,4}$
$w_{0,3} w_{1,3} w_{2,3}$	$w_{3,3} w_{4,3} w_{5,3}$	$w_{6,3} w_{7,3} w_{8,3} w_{9,3} w_{10,3} w_{11,3}$	$w_{0,3} w_{1,3} w_{2,3}$	$w_{3,3} w_{4,3} w_{5,3}$	$w_{6,3} w_{7,3} w_{8,3} w_{9,3} w_{10,3} w_{11,3}$
$w_{0,2} w_{1,2} w_{2,2}$	$w_{3,2} w_{4,2} w_{5,2}$	$w_{6,2} w_{7,2} w_{8,2} w_{9,2} w_{10,2} w_{11,2}$	$w_{0,2} w_{1,2} w_{2,2}$	$w_{3,2} w_{4,2} w_{5,2}$	$w_{6,2} w_{7,2} w_{8,2} w_{9,2} w_{10,2} w_{11,2}$
$w_{0,1} w_{1,1} w_{2,1}$	$w_{3,1} w_{4,1} w_{5,1}$	$w_{6,1} w_{7,1} w_{8,1} w_{9,1} w_{10,1} w_{11,1}$	$w_{0,1} w_{1,1} w_{2,1}$	$w_{3,1} w_{4,1} w_{5,1}$	$w_{6,1} w_{7,1} w_{8,1} w_{9,1} w_{10,1} w_{11,1}$
$w_{0,0} w_{1,0} w_{2,0}$	$w_{3,0} w_{4,0} w_{5,0}$	$w_{6,0} w_{7,0} w_{8,0} w_{9,0} w_{10,0} w_{11,0}$	$w_{0,0} w_{1,0} w_{2,0}$	$w_{3,0} w_{4,0} w_{5,0}$	$w_{6,0} w_{7,0} w_{8,0} w_{9,0} w_{10,0} w_{11,0}$

FIGURE 4. A periodic point of A . Here $w_{i,j}$ are symbols of the alphabet of W .

- The first component A is partly composed of an aperiodic East-deterministic SFT W , whose symbols will be called *white* symbols. We can take the one from J. Kari [Kar92]¹.

To obtain A from W , we add to the alphabet a new symbol **I**. With the additional forbidden patterns:

- no white symbol shall be above or below a **I**,
- two **I** cannot appear next to each other horizontally.

With this construction, a periodic point of A of period p must have columns of white symbols separated by vertical lines of **I**. This is due to the fact that W forms an aperiodic SFT.

For the moment nothing forbids more than one gray column to appear inside a period. Figure 4 shows a possible form of a periodic point at this stage.

¹Note that in the paper the SFT is described by Wang tilings and that it is NW-deterministic. However, it is straightforward to modify the rules in order to get an East-deterministic aperiodic SFT. This exact SFT will be studied later on in section 5

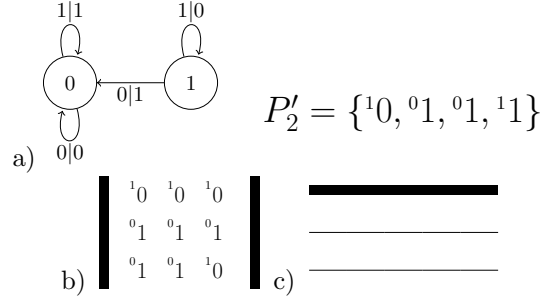


FIGURE 5. a) The transducer corresponding to $k = 2$ and the symbols of the corresponding SFT. A valid pattern for C_2 is given in b) and c): the breaker symbol in b) is from the A layer and corresponds always to the 1 symbol of C_2 .

- The second layer of symbols $C_k = P_k \times \{\blacksquare, _ \}$ will produce horizontal lines so that points of period np will consist of rectangles of size $p \times k^p$, delimited by the symbols \blacksquare and \mathbf{I} .

The idea is as follows: suppose each horizontal segment between two vertical lines is a word over the alphabet $P'_k = \{\{0, \dots, k-1\} \times \{0, 1\}\}$, that is, represents a number a between 0 and $k^{p-1} - 1$ written in base k . It is then easy with local constraints to ensure that the word on the *next* line is $a + 1 \pmod{k^{p-1}}$. The $\{0, 1\}$ component represents the carry. The alphabet P_k is composed of P'_k and of a carry 1 that will be superimposed to the symbol \mathbf{I} only. See figure 5 for a transducer in the case $k = 2$ and its realization as an SFT.

With the $\{\blacksquare, _ \}$ subcomponent, we mark the lines corresponding to the number 0, so that one line out of k^{p-1} is marked. This line is the only one where a 01 is transformed into a 10 on the right of a vertical line of \mathbf{I} .

A $_$ is forbidden to appear on the right or left of a \blacksquare : this forces each column to have counters that are resetted at the exact same moment, and thus to have the exact same size. Figure 6a shows some typical tiling at this stage: the period of a tiling is not necessarily the same as the distance between the rectangles, it may be larger. Indeed, the white symbols in two consecutive rectangles may be different.

- Component T is formed of the same alphabet as W of component A , recall that W is an East-deterministic SFT. The forbidden patterns are that two different symbols cannot be horizontal neighbors. In addition to that, we forbid of elements of component T on the right of a \mathbf{I} to be different to the ones of component W . That means that the symbols of the first column to the right of a vertical line of \mathbf{I} are exactly the same for each vertical line of \mathbf{I} .

The SFT W being East-deterministic, this means that the symbols of the columns between vertical lines of \mathbf{I} are exactly the same for each column, these are the boundaries of the rectangles.

At this stage, a periodic point necessarily has regular rectangles on all the plane, whose width correspond to the period, as shown in figure 6b.

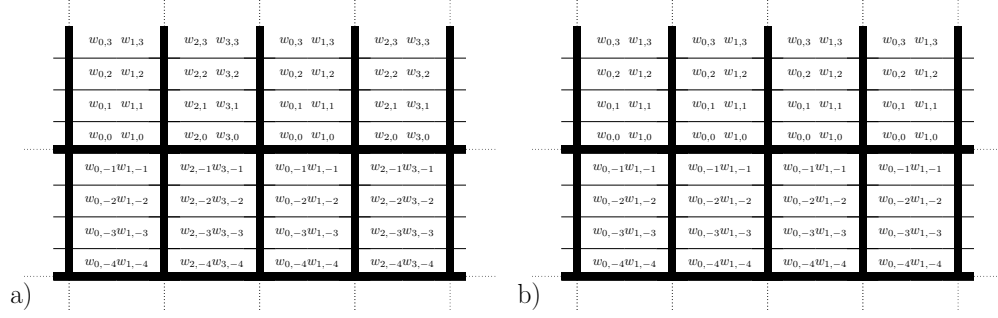


FIGURE 6. a) An example of a valid periodic point with C_2 , the distance between two consecutive vertical lines is now constant. However the period is not necessarily the width of the rectangles. b) Once we add component T , the width of the rectangles is now exactly the period. Note that we only show here the component A and the subcomponent $\{\blacksquare, _ \}$ of C_2 .

Now we prove that if $y \in Y_k$ is periodic of period n , then it necessarily is formed of vertical lines of \blacksquare at distance n of each other and of horizontal lines of $_$ at distance k^{n-1} of each other.

Consider a point of Y_k of period n :

- Due to component A , a vertical line of \blacksquare must appear. The period is a succession of vertical lines of \blacksquare and white columns.
- Due to component C_k , the vertical lines of \blacksquare are spaced by a distance of p , where $p \mid n$. Furthermore, there are horizontal lines of $_$ at distance k^{p-1} of each other.
- Due to component T , the tiling we obtain is horizontally periodic of period p , thus $p = n$.

