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ERROR ESTIMATE FOR TIME-EXPLICIT FINITE VOLUME APPROXIMATION OF STRONG SOLUTIONS TO SYSTEMS OF CONSERVATION LAWS

CLÉMENT CANCÈS, HÉLÈNE MATHIS, AND NICOLAS SEGUIN

ABSTRACT. We study the finite volume approximation of strong solutions to nonlinear systems of conservation laws. We focus on time-explicit schemes on unstructured meshes, with entropy satisfying numerical fluxes. The numerical entropy dissipation is quantified at each interface of the mesh, which enables to prove a weak- BV estimate for the numerical approximation under a strengthened CFL condition. Then we derive error estimates in the multidimensional case, using the relative entropy between the strong solution and its finite volume approximation. The error terms are carefully studied, leading to a classical $\mathcal{O}(h^{1/4})$ estimate in L^2 under this strengthened CFL condition.

Keywords. Hyperbolic systems, finite volume scheme, relative entropy, error estimate

AMS subjects classification. 35L65, 65M08, 65M12, 65M15

1. INTRODUCTION

The aim of this paper is to provide an *a priori* error estimate for time-explicit finite volume approximation on unstructured meshes of strong solutions to hyperbolic systems of conservation laws. Our proof relies on the control of perturbations coming from the discretization in the uniqueness proof proposed by R. J. DiPerna [20] and C. M. Dafermos [14] (see also [15]).

Numerous studies on error estimates for hyperbolic problems were published in the last decades. Let us first highlight some optimal convergence rates that are established in the literature. Classical first-order finite difference methods on cartesian grids for the approximation of smooth solutions of linear equations can be directly studied by estimating the truncation error, leading to an $\mathcal{O}(h)$ error estimate, where the length h is the characteristic size of the grid. Adapting S. N. Kruzhkov's doubling variable technique [35], N. N. Kuznetsov proved in [36] that finite difference schemes for nonlinear one-dimensional conservation laws converge towards the entropy weak solution with the optimal rate $\mathcal{O}(h^{1/2})$ in the space-time L^1 norm. The optimal rate $\mathcal{O}(h^{1/2})$ has been recovered by B. Merlet and J. Vovelle [41] and by F. Delarue and F. Lagoutière [18] for weak solutions to the linear transport equation approximated by the upwind finite volume scheme on unstructured grids. The rate $\mathcal{O}(h^{1/2})$ appears to stay optimal when strong solutions to linear transport equations are approximated on two-dimensional unstructured grids as shown by C. Johnson and J. Piktäranta in [30]. Since linear transport enters

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our framework, we cannot expect a better *a priori* error estimate than $\mathcal{O}(h^{1/2})$. However, if one restricts to dimensional one, an estimate in $\mathcal{O}(h)$ can be derived even for nonlinear systems of conservation laws, see D. Bouche *et al.* [4].

Many studies exist when considering entropy *weak* solutions, based on nonlinear techniques which extend in some sense N. N. Kuznetsov's article [36]. These works focus on the multidimensional case with unstructured meshes, for scalar conservation laws [10, 11, 49, 22, 7]. They mainly use the notion of error measures, see for instance [6], and lead to an error estimate in $\mathcal{O}(h^{1/4})$ (recall that the convergence has been initially addressed by A. Szepessy [44]). The key-point in these multidimensional studies is the control of the *BV* semi-norm. For unstructured meshes, one can only prove that it grows as $h^{-1/2}$ (even in the linear scalar case, cf. [19]), which actually is the main barrier to obtain a better rate of convergence. Similar tools allowed V. Jovanovic and C. Rohde to propose error estimates [31] for the finite volume approximation of the solution to Friedrich's systems (i.e., linear symmetric hyperbolic systems)

As mentioned above, we are interested in multidimensional systems of conservation laws. The solutions to such systems may develop discontinuities in finite time and, since the pioneering work of P. D. Lax [37], entropy conditions are added to select physical/admissible solutions. Recently, it has been shown by C. De Lellis and L. Székelyhidi Jr. in [16, 17] that such a criterion is not sufficient in the multidimensional case. Nonetheless, it is known since several decades (see in particular [20, 14]) that if a strong solution exists, then there exists a unique entropy weak solution corresponding to the same initial data, and that it coincides with this strong solution. Moreover, it can be shown that entropy weak solutions are stable with respect to strong solutions. Since error estimates of any approximation are based on the stability properties of the model, we restrict this study to strong solutions which are known to exist, in finite time, and to be unique [33, 39, 15]. As in [32], we use the notion of relative entropy to compare the approximate solution with a smooth solution. The mathematical techniques are basically the same as in the scalar case (we follow in particular [22, 7]): weak-*BV* estimates and error measures. The main result of this paper is an *a priori* error estimate of order $\mathcal{O}(h^{1/4})$ in the space-time L^2 norm for first-order time-explicit finite volume schemes under classical assumptions on the numerical fluxes [5, 46]. One key ingredient is an extension to the system case of the so-called weak-*BV* estimate introduced in the scalar case in [22, 7]. As in the scalar, it relies on a quantification of the numerical dissipation and requires a slightly reduced CFL condition. We finally obtain an error estimate in $\mathcal{O}(h^{1/4})$, and simplify the framework of a study of V. Jovanovic and C. Rohde [32] where time-implicit methods are considered and the weak-*BV* estimate is assumed).

Concerning higher order methods, let us mention the result [8] of C. Chainais-Hillairet who proved an error estimate of order $\mathcal{O}(h^{1/4})$ for the time-explicit second order finite volume discretization with flux limiters [48] of nonlinear scalar conservation laws. The strategy exploited in [8] consists in showing that the solution to the second order remains close to the solution to the monotone scheme without limiters. Such a strategy might be adapted in our framework but provides very under-optimal estimates. High order time-implicit discontinuous Galerkin methods have also been analyzed in details by Hildebrand and Mishra in [29]. Using appropriate weak-*BV* estimates, they prove convergence towards entropy measured-valued

solutions of multidimensional systems of conservation laws. No error estimate has been derived yet up to our knowledge.

Remark 1.1. *As it is well known, solutions to hyperbolic nonlinear systems of conservation laws may develop discontinuities after a finite time. However, the occurrence of such discontinuities is prohibited when appropriate relaxation terms are added to the systems [27, 51] (we then have hyperbolic balance laws instead of hyperbolic conservation laws). By adapting the analysis carried out by V. Jovanovic and C. Rohde in [32], such terms can be considered in the analysis. A time explicit treatment of the source terms would lead to a reduced CFL. Therefore, in the case of stiff relaxation terms, an implicit treatment of the source terms is relevant. We refer to the work [9] of C. Chainais-Hillairet and S. Champier for an error estimate in the case of a scalar balance law.*

1.1. Hyperbolic systems of conservation laws.

1.1.1. *Strong, weak, and entropy weak solutions.* We consider a system of m conservation laws

$$(1) \quad \partial_t u(x, t) + \sum_{\alpha=1}^d \partial_\alpha f_\alpha(u)(x, t) = 0.$$

System (1) is set on the whole space $x \in \mathbb{R}^d$, and for any time $t \in [0, T]$, $T > 0$. We assume that there exists a convex bounded subset of \mathbb{R}^m , denoted by Ω and called *set of the admissible states* such that

$$(2) \quad u(x, t) \in \Omega, \quad \forall (x, t) \in \mathbb{R}^d \times [0, T].$$

System (1) is complemented with the initial condition

$$(3) \quad u(x, 0) = u_0(x) \in \Omega, \quad \forall x \in \mathbb{R}^d.$$

We assume for all $\alpha \in \{1, \dots, d\}$ the functions $f_\alpha : \mathbb{R}^m \rightarrow \mathbb{R}^m$ to belong to $C^2(\overline{\Omega}; \mathbb{R}^m)$, and be such that Df_α are diagonalizable with real eigenvalues, where D denotes the differential with respect to the variables u .

System (1) is endowed with a uniformly convex entropy $\eta \in C^2(\overline{\Omega}; \mathbb{R})$ such that there exists $\beta_1 \geq \beta_0 > 0$ so that

$$(4) \quad \text{spec}(D^2\eta(u)) \subset [\beta_0; \beta_1], \quad \forall u \in \overline{\Omega},$$

and the corresponding entropy flux $\xi \in C^2(\overline{\Omega}; \mathbb{R}^d)$ satisfies for all $\alpha \in \{1, \dots, d\}$

$$(5) \quad D\xi_\alpha(u) = D\eta(u)Df_\alpha(u), \quad \forall u \in \Omega.$$

Without loss of generality, we assume that $\eta(u) \geq 0$ for all $u \in \overline{\Omega}$. The existence of the entropy flux ξ amounts to assume the integrability condition (see e.g. [25])

$$(6) \quad D^2\eta(u)Df_\alpha(u) = Df_\alpha(u)^T D^2\eta(u), \quad \forall u \in \Omega.$$

Let us introduce the quantity L_f by

$$(7) \quad L_f = \sup_{\alpha \in \{1, \dots, d\}} \sup_{(u, v) \in \Omega^2} \sup_{w \in \mathbb{R}^m \setminus \{0\}} \left| \frac{w^T D^2\eta(v) Df_\alpha(u) w}{w^T D^2\eta(v) w} \right|.$$

Remark 1.2. *Notice that, in view of (6), the matrix $Df_\alpha(u)$ is self-adjoint for the scalar product $\langle w, v \rangle_u = w^T D^2\eta(u)v$. Therefore, the Rayleigh quotient*

$$(8) \quad \sup_{w \in \mathbb{R}^m \setminus \{0\}} \left| \frac{w^T D^2\eta(u) Df_\alpha(u) w}{w^T D^2\eta(u) w} \right| = \sup_{w \in \mathbb{R}^m \setminus \{0\}} \frac{\langle w, Df_\alpha(u) w \rangle_u}{\langle w, w \rangle_u}$$

provides exactly the largest eigenvalue in absolute value of $Df_\alpha(u)$. The situation in (7) is more intricate than in (8) since u might be different of v , but the quantity L_f is bounded in view of the boundedness of Ω and of the regularity of f_α and η .

Despite it is well-known that even for smooth initial data u_0 , the solutions of (1)–(3) may develop discontinuities after a finite time, our study is restricted to the approximation of smooth solutions $u \in W^{1,\infty}(\mathbb{R}^d \times \mathbb{R}_+; \Omega)$ to (1)–(3). Such solutions are called *strong solutions*, and they satisfy the conservation of the entropy

$$(9) \quad \partial_t \eta(u) + \sum_{\alpha=1}^d \partial_\alpha \xi_\alpha(u) = 0 \quad \text{in } \mathbb{R}^d \times \mathbb{R}_+.$$

We refer for instance to [33, 39, 15] for specific results on strong solutions of systems of conservation laws.

