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ERROR ESTIMATE FOR TIME-EXPLICIT FINITE VOLUME APPROXIMATION OF STRONG SOLUTIONS TO SYSTEMS OF CONSERVATION LAWS*

CLÉMENT CANCÈS^{†‡}, HÉLÈNE MATHIS[§], AND NICOLAS SEGUIN^{†‡}

Abstract. We study the finite volume approximation of strong solutions to nonlinear systems of conservation laws. We focus on time-explicit schemes on unstructured meshes, with entropy satisfying numerical fluxes. The numerical entropy dissipation is quantified at each interface of the mesh, which enables to prove a weak- BV estimate for the numerical approximation under a strengthened CFL condition. Then we derive error estimates in the multidimensional case, using the relative entropy between the strong solution and its finite volume approximation. The error terms are carefully studied, leading to a classical $h^{1/4}$ estimate in L^2 under this strengthened CFL condition.

Key words. Hyperbolic systems, finite volume scheme, relative entropy, error estimate

AMS subject classifications. 35L65,65M08,65M12,65M15

1. Introduction. This paper deals with error estimates of the finite volume approximation of hyperbolic systems of conservation laws. The solutions to such systems may develop discontinuities in finite time and, since the pioneering work of Lax [32], entropy conditions are added to select physical/admissible solutions. However, it has been shown by De Lellis and Székelyhidi Jr in [12, 13] that such a criterion is not sufficient in the multidimensional case. Nonetheless, it is known since several decades (see in particular [15, 10]) that if a strong solution exists, then there exists a unique entropy solution corresponding to the same initial data, and that it coincides with this strong solution. Moreover, it can be shown that entropy solutions are stable with respect to strong solutions. Since error estimates of any approximation is based on the stability properties of the model, we restrict this study to strong solutions which are known to exist, in finite time, and to be unique.

The well-posedness of the scalar case has been proved by Kruzhkov in [29], in several dimensions. The first error estimate for finite volume methods on cartesian grids has been obtained by Kuznetsov [30], who provides an estimate in $h^{1/2}$ (h being the characteristic size of the grid) for initial data of bounded variations. The multidimensional case for scalar conservation laws is much more tricky as soon as unstructured meshes are considered [8, 9, 40, 18, 7]. These works use the DiPerna's uniqueness theorem [17] and lead to an $h^{1/4}$ error estimate.

In the case of nonlinear systems, only few results exist (the linear case, or more precisely the case of Friedrichs systems, are been studied in [41] and [25]). In one space dimension, the first convergence study is due to DiPerna in [16]. In the multidimensional case, if arbitrary times are considered, the convergence can be reached only towards measure-valued solutions (due to the lack of uniqueness of entropy weak solutions mentioned above), see [24] where space-time discontinuous Galerkin meth-

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[†]UPMC Univ Paris 06, UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France

[‡]CNRS, UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France (cances@ljl.math.upmc.fr, nicolas.seguin@upmc.fr)

[§]Université de Nantes, Laboratoire de Mathématiques Jean Leray, 2, Rue de la Houssinière, 44322 Nantes Cedex 03, France (helene.mathis@univ-nantes.fr)

ods are analyzed. On the other hand, an $h^{1/4}$ error estimate between approximate and strong solution has been proved in [26] for time-implicit finite volume schemes.

The key-point in all the multidimensional studies is the control of the BV seminorm. For unstructured meshes, one can only prove that it grows as $h^{-1/2}$ (even in the scalar case, cf. [14]), which actually is the main barrier to obtain expected rate of convergence. Here, we prove for first-order time-explicit finite volume schemes such an estimate, under classical assumptions on the numerical flux (see for instance [5] and [38]). As in the scalar case, this so-called weak- BV estimate requires a slightly reduced CFL condition, in order to ensure a sufficient amount of numerical entropy dissipation (see for instance [19]). We also simplify the framework of [26] and prove an error estimate in $h^{1/4}$.

REMARK 1.1. *Let us also stress that, in the one-dimensional case, more accurate results may exist. In [4], the authors prove the optimal rate of convergence in the case of strong solutions. On the other hand, Laforest in [31] is able to obtain a posteriori estimates for front-tracking methods.*

1.1. Hyperbolic systems of conservation laws.

1.1.1. Strong, weak, and entropy weak solutions. We consider a system of m conservation laws

$$(1.1) \quad \partial_t u(x, t) + \sum_{\alpha=1}^d \partial_\alpha f_\alpha(u)(x, t) = 0.$$

System (1.1) is set on the whole space $x \in \mathbb{R}^d$, and for any time $t \in [0, T]$, $T > 0$. We assume that there exists a convex bounded subset of \mathbb{R}^m , denoted by Ω and called *set of the admissible states* such that

$$(1.2) \quad u(x, t) \in \Omega, \quad \forall (x, t) \in \mathbb{R}^d \times [0, T].$$

System (1.1) is complemented with the initial condition

$$(1.3) \quad u(x, 0) = u_0(x) \in \Omega, \quad \forall x \in \mathbb{R}^d.$$

We assume for all $\alpha \in \{1, \dots, d\}$ the functions $f_\alpha : \mathbb{R}^m \rightarrow \mathbb{R}^m$ to belong to $C^2(\bar{\Omega}; \mathbb{R}^m)$, and be such that Df_α are diagonalizable with real eigenvalues, where D denotes the differential with respect to the variables u .

System (1.1) is endowed with a uniformly convex entropy $\eta \in C^2(\bar{\Omega}; \mathbb{R})$ such that there exists $\beta_1 \geq \beta_0 > 0$ so that

$$(1.4) \quad \text{spec}(D^2\eta(u)) \subset [\beta_0; \beta_1], \quad \forall u \in \bar{\Omega},$$

and the corresponding entropy flux $\xi \in C^2(\bar{\Omega}; \mathbb{R}^d)$ satisfies for all $\alpha \in \{1, \dots, d\}$

$$(1.5) \quad D\xi_\alpha(u) = D\eta(u)Df_\alpha(u), \quad \forall u \in \Omega.$$

Without loss of generality, we assume that $\eta(u) \geq 0$ for all $u \in \bar{\Omega}$. The existence of the entropy flux ξ amounts to assume the integrability condition (see e.g. [22])

$$(1.6) \quad D^2\eta(u)Df_\alpha(u) = Df_\alpha(u)^T D^2\eta(u), \quad \forall u \in \Omega.$$

Let us introduce the quantity L_f by

$$(1.7) \quad L_f = \sup_{\alpha \in \{1, \dots, d\}} \sup_{(u, v) \in \Omega^2} \sup_{w \in \mathbb{R}^m \setminus \{0\}} \left| \frac{w^T D^2\eta(v) Df_\alpha(u) w}{w^T D^2\eta(v) w} \right|.$$

REMARK 1.2. Notice that, in view of (1.6), the matrix $Df_\alpha(u)$ is self-adjoint for the scalar product $\langle w, v \rangle_u = w^T D^2\eta(u)v$. Therefore, the Rayleigh quotient

$$(1.8) \quad \sup_{w \in \mathbb{R}^m \setminus \{0\}} \left| \frac{w^T D^2\eta(u) Df_\alpha(u) w}{w^T D^2\eta(u) w} \right| = \sup_{w \in \mathbb{R}^m \setminus \{0\}} \frac{\langle w, Df_\alpha(u) w \rangle_u}{\langle w, w \rangle_u}$$

provides exactly the largest eigenvalue in absolute value of $Df_\alpha(u)$. The situation in (1.7) is more intricate than in (1.8) since u might be different of v , but the quantity L_f is bounded in view of the boundedness of Ω and of the regularity of f_α and η .

Despite it is well-known that even for smooth initial data u_0 , the solutions of (1.1)–(1.3) may develop discontinuities after a finite time, our study is restricted to the approximation of smooth solutions $u \in W^{1,\infty}(\mathbb{R}^d \times \mathbb{R}_+; \Omega)$ to (1.1)–(1.3). Such solutions are called *strong solutions*, and they satisfy the conservation of the entropy

$$(1.9) \quad \partial_t \eta(u) + \sum_{\alpha=1}^d \partial_\alpha \xi_\alpha(u) = 0 \quad \text{in } \mathbb{R}^d \times \mathbb{R}_+.$$

We refer for instance to [27, 34, 11] for specific results on strong solutions of systems of conservation laws.

Assuming that $u_0 \in L^\infty(\mathbb{R}^d; \Omega)$, a function $u \in L^\infty(\mathbb{R}^d \times \mathbb{R}_+; \Omega)$ is said to be a *weak solution* to (1.1)–(1.3) if, for all $\phi \in C_c^1(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R}^n)$, one has

$$(1.10) \quad \iint_{\mathbb{R}^d \times \mathbb{R}_+} u \partial_t \phi \, dx dt + \int_{\mathbb{R}^d} u_0 \phi(\cdot, 0) \, dx + \iint_{\mathbb{R}^d \times \mathbb{R}_+} \sum_{\alpha=1}^d f_\alpha(u) \partial_\alpha \phi \, dx dt = 0.$$

Moreover, u is said to be an *entropy weak solution* to (1.1)–(1.3) if u is a weak solution, i.e., u satisfies (1.10), and if, for all $\psi \in C_c^1(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R}_+)$, it satisfies

$$(1.11) \quad \iint_{\mathbb{R}^d \times \mathbb{R}_+} \eta(u) \partial_t \psi \, dx dt + \int_{\mathbb{R}^d} \eta(u_0) \psi(\cdot, 0) \, dx + \iint_{\mathbb{R}^d \times \mathbb{R}_+} \sum_{\alpha=1}^d \xi_\alpha(u) \partial_\alpha \psi \, dx dt \geq 0.$$

1.1.2. Relative entropy. In the scalar case, the comparison of two entropy weak solutions lies on the Kruzhkov’s paper [29], which has been extended to the comparison of an entropy weak solution with an approximate solution by Kuznetsov [30]. In the case of systems of conservation laws, these techniques no longer work. Basically, the family of entropy–entropy flux pairs (η, ξ) is not sufficiently rich to control the difference between two solutions. Nevertheless, let us assume that one of these solutions is a strong solution, u in the sequel, and introduce:

DEFINITION 1.1 (Relative entropy). Let $u, v \in \Omega$. The relative entropy of v w.r.t. u is defined by

$$H(v, u) = \eta(v) - \eta(u) - D\eta(u)(v - u),$$

and the corresponding relative entropy fluxes $Q : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^d$ are

$$Q_\alpha(v, u) = \xi_\alpha(v) - \xi_\alpha(u) - D\eta(u)(f_\alpha(v) - f_\alpha(u)), \quad \forall \alpha \in \{1, \dots, d\}.$$

The notion of relative entropy for systems of conservation laws goes back to the early works of DiPerna and Dafermos (see [15], [10] and the condensed presentation in

[11]). It has also been extensively used for the study of hydrodynamic limits of kinetic equations (see the first works [42] and [1], but also [36] for more recent results). For systems of conservation laws, one can check that, given a strong solution u and an entropy weak solution v with respective initial data u_0 and v_0 , one has

$$(1.12) \quad \partial_t H(v, u) + \sum_{\alpha=1}^d \partial_\alpha Q_\alpha(v, u) \leq - \sum_{\alpha=1}^d (\partial_\alpha u)^T Z_\alpha(v, u)$$

in the weak sense, where

$$(1.13) \quad Z_\alpha(v, u) = D^2 \eta(u) (f_\alpha(v) - f_\alpha(u) - Df_\alpha(u)(v - u)).$$

On the other hand, it follows from the definition of H that

$$(1.14) \quad H(v, u) = \int_0^1 \int_0^\theta (v - u)^T D^2 \eta(u + \gamma(v - u))(v - u) d\gamma d\theta,$$

which, together with (1.4), leads to

$$(1.15) \quad \frac{\beta_0}{2} |v - u|^2 \leq H(v, u) \leq \frac{\beta_1}{2} |v - u|^2, \quad \forall u, v \in \Omega.$$

If u is assumed to be a strong solution, its first derivative is bounded and by a classical localization procedure *à la* Kruzhkov and a Gronwall lemma, one obtains a L^2_{loc} stability estimate for any $r > 0$

$$(1.16) \quad \int_{|x| < r} |v(x, T) - u(x, T)|^2 dx \leq C(T, u) \int_{|x| < r + L_f T} |v_0(x) - u_0(x)|^2 dx,$$

where the dependence of C on u reflect the needs of smoothness on u (C blows up when u becomes discontinuous). This inequality, rigorously proved in [11], provides a weak-strong uniqueness result. Similar (but more sophisticated) ideas have been applied to other fluid systems, see for instance [35] and [20] for more recent developments.

REMARK 1.3. *In [39], Tzavaras studies the comparison of solutions of a hyperbolic system with relaxation with solutions of the associated equilibrium system of conservation laws. He also makes use of the relative entropy for strong solutions. Very similar questions have been addressed in [3, 2] for the convergence of kinetic equations towards the system of gas dynamics. Here again, only strong solutions of the Euler equations are considered. To finish the bibliographical review, let us mention the work by Leger and Vasseur [33] where the reference solution may include some particular discontinuities.*

REMARK 1.4. *For general conservation laws, the relative entropy is not symmetric, i.e., $H(u, v) \neq H(v, u)$ and $Q(u, v) \neq Q(v, u)$. In the very particular case of Friedrichs systems, i.e. when there exist symmetric matrices $A_\alpha \in \mathbb{R}^{m \times m}$ ($\alpha \in \{1, \dots, d\}$) such that $f_\alpha(u) = A_\alpha u$, then $u \mapsto |u|^2$ is an entropy and the corresponding entropy flux ξ is $\xi_\alpha(u) = u^T A_\alpha u$, ($\alpha \in \{1, \dots, d\}$). It is then easy to check that*

$$H(v, u) = H(v, u) = |u - v|^2, \quad Q_\alpha(v, u) = Q_\alpha(u, v) = (v - u)^T A_\alpha (v - u),$$

and $Z_\alpha(v, u) = 0$ for all $(u, v) \in \mathbb{R}^m$. As a consequence, inequality (1.12) becomes

$$\partial_t H(v, u) + \sum_{\alpha=1}^d \partial_\alpha Q_\alpha(v, u) \leq 0,$$

even if u is only a weak solution. This allows to make use of the doubling variable technique [29] to compare u to v , recovering the classical uniqueness result for Friedrichs systems [21].

Our aim is to replace the entropy weak solution v in (1.12) by an approximate solution provided by finite volume schemes on unstructured meshes. Following the formalism introduced in [18], this makes appear in (1.12) bounded Radon measures which can be controlled, leading to error estimates in $h^{1/4}$ between a strong solution and its finite volume approximation, h being the characteristic size of the cells of the mesh. The purpose of the following section is to define the finite volume scheme and to recall some classical properties required on the numerical fluxes.

1.2. Definition of the time-explicit finite volume scheme.

1.2.1. Space and time discretizations. Let \mathcal{T} be a mesh of \mathbb{R}^d such that \mathbb{R}^d is the union of the closure of the elements of \mathcal{T} . We denote $h = \sup\{\text{diam}(K), K \in \mathcal{T}\} < \infty$, and assume without loss of generality that $h \leq 1$. For all $K \in \mathcal{T}$, we denote by $|K|$ its d -dimensional Lebesgue measure, and by $\mathcal{N}(K)$ the set of its neighboring cells. For $L \in \mathcal{N}(K)$, the common interface (called *edge*) between K and L is denoted by σ_{KL} and $|\sigma_{KL}|$ is its $(d - 1)$ -Lebesgue measure. We denote by \mathcal{E} the set of all the edges and assume that there exists $a > 0$ such that

$$(1.17) \quad |K| \geq ah^d \quad \text{and} \quad |\partial K| := \sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| \leq \frac{h^{d-1}}{a}, \quad \forall K \in \mathcal{T}.$$

The unit normal vector to σ_{KL} from K to L is denoted n_{KL} . Let $\Delta t > 0$ be the time step and we set $t^n = n\Delta t$, $\forall n \in \mathbb{N}$. Let $T > 0$ be a given time, we introduce $N_T = \max\{n \in \mathbb{N}, n \leq T/\Delta t + 1\}$. Since we consider time-explicit methods, the time step Δt will be subject to a CFL condition which will be given later.

REMARK 1.5. *In order to avoid some additional heavy notations, we have chosen to deal with an uniform time discretization and a space discretization that does not depend on time. Nevertheless, it is possible, following the path described in [28], to adapt our study to the case of time depending space discretizations and to non-uniform time discretizations. This would be mandatory for considering a dynamic mesh adaptation procedure based on the a posteriori numerical error estimators that can be derived from our study.*

Since we will consider weak formulations and compact supported test functions in the next sections, we introduce local sets of cells and interfaces: let $r > 0$, we denote

$$(1.18) \quad \begin{aligned} \mathcal{T}_r &= \{K \in \mathcal{T} \mid K \subset B(0, r)\}, \\ \mathcal{E}_r &= \{\sigma_{KL} \in \mathcal{E} \mid (K, L) \in (\mathcal{T}_r)^2, L \in \mathcal{N}(K)\}, \\ \partial\mathcal{T}_r &= \{\sigma_{KL} \in \mathcal{E} \mid K \in \mathcal{T}_r, L \in \mathcal{N}(K), L \notin \mathcal{T}_r\}. \end{aligned}$$

In particular, $\{\sigma_{KL} \in \mathcal{E} \mid K \in \mathcal{T}_r, L \in \mathcal{N}(K)\} = \mathcal{E}_r \cup \partial\mathcal{T}_r$ and $\mathcal{E}_r \cap \partial\mathcal{T}_r = \emptyset$.

1.2.2. Numerical flux and finite volume schemes. For all $(K, L) \in \mathcal{T}^2$, $L \in \mathcal{N}(K)$, we consider numerical fluxes G_{KL} , which are Lipschitz continuous functions from Ω^2 to \mathbb{R}^m . We assume that these numerical fluxes are *conservative*, i.e.,

$$(1.19) \quad G_{KL}(u, v) = -G_{LK}(v, u), \quad \forall (u, v) \in \Omega^2,$$

We also assume that the numerical fluxes fulfill the following *consistency* condition:

$$(1.20) \quad G_{KL}(u, u) = f(u) \cdot n_{KL}, \quad \forall u \in \Omega,$$

which implies

$$(1.21) \quad \sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| G_{KL}(u, u) = 0, \quad \forall u \in \mathbb{R}^m, \forall K \in \mathcal{T}.$$

Following [5], we also assume that they *preserve the admissible states by interface*. More precisely, we assume that there exists $\lambda^* > 0$ such that, for all $\lambda > \lambda^*$, for all $K \in \mathcal{T}$, and for all $L \in \mathcal{N}(K)$,

$$(1.22) \quad u - \frac{1}{\lambda} (G_{KL}(u, v) - f(u) \cdot n_{KL}) \in \Omega, \quad \forall (u, v) \in \Omega^2.$$

In order to ensure the nonlinear stability of the scheme, we also require the existence of a *numerical entropy flux*. More precisely, we assume that for all $(K, L) \in \mathcal{E}$, there exist Lipschitz continuous functions $\xi_{KL} : \Omega \times \Omega \rightarrow \mathbb{R}$ which are conservative, i.e.,

$$(1.23) \quad \xi_{KL}(u, v) = -\xi_{LK}(v, u), \quad \forall (u, v) \in \Omega^2,$$

and satisfy the interfacial entropy inequalities: for all $\lambda \geq \lambda^* > 0$, for all $(u, v) \in \Omega^2$,

$$(1.24) \quad \xi_{KL}(u, v) - \xi(u) \cdot n_{KL} \leq -\lambda \left(\eta \left(u - \frac{1}{\lambda} (G_{KL}(u, v) - f(u) \cdot n_{KL}) \right) - \eta(u) \right).$$

In what follows, and before strengthening it in (2.9), we assume that the following CFL condition is fulfilled:

$$(1.25) \quad \frac{\Delta t}{|K|} \lambda^* \sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| \leq 1, \quad \forall K \in \mathcal{T}.$$

Note that the regularity of the mesh (1.17) implies that (1.25) holds if

$$(1.26) \quad \Delta t \leq \frac{a^2}{\lambda^*} h.$$

We have now introduced all the necessary material to define the time-explicit numerical scheme we will consider.