□

4. STRONG PERIODICITY IN SFTs AND TIME COMPLEXITY

In this section we prove theorem 1.1 on strong periods.

In a square $n \times n$ one can only embed computations ending in time inferior to n . However, given a $n \times n$ square of symbols, one cannot check that it has no forbidden patterns in less than n^2 time steps with a Turing machine.

The analogue for higher dimensions holds: any $2d$ -dimensional cube of n^{2d} symbols can embed computations in time n^d , and checking such a cube needs n^{2d} time steps. Thus the class $\text{NTIME}_1(n^d)$ for unary inputs²

can be captured by $2d$ -dimensional cubes and only d -dimensional cubes are checkable in time n^d .

The gap here is not surprising: while space complexity classes are usually model independent, this is not the case for time complexity, where the exact definition of the computational model matters. An exact characterization of strongly periodic SFTs for $d = 2$ would in fact be possible, but messy: it would involve Turing machines working in space $O(n)$ with $O(n)$ reversals, see e.g [CM06].

Here the solution comes from the fact that periodic SFTs of all dimensions would be captured by the time complexity class $\text{NP}_1 = \bigcup_{d \in \mathbb{N}} \text{NTIME}_1(n^d)$, the gap being filled by the infinite union. This is theorem 1.1, whose proof below will be, as before, divided in three parts :

- We first show that one can check whether a d dimensional SFT is strongly periodic of period p in time p^d and thus that the problem is in $\text{NTIME}_1(n^d)$ (lemma 4.1).
- For the converse, we first construct a base SFT with marked cubes (lemma 4.2) in a similar way as for horizontal periods,
- All that is left to prove then is how the Turing machines are encoded inside these cubes (lemma 4.3).

It is interesting to note that the complexity class NP_1 also characterizes spectra of first order formula, see [JS74].

Lemma 4.1. *For any d dimensional SFT X , $\text{un}(\mathfrak{P}_X) \in \text{NP}$: there exists a Turing machine $M \in \text{NP}_1$ that given p as an input, determines whether p is a strong period of X .*

Proof. It suffices to take a Turing machine that nondeterministically guesses a p^d cube and then checks whether it contains any forbidden patterns and whether p is the strong period of X : for this last part, it has to check that for all $k < n$, k is not a period. \square

Lemma 4.2. *There exists a d -dimensional SFT Y_d such that :*

- Any periodic point is strongly periodic.
- Any strongly periodic point $y \in Y_d$ of period p is constituted of adjacent d -cubes p^d with marked borders.
- Every integer $p \geq 2$ is a strong period of Y_d .

Proof. As before, the construction will be based on some aperiodic SFT, with some added symbols to break the aperiodicity and force a regular structure. The SFT Y_d is made of three layers $A \times S \times T$:

²Note that the class $\text{NTIME}_1(n^d)$ for unary inputs corresponds to the class $\text{NTIME}_2(2^{dn})$ for binary inputs and that complexity classes are usually defined on binary inputs. So NP_1 is not the famous NP class, although $\text{NP} = \text{P}$ would imply $\text{NP} = \text{P}$.

- layer A will force strongly periodic points to have marked lines,
- layer S will force strongly periodic points to have marked d -cubes,
- and finally, layer T will force the strongly periodic points to be composed of d -cubes of side p , the strong period.

We will now detail each component and the strongly periodic points thus obtained.

- Let W be a 2-dimensional NW-deterministic aperiodic SFT³, we define a 2-dimensional SFT A' :
 - the alphabet of A' is composed of the alphabet of W with the additional symbols $\mathbf{l}, \mathbf{+}, \mathbf{-}$,
 - the forbidden patterns of W are kept and the following are added:
 - * above and below a \mathbf{l} may only appear a \mathbf{l} or a $\mathbf{+}$,
 - * on the left and right of a $\mathbf{-}$ may only appear a $\mathbf{-}$ or a $\mathbf{+}$,
 - * on the left and right of a $\mathbf{+}$ there may only be a $\mathbf{-}$,
 - * above and below a $\mathbf{+}$, there may only be a \mathbf{l} .

The (strongly) periodic points of A' have necessarily one of the new symbols $\mathbf{l}, \mathbf{+}, \mathbf{-}$. That is to say they are necessarily formed of either an infinity of lines of \mathbf{l} , either an infinity of lines of $\mathbf{-}$, or by an infinity of squares with sides marked by $\mathbf{-}$ and \mathbf{l} and corners by $\mathbf{+}$.

The d -dimensional SFT A is obtained from the 2-dimensional SFT A' by keeping the alphabet, keeping the rules for the first two dimensions, and then force the symbols next to each other along all other directions to be equal. In the sequel, we will call A' the plane with the rules of A' .

- The second layer S will force the A' plane of periodic points to be formed of squares and will mark the frontiers of the d -cubes. For $2 \leq i \leq d$, we define $d - 1$ SFTs S_i with the same alphabet formed by the symbols $\mathbf{l}, \mathbf{+}, \mathbf{-}$, $\mathbf{+}, \mathbf{-}, \mathbf{l}, \mathbf{+}, \mathbf{-}$, $\mathbf{+}, \mathbf{-}, \mathbf{l}, \mathbf{+}, \mathbf{-}$:

- The adjacency rules for S_i on the plane defined by e_1, e_i are that two symbols can be next to each other iff their borders match: left/right matchings correspond to $\pm e_i$ and above/below to $\pm e_1$.
- The rules applying to the other dimensions are that if there is a symbol a at $x \in \mathbb{Z}^d$, then there must also be a symbol a at $x \pm e_k$, for $k \neq 1, i$.

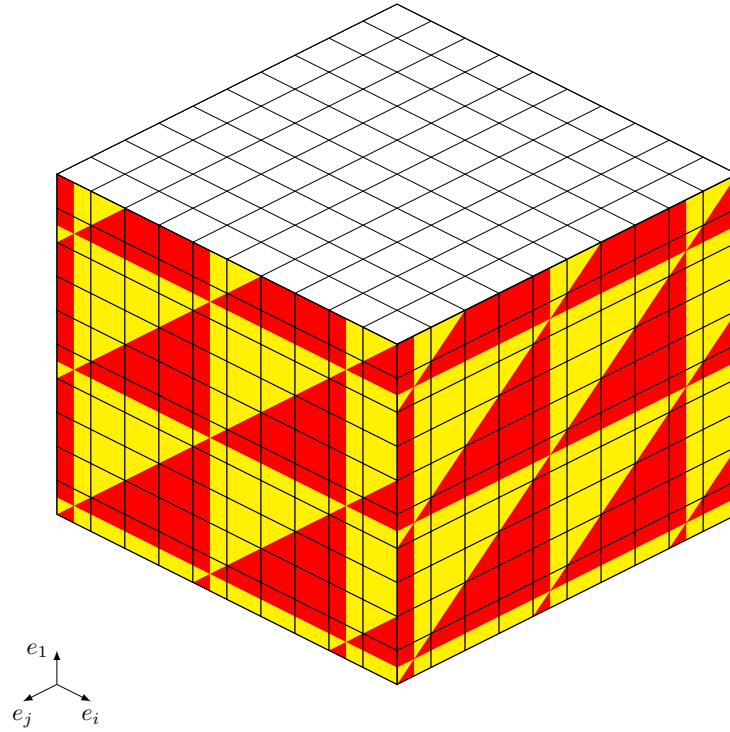
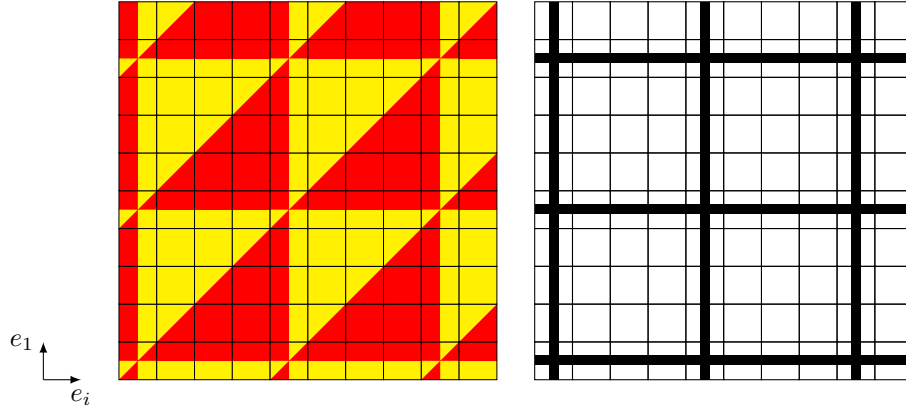
We superimpose the S_i 's in order to obtain S : at each position, the symbols on all S_i components must all be taken from only one of the sets $\{\mathbf{+}, \mathbf{-}\}$ and $\{\mathbf{l}, \mathbf{+}, \mathbf{-}\}$.

