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RELATIVE ENTROPY FOR THE FINITE VOLUME APPROXIMATION OF STRONG SOLUTIONS TO SYSTEMS OF CONSERVATION LAWS

CLÉMENT CANCÈS, HÉLÈNE MATHIS, NICOLAS SEGUIN

ABSTRACT. We study in this paper the finite volume approximation of strong solutions to systems of conservation laws. We derive error estimates in the multidimensional case, using the relative entropy between the strong solution and its finite volume approximation. The error terms are carefully studied, leading to a classical $h^{1/2}$ estimate under a BV assumption on the numerical approximation.

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1. SYSTEMS OF CONSERVATION LAWS

1.1. Strong, weak, and entropy weak solutions. We consider a system of m conservation laws

$$(1) \quad \partial_t u(x, t) + \sum_{\alpha=1}^d \partial_\alpha f_\alpha(u)(x, t) = 0.$$

The system (1) is set on the whole space $x \in \mathbb{R}^d$, and for any time $t \in [0, T]$, $T > 0$. We assume that there exists a convex bounded subset of \mathbb{R}^m , denoted by Ω and called *set of the admissible states* such that

$$(2) \quad u(x, t) \in \Omega, \quad \forall (x, t) \in \mathbb{R}^d \times [0, T].$$

The system (1) is complemented with the initial condition

$$(3) \quad u(x, 0) = u_0(x) \in \Omega, \quad \forall x \in \mathbb{R}^d.$$

We assume for all $\alpha \in \{1, \dots, d\}$ the functions $f_\alpha : \mathbb{R}^m \rightarrow \mathbb{R}^m$ to belong to $C^2(\bar{\Omega}; \mathbb{R}^m)$, and be such that $\nabla_u f_\alpha$ are diagonalizable with real eigenvalues, where

$\nabla_u \phi$ denotes the matrix of the differential of $\phi : u \mapsto \phi(u)$. Moreover, we assume that there exists $L_f > 0$ such that

$$(4) \quad \text{spec} \left(\frac{1}{2} (\nabla_u f_\alpha(u) + \nabla_u f_\alpha(u)^T) \right) \subset [-L_f, L_f], \quad \forall \alpha \in \{1, \dots, d\}, \quad \forall u \in \Omega.$$

The system (1) is endowed with a uniformly convex entropy $\eta \in C^2(\overline{\Omega}; \mathbb{R})$ such that there exists $M \geq \beta > 0$ so that

$$(5) \quad \text{spec} (\nabla_u^2 \eta(u)) \subset [\beta, M], \quad \forall u \in \overline{\Omega},$$

and the corresponding entropy flux $\xi \in C^2(\overline{\Omega}; \mathbb{R}^d)$ satisfies for all $\alpha \in \{1, \dots, d\}$

$$(6) \quad \nabla_u \xi_\alpha = (\nabla_u \eta)^T \nabla_u f_\alpha.$$

The existence of the entropy flux ξ amounts to assume the integrability condition

$$(7) \quad \nabla_u^2 \eta \nabla_u f_\alpha = (\nabla_u f_\alpha)^T \nabla_u^2 \eta.$$

Despite it is well-known that even for smooth initial data u_0 , the solutions of (1)–(3) may develop discontinuities after a finite time, our study is restricted to the approximation of smooth solutions $u \in C^1(\mathbb{R}^d \times \mathbb{R}_+; \Omega)$ of (1)–(3). Such solutions are called *strong solutions* and they automatically satisfy the conservation of the entropy

$$(8) \quad \partial_t \eta(u) + \sum_{\alpha=1}^d \partial_\alpha \xi_\alpha(u) = 0 \quad \text{in } \mathbb{R}^d \times \mathbb{R}_+.$$

We refer for example to [Daf10] for more details.

Assuming that $u_0 \in L^\infty(\mathbb{R}^d; \Omega)$, a function $u \in L^\infty(\mathbb{R}^d \times \mathbb{R}_+; \Omega)$ is said to be a *weak solution* to (1)–(3) if, for all $\phi \in C_c^1(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R}^n)$, one has

$$(9) \quad \iint_{\mathbb{R}^d \times \mathbb{R}_+} u \partial_t \phi \, dx dt + \int_{\mathbb{R}^d} u_0 \phi(\cdot, 0) \, dx + \iint_{\mathbb{R}^d \times \mathbb{R}_+} \sum_{\alpha=1}^d f_\alpha(u) \partial_\alpha \phi \, dx dt = 0.$$

Moreover, u is said to be an *entropy weak solution* to (1)–(3) if u is a weak solution, i.e., u satisfies (9), and if, for all $\psi \in C_c^1(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R}_+)$, it satisfies

$$(10) \quad \iint_{\mathbb{R}^d \times \mathbb{R}_+} \eta(u) \partial_t \psi \, dx dt + \int_{\mathbb{R}^d} \eta(u_0) \psi(\cdot, 0) \, dx + \iint_{\mathbb{R}^d \times \mathbb{R}_+} \sum_{\alpha=1}^d \xi_\alpha(u) \partial_\alpha \psi \, dx dt \geq 0.$$

1.2. Relative entropy. In the scalar case, the comparison of two entropy weak solutions lies on the Kruzhkov's paper [Kru70] which has been extended to the comparison of an entropy weak solution with an approximate solution by Kuznetsov [Kuz76]. Several improvements can be found in [CCL94, Vil94, EGGH98, BP98, CH99, EGH00]. In the case of systems of conservation laws, these techniques no longer work. Basically, the family of entropy–entropy flux pair (η, ξ) is not sufficiently rich to control the difference between two solutions. Nevertheless, let us assume that one of these solutions is a strong solution (u in the following definition) and introduce:

Definition 1.1 (Relative entropy). *Let $u, v \in \Omega$, we define the relative entropy of v w.r.t. u by*

$$H(v, u) = \eta(v) - \eta(u) - (\nabla_u \eta(u))^T (v - u),$$

while the corresponding relative entropy flux $Q : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^d$ is defined by

$$Q_\alpha(v, u) = \xi_\alpha(v) - \xi_\alpha(u) - (\nabla_u \eta(u))^T (f_\alpha(v) - f_\alpha(u)), \quad \forall \alpha \in \{1, \dots, d\}.$$

The notion of relative entropy for systems of conservation laws goes back to the early works of DiPerna and Dafermos (see [DiP79], [Daf79] and the condensed presentation in [Daf10]). It is also a powerful tool which has been extensively used for the study of hydrodynamic limits of kinetic equations (see the first works [Yau91] and [BGL93], but also [SR09] for more recent results). In the context of systems of conservation laws, it is easy to check that, given a strong solution u and an entropy weak solution v with respective initial data u_0 and v_0 , one has

$$(11) \quad \partial_t H(v, u) + \sum_{\alpha=1}^d \partial_\alpha Q_\alpha(v, u) \leq - \sum_{\alpha=1}^d (\partial_\alpha u)^T Z_\alpha(v, u)$$

in the weak sense, where

$$(12) \quad Z_\alpha(v, u) = \nabla_u^2 \eta(u) (f_\alpha(v) - f_\alpha(u) - (\nabla_u f_\alpha(u))^T (v - u)).$$

On the other hand, it follows from the definition of H that

$$(13) \quad H(v, u) = \int_0^1 \int_0^\theta (v - u)^T \nabla_u^2 \eta(u + \gamma(v - u)) (v - u) d\gamma d\theta,$$

which, together with (5), leads to

$$(14) \quad \frac{M}{2} (v - u)^2 \geq H(v, u) \geq \frac{\beta}{2} (v - u)^2, \quad \forall u, v \in \Omega.$$

If u is assumed to be a strong solution, its first and second derivatives are bounded and by a classical localization procedure *à la* Kruzhkov and a Gronwall lemma, one obtains a L_{loc}^2 stability estimate for any $r > 0$

$$(15) \quad \int_{|x| < r} |v(x, T) - u(x, T)|^2 dx \leq C(T, u) \int_{|x| < r + L_f T} |v_0(x) - u_0(x)|^2 dx$$

where the dependence of C on u reflect the needs of smoothness on u (C blows up when u becomes discontinuous). This inequality, rigorously proved in [Daf10], provides a weak–strong uniqueness result and similar (but more sophisticated) ideas have been recently applied to other fluid systems [FN12].

Remark 1.1. *For general conservation laws, the relative entropy is not symmetric, i.e. $H(u, v) \neq H(v, u)$ and $Q(u, v) \neq Q(v, u)$. In the very particular case of Friedrichs systems, i.e. when there exist symmetric matrices $A_\alpha \in \mathbb{R}^{m \times m}$ ($\alpha \in \{1, \dots, d\}$) such that $f_\alpha(u) = A_\alpha u$, then $u \mapsto |u|^2$ is an entropy and the corresponding entropy flux ξ is $\xi_\alpha(u) = u^T A_\alpha u$, ($\alpha \in \{1, \dots, d\}$). It is then easy to check that*

$$\begin{aligned} H(v, u) &= |v - u|^2 = H(u, v), \\ Q_\alpha(v, u) &= (v - u)^T A_\alpha (v - u) = Q_\alpha(v, u), \\ Z_\alpha(v, u) &= 0, \end{aligned}$$

for all $(u, v) \in \mathbb{R}^m$. As a consequence, inequality (11) becomes

$$\partial_t H(v, u) + \sum_{\alpha=1}^d \partial_\alpha Q_\alpha(v, u) \leq 0$$

even if u is a weak solution since H and Q are symmetric functions. This allows to make use of the doubling variable technique [Kru70] to compare u to v , recovering the classical uniqueness result for Friedrichs systems [Fri54].