Assuming that $u_0 \in L^\infty(\mathbb{R}^d; \Omega)$, a function $u \in L^\infty(\mathbb{R}^d \times \mathbb{R}_+; \Omega)$ is said to be a *weak solution* to (1)–(3) if, for all $\phi \in C_c^1(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R}^n)$, one has

$$(10) \quad \iint_{\mathbb{R}^d \times \mathbb{R}_+} u \partial_t \phi \, dx dt + \int_{\mathbb{R}^d} u_0 \phi(\cdot, 0) \, dx + \iint_{\mathbb{R}^d \times \mathbb{R}_+} \sum_{\alpha=1}^d f_\alpha(u) \partial_\alpha \phi \, dx dt = 0.$$

Moreover, u is said to be an *entropy weak solution* to (1)–(3) if u is a weak solution, i.e., u satisfies (10), and if, for all $\psi \in C_c^1(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R}_+)$, it satisfies

$$(11) \quad \iint_{\mathbb{R}^d \times \mathbb{R}_+} \eta(u) \partial_t \psi \, dx dt + \int_{\mathbb{R}^d} \eta(u_0) \psi(\cdot, 0) \, dx + \iint_{\mathbb{R}^d \times \mathbb{R}_+} \sum_{\alpha=1}^d \xi_\alpha(u) \partial_\alpha \psi \, dx dt \geq 0.$$

1.1.2. Relative entropy. In [35], Kruzhkov is able to compare two entropy weak solutions using the doubling variable technique. In [36], such method has been extended in order to compare an entropy weak solution with an approximate solution. In the case of systems of conservation laws, these techniques no longer work. Basically, the family of entropy–entropy flux pairs (η, ξ) is not sufficiently rich to control the difference between two solutions. Nevertheless, let us assume that one of these solutions is a strong solution, u in the sequel, and introduce:

Definition 1.1 (Relative entropy). *Let $u, v \in \Omega$. The relative entropy of v w.r.t. u is defined by*

$$H(v, u) = \eta(v) - \eta(u) - D\eta(u)(v - u),$$

and the corresponding relative entropy fluxes $Q : \Omega \times \Omega \rightarrow \mathbb{R}^d$ are

$$Q_\alpha(v, u) = \xi_\alpha(v) - \xi_\alpha(u) - D\eta(u)(f_\alpha(v) - f_\alpha(u)), \quad \forall \alpha \in \{1, \dots, d\}.$$

The notion of relative entropy for systems of conservation laws goes back to the early works of DiPerna and Dafermos (see [20], [14] and the condensed presentation in [15]). It has also been extensively used for the study of hydrodynamic limits of kinetic equations (see the first works [50] and [1], but also [43] for more recent results). For systems of conservation laws, one can check that, given a strong solution u and an entropy weak solution v with respective initial data u_0 and v_0 , one has

$$(12) \quad \partial_t H(v, u) + \sum_{\alpha=1}^d \partial_\alpha Q_\alpha(v, u) \leq - \sum_{\alpha=1}^d (\partial_\alpha u)^T Z_\alpha(v, u)$$

in the weak sense, where

$$(13) \quad Z_\alpha(v, u) = D^2\eta(u)(f_\alpha(v) - f_\alpha(u) - Df_\alpha(u)(v - u)).$$

On the other hand, it follows from the definition of H that

$$(14) \quad H(v, u) = \int_0^1 \int_0^\theta (v - u)^T D^2\eta(u + \gamma(v - u))(v - u) \, d\gamma d\theta,$$

which, together with (4), leads to

$$(15) \quad \frac{\beta_0}{2}|v - u|^2 \leq H(v, u) \leq \frac{\beta_1}{2}|v - u|^2, \quad \forall u, v \in \Omega.$$

If u is assumed to be a strong solution, its first derivative is bounded and by a classical localization procedure *à la* Kruzhkov and a Gronwall lemma, one obtains a L^2_{loc} stability estimate for any $r > 0$

$$(16) \quad \int_{|x|<r} |v(x, T) - u(x, T)|^2 dx \leq C(T, u) \int_{|x|<r+L_f T} |v_0(x) - u_0(x)|^2 dx,$$

where the dependence of C on u reflect the needs of smoothness on u (C blows up when u becomes discontinuous). This inequality, rigorously proved in [15], provides a weak–strong uniqueness result. Similar (but more sophisticated) ideas have been applied to other fluid systems, see for instance [40] and [23] for more recent developments.

Remark 1.3. In [47], Tzavaras studies the comparison of solutions of a hyperbolic system with relaxation with solutions of the associated equilibrium system of conservation laws. He also makes use of the relative entropy for strong solutions. Very similar questions have been addressed in [3, 2] for the convergence of kinetic equations towards the system of gas dynamics. Here again, only strong solutions of the Euler equations are considered. To finish the bibliographical review, let us mention the work by Leger and Vasseur [38] where the reference solution may include some particular discontinuities.

Remark 1.4. For general conservation laws, the relative entropy is not symmetric, i.e. $H(u, v) \neq H(v, u)$ and $Q(u, v) \neq Q(v, u)$. In the very particular case of Friedrichs systems, i.e. when there exist symmetric matrices $A_\alpha \in \mathbb{R}^{m \times m}$ ($\alpha \in \{1, \dots, d\}$) such that $f_\alpha(u) = A_\alpha u$, then $u \mapsto |u|^2$ is an entropy and the corresponding entropy flux ξ is $\xi_\alpha(u) = u^T A_\alpha u$, ($\alpha \in \{1, \dots, d\}$). It is then easy to check that

$$H(v, u) = H(v, u) = |u - v|^2, \quad Q_\alpha(v, u) = Q_\alpha(u, v) = (v - u)^T A_\alpha (v - u),$$

and $Z_\alpha(v, u) = 0$ for all $(u, v) \in \mathbb{R}^m$. As a consequence, inequality (12) becomes

$$\partial_t H(v, u) + \sum_{\alpha=1}^d \partial_\alpha Q_\alpha(v, u) \leq 0,$$

even if u is only a weak solution. This allows to make use of the doubling variable technique [35] to compare u to v , recovering the classical uniqueness result for Friedrichs systems [24].

Our aim is to replace the entropy weak solution v in (12) by an approximate solution provided by finite volume schemes on unstructured meshes. Following the formalism introduced in [22], this makes appear in (12) bounded Radon measures

which can be controlled, leading to error estimates in $h^{1/4}$ between a strong solution and its finite volume approximation, h being the characteristic size of the cells of the mesh. The purpose of the following section is to define the finite volume scheme and to recall some classical properties required on the numerical fluxes.

1.2. Definition of the time-explicit finite volume scheme.

1.2.1. *Space and time discretizations.* Let \mathcal{T} be a mesh of \mathbb{R}^d , defined as a family of disjoint polygonal (or polyhedral) connected subsets of \mathbb{R}^d , such that \mathbb{R}^d is the union of the closure of the elements of \mathcal{T} . We denote $h = \sup\{\text{diam}(K), K \in \mathcal{T}\} < \infty$, and assume without loss of generality that $0 < h \leq 1$. For all $K \in \mathcal{T}$, we denote by $|K|$ its d -dimensional Lebesgue measure, and by $\mathcal{N}(K)$ the set of its neighboring cells. For $L \in \mathcal{N}(K)$, the common interface (called *edge*) between K and L is denoted by σ_{KL} and $|\sigma_{KL}|$ is its $(d-1)$ -Lebesgue measure. We denote by \mathcal{E} the set of all the edges and assume that there exists $a > 0$ such that

$$(17) \quad |K| \geq ah^d \quad \text{and} \quad |\partial K| := \sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| \leq \frac{h^{d-1}}{a}, \quad \forall K \in \mathcal{T}.$$

The unit normal vector to σ_{KL} from K to L is denoted n_{KL} . Note that the elements we consider are not necessarily simplices. Let $\Delta t > 0$ be the time step and we set $t^n = n\Delta t$, $\forall n \in \mathbb{N}$. Let $T > 0$ be a given time, we introduce $N_T = \max\{n \in \mathbb{N}, n \leq T/\Delta t + 1\}$. Since we consider time-explicit methods, the time step Δt will be subject to a CFL condition which will be given later.

Remark 1.5. *In order to avoid some additional heavy notations, we have chosen to deal with a uniform time discretization and a space discretization that does not depend on time. Nevertheless, it is possible, following the path described in [34], to adapt our study to the case of time-dependent space discretizations and to non-uniform time discretizations. This would be mandatory for considering a dynamic mesh adaptation procedure based on the a posteriori numerical error estimators that can be derived from our study.*

Since we will consider weak formulations and compactly supported test functions in the next sections, we introduce local sets of cells and interfaces: let $r > 0$, we introduce the sets

$$(18) \quad \begin{aligned} \mathcal{T}_r &= \{K \in \mathcal{T} \mid K \subset B(0, r)\}, \\ \mathcal{E}_r &= \{\sigma_{KL} \in \mathcal{E} \mid (K, L) \in (\mathcal{T}_r)^2, L \in \mathcal{N}(K)\}, \\ \partial\mathcal{T}_r &= \{\sigma_{KL} \in \mathcal{E} \mid K \in \mathcal{T}_r, L \in \mathcal{N}(K), L \notin \mathcal{T}_r\}. \end{aligned}$$

In particular, $\{\sigma_{KL} \in \mathcal{E} \mid K \in \mathcal{T}_r, L \in \mathcal{N}(K)\} = \mathcal{E}_r \cup \partial\mathcal{T}_r$ and $\mathcal{E}_r \cap \partial\mathcal{T}_r = \emptyset$.