DEFINITION 1.2 (Finite volume scheme). *The finite volume scheme is defined by the discrete unknowns u_K^n , $K \in \mathcal{T}$ and $n \in \{0, \dots, N_T\}$, which satisfy*

$$(1.27) \quad \frac{u_K^{n+1} - u_K^n}{\Delta t} |K| + \sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| G_{KL}(u_K^n, u_L^n) = 0$$

together with the initial condition

$$(1.28) \quad u_K^0 = \frac{1}{|K|} \int_K u_0(x) dx, \quad \forall K \in \mathcal{T},$$

under assumptions (1.19)–(1.24) on the numerical flux G_{KL} and under the CFL condition (1.26). The approximate solution $u^h : \mathbb{R}^d \times \mathbb{R}_+ \rightarrow \mathbb{R}^m$ provided by the finite volume scheme (1.27)–(1.28) is defined by

$$(1.29) \quad u^h(x, t) = u_K^n, \quad \text{for } x \in K, \quad t^n \leq t < t^{n+1}, \quad K \in \mathcal{T}, \quad n \in \{0, \dots, N_T\}.$$

1.3. Error estimate and organization of the paper. Our aim is to provide an error estimate of the the form

$$\|u - u^h\|_{L^2(\Gamma)} \leq Ch^{1/4},$$

for all compact subset Γ of $\mathbb{R}^d \times \mathbb{R}_+$, where u stands for the unique strong solution to (1.1), (1.3) and u_h for the numerical solution (1.27)–(1.29). The rigorous statement is given in Theorem 2.7. This estimate extends to the system case the contributions of [9, 40, 18, 7] on the scalar case. In [26], which also deals with strong solutions of nonlinear systems, the assumptions are less classical than ours, in particular we do not need any ‘inverse’ CFL condition of the form $C \leq \Delta t/h$ (see also [18] for a similar comment in the scalar case).

The proof of this estimate relies on a so-called *weak-BV estimate*, that is

$$\sum_{n=0}^{N_T} \Delta t \sum_{(K,L) \in \mathcal{E}_r} |\sigma_{KL}| |G_{KL}(u_K^n, u_L^n) - f(u_K^n) \cdot n_{KL}| \leq \frac{C}{\sqrt{h}},$$

where \mathcal{E}_r is defined in (1.18). The rigorous statement of this estimate and its proof are gathered in §2.2. Up to the authors’ knowledge, this estimate is new for time-explicit finite volume schemes: in [26], only time-implicit methods are considered (see also [24]).

Let us now present the outline of the paper. In Section 2 we first briefly recall some classical properties of the finite volume scheme. Then we address the proof of the weak-BV property by introducing a new flux which depicts the entropy dissipation through the edges. Straightforward consequences are then derived. The next two sections address the proof of the error estimate. In order to compare the discrete solution u^h with the strong solution u , we write continuous weak and entropy formulations for u^h in Section 3, so that we can adapt the uniqueness proof proposed in [11]. Nevertheless, the discrete solution u^h is obviously not a weak entropy solution. Therefore, some error terms coming from the discretization have to be taken into account in the formulation, which take the form of positive locally bounded Radon measures, following [18]. A large part of Section 3 consists in making these measures explicit and in bounding them with quantities which tend to 0 with the discretization size. In Section 4, we make use of the weak and entropy weak formulations for the discrete solution (and of their corresponding error measures) to derive the error estimate. The distance between the strong solution u and the discrete solution u^h is quantified thanks to the relative entropy $H(u^h, u)$ introduced in Definition 1.1.

2. Nonlinear stability.

2.1. Preservation of admissible states and discrete entropy inequality.

We first give two classical properties of the numerical scheme (1.27) which are direct consequences of the assumptions we made in §1.2.2: preservation of the set of admissible states and discrete cell-entropy inequalities. We refer to [5] for the proofs.

LEMMA 2.1. *Assume that the initial condition satisfies (1.3) and that the assumption of preservation of the admissible states by interface (1.22) and the CFL condition (1.25) hold, then, for all $K \in \mathcal{T}$, for all $n \in \{0, \dots, N_T\}$, u_K^n belong to Ω .*

Following once again the procedure detailed in [5], we can derive entropy properties on the numerical scheme from (1.24).

PROPOSITION 2.2. *The numerical entropy flux ξ_{KL} is consistent with ξ , i.e.*

$$(2.1) \quad \xi_{KL}(u, u) = \xi(u) \cdot n_{KL}, \quad \forall u \in \Omega.$$

Moreover, under the CFL condition (1.26), the discrete solution u^h satisfies the discrete entropy inequalities: $\forall K \in \mathcal{T}, \forall n \geq 0$,

$$(2.2) \quad \frac{|K|}{\Delta t} (\eta(u_K^{n+1}) - \eta(u_K^n)) + \sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| \xi_{KL}(u_K^n, u_L^n) \leq 0.$$

Note that the consistency (2.1) of the entropy fluxes ξ_{KL} ensures that

$$(2.3) \quad \sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| \xi_{KL}(u, u) = 0, \quad \forall u \in \Omega.$$

2.2. Weak-BV inequality for systems of conservation laws. For all $(K, L) \in \mathcal{T}^2$, $L \in \mathcal{N}(K)$, we introduce the flux

$$(2.4) \quad X_{KL}(u, v) := \xi(u) \cdot n_{KL} + D\eta(u)(G_{KL}(u, v) - f(u) \cdot n_{KL}), \quad \forall (u, v) \in \Omega^2.$$

Let us remark that it is neither symmetric nor conservative. Such a quantity may provide the connection between fully discrete and semi-discrete entropy satisfying schemes, but also between entropy-conservative and entropy-stable schemes. It is in particular shown in [5] (see also [37, 38]) that the fluxes X_{KL} for $(K, L) \in \mathcal{E}$ verify

$$(2.5) \quad -X_{LK}(v, u) \leq \xi_{KL}(u, v) \leq X_{KL}(u, v), \quad \forall (u, v) \in \Omega^2.$$

Actually, inequalities (2.5) can be specified by quantifying the entropy dissipation across the edges.

PROPOSITION 2.3. *For all $\sigma_{KL} \in \mathcal{E}$ and all $(u, v) \in \Omega^2$, one has*

$$(2.6) \quad X_{KL}(u, v) - \xi_{KL}(u, v) \geq \frac{\beta_0}{2\lambda^*} |G_{KL}(u, v) - f(u) \cdot n_{KL}|^2,$$

where β_0 is defined in (1.4) and λ^* has to be such that (1.22) and (1.24) hold.

Proof. We rewrite the left-hand side of Ineq. (1.24) for $\lambda = \lambda^*$ using the definition (2.4) of the flux X_{KL} in order to obtain

$$(2.7) \quad \begin{aligned} X_{KL}(u, v) - \xi_{KL}(u, v) - D\eta(u)(G_{KL}(u, v) - f(u) \cdot n_{KL}) \\ \geq \lambda^* \left[\eta(u - \frac{1}{\lambda^*}(G_{KL}(u, v) - f(u) \cdot n_{KL})) - \eta(u) \right]. \end{aligned}$$

The uniform convexity (1.4) of η ensures that

$$(2.8) \quad \begin{aligned} \lambda^* \left[\eta(u - \frac{1}{\lambda^*}(G_{KL}(u, v) - f(u) \cdot n_{KL})) - \eta(u) \right] \\ \geq -D\eta(u)(G_{KL}(u, v) - f(u) \cdot n_{KL}) + \frac{1}{2} \frac{\beta_0}{\lambda^*} |G_{KL}(u, v) - f(u) \cdot n_{KL}|^2. \end{aligned}$$

Combining (2.7) and (2.8) leads to (2.6). \square

Thanks to the specified version (2.6) of the classical inequalities (2.5), we are now in position for proving a new stability estimate for time-explicit finite volume scheme, namely the weak-BV inequality. This inequality is obtained by quantifying the numerical diffusion of the numerical scheme. As in the scalar case (see [8, 9, 40,

18, 7]), such an equality requires a strengthened CFL condition. In our system case, we require the existence of some $\zeta \in (0, 1)$ such that

$$(2.9) \quad \Delta t \leq \frac{\beta_0}{\beta_1} \frac{a^2}{\lambda^*} (1 - \zeta) h$$

holds, where β_0 and β_1 are defined by (1.4), a and h are the mesh parameters (1.17), and where λ^* appears in the condition (1.22) and (1.24). Note that the strengthened CFL condition (2.9) implies the classical CFL condition (1.26). We are now able to obtain the following local estimate, using the notations (1.18).