Our aim is to replace the entropy weak solution v in (11) by an approximate solution provided by finite volume schemes on unstructured meshes. Following the formalism introduced in [EGGH98], this makes appear in (11) bounded Radon measures which can be controlled, leading to error estimates in $h^{1/2}$ between a

strong solution and its finite volume approximation, h being the characteristic size of the cells of the mesh.

Remark 1.2. *In the context of Friedrichs systems, such error estimates can be obtained, even in the weak setting, for the same reasons as those given in Remark 1.1, see [VV03, RJ05].*

Remark 1.3. *In [Tza05], Tzavaras studies the comparison of solutions of a hyperbolic system with relaxation with solutions of the associated equilibrium system of conservation laws. He also makes use of the relative entropy but, since he only focus on strong solutions (for both systems), he does not need to place himself in a weak setting with measure terms. Very similar questions have been addressed in [BV05, BTV09] for the convergence of kinetic equations towards the system of gas dynamics. Here again, only strong solutions of the Euler equations are considered. To finish the bibliographical review, let us mention the work by Leger and Vasseur [LV11] where the reference solution may include some particular discontinuities.*

1.3. Organization of the paper. We first introduce in Section 2 the family of numerical schemes we consider for approximating the solution to the problem (1)–(3). Some natural assumptions are made on the scheme, but also on the numerical solution on which we assume some reasonable stability property.

In order to compare the discrete solution (denoted by u^h in the sequel of the paper) with the strong solution u , we write continuous weak and entropy formulations in Section 3, so that we can adapt the uniqueness proof proposed in [Daf10]. Nevertheless, the discrete solution u^h is obviously not a weak entropy solution. Therefore, some error terms coming from the discretization have to be taken into account in the formulation, which take the form of positive locally bounded Radon measures, following [EGGH98]. A large part of Section 3 consists in making these measures explicit and to bound them with quantities tending towards 0 with the discretization size.

In Section 4, we make use of the weak and entropy weak formulations for the discrete solution (and of their corresponding error measures) to derive an error estimate. The distance between the strong solution u and the discrete solution u^h is quantified thanks to the relative entropy $H(u^h, u)$ introduced in Definition 1.1.

2. ENTROPY SATISFYING FINITE VOLUME SCHEMES

2.1. Definition of the numerical scheme. Let \mathcal{T} be a mesh of \mathbb{R}^d such that \mathbb{R}^d is the union of the closure of the elements of \mathcal{T} . We denote

$$h = \sup\{\text{diam}(K), K \in \mathcal{T}\} < \infty.$$

For all $K \in \mathcal{T}$, we denote by $|K|$ its d -dimensional Lebesgue measure, and by $\mathcal{N}(K)$ the set of its neighboring cells. For $L \in \mathcal{N}(K)$, the common interface between K and L is denoted by σ_{KL} and $|\sigma_{KL}|$ is its $(d-1)$ -Lebesgue measure. We assume that there exists $a > 0$ such that

$$(16) \quad |K| \leq ah^d \quad \text{and} \quad |\partial K| := \sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| \leq \frac{h^{d-1}}{a}, \quad \forall K \in \mathcal{T}.$$

The unit normal vector to σ_{KL} from K to L is denoted n_{KL} . Let $\Delta t > 0$ be the time step and we set $t^n = n\Delta t$, $\forall n \in \mathbb{N}$.

Remark 2.1. *In order to avoid some additional heavy notations, we have chosen to deal with an uniform time discretization and a space discretization that does not depend on time. Nevertheless, it is possible, by following the path described in [KO00], to adapt our study to the case of time depending space discretizations and to non-uniform time discretizations. This would be mandatory for considering*

a dynamic mesh adaptation procedure based on the a posteriori numerical error estimators that can be derived from our study.

For all $(K, L) \in \mathcal{T}^2$, $L \in \mathcal{N}(K)$, we consider the numerical flux $G_{KL} \in C^2(\Omega \times \Omega; \mathbb{R}^m) : (u, v) \rightarrow G_{KL}(u, v)$ which satisfies the following conditions:

$$(17) \quad G_{KL}(u, v) = -G_{LK}(v, u), \quad \forall (u, v) \in \Omega^2,$$

$$(18) \quad G_{KL}(u, u) = f(u) \cdot n_{KL}, \quad \forall u \in \Omega.$$

We assume that there exists $\lambda^* > 0$ such that, for all $\lambda > \lambda^*$, for all $K \in \mathcal{T}$, and for all $L \in \mathcal{N}(K)$,

$$(19) \quad u - \frac{1}{\lambda} [G_{KL}(u, v) - f(u) \cdot n_{KL}] \in \Omega, \quad \forall (u, v) \in \Omega^2.$$

We also assume there exists a numerical entropy flux ξ_{KL} which is conservative

$$(20) \quad \xi_{KL}(u, v) = -\xi_{LK}(v, u), \quad \forall (u, v) \in \Omega^2,$$

and such that, for all $\lambda \geq \lambda^* > 0$, the following entropy inequality by interface is satisfied:

$$(21) \quad \xi_{KL}(u, v) - \xi(u) \cdot n_{KL} \leq -\lambda \left(\eta \left(u - \frac{1}{\lambda} [G_{KL}(u, v) - f(u) \cdot n_{KL}] \right) - \eta(u) \right).$$

We additionally assume that the following CFL condition holds

$$(22) \quad \frac{\Delta t}{|K|} \lambda^* \sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| \leq 1, \quad \forall K \in \mathcal{T}.$$

In particular, the regularity of the mesh (16) implies that (22) holds if

$$(23) \quad \Delta t \leq \frac{a^2}{\lambda^*} h.$$

We have now introduced all the necessary material to define the numerical scheme we will consider in what follows.

Definition 2.1 (Finite volume scheme). *Let us introduce $N_T = \max\{n \in \mathbb{N}, n \leq T/\Delta t + 1\}$ and define the discrete unknowns u_K^n , $K \in \mathcal{T}$ and $n \in \{0, \dots, N_T\}$, by*

$$(24) \quad \frac{u_K^{n+1} - u_K^n}{\Delta t} |K| + \sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| G_{KL}(u_K^n, u_L^n) = 0$$

together with the initial condition

$$(25) \quad u_K^0 = \frac{1}{|K|} \int_K u_0(x) dx, \quad \forall K \in \mathcal{T},$$

under assumptions (17)–(21) on the numerical flux G_{KL} and under the CFL condition (23). The approximate solution $u^h \in L^\infty(\mathbb{R}^d \times \mathbb{R}_+)$ provided by the finite volume scheme (24)–(25) is defined by

$$(26) \quad u^h(x, t) = u_K^n, \quad \text{for } x \in K, \quad t^n \leq t < t^{n+1}, \quad K \in \mathcal{T}, \quad n \in \{0, \dots, N_T\}.$$

An important property of such a numerical scheme is the preservation of the set of admissible states.

Lemma 2.2. *Assume that (3), (19) and (22) holds, then, for all $K \in \mathcal{T}$, for all $n \in \{0, \dots, N_T\}$, u_K^n belong to Ω .*

Proof. Recall first that for all $K \in \mathcal{T}$, u_K^0 is defined by (25). Since, thanks to (2), u_0 takes its values in the convex set Ω , then so does its spatial mean values on the cells $K \in \mathcal{T}$. This ensures that u_K^0 belongs to Ω for all $K \in \mathcal{T}$.

Now, let us notice that assumption (18) implies

$$(27) \quad \sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| G_{KL}(u, u) = 0, \quad \forall u \in \mathbb{R}^m, \forall K \in \mathcal{T}.$$

Assume now that for all $K \in \mathcal{T}$, $u_K^n \in \Omega$. Using (24) and (27), we know that for all $K \in \mathcal{T}$ and for all $n \in \mathbb{N}$, one has

$$(28) \quad u_K^{n+1} = \sum_{L \in \mathcal{N}(K)} \frac{|\sigma_{KL}|}{|\partial K|} \left(u_K^n - \frac{\Delta t |\partial K|}{|K|} (G_{KL}(u_K^n, u_L^n) - f(u_K^n) \cdot n_{KL}) \right).$$

Thanks to (19) and (22), u_K^{n+1} is a convex combination of elements of the convex set Ω , then it also belongs to Ω . \square

Thanks to (21), we can derive entropy consistency properties on the numerical scheme.

Proposition 2.3. *Let u^h be defined by (24)–(26) (and thus, satisfying assumptions (17)–(21)). Then the numerical entropy flux ξ_{KL} is consistent with ξ , i.e.*

$$(29) \quad \xi_{KL}(u, u) = \xi(u) \cdot n_{KL}, \quad \forall u \in \Omega.$$

Moreover, under the CFL condition (23), u^h satisfies the discrete entropy inequality

$$(30) \quad \frac{m(K)}{\Delta t} (\eta(u_K^{n+1}) - \eta(u_K^n)) + \sum_{L \in \mathcal{N}(K)} \xi_{KL}(u_K^n, u_L^n) \leq 0.$$

Proof. First we focus on the consistence property (29). We write the inequality (21) for $\xi_{KL}(u_K, u_L)$ and $\xi_{LK}(u_L, u_K)$:

$$\begin{aligned} \xi_{KL}(u_K, u_L) &\leq \xi(u_K) \cdot n_{KL} \\ &\quad - \lambda \left(\eta(u_K - \frac{1}{\lambda} [G_{KL}(u_K, u_L) - f(u_K) \cdot n_{KL}]) - \eta(u_K) \right), \\ \xi_{LK}(u_L, u_K) &\leq \xi(u_L) \cdot n_{LK} \\ &\quad - \lambda \left(\eta(u_L - \frac{1}{\lambda} [G_{LK}(u_L, u_K) - f(u_L) \cdot n_{LK}]) - \eta(u_L) \right). \end{aligned}$$

Setting $u_K = u_L = u$, then using the conservation property (20) and the consistency relation (18) lead to the consistency of the numerical entropy flux (29).