See figure 7 for an example of how S_i and S_j can be superimposed.

Then to obtain $A \times S$ we add superimposition rules only with the S_2 subcomponent of S , which has its rules on the A' plane and forms squares on it. The rules are that $\mathbf{+}$ can only be superimposed to $\mathbf{+}$, $\mathbf{-}$ (resp. \mathbf{l}) can only be superimposed to $\mathbf{-}$ (resp. \mathbf{l}) and the other symbols can only be superimposed to white symbols. As a consequence, the symbols $\mathbf{l}, \mathbf{-}, \mathbf{+}$ on the A' plane must form squares on the strongly periodic points. Figure 8 shows how the A' planes of component S_2 and component A are superimposed.

The strongly periodic points of the resulting SFT are points that have d -cubes whose corners are marked by $(\mathbf{+}, \dots, \mathbf{+})$. The boundaries of the d -cubes are marked by the $\mathbf{+}$ and $\mathbf{-}$ symbols: if the side of the d -cubes

³For this lemma we can take any such SFT, however we will use Kari's [Kar92] tileset later to further the construction.

FIGURE 7. How S_i and S_j are superimposed.FIGURE 8. The A' planes of components S_2 (left) and A (right) and how they are superimposed: layer S_2 forces the symbols \blacksquare and \blacklozenge of component A to form squares.

is n , and there is a corner at $\mathbf{p} = (p_1, \dots, p_d) \in \mathbb{Z}^d$, then for any point $\mathbf{q} = (q_1, \dots, q_d) \in \mathbb{Z}^d$, a \blacksquare or \blacklozenge on component S_k is equivalent to $p_k \equiv q_k \pmod n$ and a \blacklozenge or \blacksquare is equivalent to $p_1 \equiv q_1 \pmod n$.

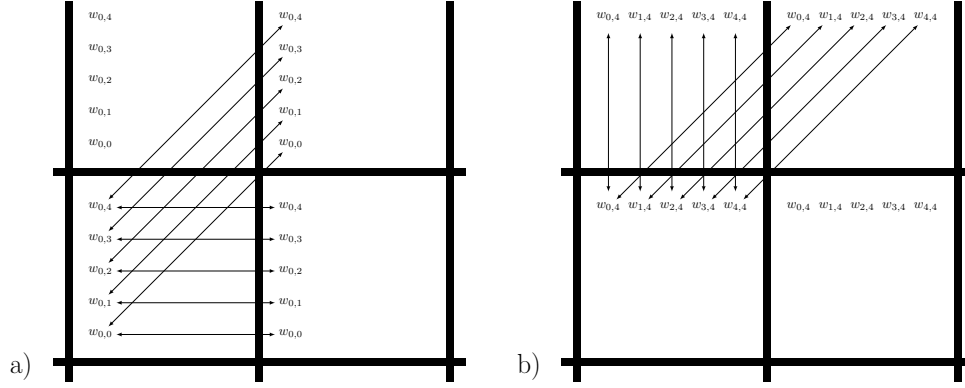








FIGURE 9. In a), the way L_{right} and L_{diag} synchronise the first column of the aperiodic background for all squares. In b), the way U_{up} and U_{diag} synchronise the top line of the aperiodic background for all squares.

However, the squares formed on the A' plane of A may not have the same aperiodic background, and thus there could be more than one square in a period. Thus the period is a multiple of the size of the d -cubes. We want now to prevent this from happening and force the strong period to be exactly the size of the d -cubes.

- Component $T = L_{right} \times L_{diag} \times U_{up} \times U_{diag}$ is here to address this last problem: by synchronizing the aperiodic background between squares on A' it will force the least distance between two $(\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}, \dots, \begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix})$ to be the strong period. To do this, since W is NW-deterministic, we only need to transmit to the neighboring squares the upper line of symbols and the leftmost one, see figure 9. Each subcomponent's alphabet is a copy of the alphabet of W and the rules are as follows:
 - On L_{right} , the symbols at \mathbf{z} are the same as the symbols at $\mathbf{z} \pm \mathbf{e}_2$. The only superimposition rule is that a symbol on the right of a \blacksquare on A' must be the same as on L_{right} . This component synchronises the leftmost column of symbols of all horizontally aligned squares.
 - On L_{diag} , the symbols at \mathbf{z} are the same as the symbols at $\mathbf{z} \pm (\mathbf{e}_1 + \mathbf{e}_2)$. The only superimposition rule is that a symbol on the right of a \blacksquare on A' must be the same as on L_{diag} . This component synchronises the leftmost column of symbols of all diagonally aligned squares.
 - On U_{up} , the symbols at \mathbf{z} are the same as the symbols at $\mathbf{z} \pm \mathbf{e}_1$. The only superimposition rule is that a symbol above a \blacksquare on A' must be the same as on U_{up} .
 - On U_{diag} , the symbols at \mathbf{z} are the same as the symbols at $\mathbf{z} \pm (\mathbf{e}_1 + \mathbf{e}_2)$. The only superimposition rule is that a symbol above a \blacksquare on A' must be the same as on U_{diag} .

The construction is now finished. Now take a strongly periodic point of Y_d with strong period p . By construction, it necessarily is constituted of identical adjacent hypercubes of side p . The boundaries of the hypercubes being marked by symbols

(, ..., ) and (, ..., ) and the corners by (, ..., ). It is straightforward to see that any $p \geq 2$ is a strong period of the constructed subshift. \square

Lemma 4.3. *Let $L \subseteq \mathbb{N}$ be a language such that $\text{un}(L) \in \text{NTIME}(n^d)$, there exists a $2d$ -dimensional SFT X such that $L = \mathfrak{P}_X$ and such that all periodic points are strongly periodic.*

Proof. Let M be a Turing machine recognizing L in nondeterministic time n^d . We need to construct an SFT X_M whose strong periods are exactly the accepted inputs of M . We Using lemma 4.2, all that is left to prove is how to restrict the periods to the integers accepted by M . In order to do this, we will encode computations of M inside the $2d$ -cubes of Y_{2d} : on a unary input, M takes at most n^d time steps to accept or reject, so a $2d$ -cube of side n has exactly the right amount of space to encode such a computation. The idea is to *fold* the space-time diagram of the Turing machine so that it fits into the cube while still preserving the local constraints. Such a folding has already been described by Borchert [Bor08] and can also be deduced from Jones/Selman [JS74]. We then have to make sure that the nondeterministic transitions are identical in all $2d$ -cubes of a point. Let us now describe this in more details.