1.2.2. *Numerical flux and finite volume schemes.* For all $(K, L) \in \mathcal{T}^2$, $L \in \mathcal{N}(K)$, we consider numerical fluxes G_{KL} , which are Lipschitz continuous functions from Ω^2 to \mathbb{R}^m . We assume that these numerical fluxes are *conservative*, i.e.,

$$(19) \quad G_{KL}(u, v) = -G_{LK}(v, u), \quad \forall (u, v) \in \Omega^2,$$

We also assume that the numerical fluxes fulfill the following *consistency* condition:

$$(20) \quad G_{KL}(u, u) = f(u) \cdot n_{KL}, \quad \forall u \in \Omega,$$

which implies

$$(21) \quad \sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| G_{KL}(u, u) = 0, \quad \forall u \in \mathbb{R}^m, \forall K \in \mathcal{T}.$$

Following [5], we assume that the numerical flux ensures the preservation of the convex set of admissible states Ω at each interface. More precisely, we assume that there exists $\lambda^* > 0$ such that, for all $\lambda > \lambda^*$, for all $K \in \mathcal{T}$, and for all $L \in \mathcal{N}(K)$,

$$(22) \quad u - \frac{1}{\lambda} (G_{KL}(u, v) - f(u) \cdot n_{KL}) \in \Omega, \quad \forall (u, v) \in \Omega^2.$$

In order to ensure the nonlinear stability of the scheme, we also require the existence of a *numerical entropy flux*. More precisely, we assume that for all $(K, L) \in \mathcal{E}$, there exist Lipschitz continuous functions $\xi_{KL} : \Omega \times \Omega \rightarrow \mathbb{R}$ which are conservative, i.e.,

$$(23) \quad \xi_{KL}(u, v) = -\xi_{LK}(v, u), \quad \forall (u, v) \in \Omega^2,$$

and satisfy the interfacial entropy inequalities: for all $\lambda \geq \lambda^* > 0$, for all $(u, v) \in \Omega^2$,

$$(24) \quad \xi_{KL}(u, v) - \xi(u) \cdot n_{KL} \leq -\lambda \left(\eta \left(u - \frac{1}{\lambda} (G_{KL}(u, v) - f(u) \cdot n_{KL}) \right) - \eta(u) \right).$$

In what follows, and before strengthening it in (38), we assume that the following CFL condition is fulfilled:

$$(25) \quad \frac{\Delta t}{|K|} \lambda^* \sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| \leq 1, \quad \forall K \in \mathcal{T}.$$

Note that the regularity of the mesh (17) implies that (25) holds if

$$(26) \quad \Delta t \leq \frac{a^2}{\lambda^*} h.$$

We have now introduced all the necessary material to define the time-explicit numerical scheme we will consider.

Definition 1.2 (Finite volume scheme). *The finite volume scheme is defined by the discrete unknowns u_K^n , $K \in \mathcal{T}$ and $n \in \{0, \dots, N_T\}$, which satisfy*

$$(27) \quad \frac{u_K^{n+1} - u_K^n}{\Delta t} |K| + \sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| G_{KL}(u_K^n, u_L^n) = 0$$

together with the initial condition

$$(28) \quad u_K^0 = \frac{1}{|K|} \int_K u_0(x) dx, \quad \forall K \in \mathcal{T},$$

under assumptions (19)–(24) on the numerical flux G_{KL} and under the CFL condition (26). The approximate solution $u^h : \mathbb{R}^d \times \mathbb{R}_+ \rightarrow \mathbb{R}^m$ provided by the finite volume scheme (27)–(28) is defined by

$$(29) \quad u^h(x, t) = u_K^n, \quad \text{for } x \in K, \ t^n \leq t < t^{n+1}, \ K \in \mathcal{T}, \ n \in \{0, \dots, N_T\}.$$

Remark 1.6. *Let us provide some examples of numerical fluxes which satisfy assumptions (22) and (24). The most classical example is the Godunov flux [26], which writes*

$$G_{KL}(u, v) = f(\mathcal{U}_{KL}(0; u, v)) \cdot n_{KL}$$

where $\mathcal{U}_{KL}(x/t; u, v)$ stands for the solution of the Riemann problem for the system of conservation laws (1) in the one-dimensional direction n_{KL} , with initial data u

and v . If λ^* is greater than all the wave speeds in the Riemann problems, then one can prove (22) and (24), with the numerical entropy flux

$$\xi_{KL}(u, v) = \xi(\mathcal{U}_{KL}(0; u, v)) \cdot n_{KL}.$$

Another classical example is the Rusanov scheme [42], which is the finite volume extension of the Lax–Friedrichs scheme. It reads

$$G_{KL}(u, v) = \frac{1}{2}(f(u) + f(v)) \cdot n_{KL} - \frac{c}{2}(v - u)$$

where $c > 0$ is a parameter (which can be defined by interface). The associated numerical entropy flux is

$$\xi_{KL}(u, v) = \frac{1}{2}(X_{KL}(u, v) - X_{LK}(v, u))$$

where $X_{KL}(u, v) = \xi(u) \cdot n_{KL} + D\eta(u)(G_{KL}(u, v) - f(u) \cdot n_{KL})$ (this function will also be introduced hereafter for the computation of weak-BV estimates). Once again, if c is greater than all the wave speeds, one can prove that this numerical entropy flux satisfies (24). Proving Assumption (22) is more difficult and overall model dependent. For the shallow-water equations, the positivity of the height of water is directly obtained (see for instance [5]). The case of Euler equations is more intricate, in particular for proving the positivity of the specific energy. This can be done using the structure of the system, see for instance [5]. For details on the proofs, more explicit CFL conditions, or for other admissible numerical fluxes, the reader can refer for instance to [28], [13], [46], [5], [12].

1.3. Error estimate and organization of the paper. Our aim is to provide an error estimate of the the form

$$\|u - u^h\|_{L^2(\Gamma)} \leq Ch^{1/4},$$

for all compact subsets Γ of $\mathbb{R}^d \times \mathbb{R}_+$, where u stands for the unique strong solution to (1), (3) and u_h for the numerical solution (27)–(29). The rigorous statement is given in Theorem 2.7. This estimate extends to the system case the contributions of [11, 49, 22, 7] on the scalar case. In [32], which also deals with strong solutions of nonlinear systems, the assumptions are less classical than ours, in particular we do not need any ‘inverse’ CFL condition of the form $C \leq \Delta t/h$ (see also [22] for a similar comment in the scalar case).

The proof of this estimate relies on a so-called *weak-BV estimate*, that is

$$\sum_{n=0}^{N_T} \Delta t \sum_{(K,L) \in \mathcal{E}_r} |\sigma_{KL}| |G_{KL}(u_K^n, u_L^n) - f(u_K^n) \cdot n_{KL}| \leq \frac{C}{\sqrt{h}},$$

where \mathcal{E}_r is defined in (18). The rigorous statement of this estimate and its proof are gathered in §2.2. Up to the authors’ knowledge, this estimate is new for time-explicit finite volume schemes: in [32], only time-implicit methods are considered (see also [29]).

Let us now present the outline of the paper. In Section 2 we first briefly recall some classical properties of the finite volume scheme. Then we address the proof of the weak-BV property by introducing a new flux which depicts the entropy dissipation through the edges. Straightforward consequences are then derived.

The next two sections address the proof of the error estimate. In order to compare the discrete solution u^h with the strong solution u , we write continuous weak

and entropy formulations for u^h in Section 3, so that we can adapt the uniqueness proof proposed in [15]. Nevertheless, the discrete solution u^h is obviously not a weak entropy solution. Therefore, some error terms coming from the discretization have to be taken into account in the formulation, which take the form of positive locally bounded Radon measures, following [22]. A large part of Section 3 consists in making these measures explicit and in bounding them with quantities which tend to 0 with the discretization size. In Section 4, we make use of the weak and entropy weak formulations for the discrete solution (and of their corresponding error measures) to derive the error estimate. The distance between the strong solution u and the discrete solution u^h is quantified thanks to the relative entropy $H(u^h, u)$ introduced in Definition 1.1.

2. NONLINEAR STABILITY

2.1. Preservation of admissible states and discrete entropy inequality.

We first give two classical properties of the numerical scheme (27) which are direct consequences of the assumptions we made in §1.2.2. We refer to [5] for the proofs.

Lemma 2.1. *Assume that the initial condition satisfies (3) and that assumption (22) and the CFL condition (25) hold, then, for all $K \in \mathcal{T}$, for all $n \in \{0, \dots, N_T\}$, u_K^n belong to Ω .*

Following once again the procedure detailed in [5], we can derive entropy properties on the numerical scheme from (24).

Proposition 2.2. *The numerical entropy flux ξ_{KL} is consistent with ξ , i.e.*

$$(30) \quad \xi_{KL}(u, u) = \xi(u) \cdot n_{KL}, \quad \forall u \in \Omega.$$

Moreover, under the CFL condition (26), the discrete solution u^h satisfies the discrete entropy inequalities: $\forall K \in \mathcal{T}, \forall n \geq 0$,

$$(31) \quad \frac{|K|}{\Delta t} (\eta(u_K^{n+1}) - \eta(u_K^n)) + \sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| \xi_{KL}(u_K^n, u_L^n) \leq 0.$$

Note that the consistency (30) of the entropy fluxes ξ_{KL} ensures that

$$(32) \quad \sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| \xi_{KL}(u, u) = 0, \quad \forall u \in \Omega.$$

2.2. Weak-BV inequality for systems of conservation laws. For all $(K, L) \in \mathcal{T}^2$, $L \in \mathcal{N}(K)$, we introduce the flux

$$(33) \quad X_{KL}(u, v) := \xi(u) \cdot n_{KL} + D\eta(u)(G_{KL}(u, v) - f(u) \cdot n_{KL}), \quad \forall (u, v) \in \Omega^2.$$

Let us remark that it is neither symmetric nor conservative. Such a quantity may provide the connection between fully discrete and semi-discrete entropy satisfying schemes, but also between entropy-conservative and entropy-stable schemes. It is in particular shown in [5] (see also [45, 46]) that the fluxes X_{KL} for $(K, L) \in \mathcal{E}$ verify

$$(34) \quad -X_{LK}(v, u) \leq \xi_{KL}(u, v) \leq X_{KL}(u, v), \quad \forall (u, v) \in \Omega^2.$$

Actually, inequalities (34) can be specified by quantifying the entropy dissipation across the edges.

Proposition 2.3. *For all $\sigma_{KL} \in \mathcal{E}$ and all $(u, v) \in \Omega^2$, one has*

$$(35) \quad X_{KL}(u, v) - \xi_{KL}(u, v) \geq \frac{\beta_0}{2\lambda^*} |G_{KL}(u, v) - f(u) \cdot n_{KL}|^2,$$

where β_0 is defined in (4) and λ^* has to be such that (22) and (24) hold.