Applying η to (28) and using its convexity yields

$$(31) \quad \eta(u_K^{n+1}) \leq \sum_{L \in \mathcal{N}(K)} \frac{|\sigma_{KL}|}{|\partial K|} \eta \left(u_K^n - \frac{\Delta t |\partial K|}{|K|} (G_{KL}(u_K^n, u_L^n) - f(u_K^n) \cdot n_{KL}) \right).$$

Now, it follows from the CFL condition (23), the interface entropy condition (21) and

$$(32) \quad \sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| \xi_{KL}(u, u) = 0, \quad \forall u \in \Omega$$

that (30) holds. \square

2.2. Stability estimates and main result. In order to carry out an error estimate, we have to assume some “regularity” on the approximate solution u^h . In practice, we assume from now on that there exists $C_{BV} > 0$ (that may depend at most on r , T , u_0 and on G_{KL}) such that

$$(33) \quad \sum_{n=0}^{N_T} \Delta t \sum_{(K,L) \in \mathcal{E}_r} |\sigma_{KL}| |G_{KL}(u_K^n, u_L^n) - f(u_K^n) \cdot n_{KL}| \leq C_{BV}.$$

Remark 2.2. Assumption (33) requires several comments.

- (1) Since we consider explicit numerical schemes, the stability estimate (33) requires a CFL condition to be satisfied in order to prevent oscillations. One can reasonably assess that

$$(34) \quad \Delta t \leq C_{stab} h$$

for some $C_{stab} > 0$ which may depend on the same quantities as C_{BV} . We then have to choose Δt small enough so that both (34) and (23) are satisfied.

- (2) Assuming (33) is weaker than assuming that $u^h \in (BV_{loc}(\mathbb{R}_+ \times \mathbb{R}^d))^m$. Indeed, consider the simple linear transport case where $f(u) = u \otimes \mathbf{e}_1$ for $\mathbf{e}_1 = (1, 0)^T$ in \mathbb{R}^2 with a grid whose edges are along \mathbf{e}_1 and $\mathbf{e}_2 = (0, 1)^T$, and define $G_{K,L}$ as the upstream flux. Then (33) only requires u^h to have a bounded variation in the direction of \mathbf{e}_1 . We refer to [Des04, Lemma 5] for an example where u^h is not uniformly bounded in $BV_{loc}(\mathbb{R}_+ \times \mathbb{R}^2)$ but where (33) still holds.
- (3) In the scalar case $m = 1$, it is possible to recover estimate (33) on cartesian grids even for nonlinear flux functions $f : \mathbb{R} \rightarrow \mathbb{R}^d$ (cf. e.g. [CH99, EGGH98, GR91]). On unstructured grids, despite (33) seems to remain true in the numerical experiments (cf. [KO00]), only a weaker inequality can be proved, namely

$$\sum_{n=0}^{N_T} \Delta t \sum_{(K,L) \in \mathcal{E}_r} |\sigma_{KL}| |G_{KL}(u_K^n, u_L^n) - f(u_K^n) \cdot n_{KL}| \leq \frac{C_{wBV}}{\sqrt{h}}$$

for some $C_{wBV} > 0$. This estimate, called weak-BV estimate in [EGGH98, CH99], relies on the quantification of the numerical diffusion introduced by the scheme [Vil94, CCL94].

Using the BV assumption (33), one may derive entropy BV estimates and time-BV estimates.

Lemma 2.4. Assume that (17)–(21) and (33) hold, then one has

$$(35) \quad \sum_{n=0}^{N_T} \Delta t \sum_{(K,L) \in \mathcal{E}_r} |\sigma_{KL}| |\xi_{KL}(u_K^n, u_L^n) - \xi(u_K) \cdot n_{KL}| \leq \|\nabla_u \eta\|_{L^\infty(\Omega)} C_{BV}.$$

Proof. This is a direct consequence of (21) in which the Lipschitz continuity of η has been taken into account. \square

Lemma 2.5 (time-BV estimates). Under Assumption (33), one has

$$(36) \quad \sum_{n=0}^{N_T} \sum_{K \in \mathcal{T}_r} |K| |u_K^{n+1} - u_K^n| \leq C_{BV},$$

$$(37) \quad \sum_{n=0}^{N_T} \sum_{K \in \mathcal{T}_r} |K| |\eta(u_K^{n+1}) - \eta(u_K^n)| \leq \|\nabla_u \eta\|_{L^\infty(\Omega)} C_{BV}.$$

Proof. Thanks to the definition (24) of the scheme and thanks to the divergence free property (27), for all $K \in \mathcal{T}$ and all $n \in \mathbb{N}$, one has

$$(u_K^{n+1} - u_K^n) |K| = -\Delta t \sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| (G_{KL}(u_K^n, u_L^n) - f(u_K^n) \cdot n_{KL}),$$

so that

$$|u_K^{n+1} - u_K^n| |K| \leq \Delta t \sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| |G_{KL}(u_K^n, u_L^n) - f(u_K^n) \cdot n_{KL}|.$$

Summing over $K \in \mathcal{T}_r$ and $n \in \{0, \dots, N_T\}$ and using (33) provides (36). Inequality (37) then follows from the Lipschitz continuity of η . \square

Let us now state our main result.

Theorem 2.6. *Assume that $u_0 \in W^{1,\infty}(\mathbb{R}^d)$ and that the solution u of the Cauchy problem (1)–(3) belongs to $W^{1,\infty}([0, T] \times \mathbb{R}^d)$. Assume also that (3), (16), (17)–(21), (23), and identities (6)–(7) hold. Let u^h defined by (24)–(26), and assume that the stability estimate (33) holds, then, for all $r > 0$ and $T > 0$ there exist C depending only on $T, r, \Omega, a, \lambda^*, u_0, G_{KL}, \eta$ and f , such that*

$$\int_0^T \int_{B(0, r+L_f(T-t))} |u - u_h|^2 dx dt \leq Ch.$$

3. CONTINUOUS WEAK AND ENTROPY FORMULATIONS FOR THE DISCRETE SOLUTION

In order to obtain the error estimate of Theorem 2.6, we aim at using the relative entropy of u^h w.r.t. u . Since u^h is only an approximate solution, it does not exactly satisfies neither the weak formulation (9) nor the entropy weak formulation (10). Some numerical error terms appear in these formulations, and thus also appear the inequality of the relative entropy

$$(38) \quad \partial_t H(u^h, u) + \sum_{\alpha=1}^d \partial_\alpha Q_\alpha(u^h, u) \leq - \sum_{\alpha=1}^d (\partial_\alpha u)^T Z_\alpha(u^h, u) + \text{numerical error terms.}$$

As usual, these terms may be described by Radon measures — see [BP98, EGGH98, CH99, KO00] in the scalar case and [RJ05] in the case of Friedrichs systems. Note that for nonlinear systems of conservation laws, a function which satisfies the entropy inequality (10) is not necessarily a weak solution (9). This leads us to consider the measures not only in the entropy inequality of u^h but also in the weak formulation of u^h . Let us first begin with the entropy formulation and the related measures.

For $X = \mathbb{R}^d$ or $X = \mathbb{R}^d \times \mathbb{R}^+$, we denote $\mathcal{M}(X)$ the set of locally bounded Radon measures on X , i.e., $\mathcal{M}(X) = (C_c(X))'$. If $\mu \in \mathcal{M}(X)$ we set

$$\langle \mu, \varphi \rangle = \int_X \varphi d\mu, \forall \varphi \in C_c(X).$$

Definition 3.1. *For $\psi \in C_c(\mathbb{R}^d)$, $\varphi \in C_c(\mathbb{R}^d \times \mathbb{R}^+)$, we define $\mu_0 \in \mathcal{M}(\mathbb{R}^d)$ and $\mu \in \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^+)$ by*

$$\begin{aligned} \langle \mu_0, \psi \rangle &= \int_{\mathbb{R}^d} |\eta(u_0(x)) - \eta(u^h(x, 0))| \psi(x) dx, \\ \langle \mu, \varphi \rangle &= \langle \mu_{\mathcal{T}}, \varphi \rangle + \sum_{n=0}^{\infty} \Delta t \sum_{(K,L) \in \mathcal{E}_r} |\sigma_{KL}| |\xi_{KL}(u_K^n, u_L^n) - \xi_{KL}(u_K^n, u_K^n)| \langle \mu_{KL}, \varphi \rangle \\ &\quad + \sum_{n=0}^{\infty} \Delta t \sum_{(K,L) \in \mathcal{E}_r} |\sigma_{KL}| |\xi_{KL}(u_K^n, u_L^n) - \xi_{KL}(u_L^n, u_L^n)| \langle \mu_{LK}, \varphi \rangle, \end{aligned}$$

where

$$\begin{aligned}\langle \mu_{\mathcal{T}}, \varphi \rangle &= \sum_{n=0}^{\infty} \sum_{K \in \mathcal{T}_r} |\eta(u_K^{n+1}) - \eta(u_K^n)| \int_{t^n}^{t^{n+1}} \int_K \varphi(x, t) dx dt, \\ \langle \mu_{KL}, \varphi \rangle &= \frac{1}{m(K)m(\sigma_{KL})(\Delta t)^2} \\ &\quad \times \int_{t^n}^{t^{n+1}} \int_K \int_{t^n}^{t^{n+1}} \int_{\sigma_{KL}} \int_0^1 (h + \Delta t) \varphi(\gamma + \theta(x - \gamma), s + \theta(t - s)) d\theta dx dt d\gamma ds, \\ \langle \mu_{LK}, \varphi \rangle &= \frac{1}{m(L)m(\sigma_{KL})(\Delta t)^2} \\ &\quad \times \int_{t^n}^{t^{n+1}} \int_L \int_{t^n}^{t^{n+1}} \int_{\sigma_{KL}} \int_0^1 (h + \Delta t) \varphi(\gamma + \theta(x - \gamma), s + \theta(t - s)) d\theta dx dt d\gamma ds.\end{aligned}$$