PROPOSITION 2.4. *Assume that the strengthened CFL condition (2.9) holds, then there exists C depending only on $T, r, a, \eta, \xi, \Omega$ and ζ (but neither on h nor on Δt) such that*

$$(2.10) \quad \sum_{n=0}^{N_T} \Delta t \sum_{(K,L) \in \mathcal{E}_r} |\sigma_{KL}| |G_{KL}(u_K^n, u_L^n) - f(u_K^n) \cdot n_{KL}|^2 \leq C.$$

Proof. Multiplying the numerical scheme (1.27) by $\Delta t D\eta(u_K^n)$ and summing over $n \in \{0, \dots, N_T\}$ and $K \in \mathcal{T}_r$ provides

$$(2.11) \quad A + B = 0,$$

where

$$\begin{aligned} A &= \sum_{n=0}^{N_T} \sum_{K \in \mathcal{T}_r} D\eta(u_K^n) (u_K^{n+1} - u_K^n) |K|, \\ B &= \sum_{n=0}^{N_T} \Delta t \sum_{K \in \mathcal{T}_r} D\eta(u_K^n) \sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| G_{KL}(u_K^n, u_L^n). \end{aligned}$$

The concavity of $u \mapsto \eta(u) - \frac{\beta_1}{2} |u - u_K^n|^2$ together with the definition (1.27) of the numerical scheme and property (1.21) provide that

$$\begin{aligned} A &\geq \sum_{K \in \mathcal{T}_r} \eta(u_K^{N_T+1}) |K| - \sum_{K \in \mathcal{T}_r} \eta(u_K^0) |K| \\ &\quad - \frac{\beta_1}{2} \sum_{n=0}^{N_T} \Delta t^2 \sum_{K \in \mathcal{T}_r} \frac{1}{|K|} \left| \sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| (G_{KL}(u_K^n, u_L^n) - f(u_K^n) \cdot n_{KL}) \right|^2. \end{aligned}$$

Using the Jensen inequality, we get

$$\sum_{K \in \mathcal{T}_r} \eta(u_K^0) |K| \leq \int_{|x| \leq B(0, R+h)} \eta(u_0(x)) dx =: C_1.$$

The positivity of the entropy η yields $\sum_{K \in \mathcal{T}_r} \eta(u_K^{N_T+1}) |K| \geq 0$. Moreover, Cauchy–

Schwarz inequality ensures that for all $K \in \mathcal{T}_r$ and all $n \in \{0, \dots, N_T\}$, one has

$$\begin{aligned} &\left| \sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| (G_{KL}(u_K^n, u_L^n) - f(u_K^n) \cdot n_{KL}) \right|^2 \\ &\leq \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| \right) \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| |G_{KL}(u_K^n, u_L^n) - f(u_K^n) \cdot n_{KL}|^2 \right). \end{aligned}$$

Then it follows from the regularity assumption (1.17) on the mesh that

$$(2.12) \quad A \geq -C_1 - \frac{\beta_1 \Delta t}{2a^2 h} \sum_{n=0}^{N_T} \Delta t \sum_{K \in \mathcal{T}_r} \sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| |G_{KL}(u_K^n, u_L^n) - f(u_K^n) \cdot n_{KL}|^2.$$

Concerning the term B , we use the definition (2.4) of the entropy flux X_{KL} to get

$$B = \sum_{n=0}^{N_T} \Delta t \sum_{K \in \mathcal{T}_r} \sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| (X_{KL}(u_K^n, u_L^n) - \xi(u_K^n) \cdot n_{KL} + D\eta(u_K^n) f(u_K^n) \cdot n_{KL}).$$

Using the property $\sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| n_{KL} = 0$ for all $K \in \mathcal{T}$, we can reorganize the term B into

$$(2.13) \quad B = B_1 + B_2,$$

where

$$B_1 = \sum_{n=0}^{N_T} \Delta t \sum_{K \in \mathcal{T}_r} \sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| (X_{KL}(u_K^n, u_L^n) - \xi_{KL}(u_K^n, u_L^n)),$$

$$B_2 = \sum_{n=0}^{N_T} \Delta t \sum_{(K,L) \in \partial \mathcal{T}_r} |\sigma_{KL}| \xi_{KL}(u_K^n, u_L^n).$$

Since ξ_{KL} is a continuous function of bounded quantities, B_2 can be bounded using the regularity of the mesh (1.17). More precisely, one gets

$$(2.14) \quad |B_2| \leq \max_{(K,L) \in \partial \mathcal{T}_r} \|\xi_{KL}\|_{L^\infty(\Omega^2)} \sum_{n=0}^{N_T} \Delta t \sum_{(K,L) \in \partial \mathcal{T}_r} |\sigma_{KL}| \leq C_2$$

for some $C_2 > 0$ depending only on T , r , a , ξ and Ω . On the other hand, it follows from Proposition 2.3 that

$$(2.15) \quad B_1 \geq \frac{\beta_0}{2\lambda^*} \sum_{n=0}^{N_T} \Delta t \sum_{K \in \mathcal{T}_r} \sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| |G_{KL}(u_K^n, u_L^n) - f(u_K^n) \cdot n_{KL}|^2.$$

Combining (2.12)–(2.15) into (2.11) leads to

$$\left(\frac{\beta_0}{2\lambda^*} - \frac{\beta_1 \Delta t}{2a^2 h} \right) \sum_{n=0}^{N_T} \Delta t \sum_{K \in \mathcal{T}_r} \sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| |G_{KL}(u_K^n, u_L^n) - f(u_K^n) \cdot n_{KL}|^2 \leq C_1 + C_2.$$

The CFL condition (2.9) has been strengthened so that

$$\left(\frac{\beta_0}{2\lambda^*} - \frac{\beta_1 \Delta t}{2a^2 h} \right) \geq \frac{\zeta \beta_0}{2\lambda^*}$$

remains uniformly bounded away from 0. Estimate (2.10) follows. \square

We state now a straightforward consequence of Proposition 2.4. Its proof relies on Cauchy–Schwarz inequality and is left to the reader.

COROLLARY 2.5. *Assume that (2.9) holds, then there exists C_{BV} depending only on T , r , a , ξ , η , u_0 , Ω and ζ such that*

$$(2.16) \quad \sum_{n=0}^{N_T} \Delta t \sum_{(K,L) \in \mathcal{E}_r} |\sigma_{KL}| |G_{KL}(u_K^n, u_L^n) - f(u_K^n) \cdot n_{KL}| \leq \frac{C_{BV}}{\sqrt{h}}.$$

2.3. Consequences of the weak–BV estimate. The weak–BV estimate (2.16) implies a similar control on entropy fluxes and the time variations of u^h .

LEMMA 2.6. *Assume that the strengthen CFL condition (2.9) holds, then*

$$(2.17) \quad \sum_{n=0}^{N_T} \Delta t \sum_{(K,L) \in \mathcal{E}_r} |\sigma_{KL}| |\xi_{KL}(u_K^n, u_L^n) - \xi(u_K) \cdot n_{KL}| \leq \|D\eta\|_\infty \frac{C_{BV}}{\sqrt{h}},$$

$$(2.18) \quad \sum_{n=0}^{N_T} \sum_{K \in \mathcal{T}_r} |K| |u_K^{n+1} - u_K^n| \leq \frac{C_{BV}}{\sqrt{h}},$$

$$(2.19) \quad \sum_{n=0}^{N_T} \sum_{K \in \mathcal{T}_r} |K| |\eta(u_K^{n+1}) - \eta(u_K^n)| \leq \|D\eta\|_\infty \frac{C_{BV}}{\sqrt{h}}.$$

Proof. Inequality (2.19) is a direct consequence of (1.24) in which the Lipschitz continuity of η has been taken into account. Thanks to definition (1.27) of the scheme and thanks to the divergence free property (1.21), one has for all $K \in \mathcal{T}$ and all $n \in \mathbb{N}$

$$|u_K^{n+1} - u_K^n| |K| \leq \Delta t \sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| |G_{KL}(u_K^n, u_L^n) - f(u_K^n) \cdot n_{KL}|.$$

Summing over $K \in \mathcal{T}_r$ and $n \in \{0, \dots, N_T\}$ and using (2.16) provides (2.18). Inequality (2.19) then follows from the Lipschitz continuity of η . \square

We now state our main result, that consists in an *a priori* error estimate between a strong solution u and a discrete solution u^h .

THEOREM 2.7. *Assume that $u_0 \in W^{1,\infty}(\mathbb{R}^d)$ and that the solution u of the Cauchy problem (1.1)–(1.3) belongs to $W^{1,\infty}([0, T] \times \mathbb{R}^d)$. Let u^h defined by the numerical scheme (1.27)–(1.29) and assume that the strengthen CFL condition (2.9) holds. Then, for all $r > 0$ and $T > 0$ there exist C depending only on $T, r, \Omega, a, \lambda^*, u_0, G_{KL}, \eta$ and f , such that*

$$\int_0^T \int_{B(0, r + L_f(T-t))} |u - u^h|^2 dx dt \leq C\sqrt{h}.$$

3. Continuous weak and entropy formulations for the discrete solution.

In order to obtain the error estimate of Theorem 2.7, we aim at using the relative entropy of u^h w.r.t. u . Since u^h is only an approximate solution, it neither satisfies exactly the weak formulation (1.10) nor the entropy weak formulation (1.11). Some numerical error terms appear in these formulations, and thus also appear the inequality of the relative entropy

$$(3.1) \quad \partial_t H(u^h, u) + \sum_{\alpha=1}^d \partial_\alpha Q_\alpha(u^h, u) \leq - \sum_{\alpha=1}^d (\partial_\alpha u)^T Z_\alpha(u^h, u) + \text{numerical error terms.}$$

As usual, these terms may be described by Radon measures, see for instance [6, 18, 7, 28, 25, 26]. Note that for nonlinear systems of conservation laws, a function which satisfies the entropy inequality (1.11) is not necessarily a weak solution (1.10). This leads us to introduce error measures for both the entropy inequality (1.11) and the

weak formulation (1.10) of u^h . Let us first begin with the entropy formulation and the related measures.

For $X = \mathbb{R}^d$ or $X = \mathbb{R}^d \times \mathbb{R}^+$, we denote $\mathcal{M}(X)$ the set of locally bounded Radon measures on X , i.e., $\mathcal{M}(X) = (C_c(X))'$ where $C_c(X)$ is the set of continuous compact supported functions on X . If $\mu \in \mathcal{M}(X)$ we set $\langle \mu, \varphi \rangle = \int_X \varphi d\mu$ for all $\varphi \in C_c(X)$.