Proof. We rewrite the left-hand side of Ineq. (24) for $\lambda = \lambda^*$ using the definition (33) of the flux X_{KL} in order to obtain

$$(36) \quad X_{KL}(u, v) - \xi_{KL}(u, v) - D\eta(u)(G_{KL}(u, v) - f(u) \cdot n_{KL}) \\ \geq \lambda^* \left[\eta(u - \frac{1}{\lambda^*}(G_{KL}(u, v) - f(u) \cdot n_{KL})) - \eta(u) \right].$$

The uniform convexity (4) of η ensures that

$$(37) \quad \lambda^* \left[\eta(u - \frac{1}{\lambda^*}(G_{KL}(u, v) - f(u) \cdot n_{KL})) - \eta(u) \right] \\ \geq -D\eta(u)(G_{KL}(u, v) - f(u) \cdot n_{KL}) + \frac{1}{2} \frac{\beta_0}{\lambda^*} |G_{KL}(u, v) - f(u) \cdot n_{KL}|^2.$$

Combining (36) and (37) leads to (35). \square

Thanks to the specified version (35) of the classical inequalities (34), we are now in position for proving a new stability estimate for time-explicit finite volume scheme, namely the weak-*BV* inequality. This inequality is obtained by quantifying the numerical diffusion of the numerical scheme. As in the scalar case (see [10, 11, 49, 22, 7]), such an equality requires a strengthened CFL condition. In our system case, we require the existence of some $\zeta \in (0, 1)$ such that

$$(38) \quad \Delta t \leq \frac{\beta_0}{\beta_1} \frac{a^2}{\lambda^*} (1 - \zeta) h$$

holds, where β_0 and β_1 are defined by (4), a and h are the mesh parameters (17), and where λ^* appears in the condition (22) and (24). Note that the strengthened CFL condition (38) implies the classical CFL condition (26). We are now able to obtain the following local estimate, using the notations (18).

Proposition 2.4. *Assume that the strengthened CFL condition (38) holds, then there exists C depending only on $T, r, a, \eta, \xi, \Omega$ and ζ (but neither on h nor on Δt) such that*

$$(39) \quad \sum_{n=0}^{N_T} \Delta t \sum_{(K,L) \in \mathcal{E}_r} |\sigma_{KL}| |G_{KL}(u_K^n, u_L^n) - f(u_K^n) \cdot n_{KL}|^2 \leq C.$$

Proof. Multiplying the numerical scheme (27) by $\Delta t D\eta(u_K^n)$ and summing over $n \in \{0, \dots, N_T\}$ and $K \in \mathcal{T}_r$ provides

$$(40) \quad A + B = 0,$$

where

$$A = \sum_{n=0}^{N_T} \sum_{K \in \mathcal{T}_r} D\eta(u_K^n) (u_K^{n+1} - u_K^n) |K|, \\ B = \sum_{n=0}^{N_T} \Delta t \sum_{K \in \mathcal{T}_r} D\eta(u_K^n) \sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| |G_{KL}(u_K^n, u_L^n)|.$$

The concavity of $u \mapsto \eta(u) - \frac{\beta_1}{2}|u - u_K^n|^2$ together with the definition (27) of the numerical scheme and property (21) provide that

$$A \geq \sum_{K \in \mathcal{T}_r} \eta(u_K^{N_T+1})|K| - \sum_{K \in \mathcal{T}_r} \eta(u_K^0)|K| - \frac{\beta_1}{2} \sum_{n=0}^{N_T} \Delta t^2 \sum_{K \in \mathcal{T}_r} \frac{1}{|K|} \left| \sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| (G_{KL}(u_K^n, u_L^n) - f(u_K^n) \cdot n_{KL}) \right|^2.$$

Using the Jensen inequality, we get

$$\sum_{K \in \mathcal{T}_r} \eta(u_K^0)|K| \leq \int_{|x| \leq B(0, R+h)} \eta(u_0(x)) dx =: C_1.$$

The positivity of the entropy η yields $\sum_{K \in \mathcal{T}_r} \eta(u_K^{N_T+1})|K| \geq 0$. Moreover, Cauchy-Schwarz inequality ensures that for all $K \in \mathcal{T}_r$ and all $n \in \{0, \dots, N_T\}$, one has

$$\begin{aligned} & \left| \sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| (G_{KL}(u_K^n, u_L^n) - f(u_K^n) \cdot n_{KL}) \right|^2 \\ & \leq \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| \right) \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| |G_{KL}(u_K^n, u_L^n) - f(u_K^n) \cdot n_{KL}|^2 \right). \end{aligned}$$

Then it follows from the regularity assumption (17) on the mesh that

$$(41) \quad A \geq -C_1 - \frac{\beta_1 \Delta t}{2a^2 h} \sum_{n=0}^{N_T} \Delta t \sum_{K \in \mathcal{T}_r} \sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| |G_{KL}(u_K^n, u_L^n) - f(u_K^n) \cdot n_{KL}|^2.$$

Concerning the term B , we use the definition (33) of the entropy flux X_{KL} to get

$$B = \sum_{n=0}^{N_T} \Delta t \sum_{K \in \mathcal{T}_r} \sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| (X_{KL}(u_K^n, u_L^n) - \xi(u_K^n) \cdot n_{KL} + D\eta(u_K^n) f(u_K^n) \cdot n_{KL}).$$

Using the property $\sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| n_{KL} = 0$ for all $K \in \mathcal{T}$, we can reorganize the term B into

$$(42) \quad B = B_1 + B_2,$$

where

$$\begin{aligned} B_1 &= \sum_{n=0}^{N_T} \Delta t \sum_{K \in \mathcal{T}_r} \sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| (X_{KL}(u_K^n, u_L^n) - \xi_{KL}(u_K^n, u_L^n)), \\ B_2 &= \sum_{n=0}^{N_T} \Delta t \sum_{(K,L) \in \partial \mathcal{T}_r} |\sigma_{KL}| \xi_{KL}(u_K^n, u_L^n). \end{aligned}$$

Since ξ_{KL} is a continuous function of bounded quantities, B_2 can be bounded using the regularity of the mesh (17). More precisely, one gets

$$(43) \quad |B_2| \leq \max_{(K,L) \in \partial \mathcal{T}_r} \|\xi_{KL}\|_{L^\infty(\Omega^2)} \sum_{n=0}^{N_T} \Delta t \sum_{(K,L) \in \partial \mathcal{T}_r} |\sigma_{KL}| \leq C_2$$

for some $C_2 > 0$ depending only on T, r, a, ξ and Ω . On the other hand, it follows from Proposition 2.3 that

$$(44) \quad B_1 \geq \frac{\beta_0}{2\lambda^*} \sum_{n=0}^{N_T} \Delta t \sum_{K \in \mathcal{T}_r} \sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| |G_{KL}(u_K^n, u_L^n) - f(u_K^n) \cdot n_{KL}|^2.$$

Combining (41)–(44) into (40) leads to

$$\left(\frac{\beta_0}{2\lambda^*} - \frac{\beta_1 \Delta t}{2a^2 h} \right) \sum_{n=0}^{N_T} \Delta t \sum_{K \in \mathcal{T}_r} \sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| |G_{KL}(u_K^n, u_L^n) - f(u_K^n) \cdot n_{KL}|^2 \leq C_1 + C_2.$$

The CFL condition (38) has been strengthened so that

$$\left(\frac{\beta_0}{2\lambda^*} - \frac{\beta_1 \Delta t}{2a^2 h} \right) \geq \frac{\zeta \beta_0}{2\lambda^*}$$

remains uniformly bounded away from 0. Estimate (39) follows. \square

We state now a straightforward consequence of Proposition 2.4. Its proof relies on the Cauchy–Schwarz inequality and is left to the reader.

Corollary 2.5. *Assume that (38) holds, then there exists C_{BV} depending only on $T, r, a, \xi, \eta, u_0, \Omega$ and ζ such that*

$$(45) \quad \sum_{n=0}^{N_T} \Delta t \sum_{(K,L) \in \mathcal{E}_r} |\sigma_{KL}| |G_{KL}(u_K^n, u_L^n) - f(u_K^n) \cdot n_{KL}| \leq \frac{C_{BV}}{\sqrt{h}}.$$

2.3. Consequences of the weak– BV estimate. The weak– BV estimate (45) implies a similar control on entropy fluxes and the time variations of u^h .

Lemma 2.6. *Assume that the strengthened CFL condition (38) holds, then*

$$(46) \quad \sum_{n=0}^{N_T} \Delta t \sum_{(K,L) \in \mathcal{E}_r} |\sigma_{KL}| |\xi_{KL}(u_K^n, u_L^n) - \xi(u_K) \cdot n_{KL}| \leq \|D\eta\|_\infty \frac{C_{BV}}{\sqrt{h}},$$

$$(47) \quad \sum_{n=0}^{N_T} \sum_{K \in \mathcal{T}_r} |K| |u_K^{n+1} - u_K^n| \leq \frac{C_{BV}}{\sqrt{h}},$$

$$(48) \quad \sum_{n=0}^{N_T} \sum_{K \in \mathcal{T}_r} |K| |\eta(u_K^{n+1}) - \eta(u_K^n)| \leq \|D\eta\|_\infty \frac{C_{BV}}{\sqrt{h}}.$$

Proof. Using the Lipschitz continuity of η in (24), one obtains inequality (48). Thanks to definition (27) of the scheme and thanks to the divergence free property (21), one has for all $K \in \mathcal{T}$ and all $n \in \mathbb{N}$

$$|u_K^{n+1} - u_K^n| |K| \leq \Delta t \sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| |G_{KL}(u_K^n, u_L^n) - f(u_K^n) \cdot n_{KL}|.$$

Summing over $K \in \mathcal{T}_r$ and $n \in \{0, \dots, N_T\}$ and using (45) provides (47). Inequality (48) then follows from the Lipschitz continuity of η . \square

We now state our main result, that consists in an *a priori* error estimate between a strong solution u and a discrete solution u^h .