The measures μ and μ_0 describe the error of approximation in the entropy formulation satisfied by u^h . Indeed, we have:

Lemma 3.2. *Assume that (3), (16), (17)–(21), (23), and identities (6)–(7) hold. Let u^h defined by (24)–(26), and assume that the stability estimate (33) holds, then, for all $r > 0$ and $T > 0$ there exist $C_{\mu_0} > 0$, depending only on u_0 , $\|\nabla_u \eta\|_{L^\infty(\Omega)} := \sup_{u \in \Omega} |\nabla_u \eta(u)|$ and r , and $C_\mu > 0$, depending only on $T, r, a, \lambda_\star, u_0, G_{KL}$ and η such that, for all $h < r$,*

$$(39) \quad \mu_0(B(0, r)) \leq C_{\mu_0} h,$$

$$(40) \quad \mu(B(0, r) \times [0, T]) \leq C_\mu h.$$

Proof. The measure μ_0 is the measure of density $|\eta(u_0(x)) - \eta(u^h(x, 0))|$ with respect to the Lebesgue measure. For all $r > 0$ we have

$$\lim_{h \rightarrow 0} \mu_0(B(0, r)) = 0.$$

For smooth $u_0 : \mathbb{R}^d \rightarrow \mathbb{R}^m$, the measure μ_0 satisfies

$$\mu_0(B(0, r)) \leq h \|\nabla_u \eta\|_{L^\infty(\Omega)} \int_{B(0, r+h)} |\nabla u_0| dx.$$

For $r > 0$ and $T > 0$ the measure $\mu_{\mathcal{T}}$ satisfies

$$\mu_{\mathcal{T}}(B(0, r) \times [0, T]) = \int_0^T \int_{B(0, r)} \sum_{n=0}^{N_T} \sum_{K \in \mathcal{T}_r} |\eta(u_K^{n+1}) - \eta(u_K^n)| \mathbf{1}_{K \times [t^n, t^{n+1}]} dx dt.$$

Then, using the time-BV estimate (37),

$$\mu_{\mathcal{T}}(B(0, r) \times [0, T]) \leq \Delta t \sum_{n=0}^{N_T} \sum_{K \in \mathcal{T}_r} |K| |\eta(u_K^{n+1}) - \eta(u_K^n)| \leq \Delta t \|\nabla_u \eta\|_{L^\infty(\Omega)} C_{BV}.$$

It follows from (23) that

$$(41) \quad \mu_{\mathcal{T}}(B(0, r) \times [0, T]) \leq C_{\mu_{\mathcal{T}}} h,$$

where

$$C_{\mu_{\mathcal{T}}} := \frac{a^2 \|\nabla_u \eta\|_{L^\infty(\Omega)}}{\lambda^\star} C_{BV}.$$

The measures μ_{KL} and μ_{LK} satisfy:

$$\mu_{KL}(\mathbb{R}^d \times \mathbb{R}^+) \leq h + \Delta t, \quad \mu_{LK}(\mathbb{R}^d \times \mathbb{R}^+) \leq h + \Delta t.$$

Therefore,

$$\begin{aligned} & \mu(B(0, r) \times [0, T]) \\ & \leq C_{\mu_T} h + (h + \Delta t) \sum_{n=0}^{N_T} \Delta t \sum_{(K, L) \in \mathcal{E}_r} |\sigma_{KL}| |\xi_{KL}(u_K^n, u_L^n) - \xi_{KL}(u_K^n, u_K^n)| \\ & \quad + (h + \Delta t) \sum_{n=0}^{N_T} \Delta t \sum_{(K, L) \in \mathcal{E}_r} |\sigma_{KL}| |\xi_{KL}(u_K^n, u_L^n) - \xi_{KL}(u_L^n, u_L^n)|. \end{aligned}$$

Hence, using Lemma 2.4, the CFL condition (23) and the bound (41) provides the following bound:

$$\mu(B(0, r) \times [0, T]) \leq C_\mu h,$$

where $C_\mu = C_{\mu_T} + 2 \left(1 + \frac{a^2}{\lambda^*}\right) \|\nabla_u \eta\|_{L^\infty(\Omega)} C_{BV}$. \square

Proposition 3.3. Assume (3), (16), (17)–(21), (23) and identities (6)–(7). Let u^h defined by (24)–(26), and assume that the stability estimate (33) holds. Let μ and μ_0 be the measures introduced in Definition 3.1, then, for all $\varphi \in C_c^1(\mathbb{R}^d \times \mathbb{R}^+; \mathbb{R}^+)$, one has

$$\begin{aligned} (42) \quad & \iint_{\mathbb{R}^d \times \mathbb{R}_+} \eta(u^h) \partial_t \varphi(x, t) + \sum_{\alpha=1}^d \xi_\alpha(u^h) \partial_\alpha \varphi(x, t) dx dt \\ & + \int_{\mathbb{R}^d} \eta(u_0(x)) \varphi(x, 0) dx \geq - \iint_{\mathbb{R}^d \times \mathbb{R}_+} (|\nabla \varphi| + |\partial_t \varphi|) d\mu(x, t) - \int_{\mathbb{R}^d} \varphi(x, 0) d\mu_0(x). \end{aligned}$$

Proof. Let $\varphi \in C_c^1(\mathbb{R}^d \times \mathbb{R}^+; \mathbb{R}^+)$. Let $T > 0$ and $r > 0$ such that $\text{supp } \varphi \subset B(0, r) \times [0, T]$. Let us multiply (30) by $\int_{t^n}^{t^{n+1}} \int_K \varphi(x, t) dx dt$ and sum over the control volumes $K \in \mathcal{T}_r$ and $n \leq N_T$. It yields

$$(43) \quad T_1 + T_2 \leq 0$$

where

$$(44) \quad T_1 = \sum_{n=0}^{N_T} \sum_{K \in \mathcal{T}_r} \frac{1}{\Delta t} (\eta(u_K^{n+1}) - \eta(u_K^n)) \int_{t^n}^{t^{n+1}} \int_K \varphi(x, t) dx dt$$

and

$$(45) \quad T_2 = \sum_{n=0}^{N_T} \sum_{K \in \mathcal{T}_r} \frac{1}{|K|} \int_{t^n}^{t^{n+1}} \int_K \varphi(x, t) dx dt \sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| \xi_{KL}(u_K^n, u_L^n).$$

The term T_1 corresponds to the discrete time derivative of $\eta(u^h)$ and T_2 to the discrete space derivative of $\xi(u^h)$. The proof relies on the comparison firstly between T_1 and T_{10} and secondly between T_2 and T_{20} , where T_{10} and T_{20} denote respectively the temporal and spatial term in (42):

$$\begin{aligned} (46) \quad T_{10} &= - \iint_{\mathbb{R}^d \times \mathbb{R}_+} \eta(u^h) \partial_t \varphi(x, t) dx dt - \int_{\mathbb{R}^d} \eta(u_0(x)) \varphi(x, 0) dx, \\ T_{20} &= - \iint_{\mathbb{R}^d \times \mathbb{R}_+} \sum_{\alpha=1}^d \xi_\alpha(u^h) \partial_\alpha \varphi(x, t) dx dt. \end{aligned}$$

Let us first focus on T_{10} . Following its definition (26), the approximate solution u^h is piecewise constant, then so does $\eta(u_h)$. This yields

$$\begin{aligned}
T_{10} &= - \sum_{n=0}^{N_T} \sum_{K \in \mathcal{T}_r} \eta(u_K^n) \int_{t^n}^{t^{n+1}} \int_K \partial_t \varphi dx dt - \int_{\mathbb{R}^d} \eta(u_0(x)) \varphi(x, 0) dx \\
&= - \sum_{n=0}^{N_T} \sum_{K \in \mathcal{T}_r} \eta(u_K^n) \int_K (\varphi(x, t^{n+1}) - \varphi(x, t^n)) dx - \int_{\mathbb{R}^d} \eta(u_0(x)) \varphi(x, 0) dx \\
&= \sum_{n=0}^{N_T} \sum_{K \in \mathcal{T}_r} (\eta(u_K^{n+1}) - \eta(u_K^n)) \int_K \varphi(x, t^{n+1}) dx \\
&\quad - \int_{\mathbb{R}^d} (\eta(u_0(x)) - \eta(u^h(x, 0))) \varphi(x, 0) dx \\
&= \sum_{n=0}^{N_T} \sum_{K \in \mathcal{T}_r} (\eta(u_K^{n+1}) - \eta(u_K^n)) \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \int_K \varphi(x, t^{n+1}) dx dt \\
&\quad - \int_{\mathbb{R}^d} (\eta(u_0(x)) - \eta(u^h(x, 0))) \varphi(x, 0) dx.
\end{aligned}$$