In a space-time diagram of M with input n , tape cells have coordinates (t, s) with $t \leq n^d, s \leq n^d$, where t is the time step and s the position in space. We now have to transform each cell (t, s) into a cell of the $2d$ -cube of size n , so that two consecutive (in time or space) cells of the space-time diagram remain adjacent cells of the cube. So we transform s and t into elements of $\llbracket 0, n-1 \rrbracket^d$ with a reflected n -ary code (also called reflected Gray-codes), see [Flo56; Knu05], this corresponds exactly to folding the time/space.

The vector $(t_0, \dots, t_{d-1}) \in \llbracket 0, n-1 \rrbracket^d$ will represent the integer $t = \sum a_i n^i$ where

$$a_i = \begin{cases} t_i & \text{when } \sum_{j>i} t_j \text{ is even} \\ (n-1) - t_i & \text{otherwise} \end{cases}$$

see [Knu05, Formula (51)]. The next position is given by the parity of the sum of the *stronger weighed* digits. In order to transform this in local constraints, it will suffice to encode parities of positions in the cube with some layers P^t and P^s for time and space respectively.

Layer P^t is made of several sublayers $P_i = \{0, 1\}$, one for each direction \mathbf{e}_i , $2 \leq i \leq d$. We now give the rules, recall that the boundaries of the cube are marked. Without loss of generality, we may suppose that there is a corner in position $\mathbf{0}$. This corner has 0 on all layers P_i . The rules are the following : if there the symbol p at position $\mathbf{z} \in \llbracket 0, n-1 \rrbracket^d$ on sublayer P_i , then there must be $p+1 \bmod 2$ at position $\mathbf{z} + \mathbf{e}_i$ and p at positions $\mathbf{z} + \mathbf{e}_j$, with $j \neq i$. These rules do not apply when the next position is at the boundary of the $2d$ -cube. The layer P^s is similar, except it is on dimensions $d+1$ to $2d$. An example for a three dimensional folding can be seen on figure 10.

Now that we have encoded the Turing machines inside the $2d$ -cubes, their size can only be one of its accepted inputs. However, recall that the Turing machines encoded are nondeterministic, therefore we have to synchronize the transitions between the different hypercubes, otherwise the periods may be multiples of accepted inputs. In order to do that, we add a new component N , which is constituted of the following sublayers, whose alphabets are each a copy of the possible transitions of M :

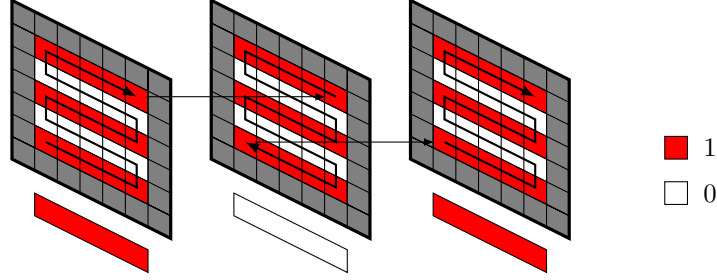


FIGURE 10. Folding of a three dimensional cube, the red on the parity layer stands for 0 and the white for 1. The direction where to look for the next cell is given by the sum of the parities.

- The first sublayer, \sqcap , will propagate the transition of a time-step to all cells of the same time-step, that is to say the rest of the tape. A cell where a transition happens imposes the symbol on \sqcap to be the transition happening. The symbol propagates along space: if there is a symbol l on \sqcap at position $\mathbf{z} \in \mathbb{Z}^d$, then there must be exactly the same symbol at position $\mathbf{z} \pm \mathbf{e}_i$, for $d+1 \leq i \leq 2d$.
- We also have a set of sublayers \sqcup_i , one for each time dimension, $1 \leq i \leq d$. Component \sqcup_i has the following rules : the symbol on \sqcup_i at position \mathbf{z} is identical to the one at position $\mathbf{z} + (\mathbf{e}_i + \mathbf{e}_{d+1})$. When the cell is on a border of the $2d$ -cube on dimension 1, the symbols on \sqcup_i and \sqcap have to be identical. For the construction of lemma 4.2, this means that S_i contains a $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$ or a $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$. Figure 11 shows how this synchronisation is done.

As in lemma 3.2, $n+4$ is a (strong) period if and only if n is accepted by M . Again, to obtain exactly n it suffices to fatten the symbols on the borders. \square

5. COUNTING THE NUMBER OF PERIODIC POINTS IN SFTs

In theorem 1.1 we have seen that the sets of strong periods of SFTs are exactly the sets of integers recognized nondeterministic polynomial time: we can go one step further and give a characterization of the sequences $p_n(X)_{n \in \mathbb{N}^*}$ where $p_n(X)$ corresponds to the number of points in X with period n .

In the previous construction this number was related to the number of accepting paths of the Turing machine : the number of possible aperiodic backgrounds possible for each square of A' makes it hard to characterize. The following theorem is a consequence of forcing the aperiodic background of squares of the same size to be unique:

Theorem 5.1. *For any SFT X , let $\mathfrak{N}_X : a^* \rightarrow \mathbb{N}$ be the function defined by $\mathfrak{N}_X(a^n) = p_n(X)/n^d$ where d is the dimension of X . We have then the following:*

$$\{\mathfrak{N}_X \mid X \text{ an SFT}\} = \#P$$

Note that the function $p_n(X)$ has been normalized by n^d , this is due to the fact that there are exactly n^d shifted versions of a same strongly periodic point of period n .

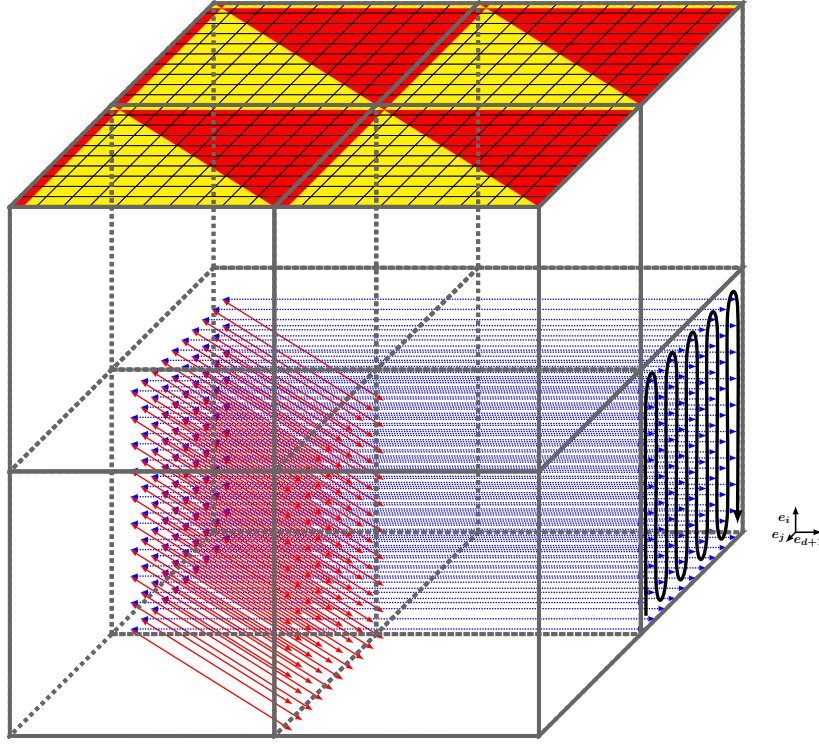


FIGURE 11. The Synchronization of nondeterministic transitions between the $2d$ -cubes : here is the projection in 3 dimensions $\mathbf{e}_{d+1}, \mathbf{e}_i, \mathbf{e}_j$, with $1 \leq i, j \leq d$. \sqcap is represented in blue and \sqcup_j is represented in red. On the top, we represented component S_j of the construction of lemma 4.2. Layers \sqcap et \sqcup_j synchronise together when we are on the side of a square on S_j .