Theorem 2.7. *Assume that $u_0 \in W^{1,\infty}(\mathbb{R}^d)$ and that the solution u of the Cauchy problem (1)–(3) belongs to $W^{1,\infty}(\mathbb{R}^d \times [0, T])$. Let u^h , with $0 < h \leq 1$, defined by the numerical scheme (27)–(29) and assume that the strengthened CFL condition (38) holds. Then, for all $r > 0$ and $T > 0$ there exist C depending only on $T, r, \Omega, a, \lambda^*, u_0, G_{KL}, \eta$ and f , such that*

$$\int_0^T \int_{B(0, r+L_f(T-t))} |u - u^h|^2 dx dt \leq C\sqrt{h}.$$

3. CONTINUOUS WEAK AND ENTROPY FORMULATIONS FOR THE DISCRETE SOLUTION

In order to obtain the error estimate of Theorem 2.7, we aim at using the relative entropy of u^h w.r.t. u . Since u^h is only an approximate solution, it neither satisfies exactly the weak formulation (10) nor the entropy weak formulation (11). Some numerical error terms appear in these formulations, and thus also appear the inequality of the relative entropy

$$(49) \quad \partial_t H(u^h, u) + \sum_{\alpha=1}^d \partial_\alpha Q_\alpha(u^h, u) \leq - \sum_{\alpha=1}^d (\partial_\alpha u)^T Z_\alpha(u^h, u) + \text{numerical error terms.}$$

As usual, these terms may be described by Radon measures, see for instance [6, 22, 7, 34, 31, 32]. Note that for nonlinear systems of conservation laws, a function which satisfies the entropy inequality (11) is not necessarily a weak solution (10). This leads us to introduce error measures for both the entropy inequality (11) and the weak formulation (10) of u^h . Let us first begin with the entropy formulation and the related measures.

For $X = \mathbb{R}^d$ or $X = \mathbb{R}^d \times \mathbb{R}^+$, we denote by $\mathcal{M}(X)$ the set of locally bounded Radon measures on X , i.e., $\mathcal{M}(X) = (C_c(X))'$ where $C_c(X)$ is the set of continuous compactly supported functions on X . If $\mu \in \mathcal{M}(X)$ we set $\langle \mu, \varphi \rangle = \int_X \varphi d\mu$ for all $\varphi \in C_c(X)$.

Definition 3.1. *For $\psi \in C_c(\mathbb{R}^d)$, $\varphi \in C_c(\mathbb{R}^d \times \mathbb{R}^+)$, we define $\mu_0 \in \mathcal{M}(\mathbb{R}^d)$ and $\mu \in \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^+)$ by*

$$\begin{aligned} \langle \mu_0, \psi \rangle &= \int_{\mathbb{R}^d} |\eta(u_0(x)) - \eta(u^h(x, 0))| \psi(x) dx, \\ \langle \mu, \varphi \rangle &= \langle \mu_{\mathcal{T}}, \varphi \rangle + \sum_{n=0}^{\infty} \Delta t \sum_{(K,L) \in \mathcal{E}_r} |\sigma_{KL}| |\xi_{KL}(u_K^n, u_L^n) - \xi_{KL}(u_K^n, u_K^n)| \langle \mu_{KL}, \varphi \rangle \\ &\quad + \sum_{n=0}^{\infty} \Delta t \sum_{(K,L) \in \mathcal{E}_r} |\sigma_{KL}| |\xi_{KL}(u_K^n, u_L^n) - \xi_{KL}(u_L^n, u_L^n)| \langle \mu_{LK}, \varphi \rangle, \end{aligned}$$

where

$$\begin{aligned} \langle \mu_{\mathcal{T}}, \varphi \rangle &= \sum_{n=0}^{\infty} \sum_{K \in \overline{\mathcal{T}}_r} |\eta(u_K^{n+1}) - \eta(u_K^n)| \int_{t^n}^{t^{n+1}} \int_K \varphi(x, t) dx dt, \\ \langle \mu_{KL}, \varphi \rangle &= \frac{1}{|K| |\sigma_{KL}| (\Delta t)^2} \\ &\quad \times \int_{t^n}^{t^{n+1}} \int_K \int_{t^n}^{t^{n+1}} \int_{\sigma_{KL}} \int_0^1 (h + \Delta t) \varphi(\gamma + \theta(x - \gamma), s + \theta(t - s)) d\theta dx dt d\gamma ds, \\ \langle \mu_{LK}, \varphi \rangle &= \frac{1}{|K| |\sigma_{KL}| (\Delta t)^2} \\ &\quad \times \int_{t^n}^{t^{n+1}} \int_L \int_{t^n}^{t^{n+1}} \int_{\sigma_{KL}} \int_0^1 (h + \Delta t) \varphi(\gamma + \theta(x - \gamma), s + \theta(t - s)) d\theta dx dt d\gamma ds. \end{aligned}$$

As it will be highlighted by Proposition 3.3 later on, the measures μ and μ_0 describe the approximation error in the entropy formulation satisfied by u^h . Let us first estimate them on compact sets.

Lemma 3.2. *Assume that the strengthened CFL condition (38) holds, then, for all $r > 0$ and $T > 0$ there exist $C_{\mu_0} > 0$, depending only on u_0 , $\|D\eta\|_{\infty}$, and r , and $C_{\mu} > 0$, depending only on $T, r, a, \lambda^*, u_0, G_{KL}$ and η such that, for all $h < r$,*

$$(50) \quad \mu_0(B(0, r)) \leq C_{\mu_0} h \quad \text{and} \quad \mu(B(0, r) \times [0, T]) \leq \frac{C_{\mu}}{\sqrt{h}}.$$

Proof. The regularity of $u_0 : \mathbb{R}^d \rightarrow \mathbb{R}^m$ yields

$$\mu_0(B(0, r)) \leq h \|D\eta\|_{\infty} \int_{B(0, r+h)} |\nabla u_0| dx.$$

For $r > 0$ and $T > 0$ the measure $\mu_{\mathcal{T}}$ satisfies

$$\mu_{\mathcal{T}}(B(0, r) \times [0, T]) = \int_0^T \int_{B(0, r)} \sum_{n=0}^{N_T} \sum_{K \in \overline{\mathcal{T}}_r} |\eta(u_K^{n+1}) - \eta(u_K^n)| \mathbf{1}_{K \times [t^n, t^{n+1}]} dx dt.$$

Then, using the time-BV estimate (48),

$$\mu_{\mathcal{T}}(B(0, r) \times [0, T]) \leq \Delta t \sum_{n=0}^{N_T} \sum_{K \in \overline{\mathcal{T}}_r} |K| |\eta(u_K^{n+1}) - \eta(u_K^n)| \leq \Delta t \|D\eta\|_{\infty} \frac{C_{BV}}{\sqrt{h}}.$$

Since Δt satisfies the CFL condition (38), one has

$$(51) \quad \mu_{\mathcal{T}}(B(0, r) \times [0, T]) \leq C_{\mu_{\mathcal{T}}} \sqrt{h},$$

where $C_{\mu_{\mathcal{T}}} := \frac{a^2 \|D\eta\|_{\infty}}{\lambda^*} C_{BV}$. The measures μ_{KL} and μ_{LK} satisfy:

$$\mu_{KL}(\mathbb{R}^d \times \mathbb{R}^+) \leq h + \Delta t, \quad \mu_{LK}(\mathbb{R}^d \times \mathbb{R}^+) \leq h + \Delta t.$$

Therefore,

$$\begin{aligned} & \mu(B(0, r) \times [0, T]) \\ & \leq C_{\mu\tau} \sqrt{h} + (h + \Delta t) \sum_{n=0}^{N_T} \Delta t \sum_{(K,L) \in \mathcal{E}_r} |\sigma_{KL}| |\xi_{KL}(u_K^n, u_L^n) - \xi_{KL}(u_K^n, u_K^n)| \\ & \quad + (h + \Delta t) \sum_{n=0}^{N_T} \Delta t \sum_{(K,L) \in \mathcal{E}_r} |\sigma_{KL}| |\xi_{KL}(u_K^n, u_L^n) - \xi_{KL}(u_L^n, u_L^n)|. \end{aligned}$$

Hence, using Lemma 2.6, the CFL condition (26) and the bound (51) provides

$$\mu(B(0, r) \times [0, T]) \leq C_\mu \sqrt{h},$$

where $C_\mu = C_{\mu\tau} + 2 \left(1 + \frac{\alpha^2}{\lambda^*}\right) \|D\eta\|_\infty C_{BV}$. \square

Proposition 3.3. *Let μ and μ_0 be the measures introduced in Definition 3.1, then, for all $\varphi \in C_c^1(\mathbb{R}^d \times \mathbb{R}^+; \mathbb{R}^+)$, one has*

$$\begin{aligned} (52) \quad & \iint_{\mathbb{R}^d \times \mathbb{R}_+} \eta(u^h) \partial_t \varphi(x, t) + \sum_{\alpha=1}^d \int_{\mathbb{R}^d} \eta(u_0(x)) \varphi(x, 0) dx \xi_\alpha(u^h) \partial_\alpha \varphi(x, t) dx dt \\ & \geq - \iint_{\mathbb{R}^d \times \mathbb{R}_+} (|\nabla \varphi| + |\partial_t \varphi|) d\mu(x, t) - \int_{\mathbb{R}^d} \varphi(x, 0) d\mu_0(x). \end{aligned}$$

Proof. Let $\varphi \in C_c^1(\mathbb{R}^d \times \mathbb{R}^+; \mathbb{R}^+)$. Let $T > 0$ and $r > 0$ such that $\text{supp } \varphi \subset B(0, r) \times [0, T]$. Let us multiply (31) by $\int_{t^n}^{t^{n+1}} \int_K \varphi(x, t) dx dt$ and sum over the control volumes $K \in \mathcal{T}_r$ and $n \leq N_T$. It yields

$$(53) \quad T_1 + T_2 \leq 0,$$

where

$$(54) \quad T_1 = \sum_{n=0}^{N_T} \sum_{K \in \mathcal{T}_r} \frac{1}{\Delta t} (\eta(u_K^{n+1}) - \eta(u_K^n)) \int_{t^n}^{t^{n+1}} \int_K \varphi(x, t) dx dt,$$

$$(55) \quad T_2 = \sum_{n=0}^{N_T} \sum_{K \in \mathcal{T}_r} \frac{1}{|K|} \int_{t^n}^{t^{n+1}} \int_K \varphi(x, t) dx dt \sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| \xi_{KL}(u_K^n, u_L^n).$$

The term T_1 corresponds to the discrete time derivative of $\eta(u^h)$ and T_2 to the discrete space derivative of $\xi(u^h)$. The proof relies on the comparison firstly between T_1 and T_{10} and secondly between T_2 and T_{20} , where T_{10} and T_{20} denote respectively the temporal and spatial term in (52):

$$\begin{aligned} T_{10} &= - \iint_{\mathbb{R}^d \times \mathbb{R}_+} \eta(u^h) \partial_t \varphi(x, t) dx dt - \int_{\mathbb{R}^d} \eta(u_0(x)) \varphi(x, 0) dx, \\ T_{20} &= - \iint_{\mathbb{R}^d \times \mathbb{R}_+} \sum_{\alpha=1}^d \xi_\alpha(u^h) \partial_\alpha \varphi(x, t) dx dt. \end{aligned}$$