Then we consider the quantity $|T_1 - T_{10}|$:

$$\begin{aligned}
|T_1 - T_{10}| &\leq \sum_{n=0}^{N_T} \sum_{K \in \mathcal{T}_r} \frac{|\eta(u_K^{n+1}) - \eta(u_K^n)|}{\Delta t} \int_{t^n}^{t^{n+1}} \int_K |\varphi(x, t) - \varphi(x, t^{n+1})| dx dt \\
&\quad + \int_{\mathbb{R}^d} |\eta(u_0(x)) - \eta(u^h(x, 0))| \varphi(x, 0) dx \\
&\leq \sum_{n=0}^{N_T} \sum_{K \in \mathcal{T}_r} |\eta(u_K^{n+1}) - \eta(u_K^n)| \int_{t^n}^{t^{n+1}} \int_K |\partial_t \varphi| dx dt \\
&\quad + \int_{\mathbb{R}^d} |\eta(u_0(x)) - \eta(u^h(x, 0))| \varphi(x, 0) dx.
\end{aligned}$$

Then, accounting from Definition 3.1, the inequality reads:

$$(47) \quad |T_1 - T_{10}| \leq \iint_{\mathbb{R}^d \times \mathbb{R}_+} |\partial_t \varphi| d\mu_{\mathcal{T}}(x, t) + \int_{\mathbb{R}^d} \varphi(x, 0) d\mu_0(x).$$

We now consider the terms T_2 and T_{20} . Performing a discrete integration by parts by reorganizing the sum, and using the properties (32) and (20) lead to

$$(48) \quad T_2 = T_{2,1} + T_{2,2},$$

with

$$\begin{aligned}
T_{2,1} &= \sum_{n=0}^{N_T} \sum_{(K,L) \in \mathcal{E}_r} \frac{|\sigma_{KL}|}{|K|} \int_{t^n}^{t^{n+1}} \int_K \varphi(x, t) (\xi_{KL}(u_K^n, u_L^n) - \xi_{KL}(u_K^n, u_K^n)) dx dt, \\
T_{2,2} &= \sum_{n=0}^{N_T} \sum_{(K,L) \in \mathcal{E}_r} \frac{|\sigma_{KL}|}{|L|} \int_{t^n}^{t^{n+1}} \int_L \varphi(x, t) (\xi_{LK}(u_L^n, u_K^n) - \xi_{LK}(u_L^n, u_L^n)) dx dt.
\end{aligned}$$

Gathering terms of T_{20} by edges yields

$$T_{20} = T_{20,1} + T_{20,2},$$

where, thanks to (20), we have set

$$\begin{aligned} T_{20,1} &= \sum_{n=0}^{N_T} \sum_{(K,L) \in \mathcal{E}_r} \int_{t^n}^{t^{n+1}} \int_{\sigma_{KL}} (\xi_{KL}(u_K^n, u_L^n) - \xi(u_K^n) \cdot n_{KL}) \varphi(\gamma, t) d\gamma dt, \\ T_{20,2} &= \sum_{n=0}^{N_T} \sum_{(K,L) \in \mathcal{E}_r} \int_{t^n}^{t^{n+1}} \int_{\sigma_{KL}} (\xi_{LK}(u_L^n, u_K^n) - \xi(u_L^n) \cdot n_{LK}) \varphi(\gamma, t) d\gamma dt. \end{aligned}$$

Let us now compare the terms $T_{2,1}$ and $T_{20,1}$:

$$\begin{aligned} T_{2,1} - T_{20,1} &= \sum_{n=0}^{N_T} \Delta t \sum_{(K,L) \in \mathcal{E}_r} |\sigma_{KL}| (\xi_{KL}(u_K^n, u_L^n) - \xi(u_K^n) \cdot n_{KL}) \\ &\quad \times \left(\frac{1}{|K| \Delta t} \int_{t^n}^{t^{n+1}} \int_K \varphi(x, t) dx dt \right. \\ &\quad \left. - \frac{1}{|\sigma_{KL}| \Delta t} \int_{t^n}^{t^{n+1}} \int_{\sigma_{KL}} \varphi(\gamma, s) d\gamma ds \right). \end{aligned}$$

In order to handle the mean value of φ over the cell K and its mean value over the edge σ_{KL} , we write

$$\begin{aligned} &\frac{1}{|K|} \int_{t^n}^{t^{n+1}} \int_K \varphi(x, t) dx dt \\ &= \frac{1}{|K| |\sigma_{KL}| \Delta t} \int_{t^n}^{t^{n+1}} \int_K \int_{t^n}^{t^{n+1}} \int_{\sigma_{KL}} \varphi(x, t) d\gamma ds dx dt, \\ &\frac{1}{|\sigma_{KL}|} \int_{t^n}^{t^{n+1}} \int_{\sigma_{KL}} \varphi(\gamma, t) d\gamma dt \\ &= \frac{1}{|K| |\sigma_{KL}| \Delta t} \int_{t^n}^{t^{n+1}} \int_K \int_{t^n}^{t^{n+1}} \int_{\sigma_{KL}} \varphi(\gamma, s) d\gamma ds dx dt. \end{aligned}$$

Thus one has

$$\begin{aligned} T_{2,1} - T_{20,1} &= \sum_{n=0}^{N_T} \Delta t \sum_{(K,L) \in \mathcal{E}_r} |\sigma_{KL}| (\xi_{KL}(u_K^n, u_L^n) - \xi_{KL}(u_K^n, u_L^n)) \\ &\quad \times \frac{1}{|K| |\sigma_{KL}| (\Delta t)^2} \int_{t^n}^{t^{n+1}} \int_K \int_{t^n}^{t^{n+1}} \int_{\sigma_{KL}} (\varphi(x, t) - \varphi(\gamma, s)) d\gamma ds dx dt. \end{aligned}$$

Writing a Taylor's expansion of φ with the integral form of the remainder provides for all $(x, \gamma, t, s) \in K \times \sigma_{KL} \times [t^n, t^{n+1}]^2$

$$(49) \quad |\varphi(x, t) - \varphi(\gamma, s)| \leq \int_0^1 (h + \Delta t) (|\nabla \varphi| + |\partial_t \varphi|) (\gamma + \theta(x - \gamma), s + \theta(t - s)) d\theta.$$

Then using Definition 3.1 of the measure μ_{KL} we obtain the following estimate:

$$(50) \quad |T_{2,1} - T_{20,1}| \leq \sum_{n=0}^{N_T} \Delta t \sum_{(K,L) \in \mathcal{E}_r} |\sigma_{KL}| |\xi_{KL}(u_K^n, u_L^n) - \xi_{KL}(u_K^n, u_L^n)| \langle \mu_{KL}, |\nabla \varphi| + |\partial_t \varphi| \rangle.$$

Similarly, one obtains that

$$(51) \quad |T_{2,2} - T_{20,2}| \leq \sum_{n=0}^{N_T} \Delta t \sum_{(K,L) \in \mathcal{E}_r} |\sigma_{KL}| |\xi_{LK}(u_L^n, u_K^n) - \xi_{LK}(u_L^n, u_K^n)| \langle \mu_{LK}, |\nabla \varphi| + |\partial_t \varphi| \rangle,$$

where the measure $\mu_{LK} \in \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^+)$ is given by Definition 3.1.

Bearing in mind the definition of $\mu \in \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^+)$ given in Definition 3.1, inequalities (43), (47), (48), (50) and (51), one has

$$-T_{10} - T_{20} \geq - \iint_{\mathbb{R}^d \times \mathbb{R}_+} (|\nabla \varphi| + |\partial_t \varphi|) d\mu(x, t) - \int_{\mathbb{R}^d} \varphi(x, 0) d\mu_0(x)$$

that concludes the proof. \square

The same kind of tools can be used to deduce how close is u^h w.r.t. to a weak solution. For that purpose we define the following measures.