DEFINITION 3.1. For $\psi \in C_c(\mathbb{R}^d)$, $\varphi \in C_c(\mathbb{R}^d \times \mathbb{R}^+)$, we define $\mu_0 \in \mathcal{M}(\mathbb{R}^d)$ and $\mu \in \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^+)$ by

$$\begin{aligned} \langle \mu_0, \psi \rangle &= \int_{\mathbb{R}^d} |\eta(u_0(x)) - \eta(u^h(x, 0))| \psi(x) dx, \\ \langle \mu, \varphi \rangle &= \langle \mu_{\mathcal{T}}, \varphi \rangle + \sum_{n=0}^{\infty} \Delta t \sum_{(K,L) \in \mathcal{E}_r} |\sigma_{KL}| |\xi_{KL}(u_K^n, u_L^n) - \xi_{KL}(u_K^n, u_K^n)| \langle \mu_{KL}, \varphi \rangle \\ &\quad + \sum_{n=0}^{\infty} \Delta t \sum_{(K,L) \in \mathcal{E}_r} |\sigma_{KL}| |\xi_{KL}(u_K^n, u_L^n) - \xi_{KL}(u_L^n, u_L^n)| \langle \mu_{LK}, \varphi \rangle, \end{aligned}$$

where

$$\begin{aligned} \langle \mu_{\mathcal{T}}, \varphi \rangle &= \sum_{n=0}^{\infty} \sum_{K \in \mathcal{T}_r} |\eta(u_K^{n+1}) - \eta(u_K^n)| \int_{t^n}^{t^{n+1}} \int_K \varphi(x, t) dx dt, \\ \langle \mu_{KL}, \varphi \rangle &= \frac{1}{m(K)m(\sigma_{KL})(\Delta t)^2} \\ &\quad \times \int_{t^n}^{t^{n+1}} \int_K \int_{t^n}^{t^{n+1}} \int_{\sigma_{KL}} \int_0^1 (h + \Delta t) \varphi(\gamma + \theta(x - \gamma), s + \theta(t - s)) d\theta dx dt d\gamma ds, \\ \langle \mu_{LK}, \varphi \rangle &= \frac{1}{m(L)m(\sigma_{KL})(\Delta t)^2} \\ &\quad \times \int_{t^n}^{t^{n+1}} \int_L \int_{t^n}^{t^{n+1}} \int_{\sigma_{KL}} \int_0^1 (h + \Delta t) \varphi(\gamma + \theta(x - \gamma), s + \theta(t - s)) d\theta dx dt d\gamma ds. \end{aligned}$$

As it will be highlighted by Proposition 3.3 later on, the measures μ and μ_0 describe the approximation error in the entropy formulation satisfied by u^h . Let us first estimate them on compact sets.

LEMMA 3.2. Assume that the strengthened CFL condition (2.9) holds, then, for all $r > 0$ and $T > 0$ there exist $C_{\mu_0} > 0$, depending only on u_0 , $\|D\eta\|_{\infty}$, and r , and $C_{\mu} > 0$, depending only on $T, r, a, \lambda^*, u_0, G_{KL}$ and η such that, for all $h < r$,

$$(3.2) \quad \mu_0(B(0, r)) \leq C_{\mu_0} h \quad \text{and} \quad \mu(B(0, r) \times [0, T]) \leq \frac{C_{\mu}}{\sqrt{h}}.$$

Proof. The regularity of $u_0 : \mathbb{R}^d \rightarrow \mathbb{R}^m$ yields

$$\mu_0(B(0, r)) \leq h \|D\eta\|_{\infty} \int_{B(0, r+h)} |\nabla u_0| dx.$$

For $r > 0$ and $T > 0$ the measure $\mu_{\mathcal{T}}$ satisfies

$$\mu_{\mathcal{T}}(B(0, r) \times [0, T]) = \int_0^T \int_{B(0, r)} \sum_{n=0}^{N_T} \sum_{K \in \mathcal{T}_r} |\eta(u_K^{n+1}) - \eta(u_K^n)| \mathbf{1}_{K \times [t^n, t^{n+1}]} dx dt.$$

Then, using the time–BV estimate (2.19),

$$\mu_{\mathcal{T}}(B(0, r) \times [0, T]) \leq \Delta t \sum_{n=0}^{N_T} \sum_{K \in \mathcal{T}_r} |K| |\eta(u_K^{n+1}) - \eta(u_K^n)| \leq \Delta t \|D\eta\|_{\infty} \frac{C_{BV}}{\sqrt{h}}.$$

Since Δt satisfies the CFL condition (2.9), one has

$$(3.3) \quad \mu_{\mathcal{T}}(B(0, r) \times [0, T]) \leq C_{\mu_{\mathcal{T}}} \sqrt{h},$$

where $C_{\mu_{\mathcal{T}}} := \frac{a^2 \|D\eta\|_{\infty}}{\lambda^*} C_{BV}$. The measures μ_{KL} and μ_{LK} satisfy:

$$\mu_{KL}(\mathbb{R}^d \times \mathbb{R}^+) \leq h + \Delta t, \quad \mu_{LK}(\mathbb{R}^d \times \mathbb{R}^+) \leq h + \Delta t.$$

Therefore,

$$\begin{aligned} & \mu(B(0, r) \times [0, T]) \\ & \leq C_{\mu_{\mathcal{T}}} \sqrt{h} + (h + \Delta t) \sum_{n=0}^{N_T} \Delta t \sum_{(K,L) \in \mathcal{E}_r} |\sigma_{KL}| |\xi_{KL}(u_K^n, u_L^n) - \xi_{KL}(u_K^n, u_K^n)| \\ & \quad + (h + \Delta t) \sum_{n=0}^{N_T} \Delta t \sum_{(K,L) \in \mathcal{E}_r} |\sigma_{KL}| |\xi_{KL}(u_K^n, u_L^n) - \xi_{KL}(u_L^n, u_L^n)|. \end{aligned}$$

Hence, using Lemma 2.6, the CFL condition (1.26) and the bound (3.3) provides

$$\mu(B(0, r) \times [0, T]) \leq C_{\mu} \sqrt{h},$$

where $C_{\mu} = C_{\mu_{\mathcal{T}}} + 2 \left(1 + \frac{a^2}{\lambda^*}\right) \|D\eta\|_{\infty} C_{BV}$. \square

PROPOSITION 3.3. *Let μ and μ_0 be the measures introduced in Definition 3.1, then, for all $\varphi \in C_c^1(\mathbb{R}^d \times \mathbb{R}^+; \mathbb{R}^+)$, one has*

$$(3.4) \quad \begin{aligned} & \iint_{\mathbb{R}^d \times \mathbb{R}_+} \eta(u^h) \partial_t \varphi(x, t) + \sum_{\alpha=1}^d \int_{\mathbb{R}^d} \eta(u_0(x)) \varphi(x, 0) dx \xi_{\alpha}(u^h) \partial_{\alpha} \varphi(x, t) dx dt \\ & \geq - \iint_{\mathbb{R}^d \times \mathbb{R}_+} (|\nabla \varphi| + |\partial_t \varphi|) d\mu(x, t) - \int_{\mathbb{R}^d} \varphi(x, 0) d\mu_0(x). \end{aligned}$$

Proof. Let $\varphi \in C_c^1(\mathbb{R}^d \times \mathbb{R}^+; \mathbb{R}^+)$. Let $T > 0$ and $r > 0$ such that $\text{supp } \varphi \subset B(0, r) \times [0, T]$. Let us multiply (2.2) by $\int_{t^n}^{t^{n+1}} \int_K \varphi(x, t) dx dt$ and sum over the control volumes $K \in \mathcal{T}_r$ and $n \leq N_T$. It yields

$$(3.5) \quad T_1 + T_2 \leq 0,$$

where

$$(3.6) \quad T_1 = \sum_{n=0}^{N_T} \sum_{K \in \mathcal{T}_r} \frac{1}{\Delta t} (\eta(u_K^{n+1}) - \eta(u_K^n)) \int_{t^n}^{t^{n+1}} \int_K \varphi(x, t) dx dt,$$

$$(3.7) \quad T_2 = \sum_{n=0}^{N_T} \sum_{K \in \mathcal{T}_r} \frac{1}{|K|} \int_{t^n}^{t^{n+1}} \int_K \varphi(x, t) dx dt \sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| \xi_{KL}(u_K^n, u_L^n).$$

The term T_1 corresponds to the discrete time derivative of $\eta(u^h)$ and T_2 to the discrete space derivative of $\xi(u^h)$. The proof relies on the comparison firstly between T_1 and T_{10} and secondly between T_2 and T_{20} , where T_{10} and T_{20} denote respectively the temporal and spatial term in (3.4):

$$\begin{aligned} T_{10} &= - \iint_{\mathbb{R}^d \times \mathbb{R}_+} \eta(u^h) \partial_t \varphi(x, t) dx dt - \int_{\mathbb{R}^d} \eta(u_0(x)) \varphi(x, 0) dx, \\ T_{20} &= - \iint_{\mathbb{R}^d \times \mathbb{R}_+} \sum_{\alpha=1}^d \xi_\alpha(u^h) \partial_\alpha \varphi(x, t) dx dt. \end{aligned}$$

Let us first focus on T_{10} . Following its definition (1.29), the approximate solution u^h is piecewise constant, then so does $\eta(u^h)$. Therefore, we can rewrite

$$\begin{aligned} T_{10} &= \sum_{n=0}^{N_T} \sum_{K \in \mathcal{T}_r} (\eta(u_K^{n+1}) - \eta(u_K^n)) \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \int_K \varphi(x, t^{n+1}) dx dt \\ &\quad - \int_{\mathbb{R}^d} (\eta(u_0(x)) - \eta(u^h(x, 0))) \varphi(x, 0) dx. \end{aligned}$$

It is now easy to verify that

$$\begin{aligned} |T_1 - T_{10}| &\leq \sum_{n=0}^{N_T} \sum_{K \in \mathcal{T}_r} |\eta(u_K^{n+1}) - \eta(u_K^n)| \int_{t^n}^{t^{n+1}} \int_K |\partial_t \varphi| dx dt \\ &\quad + \int_{\mathbb{R}^d} |\eta(u_0(x)) - \eta(u^h(x, 0))| \varphi(x, 0) dx. \end{aligned}$$

Then, accounting from Definition 3.1, the inequality reads

$$(3.8) \quad |T_1 - T_{10}| \leq \iint_{\mathbb{R}^d \times \mathbb{R}_+} |\partial_t \varphi| d\mu_{\mathcal{T}}(x, t) + \int_{\mathbb{R}^d} \varphi(x, 0) d\mu_0(x).$$