Proof. To prove that $\#P \subseteq \{\mathfrak{N}_X \mid X \text{ an SFT}\}$ we have to fix the aperiodic background for the squares of our previous construction: the number of strongly periodic points of period n with a corner at $\mathbf{0}$ will then be exactly the number of accepting paths of the Turing machine M ran on n .

In order to do that, instead of taking any NW-deterministic aperiodic SFT W , we will take Kari's SFT [Kar92] and show that it is easy to fix the top and leftmost borders of the squares: this will determine the rest of the square.

Kari's set of tiles is exactly the same as Robinson's [Rob71], see figure 12, except that it has one supplementary layer with diagonal arrows, see figure 13. A valid tiling with this tileset can be seen on figure 14, the top and left borders determine the whole square. These borders are almost trivial and can be extended to any length, still forcing an admissible pattern.

Conversely $\{\mathfrak{N}_X \mid X \text{ an SFT}\} \subseteq \#P$: checking whether $n \in \mathfrak{P}_X$ is NP_1 . The number of accepting paths of the Turing machine of the proof of lemma 4.1 is exactly the number of periodic points of period n . In order to normalize by n^d , we also force this machine to only keep the guessed "fillings" of the n^d cube that are the smallest among their translates for the lexicographic order. \square

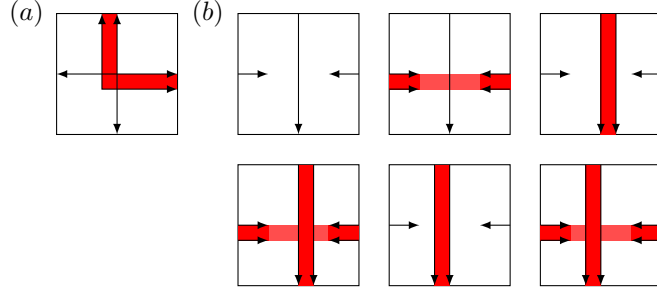


FIGURE 12. Robinson's aperiodic tileset : a) a cross and b) arms. The tileset also includes the rotates of these tiles. If the main arrow of an arm is horizontal (resp. vertical) we will call it a horizontal (resp. vertical) arm. Two tiles can be neighbors if and only if outgoing arrows match incoming ones.

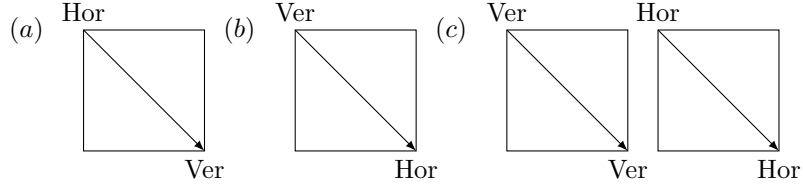


FIGURE 13. Kari's addition to make it NW-deterministic : a new layer with diagonal arrows that have to match at their extremities. On horizontal arms only, we superimpose the tile (a) and on vertical ones only, the tile (b). The tiles (c) are superimposed on crosses only.

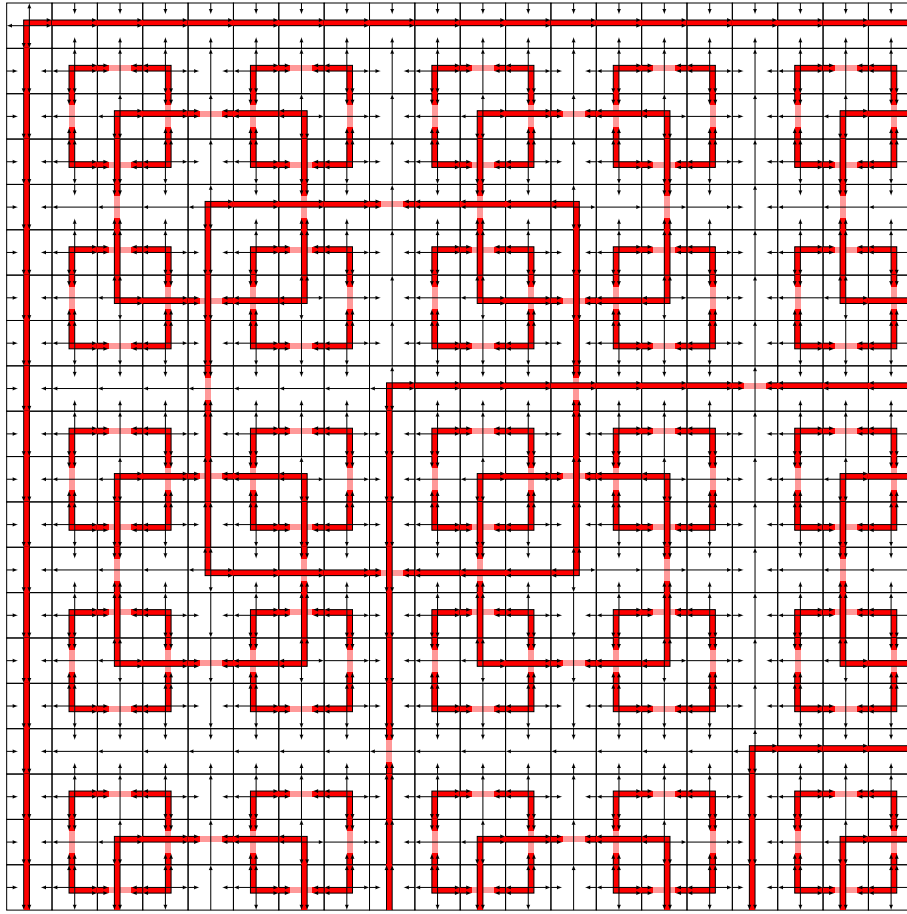


FIGURE 14. A valid tiling by Kari's NW-deterministic tileset. The top and left borders determine the whole square. The diagonal arrows are not represented but can be easily deduced.

6. 1-PERIODIC POINTS IN SFTs OF DIMENSION 2

We now go back to bidimensional SFTs, and focus on 1-periodicity. Recall that a point is 1-periodic when it only has colinear vectors of periodicity. We prove theorem 1.2: the sets of 1-periods of SFTs are exactly the sets of vectors of $\mathbb{N} \times \mathbb{Z} \setminus \{0\}$ that are in $\mathbf{NSPACE}_1(n)$.

Lemma 6.1. *Let X be an SFT of dimension 2, then $un(\mathfrak{P}_X^1) \in \mathbf{NSPACE}(n)$.*

Proof. We have to construct a Turing machine M which on input $\mathbf{v} = (m, n) \in \mathbb{N} \times \mathbb{Z} \setminus \{0\}$ decides in space $|\mathbf{v}|$ if \mathbf{v} is a 1-period. We have seen in subsection 2.1.2 how to construct the graph $G_{\mathbf{v}}(X)$ and that \mathbf{v} is a 1-period iff this graph contains two mutually accessible cycles. This graph does not fit, however, in space $|\mathbf{v}|$. The algorithm that we will use is similar to the one introduced in lemma 3.1, it will just need to check the existence of two different completions that can be “glued” together. We suppose without loss of generality that $m \geq |n|$ and that r is the radius of the SFT X .