Definition 3.4. For $\psi \in C_c(\mathbb{R}^d)$ and $\varphi \in C_c(\mathbb{R}^d \times \mathbb{R}^+)$, we set

$$\begin{aligned} \langle \bar{\mu}_0, \psi \rangle &= \int_{\mathbb{R}^d} |u_0(x) - u^h(x, 0)| \psi(x) dx, \\ \langle \bar{\mu}, \varphi \rangle &= \langle \bar{\mu}_{\mathcal{T}}, \varphi \rangle + \sum_{n=0}^{\infty} \Delta t \sum_{(K,L) \in \mathcal{E}_r} |\sigma_{KL}| |G_{KL}(u_K^n, u_L^n) - G_{KL}(u_K^n, u_L^n)| \langle \bar{\mu}_{KL}, \varphi \rangle \\ &\quad + \sum_{n=0}^{\infty} \Delta t \sum_{(K,L) \in \mathcal{E}_r} |\sigma_{KL}| |G_{KL}(u_K^n, u_L^n) - G_{KL}(u_L^n, u_K^n)| \langle \bar{\mu}_{LK}, \varphi \rangle, \end{aligned}$$

where

$$\begin{aligned} \langle \bar{\mu}_{\mathcal{T}}, \varphi \rangle &= \sum_{n=0}^{\infty} \sum_{K \in \mathcal{T}_r} |u_K^{n+1} - u_K^n| \int_{t^n}^{t^{n+1}} \int_K \varphi(x, t) dx dt, \\ \langle \bar{\mu}_{KL}, \varphi \rangle &= \frac{1}{|K| |\sigma_{KL}| \Delta t^2} \times \\ &\quad \int_{t^n}^{t^{n+1}} \int_K \int_{t^n}^{t^{n+1}} \int_{\sigma_{KL}} \int_0^1 (h + \Delta t) \varphi(\gamma + \theta(x - \gamma), s + \theta(t - s)) d\theta dx dt d\gamma ds, \\ \langle \bar{\mu}_{LK}, \varphi \rangle &= \frac{1}{|L| |\sigma_{KL}| \Delta t^2} \times \\ &\quad \int_{t^n}^{t^{n+1}} \int_L \int_{t^n}^{t^{n+1}} \int_{\sigma_{KL}} \int_0^1 (h + \Delta t) \varphi(\gamma + \theta(x - \gamma), s + \theta(t - s)) d\theta dx dt d\gamma ds. \end{aligned}$$

Remark 3.1. It follows from the definitions of the measures μ and $\bar{\mu}$ that they can be extended (in a unique way) into continuous linear forms defined on the set

$$E := \left\{ \varphi \in L^\infty(\mathbb{R}^d \times \mathbb{R}^+; \mathbb{R}) \mid \text{supp}(\varphi) \text{ is compact, and } \nabla \varphi \in \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^+)^d \right\}.$$

Indeed, any $\varphi \in E$ admits traces on σ_{KL} , so that the quantities $\langle \mu_{KL}, \varphi \rangle$, $\langle \mu_{LK}, \varphi \rangle$, $\langle \bar{\mu}_{KL}, \varphi \rangle$ and $\langle \bar{\mu}_{LK}, \varphi \rangle$ are well defined. Moreover, one has

$$|\langle \mu, \varphi \rangle| \leq \|\varphi\|_{L^\infty} \mu(\{\varphi \neq 0\}), \quad |\langle \bar{\mu}, \varphi \rangle| \leq \|\varphi\|_{L^\infty} \bar{\mu}(\{\varphi \neq 0\}), \quad \forall \varphi \in E.$$

We now state a lemma and a proposition whose proofs are left to the reader, since they are similar to the proofs of Lemma 3.2 and Proposition 3.3 respectively as one uses the estimates (33) and (36) instead of (35) and (37).

Lemma 3.5. Assume that (3), (16), (17)–(21), (23), and identities (6)–(7) hold. Let u^h defined by (24)–(26), and assume that the stability estimate (33) holds, then, for all $r > 0$ and $T > 0$ there exist $C_{\bar{\mu}_0} > 0$, depending only on u_0 and r , and $C_{\bar{\mu}} > 0$, depending only on $T, r, a, \lambda_\star, u_0, G_{KL}$ such that, for all $h < r$,

$$(52) \quad \begin{aligned} \bar{\mu}_0(B(0, r)) &\leq C_{\bar{\mu}_0} h, \\ \bar{\mu}(B(0, r) \times [0, T]) &\leq C_{\bar{\mu}} h. \end{aligned}$$

The quantities $C_{\bar{\mu}_0}$ and $C_{\bar{\mu}}$ are related to C_{μ_0} and C_{μ} by the relations

$$C_{\mu_0} = \|\nabla_u \eta\|_{L^\infty(\Omega)} C_{\bar{\mu}_0}, \quad C_{\mu} = \|\nabla_u \eta\|_{L^\infty(\Omega)} C_{\bar{\mu}},$$

where C_{μ_0} and C_{μ} have been explicitly calculated during the proof of Lemma 3.2.

We are now in position to provide the approximate weak formulation satisfied by u^h .

Proposition 3.6. *Assume (3), (16), (17)–(21), (23) and identities (6)–(7). Let u^h defined by (24)–(26), and assume that the stability estimate (33) holds. Let μ and μ_0 be the measures introduced in Definition 3.1, then, for all $\varphi \in C_c^1(\mathbb{R}^d \times \mathbb{R}^+; \mathbb{R}^m)$, one has*

$$(53) \quad \left| \iint_{\mathbb{R}^d \times \mathbb{R}_+} (u^h)^T \partial_t \varphi(x, t) + \sum_{\alpha=1}^d f_\alpha(u^h)^T \partial_\alpha \varphi(x, t) dx dt + \int_{\mathbb{R}^d} u_0(x)^T \varphi(x, 0) dx \right| \\ \leq \iint_{\mathbb{R}^d \times \mathbb{R}_+} (|\nabla \varphi| + |\partial_t \varphi|) d\bar{\mu}(x, t) + \int_{\mathbb{R}^d} |\varphi(x, 0)| d\bar{\mu}_0(x).$$

Let us stress that φ is a vector-valued function. This means that

$$|\nabla \varphi| = \max_{\alpha \in \{1, \dots, d\}} |\partial_\alpha \varphi|.$$

Nevertheless, the proof of Proposition 3.6 follows the same guidelines as the proof of Proposition 3.3 and is left to the reader.

4. ERROR ESTIMATE USING THE RELATIVE ENTROPY

Thanks to the measures defined above and Propositions 3.3 and 3.6, we are now in position to provide the precise meaning of inequality (38) satisfied by the relative entropy $H(u^h, u)$ and conclude the proof of Theorem 2.6.

4.1. Relative entropy for approximate solutions.

Proposition 4.1. *Assume (3), (16), (17)–(21), (23) and identities (6)–(7). Let u^h defined by (24)–(26), and assume that the stability estimate (33) holds. Let μ and μ_0 be the measures introduced in Definition 3.1, and let $\bar{\mu}$ and $\bar{\mu}_0$ be the measures introduced in Definition 3.4 then, for all $\varphi \in C_c^1(\mathbb{R}^d \times \mathbb{R}^+; \mathbb{R}^+)$, one has*

$$(54) \quad \iint_{\mathbb{R}^d \times \mathbb{R}_+} \left(H(u^h, u) \partial_t \varphi(x, t) + \sum_{\alpha=1}^d Q_\alpha(u^h, u) \partial_\alpha \varphi(x, t) \right) dx dt \geq \\ - \iint_{\mathbb{R}^d \times \mathbb{R}_+} (|\nabla \varphi| + |\partial_t \varphi|) d\mu(x, t) - \int_{\mathbb{R}^d} \varphi(x, 0) d\mu_0(x) \\ - \iint_{\mathbb{R}^d \times \mathbb{R}_+} (|\nabla [\varphi \nabla_u \eta(u)]| + |\partial_t [\varphi \nabla_u \eta(u)]|) d\bar{\mu}(x, t) \\ - \int_{\mathbb{R}^d} [\varphi \nabla_u \eta(u)](x, 0) d\bar{\mu}_0(x) + \iint_{\mathbb{R}^d \times \mathbb{R}_+} \varphi \sum_{\alpha=1}^d \partial_\alpha u^T Z_\alpha(u^h, u) dx dt,$$

where $Z_\alpha(u^h, u) = \nabla_u^2 \eta(u) (f_\alpha(u^h) - f_\alpha(u) - (\nabla_u f_\alpha(u))^T (u^h - u))$.

Proof. Let φ be any nonnegative Lipschitz continuous test function with compact support in $\mathbb{R}^d \times [0, T]$. Since u is a classical solution of (1)–(3), it satisfies

$$\iint_{\mathbb{R}^d \times \mathbb{R}_+} \eta(u) \partial_t \varphi(x, t) + \sum_{\alpha=1}^d \xi_\alpha(u) \partial_\alpha \varphi(x, t) dx dt + \int_{\mathbb{R}^d} \eta(u_0) \varphi(x, 0) dx = 0.$$

Subtracting this identity to (42) yields

$$(55) \quad \begin{aligned} & \iint_{\mathbb{R}^d \times \mathbb{R}_+} (\eta(u^h) - \eta(u)) \partial_t \varphi(x, t) + \sum_{\alpha=1}^d (\xi_\alpha(u^h) - \xi_\alpha(u)) \partial_\alpha \varphi(x, t) dx dt \\ & \geq \iint_{\mathbb{R}^d \times \mathbb{R}_+} (|\nabla \varphi| + |\partial_t \varphi|) d\mu(x, t) - \int_{\mathbb{R}^d} \varphi(x, 0) d\mu_0(x). \end{aligned}$$

We now exhibit the relative entropy-relative entropy flux pair in the inequality (55) and obtain

$$(56) \quad \begin{aligned} & \iint_{\mathbb{R}^d \times \mathbb{R}_+} \left(H(u^h, u) \partial_t \varphi + \sum_{\alpha=1}^d Q_\alpha(u^h, u) \partial_\alpha \varphi \right) dx dt \geq \\ & - \iint_{\mathbb{R}^d \times \mathbb{R}_+} |\nabla \varphi| + |\partial_t \varphi| d\mu(x, t) - \int_{\mathbb{R}^d} \varphi(x, 0) d\mu_0(x) \\ & - \iint_{\mathbb{R}^d \times \mathbb{R}_+} (\nabla_u \eta(u))^T \left((u^h - u) \partial_t \varphi + \sum_{\alpha=1}^d (f_\alpha(u^h) - f_\alpha(u)) \partial_\alpha \varphi \right) dx dt. \end{aligned}$$