We now consider the terms T_2 and T_{20} . Performing a discrete integration by parts by reorganizing the sum, and using the properties (2.3) and (1.23) lead to

$$(3.9) \quad T_2 = T_{2,1} + T_{2,2},$$

with

$$\begin{aligned} T_{2,1} &= \sum_{n=0}^{N_T} \sum_{(K,L) \in \mathcal{E}_r} \frac{|\sigma_{KL}|}{|K|} \int_{t^n}^{t^{n+1}} \int_K \varphi(x, t) (\xi_{KL}(u_K^n, u_L^n) - \xi_{KL}(u_K^n, u_K^n)) dx dt, \\ T_{2,2} &= \sum_{n=0}^{N_T} \sum_{(K,L) \in \mathcal{E}_r} \frac{|\sigma_{KL}|}{|L|} \int_{t^n}^{t^{n+1}} \int_L \varphi(x, t) (\xi_{LK}(u_L^n, u_K^n) - \xi_{LK}(u_L^n, u_L^n)) dx dt. \end{aligned}$$

Gathering terms of T_{20} by edges yields

$$T_{20} = T_{20,1} + T_{20,2},$$

where, thanks to (1.23), we have set

$$T_{20,1} = \sum_{n=0}^{N_T} \sum_{(K,L) \in \mathcal{E}_r} \int_{t^n}^{t^{n+1}} \int_{\sigma_{KL}} (\xi_{KL}(u_K^n, u_L^n) - \xi(u_K^n) \cdot n_{KL}) \varphi(\gamma, t) d\gamma dt,$$

$$T_{20,2} = \sum_{n=0}^{N_T} \sum_{(K,L) \in \mathcal{E}_r} \int_{t^n}^{t^{n+1}} \int_{\sigma_{KL}} (\xi_{LK}(u_L^n, u_K^n) - \xi(u_L^n) \cdot n_{LK}) \varphi(\gamma, t) d\gamma dt.$$

It is easy to verify

$$T_{2,1} - T_{20,1} = \sum_{n=0}^{N_T} \Delta t \sum_{(K,L) \in \mathcal{E}_r} |\sigma_{KL}| (\xi_{KL}(u_K^n, u_L^n) - \xi_{KL}(u_K^n, u_K^n))$$

$$\times \frac{1}{|K| |\sigma_{KL}| (\Delta t)^2} \int_{t^n}^{t^{n+1}} \int_K \int_{t^n}^{t^{n+1}} \int_{\sigma_{KL}} (\varphi(x, t) - \varphi(\gamma, s)) d\gamma ds dx dt.$$

Then using the definition of μ_{KL} in Definition 3.1, we obtain the following estimate:

$$(3.10) \quad |T_{2,1} - T_{20,1}|$$

$$\leq \sum_{n=0}^{N_T} \Delta t \sum_{(K,L) \in \mathcal{E}_r} |\sigma_{KL}| |\xi_{KL}(u_K^n, u_L^n) - \xi_{KL}(u_K^n, u_K^n)| \langle \mu_{KL}, |\nabla \varphi| + |\partial_t \varphi| \rangle.$$

Similarly, one obtains

$$(3.11) \quad |T_{2,2} - T_{20,2}|$$

$$\leq \sum_{n=0}^{N_T} \Delta t \sum_{(K,L) \in \mathcal{E}_r} |\sigma_{KL}| |\xi_{LK}(u_L^n, u_K^n) - \xi_{LK}(u_L^n, u_L^n)| \langle \mu_{LK}, |\nabla \varphi| + |\partial_t \varphi| \rangle,$$

the measure $\mu_{LK} \in \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^+)$ being given by Definition 3.1. Bearing in mind the definition of $\mu \in \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^+)$ given in Definition 3.1, inequalities (3.5), (3.8), (3.9), (3.10) and (3.11), one has

$$-T_{10} - T_{20} \geq - \iint_{\mathbb{R}^d \times \mathbb{R}^+} (|\nabla \varphi| + |\partial_t \varphi|) d\mu(x, t) - \int_{\mathbb{R}^d} \varphi(x, 0) d\mu_0(x),$$

which concludes the proof of Proposition 3.3. \square

Similar calculations can be used to estimate how close u^h is to a weak solution. For that purpose we define the following measures.

DEFINITION 3.4. *For $\psi \in C_c(\mathbb{R}^d)$ and $\varphi \in C_c(\mathbb{R}^d \times \mathbb{R}^+)$, we set*

$$\langle \bar{\mu}_0, \psi \rangle = \int_{\mathbb{R}^d} |u_0(x) - u^h(x, 0)| \psi(x) dx,$$

$$\langle \bar{\mu}, \varphi \rangle = \langle \bar{\mu}_T, \varphi \rangle + \sum_{n=0}^{\infty} \Delta t \sum_{(K,L) \in \mathcal{E}_r} |\sigma_{KL}| |G_{KL}(u_K^n, u_L^n) - G_{KL}(u_K^n, u_K^n)| \langle \bar{\mu}_{KL}, \varphi \rangle$$

$$+ \sum_{n=0}^{\infty} \Delta t \sum_{(K,L) \in \mathcal{E}_r} |\sigma_{KL}| |G_{KL}(u_K^n, u_L^n) - G_{KL}(u_L^n, u_L^n)| \langle \bar{\mu}_{LK}, \varphi \rangle,$$

where

$$\begin{aligned} \langle \bar{\mu}_{\mathcal{T}}, \varphi \rangle &= \sum_{n=0}^{\infty} \sum_{K \in \mathcal{T}_r} |u_K^{n+1} - u_K^n| \int_{t^n}^{t^{n+1}} \int_K \varphi(x, t) dx dt, \\ \langle \bar{\mu}_{KL}, \varphi \rangle &= \frac{1}{|K| |\sigma_{KL}| \Delta t^2} \times \\ &\quad \int_{t^n}^{t^{n+1}} \int_K \int_{t^n}^{t^{n+1}} \int_{\sigma_{KL}} \int_0^1 (h + \Delta t) \varphi(\gamma + \theta(x - \gamma), s + \theta(t - s)) d\theta dx dt d\gamma ds, \\ \langle \bar{\mu}_{LK}, \varphi \rangle &= \frac{1}{|L| |\sigma_{KL}| \Delta t^2} \times \\ &\quad \int_{t^n}^{t^{n+1}} \int_L \int_{t^n}^{t^{n+1}} \int_{\sigma_{KL}} \int_0^1 (h + \Delta t) \varphi(\gamma + \theta(x - \gamma), s + \theta(t - s)) d\theta dx dt d\gamma ds. \end{aligned}$$

REMARK 3.1. *It follows from the definitions of the measures μ and $\bar{\mu}$ that they can be extended (in a unique way) into continuous linear forms defined on the set*

$$E := \left\{ \varphi \in L^\infty(\mathbb{R}^d \times \mathbb{R}^+; \mathbb{R}) \mid \text{supp}(\varphi) \text{ is compact, and } \nabla \varphi \in L^1_{\text{loc}}(\mathbb{R}^d \times \mathbb{R}^+)^d \right\}.$$

Indeed, any $\varphi \in E$ admit a unique trace on σ_{KL} , so that the quantities $\langle \mu_{KL}, \varphi \rangle$, $\langle \mu_{LK}, \varphi \rangle$, $\langle \bar{\mu}_{KL}, \varphi \rangle$ and $\langle \bar{\mu}_{LK}, \varphi \rangle$ are well defined. Moreover, one has

$$|\langle \mu, \varphi \rangle| \leq \|\varphi\|_{L^\infty} \mu(\{\varphi \neq 0\}), \quad |\langle \bar{\mu}, \varphi \rangle| \leq \|\varphi\|_{L^\infty} \bar{\mu}(\{\varphi \neq 0\}), \quad \forall \varphi \in E.$$

We now state a lemma and a proposition whose proofs are left to the reader, since they are similar to the proofs of Lemma 3.2 and Proposition 3.3 respectively as one uses the estimates (2.16) and (2.18) instead of (2.17) and (2.19).

LEMMA 3.5. *Let u^h defined by (1.27)–(1.29). Assume that (2.9) holds, then, for all $r > 0$ and $T > 0$ there exist $C_{\bar{\mu}_0} > 0$, depending only on u_0 and r , and $C_{\bar{\mu}} > 0$, depending only on $T, r, a, \lambda^*, u_0, G_{KL}$ such that, for all $h < r$,*

$$(3.12) \quad \bar{\mu}_0(B(0, r)) \leq C_{\bar{\mu}_0} h \quad \text{and} \quad \bar{\mu}(B(0, r) \times [0, T]) \leq C_{\bar{\mu}} \sqrt{h},$$

where $C_{\bar{\mu}_0} = C_{\mu_0} / \|D\eta\|_\infty$ and $C_{\bar{\mu}} = C_\mu / \|D\eta\|_\infty$ (see the proof of Lemma 3.2).

We are now in position to provide the approximate weak formulation satisfied by u^h . In the statement below, φ is a vector-valued function, and we adopted the notation $|\nabla \varphi| = \max_{\alpha \in \{1, \dots, d\}} |\partial_\alpha \varphi|$. The proof of Proposition 3.6 follows the same guidelines as the proof of Proposition 3.3 and is left to the reader.

PROPOSITION 3.6. *Let μ and μ_0 be the measures introduced in Definition 3.1, then, for all $\varphi \in C_c^1(\mathbb{R}^d \times \mathbb{R}^+; \mathbb{R}^m)$, one has*

$$\begin{aligned} &\left| \iint_{\mathbb{R}^d \times \mathbb{R}_+} \left[(u^h)^T \partial_t \varphi(x, t) + \sum_{\alpha=1}^d f_\alpha(u^h)^T \partial_\alpha \varphi(x, t) \right] dx dt + \int_{\mathbb{R}^d} u_0(x)^T \varphi(x, 0) dx \right| \\ &\quad \leq \iint_{\mathbb{R}^d \times \mathbb{R}_+} (|\nabla \varphi| + |\partial_t \varphi|) d\bar{\mu}(x, t) + \int_{\mathbb{R}^d} |\varphi(x, 0)| d\bar{\mu}_0(x). \end{aligned}$$

4. Error estimate using the relative entropy. With the error measures μ , μ_0 , $\bar{\mu}$, and $\bar{\mu}_0$ at hand, we are now in position to precise inequality (3.1) satisfied by the relative entropy $H(u^h, u)$ and then to conclude the proof of Theorem 2.7.