Let us first focus on T_{10} . Following its definition (29), the approximate solution u^h is piecewise constant, then so does $\eta(u^h)$. Therefore, we can rewrite

$$\begin{aligned} T_{10} &= \sum_{n=0}^{N_T} \sum_{K \in \mathcal{T}_r} (\eta(u_K^{n+1}) - \eta(u_K^n)) \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \int_K \varphi(x, t^{n+1}) dx dt \\ &\quad - \int_{\mathbb{R}^d} (\eta(u_0(x)) - \eta(u^h(x, 0))) \varphi(x, 0) dx. \end{aligned}$$

It is now easy to verify that

$$\begin{aligned} |T_1 - T_{10}| &\leq \sum_{n=0}^{N_T} \sum_{K \in \mathcal{T}_r} |\eta(u_K^{n+1}) - \eta(u_K^n)| \int_{t^n}^{t^{n+1}} \int_K |\partial_t \varphi| dx dt \\ &\quad + \int_{\mathbb{R}^d} |\eta(u_0(x)) - \eta(u^h(x, 0))| \varphi(x, 0) dx. \end{aligned}$$

Then, accounting from Definition 3.1, the inequality reads

$$(56) \quad |T_1 - T_{10}| \leq \iint_{\mathbb{R}^d \times \mathbb{R}_+} |\partial_t \varphi| d\mu_{\mathcal{T}}(x, t) + \int_{\mathbb{R}^d} \varphi(x, 0) d\mu_0(x).$$

We now consider the terms T_2 and T_{20} . Performing a discrete integration by parts by reorganizing the sum, and using the properties (32) and (23) lead to

$$(57) \quad T_2 = T_{2,1} + T_{2,2},$$

with

$$\begin{aligned} T_{2,1} &= \sum_{n=0}^{N_T} \sum_{(K,L) \in \mathcal{E}_r} \frac{|\sigma_{KL}|}{|K|} \int_{t^n}^{t^{n+1}} \int_K \varphi(x, t) (\xi_{KL}(u_K^n, u_L^n) - \xi_{KL}(u_K^n, u_K^n)) dx dt, \\ T_{2,2} &= \sum_{n=0}^{N_T} \sum_{(K,L) \in \mathcal{E}_r} \frac{|\sigma_{KL}|}{|L|} \int_{t^n}^{t^{n+1}} \int_L \varphi(x, t) (\xi_{LK}(u_L^n, u_K^n) - \xi_{LK}(u_L^n, u_L^n)) dx dt. \end{aligned}$$

Gathering terms of T_{20} by edges yields

$$T_{20} = T_{20,1} + T_{20,2},$$

where, thanks to (23), we have set

$$\begin{aligned} T_{20,1} &= \sum_{n=0}^{N_T} \sum_{(K,L) \in \mathcal{E}_r} \int_{t^n}^{t^{n+1}} \int_{\sigma_{KL}} (\xi_{KL}(u_K^n, u_L^n) - \xi(u_K^n) \cdot n_{KL}) \varphi(\gamma, t) d\gamma dt, \\ T_{20,2} &= \sum_{n=0}^{N_T} \sum_{(K,L) \in \mathcal{E}_r} \int_{t^n}^{t^{n+1}} \int_{\sigma_{KL}} (\xi_{LK}(u_L^n, u_K^n) - \xi(u_L^n) \cdot n_{LK}) \varphi(\gamma, t) d\gamma dt. \end{aligned}$$

It is easy to verify

$$\begin{aligned} T_{2,1} - T_{20,1} &= \sum_{n=0}^{N_T} \Delta t \sum_{(K,L) \in \mathcal{E}_r} |\sigma_{KL}| (\xi_{KL}(u_K^n, u_L^n) - \xi_{KL}(u_K^n, u_K^n)) \\ &\quad \times \frac{1}{|K| |\sigma_{KL}| (\Delta t)^2} \int_{t^n}^{t^{n+1}} \int_K \int_{t^n}^{t^{n+1}} \int_{\sigma_{KL}} (\varphi(x, t) - \varphi(\gamma, s)) d\gamma ds dx dt. \end{aligned}$$

Then using the definition of μ_{KL} in Definition 3.1, we obtain the following estimate:

$$(58) \quad |T_{2,1} - T_{20,1}| \leq \sum_{n=0}^{N_T} \Delta t \sum_{(K,L) \in \mathcal{E}_r} |\sigma_{KL}| |\xi_{KL}(u_K^n, u_L^n) - \xi_{KL}(u_K^n, u_L^n)| \langle \mu_{KL}, |\nabla \varphi| + |\partial_t \varphi| \rangle.$$

Similarly, one obtains

$$(59) \quad |T_{2,2} - T_{20,2}| \leq \sum_{n=0}^{N_T} \Delta t \sum_{(K,L) \in \mathcal{E}_r} |\sigma_{KL}| |\xi_{LK}(u_L^n, u_K^n) - \xi_{LK}(u_L^n, u_K^n)| \langle \mu_{LK}, |\nabla \varphi| + |\partial_t \varphi| \rangle,$$

the measure $\mu_{LK} \in \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^+)$ being given by Definition 3.1. Bearing in mind the definition of $\mu \in \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^+)$ given in Definition 3.1, inequalities (53), (56), (57), (58) and (59), one has

$$-T_{10} - T_{20} \geq - \iint_{\mathbb{R}^d \times \mathbb{R}^+} (|\nabla \varphi| + |\partial_t \varphi|) d\mu(x, t) - \int_{\mathbb{R}^d} \varphi(x, 0) d\mu_0(x),$$

which concludes the proof of Proposition 3.3. \square

Similar calculations can be used to estimate how close u^h is to a weak solution. For that purpose we define the following measures.

Definition 3.4. For $\psi \in C_c(\mathbb{R}^d)$ and $\varphi \in C_c(\mathbb{R}^d \times \mathbb{R}^+)$, we set

$$\begin{aligned} \langle \bar{\mu}_0, \psi \rangle &= \int_{\mathbb{R}^d} |u_0(x) - u^h(x, 0)| \psi(x) dx, \\ \langle \bar{\mu}, \varphi \rangle &= \langle \bar{\mu}_{\mathcal{T}}, \varphi \rangle + \sum_{n=0}^{\infty} \Delta t \sum_{(K,L) \in \mathcal{E}_r} |\sigma_{KL}| |G_{KL}(u_K^n, u_L^n) - G_{KL}(u_K^n, u_L^n)| \langle \bar{\mu}_{KL}, \varphi \rangle \\ &\quad + \sum_{n=0}^{\infty} \Delta t \sum_{(K,L) \in \mathcal{E}_r} |\sigma_{KL}| |G_{KL}(u_K^n, u_L^n) - G_{KL}(u_L^n, u_K^n)| \langle \bar{\mu}_{LK}, \varphi \rangle, \end{aligned}$$

where

$$\begin{aligned} \langle \bar{\mu}_{\mathcal{T}}, \varphi \rangle &= \sum_{n=0}^{\infty} \sum_{K \in \mathcal{T}_r} |u_K^{n+1} - u_K^n| \int_{t^n}^{t^{n+1}} \int_K \varphi(x, t) dx dt, \\ \langle \bar{\mu}_{KL}, \varphi \rangle &= \frac{1}{|K| |\sigma_{KL}| \Delta t^2} \times \\ &\quad \int_{t^n}^{t^{n+1}} \int_K \int_{t^n}^{t^{n+1}} \int_{\sigma_{KL}} \int_0^1 (h + \Delta t) \varphi(\gamma + \theta(x - \gamma), s + \theta(t - s)) d\theta dx dt d\gamma ds, \\ \langle \bar{\mu}_{LK}, \varphi \rangle &= \frac{1}{|L| |\sigma_{KL}| \Delta t^2} \times \\ &\quad \int_{t^n}^{t^{n+1}} \int_L \int_{t^n}^{t^{n+1}} \int_{\sigma_{KL}} \int_0^1 (h + \Delta t) \varphi(\gamma + \theta(x - \gamma), s + \theta(t - s)) d\theta dx dt d\gamma ds. \end{aligned}$$

Remark 3.1. *It follows from the definitions of the measures μ and $\bar{\mu}$ that they can be extended (in a unique way) into continuous linear forms defined on the set*

$$E := \left\{ \varphi \in L^\infty(\mathbb{R}^d \times \mathbb{R}^+; \mathbb{R}) \mid \text{supp}(\varphi) \text{ is compact, and } \nabla\varphi \in L^1_{\text{loc}}(\mathbb{R}^d \times \mathbb{R}^+)^d \right\}.$$

Indeed, any $\varphi \in E$ admit a unique trace on σ_{KL} , so that the quantities $\langle \mu_{KL}, \varphi \rangle$, $\langle \mu_{LK}, \varphi \rangle$, $\langle \bar{\mu}_{KL}, \varphi \rangle$ and $\langle \bar{\mu}_{LK}, \varphi \rangle$ are well defined. Moreover, one has

$$|\langle \mu, \varphi \rangle| \leq \|\varphi\|_{L^\infty} \mu(\{\varphi \neq 0\}), \quad |\langle \bar{\mu}, \varphi \rangle| \leq \|\varphi\|_{L^\infty} \bar{\mu}(\{\varphi \neq 0\}), \quad \forall \varphi \in E.$$

We now state a lemma and a proposition whose proofs are left to the reader, since they are similar to the proofs of Lemma 3.2 and Proposition 3.3 respectively as one uses the estimates (45) and (47) instead of (46) and (48).

Lemma 3.5. *Let u^h defined by (27)–(29). Assume that (38) holds, then, for all $r > 0$ and $T > 0$ there exist $C_{\bar{\mu}_0} > 0$, depending only on u_0 and r , and $C_{\bar{\mu}} > 0$, depending only on $T, r, a, \lambda^*, u_0, G_{KL}$ such that, for all $h < r$,*

$$(60) \quad \bar{\mu}_0(B(0, r)) \leq C_{\bar{\mu}_0} h \quad \text{and} \quad \bar{\mu}(B(0, r) \times [0, T]) \leq C_{\bar{\mu}} \sqrt{h},$$

where $C_{\bar{\mu}_0} = C_{\mu_0} / \|D\eta\|_\infty$ and $C_{\bar{\mu}} = C_\mu / \|D\eta\|_\infty$ (see the proof of Lemma 3.2).