Since u is a strong solution of (1)–(3), it satisfies the following weak identity, $\forall \psi \in C_c(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R}^m)$

$$(57) \quad \iint_{\mathbb{R}^d \times \mathbb{R}_+} u \partial_t \psi(x, t) + \sum_{\alpha=1}^d f_\alpha(u) \partial_\alpha \psi(x, t) dx dt + \int_{\mathbb{R}^d} u_0(x) \psi(x, 0) dx = 0.$$

Then we combine (57) and (53) using the Lipschitz continuous vector field $[\varphi \nabla_u \eta(u)]$ as test function to get

$$(58) \quad \begin{aligned} & - \iint_{\mathbb{R}^d \times \mathbb{R}_+} (\nabla_u \eta(u))^T \left[(u^h - u) \partial_t \varphi + \sum_{\alpha=1}^d (f_\alpha(u^h) - f_\alpha(u)) \partial_\alpha \varphi \right] dx dt \geq \\ & - \iint_{\mathbb{R}^d \times \mathbb{R}_+} |\nabla[\varphi \nabla_u \eta(u)]| + |\partial_t[\varphi \nabla_u \eta(u)]| d\bar{\mu}(x, t) - \int_{\mathbb{R}^d} [\varphi \nabla_u \eta(u)](x, 0) d\bar{\mu}_0(x) \\ & + \iint_{\mathbb{R}^d \times \mathbb{R}_+} (u^h - u) \varphi \partial_t (\nabla_u \eta(u)) + \sum_{\alpha=1}^d (f_\alpha(u^h) - f_\alpha(u)) \varphi \partial_\alpha (\nabla_u \eta(u)) dx dt. \end{aligned}$$

Moreover identity (7) together with (1) gives

$$(59) \quad \begin{aligned} \partial_t (\nabla_u \eta(u)) &= \partial_t u^T \nabla_u^2 \eta(u) \\ &= - \sum_{\alpha=1}^d \partial_\alpha (f_\alpha(u))^T \nabla_u^2 \eta(u) \\ &= - \sum_{\alpha=1}^d \partial_\alpha u^T \nabla_u f_\alpha(u)^T \nabla_u^2 \eta(u) \\ &= - \sum_{\alpha=1}^d \partial_\alpha u^T \nabla_u^2 \eta(u) \nabla_u f_\alpha(u). \end{aligned}$$

Injecting (58) and (59) into (56) leads to conclusion. \square

Lemma 4.2. *There exists C_Z depending only on f, η and Ω such that, for all $\alpha \in \{1, \dots, d\}$,*

$$(60) \quad |Z_\alpha(u^h, u)| \leq C_Z |u^h - u|^2.$$

Proof. For $\mathbf{M} : \Omega \rightarrow \mathbb{R}^{m \times m}$ and $\Upsilon : \Omega \rightarrow \mathcal{L}(\mathbb{R}^m; \mathbb{R}^{m \times m})$, we set

$$\|\mathbf{M}\|_{\infty, \infty} = \sup_{u \in \Omega} |\mathbf{M}(u)|_{\infty}, \quad \|\Upsilon\|_{\infty, 2} = \sup_{u \in \Omega} \left(\sup_{v \in \mathbb{R}^m, |v|=1} |\Upsilon(u) \cdot v|_2 \right),$$

where $|\cdot|_2$ and $|\cdot|_{\infty}$ denote the usual matrix 2- and ∞ -norms respectively. Using the Taylor expansion of f_{α} around u , we get that

$$\left| f_{\alpha}(u^h) - f_{\alpha}(u) - (\nabla_u f_{\alpha}(u))^T (u^h - u) \right| \leq \frac{1}{2} \|\nabla_u^2 f_{\alpha}\|_{\infty, 2} |u^h - u|^2,$$

then, estimate (60) holds for $C_Z = \frac{1}{2} \|\nabla_u^2 \eta\|_{\infty, \infty} \|\nabla_u^2 f_{\alpha}\|_{\infty, 2}$. \square

We now prove the following lemma on the finite speed of propagation, which is usually assumed to be true in the literature [DiP79, Tza05, Daf10]. Actually, this result is the reason why we make assumption (4).

Lemma 4.3. *For all $s \geq L_f$, one has*

$$(61) \quad sH(u^h, u) + \sum_{\alpha=1}^d \frac{x_{\alpha}}{|x|} Q_{\alpha}(u^h, u) \geq 0.$$

Proof. Denote by $w^h := u^h - u$, then it follows from the characterization (13) of the relative entropy H that

$$(62) \quad H = \int_0^1 \int_0^{\theta} (w^h)^T \nabla_u^2 \eta(u + \gamma w^h) w^h d\gamma d\theta.$$

Denoting by \mathbb{A}_{γ} the symmetric definite positive matrix $\nabla_u^2 \eta(u + \gamma w^h)$, and by $\langle \cdot, \cdot \rangle_{\mathbb{A}_{\gamma}}$ the scalar product on \mathbb{R}^n defined by

$$\langle u, v \rangle_{\mathbb{A}_{\gamma}} = u^T \mathbb{A}_{\gamma} v, \quad \forall u, v \in \mathbb{R}^n,$$

the relation (62) can be rewritten

$$(63) \quad H = \int_0^1 \int_0^{\theta} \langle w^h, w^h \rangle_{\mathbb{A}_{\gamma}} d\gamma d\theta.$$

On the other hand, it follows from the definition (6) of the entropy flux ξ that, for all $\alpha \in \{1, \dots, d\}$,

$$\begin{aligned} Q_{\alpha} &= \int_0^1 (\nabla_u \eta(u + \theta w^h) - \nabla_u \eta(u))^T (\nabla_u f_{\alpha}(u + \theta w^h))^T w^h d\theta \\ &= \int_0^1 \int_0^{\theta} (w^h)^T \mathbb{A}_{\gamma} (\nabla_u f_{\alpha}(u + \theta w^h))^T w^h d\gamma d\theta \\ &= \int_0^1 \int_0^{\theta} \langle w^h, (\nabla_u f_{\alpha}(u + \theta w^h))^T w^h \rangle_{\mathbb{A}_{\gamma}} d\gamma d\theta. \end{aligned}$$

The Rayleigh quotient

$$\begin{aligned} \langle w^h, (\nabla_u f_{\alpha}(u + \theta w^h))^T w^h \rangle_{\mathbb{A}_{\gamma}} &= \langle w^h, \nabla_u f_{\alpha}(u + \theta w^h) w^h \rangle_{\mathbb{A}_{\gamma}} \\ &= \langle w^h, \frac{1}{2} \left((\nabla_u f_{\alpha}(u + \theta w^h))^T + \nabla_u f_{\alpha}(u + \theta w^h) \right) w^h \rangle_{\mathbb{A}_{\gamma}} \end{aligned}$$

satisfies, thanks to (4),

$$\left| \langle w^h, (\nabla_u f_{\alpha}(u + \theta w^h))^T w^h \rangle_{\mathbb{A}_{\gamma}} \right| \leq L_f \langle w^h, w^h \rangle_{\mathbb{A}_{\gamma}}.$$

Therefore, we obtain that

$$\begin{aligned}
 |Q_\alpha| &\leq \int_0^1 \int_0^\theta \left| \langle w^h, (\nabla_u f_\alpha(u + \theta w^h))^T w^h \rangle_{\mathbb{A}_\gamma} \right| d\gamma d\theta \\
 (64) \quad &\leq L_f \int_0^1 \int_0^\theta \langle w^h, w^h \rangle_{\mathbb{A}_\gamma} d\gamma d\theta = L_f H.
 \end{aligned}$$

The fact that (61) holds is a straightforward consequence of (64). \square

4.2. End of the proof of Theorem 2.6. We now have all the preliminary tools required for comparing u^h to u via the relative entropy $H(u^h, u)$.

Let $\delta \in (0, T)$ be a parameter to be fixed later, and, for $k \in \mathbb{N}$, we define the nonincreasing Lipschitz continuous function $\theta_k : \mathbb{R}^+ \rightarrow [0, 1]$ by

$$\theta_k(t) = \min \left(1, \max \left(0, \frac{(k+1)\delta - t}{\delta} \right) \right), \quad \forall t \geq 0.$$

Let us also introduce the Lipschitz continuous function $\psi : \mathbb{R}^d \times \mathbb{R}_+ \rightarrow [0, 1]$ defined by

$$\psi(x, t) = 1 - \min(1, \max(0, |x| - r - L_f(T - t) + 1)),$$

where L_f is so that (4) holds. The function $\varphi_k : (x, t) \in \mathbb{R}^d \times \mathbb{R}_+ \mapsto \theta_k(t)\psi(x, t) \in [0, 1]$ can be considered as a test function in (54). Indeed, denoting by

$$\mathcal{I}_k^\delta = [k\delta, (k+1)\delta], \quad \mathcal{C}_{r,T}(t) = \{(x, t) \mid |x| \in [r + L_f(T - t), r + L_f(T - t) + 1]\},$$

one has

$$\begin{aligned}
 \partial_t \varphi_k(x, t) &= -\frac{1}{\delta} \mathbf{1}_{\mathcal{I}_k^\delta}(t) \psi(x, t) - L_f \theta_k(t) \mathbf{1}_{\mathcal{C}_{r,T}(t)}(x), \\
 \nabla \varphi_k(x, t) &= -\frac{x}{|x|} \theta_k(t) \mathbf{1}_{\mathcal{C}_{r,T}(t)}(x),
 \end{aligned}$$

so that both $\partial_t \varphi_k$ and $|\nabla \varphi_k|$ belong to the set E defined in Remark 3.1. Then taking φ_k as test function in (54) yields