4.1. Relative entropy for approximate solutions. PROPOSITION 4.1. *Let μ and μ_0 be the measures introduced in Definition 3.1, and let $\bar{\mu}$ and $\bar{\mu}_0$ be the measures introduced in Definition 3.4, then, for all $\varphi \in C_c^1(\mathbb{R}^d \times \mathbb{R}^+; \mathbb{R}^+)$, one has*

$$(4.1) \quad \begin{aligned} & \iint_{\mathbb{R}^d \times \mathbb{R}_+} \left(H(u^h, u) \partial_t \varphi(x, t) + \sum_{\alpha=1}^d Q_\alpha(u^h, u) \partial_\alpha \varphi(x, t) \right) dx dt \geq \\ & - \iint_{\mathbb{R}^d \times \mathbb{R}_+} (|\nabla \varphi| + |\partial_t \varphi|) d\mu(x, t) - \int_{\mathbb{R}^d} \varphi(x, 0) d\mu_0(x) \\ & - \iint_{\mathbb{R}^d \times \mathbb{R}_+} (|\nabla [\varphi D\eta(u)]| + |\partial_t [\varphi D\eta(u)]|) d\bar{\mu}(x, t) \\ & - \int_{\mathbb{R}^d} [\varphi D\eta(u)](x, 0) d\bar{\mu}_0(x) + \iint_{\mathbb{R}^d \times \mathbb{R}_+} \varphi \sum_{\alpha=1}^d \partial_\alpha u^T Z_\alpha(u^h, u) dx dt, \end{aligned}$$

where $Z_\alpha(u^h, u) = D^2\eta(u)(f_\alpha(u^h) - f_\alpha(u) - (Df_\alpha(u))(u^h - u))$.

Proof. Let φ be any nonnegative Lipschitz continuous test function with compact support in $\mathbb{R}^d \times [0, T]$. Since u is a classical solution of (1.1)–(1.3), it satisfies

$$\iint_{\mathbb{R}^d \times \mathbb{R}_+} \eta(u) \partial_t \varphi(x, t) + \sum_{\alpha=1}^d \xi_\alpha(u) \partial_\alpha \varphi(x, t) dx dt + \int_{\mathbb{R}^d} \eta(u_0) \varphi(x, 0) dx = 0.$$

Subtracting this identity to (3.4) yields

$$(4.2) \quad \begin{aligned} & \iint_{\mathbb{R}^d \times \mathbb{R}_+} (\eta(u^h) - \eta(u)) \partial_t \varphi(x, t) + \sum_{\alpha=1}^d (\xi_\alpha(u^h) - \xi_\alpha(u)) \partial_\alpha \varphi(x, t) dx dt \\ & \geq \iint_{\mathbb{R}^d \times \mathbb{R}_+} (|\nabla \varphi| + |\partial_t \varphi|) d\mu(x, t) - \int_{\mathbb{R}^d} \varphi(x, 0) d\mu_0(x). \end{aligned}$$

We now exhibit the relative entropy-relative entropy flux pair in the inequality (4.2) and obtain

$$(4.3) \quad \begin{aligned} & \iint_{\mathbb{R}^d \times \mathbb{R}_+} \left(H(u^h, u) \partial_t \varphi + \sum_{\alpha=1}^d Q_\alpha(u^h, u) \partial_\alpha \varphi \right) dx dt \geq \\ & - \iint_{\mathbb{R}^d \times \mathbb{R}_+} |\nabla \varphi| + |\partial_t \varphi| d\mu(x, t) - \int_{\mathbb{R}^d} \varphi(x, 0) d\mu_0(x) \\ & - \iint_{\mathbb{R}^d \times \mathbb{R}_+} (D\eta(u))^T \left((u^h - u) \partial_t \varphi + \sum_{\alpha=1}^d (f_\alpha(u^h) - f_\alpha(u)) \partial_\alpha \varphi \right) dx dt. \end{aligned}$$

Since u is a strong solution of (1.1)–(1.3), it satisfies the following weak identity, $\forall \psi \in C_c(\mathbb{R}^d \times \mathbb{R}^+; \mathbb{R}^m)$

$$(4.4) \quad \iint_{\mathbb{R}^d \times \mathbb{R}_+} \left[u \partial_t \psi(x, t) + \sum_{\alpha=1}^d f_\alpha(u) \partial_\alpha \psi(x, t) \right] dx dt + \int_{\mathbb{R}^d} u_0(x) \psi(x, 0) dx = 0.$$

Then we combine Proposition 3.6 with (4.4), so that using the Lipschitz continuous vector field $[\varphi D\eta(u)]$ as test function leads to

$$(4.5) \quad - \iint_{\mathbb{R}^d \times \mathbb{R}_+} (D\eta(u))^T \left[(u^h - u) \partial_t \varphi + \sum_{\alpha=1}^d (f_\alpha(u^h) - f_\alpha(u)) \partial_\alpha \varphi \right] dx dt \geq \\ - \iint_{\mathbb{R}^d \times \mathbb{R}_+} |\nabla[\varphi D\eta(u)]| + |\partial_t[\varphi D\eta(u)]| d\bar{\mu}(x, t) - \int_{\mathbb{R}^d} [\varphi D\eta(u)](x, 0) d\bar{\mu}_0(x) \\ + \iint_{\mathbb{R}^d \times \mathbb{R}_+} (u^h - u) \varphi \partial_t (D\eta(u)) + \sum_{\alpha=1}^d (f_\alpha(u^h) - f_\alpha(u)) \varphi \partial_\alpha (D\eta(u)) dx dt.$$

Moreover identity (1.6) together with (1.1) gives

$$(4.6) \quad \partial_t (D\eta(u)) = \partial_t u^T D^2 \eta(u) = - \sum_{\alpha=1}^d \partial_\alpha (f_\alpha(u))^T D^2 \eta(u) \\ = - \sum_{\alpha=1}^d \partial_\alpha u^T D f_\alpha(u)^T D^2 \eta(u) = - \sum_{\alpha=1}^d \partial_\alpha u^T D^2 \eta(u) D f_\alpha(u).$$

Injecting (4.5) and (4.6) into (4.3) leads to conclusion. \square

LEMMA 4.2. *There exists C_Z depending only on f, η and Ω such that, for all $\alpha \in \{1, \dots, d\}$,*

$$(4.7) \quad |Z_\alpha(u^h, u)| \leq C_Z |u^h - u|^2.$$

Proof. For $M : \Omega \rightarrow \mathbb{R}^{m \times m}$ and $\Upsilon : \Omega \rightarrow \mathcal{L}(\mathbb{R}^m; \mathbb{R}^{m \times m})$, we set

$$\|M\|_{\infty, \infty} = \sup_{u \in \Omega} |M(u)|_\infty, \quad \|\Upsilon\|_{\infty, 2} = \sup_{u \in \Omega} \left(\sup_{v \in \mathbb{R}^m, |v|=1} |\Upsilon(u) \cdot v|_2 \right),$$

where $|\cdot|_2$ and $|\cdot|_\infty$ denote the usual matrix 2- and ∞ -norms respectively. Using the Taylor expansion of f_α around u , we get that

$$|f_\alpha(u^h) - f_\alpha(u) - (Df_\alpha(u))(u^h - u)| \leq \frac{1}{2} \|D^2 f_\alpha\|_{\infty, 2} |u^h - u|^2,$$

then, estimate (4.7) holds for $C_Z = \frac{1}{2} \|D^2 \eta\|_{\infty, \infty} \|D^2 f_\alpha\|_{\infty, 2} \cdot \square$

We now prove the following lemma on the finite speed of propagation.

LEMMA 4.3. *Let L_f be defined by (1.7), then, for all $s \geq L_f$, one has*

$$(4.8) \quad sH(u^h, u) + \sum_{\alpha=1}^d \frac{x_\alpha}{|x|} Q_\alpha(u^h, u) \geq 0.$$

Proof. Denote by $w^h := u^h - u$, then it follows from the characterization (1.14) of the relative entropy H that

$$(4.9) \quad H = \int_0^1 \int_0^\theta (w^h)^T D_u^2 \eta(u + \gamma w^h) w^h d\gamma d\theta.$$

Denoting by \mathbb{A}_γ the symmetric definite positive matrix $D_u^2\eta(u + \gamma w^h)$, and by $\langle \cdot, \cdot \rangle_{\mathbb{A}_\gamma}$ the scalar product on \mathbb{R}^n defined by $\langle v_1, v_2 \rangle_{\mathbb{A}_\gamma} = v_1^T \mathbb{A}_\gamma v_2$, the relation (4.9) can be rewritten

$$(4.10) \quad H = \int_0^1 \int_0^\theta \langle w^h, w^h \rangle_{\mathbb{A}_\gamma} d\gamma d\theta.$$

On the other hand, it follows from the definition (1.5) of the entropy flux ξ that

$$\begin{aligned} Q_\alpha &= \int_0^1 (D\eta(u + \theta w^h) - D\eta(u)) (Df_\alpha(u + \theta w^h))^T w^h d\theta \\ &= \int_0^1 \int_0^\theta \langle w^h, (Df_\alpha(u + \theta w^h))^T w^h \rangle_{\mathbb{A}_\gamma} d\gamma d\theta \end{aligned}$$

for all $\alpha \in \{1, \dots, d\}$. The quantity L_f introduced in (1.7) has been designed so that $|\langle w^h, (Df_\alpha(u + \theta w^h))^T w^h \rangle_{\mathbb{A}_\gamma}| \leq L_f \langle w^h, w^h \rangle_{\mathbb{A}_\gamma}$. Therefore, we obtain

$$(4.11) \quad |Q_\alpha| \leq L_f \int_0^1 \int_0^\theta \langle w^h, w^h \rangle_{\mathbb{A}_\gamma} d\gamma d\theta = L_f H.$$

The fact that (4.8) holds is a straightforward consequence of (4.11). \square

4.2. End of the proof of Theorem 2.7. We now have at hand all the tools needed for comparing u^h to u via the relative entropy $H(u^h, u)$.