- Initialize an array P of size $\max(m, n)$ such that $P[i] = 1$ for all i 's and a boolean D to false.
- Nondeterministically sizes $t_1, t_2 \leq 3m|\Sigma|^{4rm}$, the sizes of the two cycles.
- Parallely choose $2 \times 2r$ horizontal lines $(l_i)_{0 \leq i \leq r-1}$ and $(l'_i)_{0 \leq i \leq r-1}$ of length m , each sequence of lines forms a rectangle of $2r \times m$ symbols of Σ .
- We now do these steps in parallel:
 - For all $2r < i \leq t_1$, nondeterministically choose a line l_i of length m and check that there is no forbidden pattern in the lines l_i, \dots, l_{i-2r-n} . For this step, it suffices to keep the $2r$ last lines l_i, \dots, l_{i-2r} and $2r$ symbols on each of the preceeding n lines.
 - We also nondeterministically choose lines l'_j for $2r < j < t_2$ at the same time and check if they form an invalid pattern. It is important to choose line l_i at the same time as line l'_j until $\min(t_1, t_2)$ is reached: whenever l_i is different from l'_i , assign true to D .
 - At each of these steps, check if there is a periodicity vector (m', n') such that $(m', n') \cdot k = (m, n)$, if it is not the case then $P[k] \leftarrow 0$.
 - Once the last lines l_{t_1} and l'_{t_2} have been guessed, check that there is no forbidden pattern on the patterns formed by the symbols remembered on lines:

$$\begin{aligned}
 & * l_{t_1-2r-n}, \dots, l_{t_1}, l_0, \dots, l_{2r} \\
 & * l'_{t_2-2r-n}, \dots, l'_{t_2}, l'_0, \dots, l'_{2r} \\
 & * l_{t_1-2r-n}, \dots, l_{t_1}, l'_0, \dots, l'_{2r} \\
 & * l'_{t_2-2r-n}, \dots, l'_{t_2}, l_0, \dots, l_{2r}
 \end{aligned}$$

This checks that the two cycles found are mutually accessible.

- If any of the steps before failed or if there exists $k, m', n' < m$ such that $k \cdot (m', n') = (m, n)$ and $P[k] = 1$, or if D is false then reject. Accept otherwise.

This algorithm only needs to keep $4rm + 2rn$ symbols of Σ , as well as P and D in memory. \square

Lemma 6.2. *For any constant $k \in \mathbb{N}^*$, there exists a 2-dimensional SFT Y_k such that any one periodic point of period (m, n) is formed of $m \times k^{m-1}$ rectangles with*

marked borders, as in figure 15. Furthermore, Y_k admits as 1-periods any (m, n) such that $0 < n < m$.

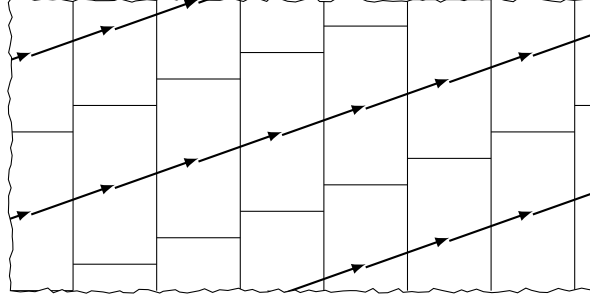


FIGURE 15. The marked rectangles of the 1-periodic configurations.

Proof. As in the preceeding proofs, the construction will be done in successive steps, by superimposition of several layers A, C_c, R, S :

- Again, the first component A is based on an aperiodic East-deterministic SFT W . The alphabet of A is $\Sigma_A = (\Sigma_W \times \{\cdot, \blacksquare\}) \cup \{\blacksquare, \blacksquare, \blacksquare, \blacksquare\}$, we call the symbols of Σ_W the *white* symbols. Again, the other symbols allow to break periodicity, the rules are the following:
 - The rules between the symbols of Σ_W remain unchanged.
 - White symbols may have a \blacksquare or another white symbol above, but two \blacksquare may not be above/below each other.
 - The constraints on whites “are transmitted over the \blacksquare symbols: removing a line of \blacksquare and gluing the white symbols above and below must not produce any forbidden pattern.
 - The rules between the symbols $\{\blacksquare, \blacksquare, \blacksquare, \blacksquare\}$ are Wang rules.
 - Only the white sides of symbols $\{\blacksquare, \blacksquare, \blacksquare, \blacksquare\}$ may touch a white symbol.

At this stage, the configurations with a periodicity vector necessarily have an infinity of vertical lines. Vertical lines may eventually be met by extremities of finite horizontal lines, see figure 16. An infinity of horizontal lines leads to an aperiodic configuration.

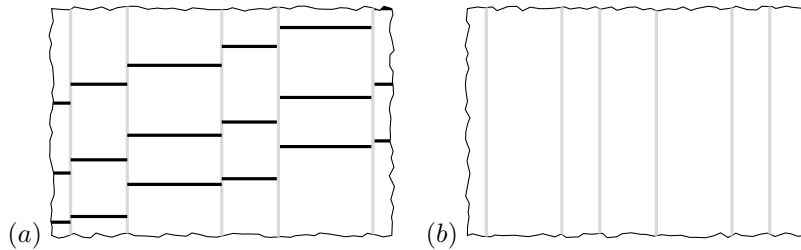


FIGURE 16. There are several possibilities for 1-periodic points of A : an infinity of vertical lines (b), or an infinity of vertical lines separated by finite horizontal lines (a).

- Component C_k is a k -ary counter, exactly as in lemma 3.3. Now that this counter has been added, the points having a periodicity vector necessarily contain an infinity of vertical lines, necessarily joined by horizontal lines at distance k^{n-1} when they are distant by n .
- In points with a periodicity vector, component R forces columns formed by the nearest vertical lines to all be of the same width, and the offset between horizontal lines of two neighboring columns to always be the same. The first is done by first projecting each horizontal line to the left and to the right until it reaches the next vertical line. Between projections on the same side, we again put a counter C_k , this forces the sizes of the two columns to be identical.

To make the offset between horizontal lines constant for all columns, we project two signals of slope 1 on both the left and the right of the horizontal lines. These signals propagate normally on white symbols and cannot touch a vertical line directly: they have to first pass on a horizontal line. As they do this, their direction changes and their slope becomes -1 . They must then touch the next vertical line at the exact same time as the projection from two columns to the left/right, at which time the signal stops, see figure 17.

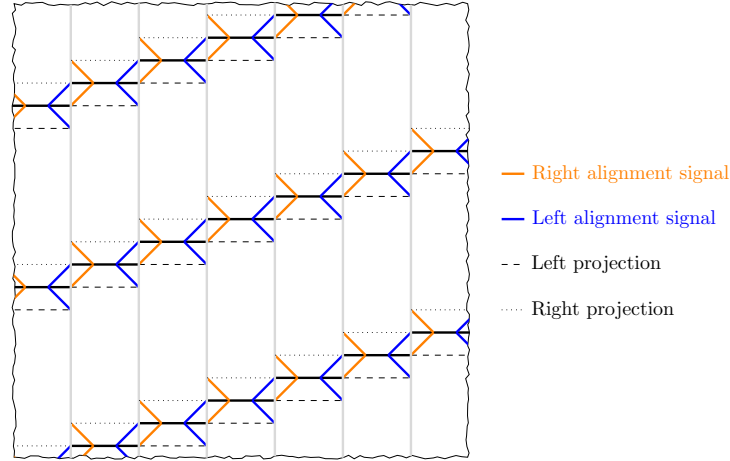


FIGURE 17. How component R operates. Each horizontal line sends a projection to the next vertical line on each side. On each side, between projections, there is a counter C_k . Another signal is sent diagonally, with slope 1, it can only touch a horizontal line, at which point it changes its direction and needs to rereach the same vertical line at the same time as the projection from two columns to the left/right.