$$\begin{aligned}
 &\frac{1}{\delta} \int_{\mathcal{I}_k^\delta} \int_{\mathbb{R}^d} H \psi dx dt + \int_0^T \theta_k(t) \int_{\mathbb{R}^d} \sum_{\alpha=1}^d \partial_\alpha u^T Z_\alpha(u^h, u) \psi dx dt \\
 &\leq - \int_0^T \theta_k(t) \int_{\mathcal{C}_{r,T}(t)} \left(L_f H + \sum_{\alpha=1}^d Q_\alpha \frac{x_\alpha}{|x|} \right) dx dt + R_1 + R_2 + R_3 + R_4,
 \end{aligned}$$

where

$$\begin{aligned}
 R_1 &= \iint_{\mathbb{R}^d \times [0, T]} (|\nabla \varphi_k(x, t)| + |\partial_t \varphi_k(x, t)|) d\mu(x, t), \\
 R_2 &= \int_{\mathbb{R}^d} \psi(x, 0) d\mu_0(x), \\
 R_3 &= \iint_{\mathbb{R}^d \times [0, T]} |\nabla_u \eta(u)| (|\nabla \varphi_k(x, t)| + |\partial_t \varphi_k(x, t)|) d\bar{\mu}(x, t) \\
 &\quad + \iint_{\mathbb{R}^d \times [0, T]} \varphi_k(x, t) |\nabla_u^2 \eta(u)(x, t)|_\infty (|\partial_t u| + |\nabla u|) d\bar{\mu}(x, t), \\
 R_4 &= \int_{\mathbb{R}^d} \psi(x, 0) |\nabla_u \eta(u_0)| d\bar{\mu}_0(x).
 \end{aligned}$$

Thanks to Lemma 4.3, one has

$$(65) \quad \frac{1}{\delta} \int_{\mathcal{I}_k^\delta} \int_{\mathbb{R}^d} H \psi dx dt + \int_0^T \theta_k(t) \int_{\mathbb{R}^d} \sum_{\alpha=1}^d \partial_\alpha u^T Z_\alpha(u^h, u) \psi dx dt \leq R_1 + R_2 + R_3 + R_4.$$

In view of the definition of φ_k , one has

$$\begin{aligned} \text{supp}(\varphi_k) &\subset \bigcup_{t \in [0, (k+1)\delta]} B(0, r + L_f(T-t) + 1) \times \{t\}, \\ \|\varphi_k\|_\infty &= 1, \quad \|\nabla \varphi_k\|_\infty \leq 1, \quad \|\partial_t \varphi_k\|_\infty \leq \frac{1}{\delta} + L_f. \end{aligned}$$

This leads to

$$R_1 \leq \left(\frac{1}{\delta} + L_f + 1 \right) \mu(\text{supp}(\nabla \varphi_k) \cup \text{supp}(\partial_t \varphi_k)).$$

Thanks to Lemma 3.2, we obtain that there exists C_μ^k (depending on $k, r, T, \delta, L_f, a, \lambda^*, u_0, G_{KL}$ and η) such that

$$(66) \quad R_1 \leq C_\mu^k \left(\frac{1}{\delta} + L_f + 1 \right) h.$$

It follows from similar arguments that there exists $C_{\mu_0}^k$ (depending on k, η, u_0, r, L_f, T and δ) such that

$$(67) \quad R_2 \leq C_{\mu_0}^k h,$$

and, thanks to Lemma 3.5, we obtain that there exists $C_{\mu_0}^k$ (depending on k, r, u_0, L_f, T and δ) such that

$$(68) \quad R_4 \leq C_{\mu_0}^k \|\nabla_u \eta(u_0)\|_\infty h.$$

Similarly, there exists C_μ^k (depending on $k, T, r, L_f, a, \lambda^*, u_0, G_{KL}$ and δ) such that

$$(69) \quad R_3 \leq C_\mu^k \left(\|\nabla_u \eta(u)\|_\infty \left(\frac{1}{\delta} + L_f + 1 \right) + \|\nabla_u^2 \eta\|_{\infty, \infty} (\|\partial_t u\|_\infty + \|\nabla u\|_\infty) \right) h.$$

By using Lemma 4.2, and using that $0 \leq \theta_k(t) \leq 1$, we obtain

$$(70) \quad \begin{aligned} &\int_0^T \theta_k(t) \int_{\mathbb{R}^d} \sum_{\alpha=1}^d \partial_\alpha u^T Z_\alpha(u^h, u) \psi dx dt \\ &\geq -C_Z \|\nabla u\|_\infty \iint_{\mathbb{R}^d \times [0, (k+1)\delta]} |u^h(x, t) - u(x, t)|^2 \psi(x, t) dx dt. \end{aligned}$$

Since the entropy η is supposed to be β -convex, we have

$$(71) \quad H(x, t) \geq \frac{\beta}{2} |u^h(x, t) - u(x, t)|^2.$$

Putting (66)–(70) together with (65) provides that

$$(72) \quad \begin{aligned} &\left(\frac{\beta}{2\delta} - C_Z \|\nabla u\|_\infty \right) \int_{\mathcal{I}_k^\delta} \int_{\mathbb{R}^d} |u^h - u|^2 \psi dx dt \\ &\leq C_Z \|\nabla u\|_\infty \iint_{\mathbb{R}^d \times [0, k\delta]} |u^h - u|^2 \psi dx dt + C_k h, \end{aligned}$$

where

$$\begin{aligned} C_k &= C_\mu^k \left(\frac{1}{\delta} + L_f + 1 \right) + C_{\mu_0}^k + C_{\mu_0}^k \|\nabla_u \eta(u_0)\|_\infty \\ &\quad + C_\mu^k \left(\|\nabla_u \eta(u)\|_\infty \left(\frac{1}{\delta} + L_f + 1 \right) + \|\nabla_u^2 \eta\|_{\infty, \infty} (\|\partial_t u\|_\infty + \|\nabla u\|_\infty) \right). \end{aligned}$$

Choose now

$$\delta = \frac{T}{p^* + 1}, \quad \text{where } p^* = \min \left\{ p \in \mathbb{N}^* \mid \frac{T}{p+1} \leq \frac{\beta}{2C_Z \|\nabla u\|_\infty + 2} \right\}$$

(note that neither δ nor p^* depend on h), so that (72) turns to

$$(73) \quad e_k \leq \omega \sum_{i=0}^{k-1} e_i + C_k h,$$

where

$$e_k = \int_{I_k^\delta} \int_{\mathbb{R}^d} |u^h - u|^2 \psi \, dx dt, \quad \omega = C_Z \|\nabla u\|_\infty.$$

Denoting by $\mathbf{e} = (e_0, \dots, e_{p^*})^T$, and $\mathbf{c} = (C_0, \dots, C_{p^*})^T$, we deduce from (73) that

$$\mathbb{M} \mathbf{e} \leq h \mathbf{c}, \quad \text{where} \quad \mathbb{M} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ -\omega & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ -\omega & \dots & -\omega & 1 \end{pmatrix}.$$

It is easy to verify that

$$(\mathbb{M}^{-1})_{i,j} = \begin{cases} 0 & \text{if } j > i, \\ 1 & \text{if } j = i, \\ \omega(1 + \omega)^{i-j-1} & \text{if } j < i, \end{cases}$$

and then, since $(\mathbb{M}^{-1})_{i,j} \geq 0$, that

$$(74) \quad e_k \leq \left(C_k + \omega \sum_{j=0}^{k-1} (1 + \omega)^{k-j} C_j \right) h.$$

Define $\mathbf{v} = (1, \dots, 1)^T \in \mathbb{R}^{p^*+1}$, then

$$\sum_{k=0}^{p^*} e_k = (\mathbf{e} \mid \mathbf{v}) \leq h (\mathbb{M}^{-1} \mathbf{c} \mid \mathbf{v}) = h (\mathbf{c} \mid (\mathbb{M}^{-1})^T \mathbf{v}).$$

Since

$$\left((\mathbb{M}^{-1})^T \mathbf{v} \right)_k = 1 + \omega \sum_{i=0}^{p^*-k-1} (1 + \omega)^i = (1 + \omega)^{p^*-k+1} - \omega,$$

we obtain that

$$(75) \quad \sum_{k=0}^{p^*} e_k = h \sum_{k=0}^{p^*} C_k \left((1 + \omega)^{p^*-k+1} - \omega \right).$$

Noticing that $\psi(x, t) = 1$ if $x \in B(0, r + L_f(T - t))$, and that $\psi(x, t) \geq 0$ for all $(x, t) \in \mathbb{R}^d \times (0, T)$, ensure that

$$(76) \quad \int_0^T \int_{B(0, r-st)} |u - u_h|^2 \, dx dt \leq \int_0^T \int_{\mathbb{R}^d} |u - u_h|^2 \psi(x, t) \, dx dt = \sum_{k=0}^{p^*} e_k.$$

We conclude the proof by using using (75) in (76).

5. CONCLUDING REMARKS

Our goal was to derive an error estimate for the finite volume approximation u^h . As mentioned above, we use the relative entropy which is not symmetric and we choose to use $H(u^h, u)$. If one decide to choose $H(u, u^h)$, the solution u can be an entropy weak solution instead of a strong solution. Moreover, u^h can be approximated by a continuous function \tilde{u}^h such that $\tilde{u}^h(x_K, t) = u^h(x_K, t)$ for all cell $K \in \mathcal{T}$ where x_K is the center of K . However, in that case, the right hand side of inequality (11) blows up when $h \rightarrow 0$ because of the space derivatives of \tilde{u}^h .

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