Let $\delta \in (0, T)$ be a parameter to be fixed later on, and, for $k \in \mathbb{N}$, we define the nonincreasing Lipschitz continuous function $\theta_k : \mathbb{R}^+ \rightarrow [0, 1]$ by

$$\theta_k(t) = \min \left(1, \max \left(0, \frac{(k+1)\delta - t}{\delta} \right) \right), \quad \forall t \geq 0.$$

Let us also introduce the Lipschitz continuous function $\psi : \mathbb{R}^d \times \mathbb{R}_+ \rightarrow [0, 1]$ defined by $\psi(x, t) = 1 - \min(1, \max(0, |x| - r - L_f(T - t) + 1))$, where L_f is defined by (1.7). The function $\varphi_k : (x, t) \in \mathbb{R}^d \times \mathbb{R}_+ \mapsto \theta_k(t)\psi(x, t) \in [0, 1]$ can be considered as a test function in (4.1). Indeed, denoting by

$$\mathcal{I}_k^\delta = [k\delta, (k+1)\delta], \quad \mathcal{C}_{r,T}(t) = \{(x, t) \mid |x| \in [r + L_f(T - t), r + L_f(T - t) + 1]\},$$

one has

$$\begin{aligned} \partial_t \varphi_k(x, t) &= -\frac{1}{\delta} \mathbf{1}_{\mathcal{I}_k^\delta}(t) \psi(x, t) - L_f \theta_k(t) \mathbf{1}_{\mathcal{C}_{r,T}(t)}(x), \\ \nabla \varphi_k(x, t) &= -\frac{x}{|x|} \theta_k(t) \mathbf{1}_{\mathcal{C}_{r,T}(t)}(x), \end{aligned}$$

so that both $\partial_t \varphi_k$ and $|\nabla \varphi_k|$ belong to the set E defined in Remark 3.1. Then taking φ_k as test function in (4.1) yields

$$\begin{aligned} &\frac{1}{\delta} \int_{\mathcal{I}_k^\delta} \int_{\mathbb{R}^d} H \psi dx dt + \int_0^T \theta_k(t) \int_{\mathbb{R}^d} \sum_{\alpha=1}^d \partial_\alpha u^T Z_\alpha(u^h, u) \psi dx dt \\ &\leq - \int_0^T \theta_k(t) \int_{\mathcal{C}_{r,T}(t)} \left(L_f H + \sum_{\alpha=1}^d Q_\alpha \frac{x_\alpha}{|x|} \right) dx dt + R_1 + R_2 + R_3 + R_4, \end{aligned}$$

where

$$\begin{aligned}
R_1 &= \iint_{\mathbb{R}^d \times [0, T]} (|\nabla \varphi_k(x, t)| + |\partial_t \varphi_k(x, t)|) d\mu(x, t), \\
R_2 &= \int_{\mathbb{R}^d} \psi(x, 0) d\mu_0(x), \\
R_3 &= \iint_{\mathbb{R}^d \times [0, T]} |D\eta(u)| (|\nabla \varphi_k(x, t)| + |\partial_t \varphi_k(x, t)|) d\bar{\mu}(x, t) \\
&\quad + \iint_{\mathbb{R}^d \times [0, T]} \varphi_k(x, t) |D^2\eta(u)(x, t)|_\infty (|\partial_t u| + |\nabla u|) d\bar{\mu}(x, t), \\
R_4 &= \int_{\mathbb{R}^d} \psi(x, 0) |D\eta(u_0)| d\bar{\mu}_0(x).
\end{aligned}$$

Thanks to Lemma 4.3, one has

$$(4.12) \quad \frac{1}{\delta} \int_{\mathcal{I}_k^\delta} \int_{\mathbb{R}^d} H \psi dx dt + \int_0^T \theta_k(t) \int_{\mathbb{R}^d} \sum_{\alpha=1}^d \partial_\alpha u^T Z_\alpha(u^h, u) \psi dx dt \leq R_1 + R_2 + R_3 + R_4.$$

The definition of φ_k ensure that $\|\varphi_k\|_\infty = 1$, $\|\nabla \varphi_k\|_\infty \leq 1$, $\|\partial_t \varphi_k\|_\infty \leq \frac{1}{\delta} + L_f$ and

$$\text{supp}(\varphi_k) \subset \bigcup_{t \in [0, (k+1)\delta]} B(0, r + L_f(T-t) + 1) \times \{t\},$$

This leads to

$$R_1 \leq \left(\frac{1}{\delta} + L_f + 1 \right) \mu(\text{supp}(\nabla \varphi_k) \cup \text{supp}(\partial_t \varphi_k)).$$

Thanks to Lemma 3.2, we obtain that there exists C_μ^k (depending on $k, r, T, \delta, L_f, a, \lambda^*, u_0, G_{KL}$ and η) such that

$$(4.13) \quad R_1 \leq C_\mu^k \left(\frac{1}{\delta} + L_f + 1 \right) \sqrt{h}.$$

It follows from similar arguments that there exists $C_{\mu_0}^k$ (depending on k, η, u_0, r, L_f, T and δ) such that

$$(4.14) \quad R_2 \leq C_{\mu_0}^k h,$$

and, thanks to Lemma 3.5, we obtain that there exists $C_{\bar{\mu}_0}^k$ (depending on k, r, u_0, L_f, T and δ) such that

$$(4.15) \quad R_4 \leq C_{\bar{\mu}_0}^k \|D\eta(u_0)\|_\infty h.$$

Similarly, there exists $C_{\bar{\mu}}^k$ (depending on $k, T, r, L_f, a, \lambda^*, u_0, G_{KL}$ and δ) such that

$$(4.16) \quad R_3 \leq C_{\bar{\mu}}^k \left(\|D\eta(u)\|_\infty \left(\frac{1}{\delta} + L_f + 1 \right) + \|D^2\eta\|_{\infty, \infty} (\|\partial_t u\|_\infty + \|\nabla u\|_\infty) \right) \sqrt{h}.$$

By using Lemma 4.2 and $0 \leq \theta_k(t) \leq 1$, we obtain

$$(4.17) \quad \int_0^T \theta_k(t) \int_{\mathbb{R}^d} \sum_{\alpha=1}^d \partial_\alpha u^T Z_\alpha(u^h, u) \psi dx dt \\ \geq -C_Z \|\nabla u\|_\infty \iint_{\mathbb{R}^d \times [0, (k+1)\delta]} |u^h(x, t) - u(x, t)|^2 \psi(x, t) dx dt.$$

Since the entropy η is supposed to be β_0 -convex, we have

$$(4.18) \quad H(x, t) \geq \frac{\beta_0}{2} |u^h(x, t) - u(x, t)|^2.$$

Putting (4.13)–(4.18) together with (4.12) provides

$$(4.19) \quad \left(\frac{\beta_0}{2\delta} - C_Z \|\nabla u\|_\infty \right) \int_{\mathcal{I}_k^\delta} \int_{\mathbb{R}^d} |u^h - u|^2 \psi dx dt \\ \leq C_Z \|\nabla u\|_\infty \iint_{\mathbb{R}^d \times [0, k\delta]} |u^h - u|^2 \psi dx dt + C_k \sqrt{h},$$

where (recall that $h \leq 1$)

$$C_k = C_\mu^k \left(\frac{1}{\delta} + L_f + 1 \right) + C_{\mu_0}^k + C_{\mu_0}^k \|D\eta(u_0)\|_\infty \\ + C_\mu^k \left(\|D\eta(u)\|_\infty \left(\frac{1}{\delta} + L_f + 1 \right) + \|D^2\eta\|_{\infty, \infty} (\|\partial_t u\|_\infty + \|\nabla u\|_\infty) \right).$$

Choose now $\delta = \frac{T}{p^*+1}$ with $p^* = \min \left\{ p \in \mathbb{N}^* \mid \frac{T}{p+1} \leq \frac{\beta_0}{2C_Z \|\nabla u\|_\infty + 2} \right\}$ (note that neither δ nor p^* depend on h), so that (4.19) becomes

$$(4.20) \quad e_k \leq \omega \sum_{i=0}^{k-1} e_i + C_k \sqrt{h},$$

where $e_k = \int_{\mathcal{I}_k^\delta} \int_{\mathbb{R}^d} |u^h - u|^2 \psi dx dt$ and $\omega = C_Z \|\nabla u\|_\infty$. Hence, few algebraic calculations allow us to claim that

$$(4.21) \quad \sum_{k=0}^{p^*} e_k \leq \sqrt{h} \sum_{k=0}^{p^*} C_k \left((1 + \omega)^{p^* - k + 1} - \omega \right).$$

Noticing that $\psi(x, t) = 1$ if $x \in B(0, r + L_f(T - t))$, and that $\psi(x, t) \geq 0$ for all $(x, t) \in \mathbb{R}^d \times (0, T)$, one finally has

$$(4.22) \quad \int_0^T \int_{B(0, r-st)} |u - u^h|^2 dx dt \leq \int_0^T \int_{\mathbb{R}^d} |u - u^h|^2 \psi(x, t) dx dt = \sum_{k=0}^{p^*} e_k.$$

We conclude the proof using (4.21) in (4.22).

5. Conclusion. We analyzed the convergence of first order finite volume schemes entering the framework detailed in [5] and summarized in §1.2.2. In §2.2, we derived a so-called *weak-BV* estimate based on the quantification of the numerical entropy dissipation. This estimate is new in the case of time-explicit finite volume schemes. It allows to prove some error estimate between a numerical solution and a strong solution of order $h^{1/4}$ in the space-time L^2 -norm. Let us also mention that one could use the weak-BV estimate to prove convergence to entropy measure-valued solutions, following [17] (see also [24]). On the other hand, strong solutions are global if one adds some suitable entropy-dissipating relaxation term [23, 43]), and our work could be extended to this situation without any major difficulty by mixing our result with the one proposed in [26].

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