At this stage, the points with a periodicity vector are necessarily constituted of rectangles of identical size, “translated” by a vector (m', n') from one column to another. (m', n') is not necessarily a periodicity vector, however there must exist k such that $k(m', n')$ is.

- We now arrange for the width m of the rectangles and their offset n to form the periodicity vector: we set the aperiodic background in the columns to be

identical. In order to do that, since W is east-deterministic, we synchronize the first vertical line of white symbols of a column with the first vertical line of white symbols of the next column, shifting it by the offset. This is what the last component S does. S is constituted of two sublayers W' whose alphabet is the same as W 's and a second one S' which allows to know which neighboring symbols of W' have to be equal, figure 18 explains S' in sufficient detail to infer the rules. As before, the symbols on W' and A to the right of a vertical line need to be identical on both layers.



FIGURE 18. The symbols \blacksquare are superimposed only to the left alignment signals, and on the bottom line, \cdot . The rules for W' are determined by the arrows : a symbol on W' must be equal the symbol on W' to which the arrow points on S' .

- The last step now forces the existence of 1-periodic points, all points with a vector of periodicity at the end of the previous step were also periodic. To do this, it suffices to add two colors, yellow and blue, to the rectangles formed by the vertical lines and horizontal lines. The color of a rectangle is transmitted to its (m, n) translate, where m is the width of the rectangle and n the offset between columns.

Let us now check that the 1-periodic points are necessarily of the expected shape. Let x be a 1-periodic point of 1-period (m, n) with $0 < n < m$. Then x necessarily has an infinity of vertical lines due to A . Thanks to C_k , two nearest vertical lines at distance m' are joined by horizontal lines at distance $k^{m'-1}$. R forces all nearest lines to always be spaced by the same distance m' . It also forces the offset between horizontal lines of any two neighboring columns to be some n' . Furthermore, S forces the aperiodic background of each column to be identical, up to the offset n' . So x is periodic along (m', n') , which is the smallest vector of periodicity, and since x is 1-periodic $(m', n') = (m, n)$.

Conversely, given (m, n) it is easy to exhibit a point with 1-period (m, n) : a well formed configuration of rectangles of size $m \times k^{m-1}$ where only one of the alignments of rectangles of direction (m, n) is blue and all others are yellow. \square

Lemma 6.3. *Let $L \in \mathbb{N} \times \mathbb{Z}$ be a language such that $un(L) \in \text{NSPACE}(n)$, then there exists an SFT X such that $L = \mathfrak{P}_X^1$.*

Proof. Let M be a Turing machine recognizing L in time c^n for some constant c , n being the size of the input. With lemma 6.2, it suffices to encode M in the rectangles

of Y_c while taking care of synchronizing the nondeterministic transitions between the shifted rectangles, as usual: this allows to realize the subset of L of vectors (m, n) with $0 < n, m$. Almost identical constructions allow to do the remaining cases: $0 < m < n$, $m = 0, n = 0$, $n = m$ and $n < 0 < m$ (there are then two subcases $|n| < |m|$ and $|m| < |n|$). The disjoint union of all these SFTs form the end SFT realizing L as its 1-periods. \square

There are two ways to generalize theorem 1.2 to higher dimensions, since in dimension 2 1-periodicity is both “ $(d-1)$ -periodicity” and 1-periodicity. To obtain a similar characterization, one would have to consider $(d-1)$ -periodicity in higher dimensions because it would again come to the problem of whether some graph has two mutually accessible cycles. 1-periodicity on the other hand would this time not admit a characterization in terms of some complexity class but rather some computability class, as it would be undecidable to decide whether some vector is a 1-period or even a periodicity vector.

7. PERIODICITY IN SOFIC AND EFFECTIVE SUBSHIFTS

For periodic and sofic subshifts it is not decidable anymore whether some pattern is admissible, this means in particular that it is not decidable anymore whether some n is a period/strong period.

The first thing that we may see however is the following lemma:

Lemma 7.1. *Given an effective/sofic subshift X of dimension d and d vectors $\mathbf{v}_1, \dots, \mathbf{v}_d$ it is co-recursively enumerable to decide whether there exists $x \in X$ such that $\Gamma_x = \mathbf{v}_1\mathbb{Z} \oplus \dots \oplus \mathbf{v}_d\mathbb{Z}$.*

Proof. It suffices to guess all fillings of the volume defined by the vectors $\mathbf{v}_1, \dots, \mathbf{v}_d$, we need then to check if the generated configurations contain some forbidden pattern. For this, we enumerate the forbidden patterns and check whether they appear in every guessed filling (eventually repeated). The machine stops iff all fillings generate a point containing a forbidden pattern. \square

The realization counterpart of this lemma for effective subshifts is quite straightforward:

Lemma 7.2. *Let L be a co-recursively enumerable set of rank d sublattices of \mathbb{Z}^d , then there exists an effective subshift X such that $L = \{\Gamma_x \mid x \in X\}$.*

Proof. It is easy to construct such an SFT with the same methods as before: taking an aperiodic SFT and breaking it with new symbols. The forbidden patterns are all volumes generated by the d -uples of vectors of L , they also make sure that all such volumes have the same aperiodic background. \square

For sofic subshifts we did not manage to obtain a construction allowing us to realize all co-r.e. sets of rank d lattices. But we managed to realize certain types of lattices, the “rectangular” lattices, that is to say the lattices of the form $\Gamma = n_1\mathbb{Z} \times \dots \times n_d\mathbb{Z}$ with $n_1, \dots, n_d \in \mathbb{N}^*$. We will note \mathfrak{P}_X^r the sets of d -uples such that there exists $x \in X$ with $\Gamma_x = n_1\mathbb{Z} \times \dots \times n_d\mathbb{Z}$.

Lemma 7.3. *Let $L \subseteq \mathbb{N}^{*d}$ be a co-recursively enumerable set, there exists a d -dimensional sofic subshift X such that $\mathfrak{P}_X^r = L$.*

In order to prove the lemma, we will need the following result:

Theorem 7.1 (Aubrun and Sablik [AS10]; Durand, Romashchenko, and Shen [DRS10]). *Let X be a d dimensional subshift, and X' be the $d+1$ dimension subshift obtained by adding a dimension to X and keeping symbols identical on it. X is an effective subshift if and only if X' is sofic.*

This result will be used to generate some sofic subshift by only giving the lower dimensional effective subshift corresponding to it.

Proof of lemma 7.3. We give the proof for dimension 2, one can easily generalize it for higher dimensions. We have a co-r.e. language $L \subseteq \mathbb{N}^{*2}$ and we want to construct a sofic subshift X such that $\mathfrak{P}_X^r = L$. We now detail the construction realizing the subset $\{(n_1, n_2) \mid (n_1, n_2) \in L \text{ and } n_1 > n_2\}$ of L . The construction will once again use an aperiodic base that will be broken by new symbols, except this time the shape will be determined by layers that we put on top and that will not be projected to obtain the sofic subshift.

We describe now the SFT that will be projected:

- The first component A consists of an aperiodic NW-deterministic SFT W , the white symbols, to which we add three symbols \blacksquare , \blacklozenge and \blacksquare .

The rules are almost the same as for A' of component A of lemma 4.2: above a \blacksquare or a \blacklozenge there may only be a \blacksquare or a \blacklozenge . On the left of a \blacksquare or a \blacklozenge , there may only be a \blacklozenge or a \blacksquare . For this layer to be periodic, a point needs to have an infinity of lines of \blacksquare , \blacksquare or \blacklozenge . If there is an infinity of lines of \blacksquare and \blacklozenge , then they cross with the \blacklozenge symbols and do not necessarily form rectangles of the same size.