We are now in position to provide the approximate weak formulation satisfied by u^h . In the statement below, φ is a vector-valued function, and we adopted the notation $|\nabla\varphi| = \max_{\alpha \in \{1, \dots, d\}} |\partial_\alpha \varphi|$. The proof of Proposition 3.6 follows the same guidelines as the proof of Proposition 3.3 and is left to the reader.

Proposition 3.6. *Let μ and μ_0 be the measures introduced in Definition 3.1, then, for all $\varphi \in C_c^1(\mathbb{R}^d \times \mathbb{R}^+; \mathbb{R}^m)$, one has*

$$\begin{aligned} & \left| \iint_{\mathbb{R}^d \times \mathbb{R}_+} \left[(u^h)^T \partial_t \varphi(x, t) + \sum_{\alpha=1}^d f_\alpha(u^h)^T \partial_\alpha \varphi(x, t) \right] dx dt + \int_{\mathbb{R}^d} u_0(x)^T \varphi(x, 0) dx \right| \\ & \leq \iint_{\mathbb{R}^d \times \mathbb{R}_+} (|\nabla\varphi| + |\partial_t \varphi|) d\bar{\mu}(x, t) + \int_{\mathbb{R}^d} |\varphi(x, 0)| d\bar{\mu}_0(x). \end{aligned}$$

4. ERROR ESTIMATE USING THE RELATIVE ENTROPY

With the error measures μ , μ_0 , $\bar{\mu}$, and $\bar{\mu}_0$ at hand, we are now in position to precise inequality (49) satisfied by the relative entropy $H(u^h, u)$ and then to conclude the proof of Theorem 2.7.

4.1. Relative entropy for approximate solutions.

Proposition 4.1. *Let μ and μ_0 be the measures introduced in Definition 3.1, and let $\bar{\mu}$ and $\bar{\mu}_0$ be the measures introduced in Definition 3.4, then, for all $\varphi \in C_c^1(\mathbb{R}^d \times$*

$\mathbb{R}^+; \mathbb{R}^+$), one has

$$(61) \quad \iint_{\mathbb{R}^d \times \mathbb{R}_+} \left(H(u^h, u) \partial_t \varphi(x, t) + \sum_{\alpha=1}^d Q_\alpha(u^h, u) \partial_\alpha \varphi(x, t) \right) dx dt \geq \\ - \iint_{\mathbb{R}^d \times \mathbb{R}_+} (|\nabla \varphi| + |\partial_t \varphi|) d\mu(x, t) - \int_{\mathbb{R}^d} \varphi(x, 0) d\mu_0(x) \\ - \iint_{\mathbb{R}^d \times \mathbb{R}_+} (|\nabla [\varphi D\eta(u)]| + |\partial_t [\varphi D\eta(u)]|) d\bar{\mu}(x, t) \\ - \int_{\mathbb{R}^d} [\varphi D\eta(u)](x, 0) d\bar{\mu}_0(x) + \iint_{\mathbb{R}^d \times \mathbb{R}_+} \varphi \sum_{\alpha=1}^d \partial_\alpha u^T Z_\alpha(u^h, u) dx dt,$$

where $Z_\alpha(u^h, u) = D^2\eta(u)(f_\alpha(u^h) - f_\alpha(u) - (Df_\alpha(u))(u^h - u))$.

Proof. Let φ be any nonnegative Lipschitz continuous test function with compact support in $\mathbb{R}^d \times [0, T]$. Since u is a classical solution of (1)–(3), it satisfies

$$\iint_{\mathbb{R}^d \times \mathbb{R}_+} \eta(u) \partial_t \varphi(x, t) + \sum_{\alpha=1}^d \xi_\alpha(u) \partial_\alpha \varphi(x, t) dx dt + \int_{\mathbb{R}^d} \eta(u_0) \varphi(x, 0) dx = 0.$$

Subtracting this identity to (52) yields

$$(62) \quad \iint_{\mathbb{R}^d \times \mathbb{R}_+} (\eta(u^h) - \eta(u)) \partial_t \varphi(x, t) + \sum_{\alpha=1}^d (\xi_\alpha(u^h) - \xi_\alpha(u)) \partial_\alpha \varphi(x, t) dx dt \\ - \geq \iint_{\mathbb{R}^d \times \mathbb{R}_+} (|\nabla \varphi| + |\partial_t \varphi|) d\mu(x, t) - \int_{\mathbb{R}^d} \varphi(x, 0) d\mu_0(x).$$

We now exhibit the relative entropy-relative entropy flux pair in the inequality (62) and obtain

$$(63) \quad \iint_{\mathbb{R}^d \times \mathbb{R}_+} \left(H(u^h, u) \partial_t \varphi + \sum_{\alpha=1}^d Q_\alpha(u^h, u) \partial_\alpha \varphi \right) dx dt \geq \\ - \iint_{\mathbb{R}^d \times \mathbb{R}_+} |\nabla \varphi| + |\partial_t \varphi| d\mu(x, t) - \int_{\mathbb{R}^d} \varphi(x, 0) d\mu_0(x) \\ - \iint_{\mathbb{R}^d \times \mathbb{R}_+} (D\eta(u))^T \left((u^h - u) \partial_t \varphi + \sum_{\alpha=1}^d (f_\alpha(u^h) - f_\alpha(u)) \partial_\alpha \varphi \right) dx dt.$$

Since u is a strong solution of (1)–(3), it satisfies the following weak identity, $\forall \psi \in C_c(\mathbb{R}^d \times \mathbb{R}^+; \mathbb{R}^m)$

$$(64) \quad \iint_{\mathbb{R}^d \times \mathbb{R}_+} \left[u \partial_t \psi(x, t) + \sum_{\alpha=1}^d f_\alpha(u) \partial_\alpha \psi(x, t) \right] dx dt + \int_{\mathbb{R}^d} u_0(x) \psi(x, 0) dx = 0.$$

Then we combine Proposition 3.6 with (64), so that using the Lipschitz continuous vector field $[\varphi D\eta(u)]$ as test function leads to

$$(65) \quad - \iint_{\mathbb{R}^d \times \mathbb{R}_+} (D\eta(u))^T \left[(u^h - u) \partial_t \varphi + \sum_{\alpha=1}^d (f_\alpha(u^h) - f_\alpha(u)) \partial_\alpha \varphi \right] dx dt \geq \\ - \iint_{\mathbb{R}^d \times \mathbb{R}_+} |\nabla[\varphi D\eta(u)]| + |\partial_t[\varphi D\eta(u)]| d\bar{\mu}(x, t) - \int_{\mathbb{R}^d} [\varphi D\eta(u)](x, 0) d\bar{\mu}_0(x) \\ + \iint_{\mathbb{R}^d \times \mathbb{R}_+} (u^h - u) \varphi \partial_t (D\eta(u)) + \sum_{\alpha=1}^d (f_\alpha(u^h) - f_\alpha(u)) \varphi \partial_\alpha (D\eta(u)) dx dt.$$

Moreover identity (6) together with (1) gives

$$(66) \quad \partial_t (D\eta(u)) = \partial_t u^T D^2 \eta(u) = - \sum_{\alpha=1}^d \partial_\alpha (f_\alpha(u))^T D^2 \eta(u) \\ = - \sum_{\alpha=1}^d \partial_\alpha u^T D f_\alpha(u)^T D^2 \eta(u) = - \sum_{\alpha=1}^d \partial_\alpha u^T D^2 \eta(u) D f_\alpha(u).$$

Injecting (65) and (66) into (63) leads to the conclusion. \square

Lemma 4.2. *There exists C_Z depending only on f, η and Ω such that, for all $\alpha \in \{1, \dots, d\}$,*

$$(67) \quad |Z_\alpha(u^h, u)| \leq C_Z |u^h - u|^2.$$

Proof. For $M : \Omega \rightarrow \mathbb{R}^{m \times m}$ and $\Upsilon : \Omega \rightarrow \mathcal{L}(\mathbb{R}^m; \mathbb{R}^{m \times m})$, we set

$$\|M\|_{\infty, \infty} = \sup_{u \in \Omega} |M(u)|_\infty, \quad \|\Upsilon\|_{\infty, 2} = \sup_{u \in \Omega} \left(\sup_{v \in \mathbb{R}^m, |v|=1} |\Upsilon(u) \cdot v|_2 \right),$$

where $|\cdot|_2$ and $|\cdot|_\infty$ denote the usual matrix 2- and ∞ -norms respectively. Using the Taylor expansion of f_α around u , we get that

$$|f_\alpha(u^h) - f_\alpha(u) - (Df_\alpha(u))(u^h - u)| \leq \frac{1}{2} \|D^2 f_\alpha\|_{\infty, 2} |u^h - u|^2,$$

then, estimate (67) holds for $C_Z = \frac{1}{2} \|D^2 \eta\|_{\infty, \infty} \|D^2 f_\alpha\|_{\infty, 2}$. \square

We now prove the following lemma on the finite speed of propagation.

Lemma 4.3. *Let L_f be defined by (7), then, for all $s \geq L_f$, one has*

$$(68) \quad sH(u^h, u) + \sum_{\alpha=1}^d \frac{x_\alpha}{|x|} Q_\alpha(u^h, u) \geq 0.$$

Proof. Denote by $w^h := u^h - u$, then it follows from the characterization (14) of the relative entropy H that

$$(69) \quad H = \int_0^1 \int_0^\theta (w^h)^T D_u^2 \eta(u + \gamma w^h) w^h d\gamma d\theta.$$

Denoting by \mathbb{A}_γ the symmetric definite positive matrix $D_u^2 \eta(u + \gamma w^h)$, and by $\langle \cdot, \cdot \rangle_{\mathbb{A}_\gamma}$ the scalar product on \mathbb{R}^n defined by $\langle v_1, v_2 \rangle_{\mathbb{A}_\gamma} = v_1^T \mathbb{A}_\gamma v_2$, the relation (69)

can be rewritten

$$(70) \quad H = \int_0^1 \int_0^\theta \langle w^h, w^h \rangle_{\mathbb{A}_\gamma} d\gamma d\theta.$$

On the other hand, it follows from the definition (5) of the entropy flux ξ that

$$\begin{aligned} Q_\alpha &= \int_0^1 (D\eta(u + \theta w^h) - D\eta(u)) (Df_\alpha(u + \theta w^h))^T w^h d\theta \\ &= \int_0^1 \int_0^\theta \langle w^h, (Df_\alpha(u + \theta w^h))^T w^h \rangle_{\mathbb{A}_\gamma} d\gamma d\theta \end{aligned}$$

for all $\alpha \in \{1, \dots, d\}$. The quantity L_f introduced in (7) has been designed so that $|\langle w^h, (Df_\alpha(u + \theta w^h))^T w^h \rangle_{\mathbb{A}_\gamma}| \leq L_f \langle w^h, w^h \rangle_{\mathbb{A}_\gamma}$. Therefore, we obtain

$$(71) \quad |Q_\alpha| \leq L_f \int_0^1 \int_0^\theta \langle w^h, w^h \rangle_{\mathbb{A}_\gamma} d\gamma d\theta = L_f H.$$

The fact that (68) holds is a straightforward consequence of (71). \square

4.2. End of the proof of Theorem 2.7. We now have at hand all the tools needed for comparing u^h to u via the relative entropy $H(u^h, u)$.