- The second component P will give the information on the dimensions that the rectangles can have and force their width to be identical. To do this, we use theorem 7.1 in order to obtain identical lines composed of concatenations of words $b^m a^{n-m}$, with $n > m$ and $(n, m) \in L$:

$$\dots b^m a^{n-m} b^m a^{n-m} b^m a^{n-m} \dots$$

These lines can be generated by an effective subshift of dimension 1, which by compactness will contain the uniform lines $\dots aaa \dots$ and $\dots bbb \dots$. So using theorem 7.1 we have an SFT that projects onto the shift with identical lines, we may assume that it has a component equal to these lines.

We now give the superimposition rules of the symbols: a b appearing after an a needs to be superimposed to a \blacksquare or a \blacklozenge , and conversely.

At this stage, component A can only be periodic in two cases: either component P is uniform and contains only a 's or b 's and it has an infinity of horizontal lines of \blacksquare , or it contains an infinity of vertical lines of \blacksquare distant by n , with n such that there is some $m < n$ with $(n, m) \in L$.

- Component R forces the vertical lines of \blacksquare to appear whenever there is an infinity of vertical lines of \blacksquare . R will be formed of the following Wang tiles: $\{\begin{smallmatrix} \text{red} & \text{blue} & \text{blue} & \text{red} & \text{blue} & \text{red} & \text{blue} & \text{red} & \text{blue} & \text{red} \\ \text{blue} & \text{red} & \text{blue} & \text{red} & \text{blue} & \text{red} & \text{blue} & \text{red} & \text{blue} & \text{red} \end{smallmatrix}\}$. The superimposition rules are the following:
 - $\begin{smallmatrix} \text{red} & \text{blue} \\ \text{blue} & \text{red} \end{smallmatrix}$ can only be superimposed to \blacksquare .
 - $\begin{smallmatrix} \text{red} & \text{blue} \\ \text{red} & \text{blue} \end{smallmatrix}$ can only be superimposed to \blacklozenge .
 - $\begin{smallmatrix} \text{red} & \text{blue} \\ \text{red} & \text{red} \end{smallmatrix}$ and $\begin{smallmatrix} \text{blue} & \text{red} \\ \text{blue} & \text{blue} \end{smallmatrix}$ can only be superimposed to an a on the right of a b .
 - $\begin{smallmatrix} \text{red} & \text{blue} \\ \text{red} & \text{red} \end{smallmatrix}$ can only be superimposed to \blacksquare .
 - $\begin{smallmatrix} \text{red} & \text{red} \\ \text{blue} & \text{red} \end{smallmatrix}$ can only be superimposed to b .
 - $\begin{smallmatrix} \text{blue} & \text{red} \\ \text{blue} & \text{blue} \end{smallmatrix}$ can only be superimposed to a .

b b b a a b b b a a b b b a a b b b a a
b b b a a b b b a a b b b a a b b b a a
b b b a a b b b a a b b b a a b b b a a
b b b a a b b b a a b b b a a b b b a a
b b b a a b b b a a b b b a a b b b a a
b b b a a b b b a a b b b a a b b b a a
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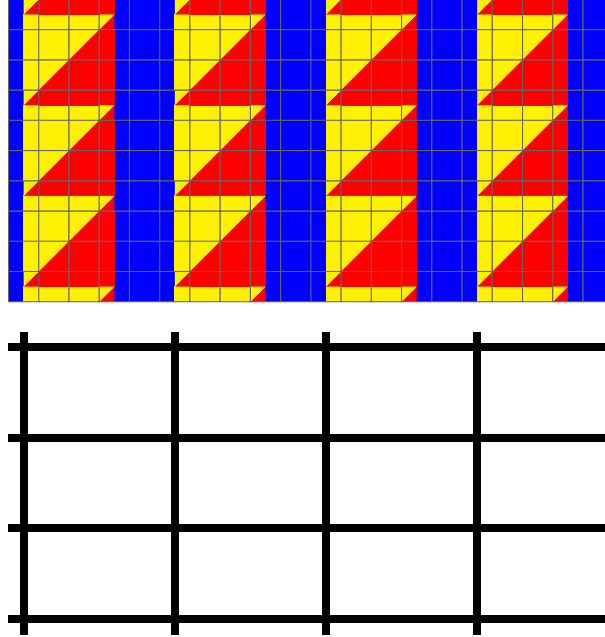


FIGURE 19. How components A , P and R are superimposed: component P is on the top, component R is below and on the bottom sits component A . The vertical lines of \mathbf{I} are superimposed to the first symbols b appearing after an a . Component R allows to adjust the height of the rectangles according to the number of b 's.

Figure 19 shows how component R is superimposed to the first components and shapes the rectangles.

The possible cases now when component A is periodic are the following: there are rectangles, all of the same size $n \times m$, with $n > m$ and $(n, m) \in L$, or there is an infinity of horizontal lines of \blacksquare . In the case when it is formed of rectangles, the vectors $(n, 0)$ and $(0, m)$ do not necessarily generate Γ_x , as the aperiodic background of the rectangles are not necessarily the same.

- Component V forces the first column of white symbols of the rectangles on component A to be identical. The one-dimensional subshift on the alphabet

$\Sigma_W \cup \{\#\}$ whose points are of the form:

$$\cdots \#w_1 \dots w_k \#w_1 \dots w_k \#w_1 \dots w_k \# \cdots$$

that is to say the periodic points formed by the repetition of some word on Σ_W where each occurrence is separated by a $\#$. Again, using theorem 7.1 we may use the subshift formed of these lines vertically. Now, $\#$ can only be superimposed to a \blacktriangle or a \blacksquare and no other symbol can be superimposed to them. We also force the symbols of W exactly on the right of a \blacksquare to be identical to the ones on V .

- H is the last component, and is similar to V . Consider the effective subshift formed by the following points:

$$\cdots \#(w_1, t_1) \dots (w_k, t_k) \#(w_1, t_1) \dots (w_k, t_k) \#(w_1, t_1) \dots (w_k, t_k) \# \cdots$$

Where the w_i 's are symbols of Σ_W and the t_i 's are symbols of some aperiodic effective subshift M of dimension 1: for example the subshift where we forbid the words $awawa$ with a a letter and w a word, which is aperiodic and non-empty, see Thue [Thu12]; Morse [Mor21]. We keep the rules of M for the words $t_1 \dots t_k$ that are not cut by some $\#$. So if no $\#$ appears in a point, then the point cannot be periodic. We can now use theorem 7.1 and use the subshift formed of these lines horizontally.

We force the $\#$ symbols to be superimposed to \blacktriangle 's or \blacksquare 's only and the symbols of component A appearing on just below a \blacksquare to be identical as the ones on H .

At this point, if we look the possible periodic points for the restriction to component A , we have the following:

- The points x formed of $n \times m$ rectangles with $n > m$ and $(n, m) \in L$, they are periodic and $\Gamma_x = n\mathbb{Z} \times m\mathbb{Z}$
- The points formed by an infinity of horizontal lines of \blacksquare , without any vertical line of \blacksquare .

The end subshift is obtained by projecting component A and H only: projecting H allows to get rid of the second class of points, as they not have any periodicity vector of the form $(0, m)$, because of the aperiodic one-dimensional subshift. \square

So we have the following characterizations for sofic and effective subshifts:

Theorem 7.2. *The sets of rank d periodicity lattices of effective subshifts of dimension d are exactly the co-r.e. sets of sublattices of \mathbb{Z}^d .*

Theorem 7.3. *For any $L \subseteq \mathbb{N}^{*d}$, there exists a d -dimensional sofic subshift X such that $\mathfrak{P}_X^r = L$ if and only if L is co-r.e.*

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