Let $\delta \in (0, T)$ be a parameter to be fixed later on, and, for $k \in \mathbb{N}$, we define the nonincreasing Lipschitz continuous function $\theta_k : \mathbb{R}^+ \rightarrow [0, 1]$ by

$$\theta_k(t) = \min \left(1, \max \left(0, \frac{(k+1)\delta - t}{\delta} \right) \right), \quad \forall t \geq 0.$$

Let us also introduce the Lipschitz continuous function $\psi : \mathbb{R}^d \times \mathbb{R}_+ \rightarrow [0, 1]$ defined by $\psi(x, t) = 1 - \min(1, \max(0, |x| - r - L_f(T - t) + 1))$, where L_f is defined by (7). The function $\varphi_k : (x, t) \in \mathbb{R}^d \times \mathbb{R}^+ \mapsto \theta_k(t)\psi(x, t) \in [0, 1]$ can be considered as a test function in (61). Indeed, denoting by

$$\mathcal{I}_k^\delta = [k\delta, (k+1)\delta], \quad \mathcal{C}_{r,T}(t) = \{(x, t) \mid |x| \in [r + L_f(T - t), r + L_f(T - t) + 1]\},$$

one has

$$\begin{aligned} \partial_t \varphi_k(x, t) &= -\frac{1}{\delta} \mathbf{1}_{\mathcal{I}_k^\delta}(t) \psi(x, t) - L_f \theta_k(t) \mathbf{1}_{\mathcal{C}_{r,T}(t)}(x), \\ \nabla \varphi_k(x, t) &= -\frac{x}{|x|} \theta_k(t) \mathbf{1}_{\mathcal{C}_{r,T}(t)}(x), \end{aligned}$$

so that both $\partial_t \varphi_k$ and $|\nabla \varphi_k|$ belong to the set E defined in Remark 3.1. Then taking φ_k as test function in (61) yields

$$\begin{aligned} &\frac{1}{\delta} \int_{\mathcal{I}_k^\delta} \int_{\mathbb{R}^d} H \psi dx dt + \int_0^T \theta_k(t) \int_{\mathbb{R}^d} \sum_{\alpha=1}^d \partial_\alpha u^T Z_\alpha(u^h, u) \psi dx dt \\ &\leq - \int_0^T \theta_k(t) \int_{\mathcal{C}_{r,T}(t)} \left(L_f H + \sum_{\alpha=1}^d Q_\alpha \frac{x_\alpha}{|x|} \right) dx dt + R_1 + R_2 + R_3 + R_4, \end{aligned}$$

where

$$\begin{aligned}
R_1 &= \iint_{\mathbb{R}^d \times [0, T]} (|\nabla \varphi_k(x, t)| + |\partial_t \varphi_k(x, t)|) d\mu(x, t), \\
R_2 &= \int_{\mathbb{R}^d} \psi(x, 0) d\mu_0(x), \\
R_3 &= \iint_{\mathbb{R}^d \times [0, T]} |D\eta(u)| (|\nabla \varphi_k(x, t)| + |\partial_t \varphi_k(x, t)|) d\bar{\mu}(x, t) \\
&\quad + \iint_{\mathbb{R}^d \times [0, T]} \varphi_k(x, t) |D^2\eta(u)(x, t)|_\infty (|\partial_t u| + |\nabla u|) d\bar{\mu}(x, t), \\
R_4 &= \int_{\mathbb{R}^d} \psi(x, 0) |D\eta(u_0)| d\bar{\mu}_0(x).
\end{aligned}$$

Thanks to Lemma 4.3, one has

$$(72) \quad \frac{1}{\delta} \int_{\mathcal{I}_k^\delta} \int_{\mathbb{R}^d} H\psi dx dt + \int_0^T \theta_k(t) \int_{\mathbb{R}^d} \sum_{\alpha=1}^d \partial_\alpha u^T Z_\alpha(u^h, u) \psi dx dt \leq R_1 + R_2 + R_3 + R_4.$$

The definition of φ_k ensures that $\|\varphi_k\|_\infty = 1$, $\|\nabla \varphi_k\|_\infty \leq 1$, $\|\partial_t \varphi_k\|_\infty \leq \frac{1}{\delta} + L_f$ and

$$\text{supp}(\varphi_k) \subset \bigcup_{t \in [0, (k+1)\delta]} B(0, r + L_f(T - t) + 1) \times \{t\},$$

This leads to

$$R_1 \leq \left(\frac{1}{\delta} + L_f + 1 \right) \mu(\text{supp}(\nabla \varphi_k) \cup \text{supp}(\partial_t \varphi_k)).$$

Thanks to Lemma 3.2, we obtain that there exists C_μ^k (depending on $k, r, T, \delta, L_f, a, \lambda^*, u_0, G_{KL}$ and η) such that

$$(73) \quad R_1 \leq C_\mu^k \left(\frac{1}{\delta} + L_f + 1 \right) \sqrt{h}.$$

It follows from similar arguments that there exists $C_{\mu_0}^k$ (depending on k, η, u_0, r, L_f, T and δ) such that

$$(74) \quad R_2 \leq C_{\mu_0}^k h,$$

and, thanks to Lemma 3.5, we obtain that there exists $C_{\mu_0}^k$ (depending on k, r, u_0, L_f, T and δ) such that

$$(75) \quad R_4 \leq C_{\mu_0}^k \|D\eta(u_0)\|_\infty h.$$

Similarly, there exists C_μ^k (depending on $k, T, r, L_f, a, \lambda^*, u_0, G_{KL}$ and δ) such that

$$(76) \quad R_3 \leq C_\mu^k \left(\|D\eta(u)\|_\infty \left(\frac{1}{\delta} + L_f + 1 \right) + \|D^2\eta\|_{\infty, \infty} (\|\partial_t u\|_\infty + \|\nabla u\|_\infty) \right) \sqrt{h}.$$

By using Lemma 4.2 and $0 \leq \theta_k(t) \leq 1$, we obtain

$$\begin{aligned}
(77) \quad & \int_0^T \theta_k(t) \int_{\mathbb{R}^d} \sum_{\alpha=1}^d \partial_\alpha u^T Z_\alpha(u^h, u) \psi dx dt \\
& \geq -C_Z \|\nabla u\|_\infty \iint_{\mathbb{R}^d \times [0, (k+1)\delta]} |u^h(x, t) - u(x, t)|^2 \psi(x, t) dx dt.
\end{aligned}$$

Since the entropy η is supposed to be β_0 -convex, we have

$$(78) \quad H(x, t) \geq \frac{\beta_0}{2} |u^h(x, t) - u(x, t)|^2.$$

Putting (73)–(78) together with (72) provides

$$(79) \quad \left(\frac{\beta_0}{2\delta} - C_Z \|\nabla u\|_\infty \right) \int_{\mathcal{I}_k^\delta} \int_{\mathbb{R}^d} |u^h - u|^2 \psi \, dxdt \\ \leq C_Z \|\nabla u\|_\infty \iint_{\mathbb{R}^d \times [0, k\delta]} |u^h - u|^2 \psi \, dxdt + C_k \sqrt{h},$$

where (recall that $h \leq 1$)

$$C_k = C_\mu^k \left(\frac{1}{\delta} + L_f + 1 \right) + C_{\mu_0}^k + C_{\mu_0}^k \|D\eta(u_0)\|_\infty \\ + C_{\mu}^k \left(\|D\eta(u)\|_\infty \left(\frac{1}{\delta} + L_f + 1 \right) + \|D^2\eta\|_{\infty, \infty} (\|\partial_t u\|_\infty + \|\nabla u\|_\infty) \right).$$

Choose now $\delta = \frac{T}{p^*+1}$ with $p^* = \min \left\{ p \in \mathbb{N}^* \mid \frac{T}{p+1} \leq \frac{\beta_0}{2C_Z \|\nabla u\|_\infty + 2} \right\}$ (note that neither δ nor p^* depend on h), so that (79) becomes

$$(80) \quad e_k \leq \omega \sum_{i=0}^{k-1} e_i + C_k \sqrt{h},$$

where $e_k = \int_{\mathcal{I}_k^\delta} \int_{\mathbb{R}^d} |u^h - u|^2 \psi \, dxdt$ and $\omega = C_Z \|\nabla u\|_\infty$. Hence, a few algebraic calculations allow us to claim that

$$(81) \quad \sum_{k=0}^{p^*} e_k \leq \sqrt{h} \sum_{k=0}^{p^*} C_k \left((1 + \omega)^{p^* - k + 1} - \omega \right).$$

Noticing that $\psi(x, t) = 1$ if $x \in B(0, r + L_f(T - t))$, and that $\psi(x, t) \geq 0$ for all $(x, t) \in \mathbb{R}^d \times (0, T)$, one finally has

$$(82) \quad \int_0^T \int_{B(0, r-st)} |u - u^h|^2 \, dxdt \leq \int_0^T \int_{\mathbb{R}^d} |u - u^h|^2 \psi(x, t) \, dxdt = \sum_{k=0}^{p^*} e_k.$$

We conclude the proof using (81) in (82).

5. CONCLUSION

We analyzed the convergence of first order finite volume schemes entering the framework detailed in [5] and summarized in §1.2.2. In §2.2, we derived a so-called *weak-BV* estimate based on the quantification of the numerical entropy dissipation. This estimate is new in the case of time-explicit finite volume schemes. It allows to prove some error estimate between a numerical solution and a strong solution of order $h^{1/4}$ in the space-time L^2 -norm. Let us also mention that one could use the weak-BV estimate to prove to convergence to entropy measure-valued solutions, following [21] (see also [29]). On the other hand, strong solutions are global if one adds some suitable entropy-dissipating relaxation term [27, 51]), and our work could be extended to this situation without any major difficulty by mixing our result with the one proposed in [32].

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