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## On McDiarmid's concentration inequality

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#### Abstract

In this paper we improve the rate function in the McDiarmid concentration inequality for separately Lipschitz functions of independent random variables. In particular the rate function tends to infinity at the boundary. We also prove that in some cases the usual normalization factor is not adequate and may be improved.


Mathematics subject classification (2000) : 60E15.
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## 1. Introduction

Throughout the paper $\left(E_{1}, d_{1}\right), \ldots,\left(E_{n}, d_{n}\right)$ is a finite sequence of separable metric spaces with positive finite diameters $\Delta_{1}, \ldots, \Delta_{n}$. Let $E^{n}=E_{1} \times \cdots \times E_{n}$. A function $f$ from $E^{n}$ into $\mathbb{R}$ is said to be separately 1 -Lipschitz if

$$
\left|f\left(x_{1}, \ldots, x_{n}\right)-f\left(y_{1}, \ldots, y_{n}\right)\right| \leq d_{1}\left(x_{1}, y_{1}\right)+\cdots+d_{n}\left(x_{n}, y_{n}\right)
$$

Let $(\Omega, \mathcal{T}, \mathbb{P})$ be a probability space and $X=\left(X_{1}, \ldots, X_{n}\right)$ be a random vector with independent components, with values in $E^{n}$. Let $f$ be any separately 1-Lipschitz function from $E^{n}$ into $\mathbb{R}$. Set $Z=f(X)$. Let the McDiarmid diameter $\sigma_{n}$ be defined by

$$
\begin{equation*}
\sigma_{n}^{2}=\Delta_{1}^{2}+\Delta_{2}^{2}+\cdots+\Delta_{n}^{2} \tag{1.1}
\end{equation*}
$$

McDiarmid $(1989,1998)$ proved that, for any positive $x$,

$$
\begin{equation*}
\mathbb{P}\left(Z-\mathbb{E}(Z) \geq \sigma_{n} x\right) \leq \exp \left(-2 x^{2}\right) \tag{1.2}
\end{equation*}
$$

This inequality is an extension of Theorem 2 in Hoeffding (1963). We refer to Devroye and Lugosi (2001), Chapter 2, for more about concentration inequalities. Later Bentkus (2007, paper submitted on August 17, 2001) and Pinelis (2006) replaced the upper bound in (1.2) by a Gaussian tail function. They proved that

$$
\begin{equation*}
\mathbb{P}\left(Z-\mathbb{E}(Z) \geq \sigma_{n} x\right) \leq c \mathbb{P}(Y \geq 2 x), \quad \text { with } Y \stackrel{D}{=} N(0,1) \tag{1.3}
\end{equation*}
$$

The best known constant in (1.3) is $c=5.70$, due to Pinelis (2006).
We now comment these results. Since $f$ is separately 1-Lipschitz and the spaces $E_{i}$ have a finite diameter $\Delta_{i}$, the function $f$ is uniformly bounded over $E^{n}$. Furthermore if $M=\sup _{E^{n}} f$ and $m=\inf _{E^{n}} f$, then

$$
\begin{equation*}
m \leq Z \leq M \text { and } M-m \leq \Delta_{1}+\Delta_{2}+\cdots+\Delta_{n}:=D_{n} \tag{1.4}
\end{equation*}
$$

From (1.4) it follows that

$$
\begin{equation*}
\mathbb{P}\left(Z-\mathbb{E}(Z) \geq D_{n}\right)=\mathbb{P}(Z=M \text { and } \mathbb{E}(Z)=m)=0 \tag{1.5}
\end{equation*}
$$

Now (1.5) cannot be deduced from either (1.2) or (1.3). Hence it seems clear that the rate function $2 x^{2}$ in the Mc-Diarmid inequality (1.2) is suboptimal for large values of $x$. The aim of this paper is to improve the rate function in (1.2). In Section 2, we give a more efficient large deviations rate function in the case $\Delta_{1}=\Delta_{2}=\cdots=\Delta_{n}=1$. In particular we prove that, for any $x$ in $[0,1]$,

$$
\begin{equation*}
\mathbb{P}(Z-\mathbb{E}(Z) \geq n(1-x)) \leq x^{n\left(1-x^{2}\right)} \tag{1.6}
\end{equation*}
$$

This inequality implies (1.2) and yields (1.5). Next, in Section 3, we extend the results of Section 2 to the case of distinct diameters, for small values or large values of the deviation. Let us recall the known lower bounds for large values of the deviations. Take $E_{i}=\left[0, \Delta_{i}\right]$ and set $\Delta=\left(\Delta_{1}, \Delta_{2}, \ldots, \Delta_{n}\right)$. Let $P_{M c D}(z, \Delta)$ denote the maximal value of $\mathbb{P}(Z-\mathbb{E}(Z) \geq z)$ over all the separately 1-Lipschitz functions and all the random vectors $X$ with values in $E$ and with independent components. By Proposition 5.7 in Ohwadi et al. (2012),

$$
\begin{equation*}
P_{M c D}\left(D_{n}-n x, \Delta\right) \geq x^{n} /\left(\Delta_{1} \Delta_{2} \ldots \Delta_{n}\right) \text { for any } x \leq \min \left(\Delta_{1}, \Delta_{2}, \ldots, \Delta_{n}\right) \tag{1.7}
\end{equation*}
$$

In Theorem 3.2 of Section 3, we prove the converse inequality, with $D_{n}-(56 / 67) n x$ instead of $D_{n}-n x$. For small values of the deviation we obtain in Theorem 3.1 the following extension of (1.6): for any $x$ in $[0,1]$,

$$
\begin{equation*}
\mathbb{P}\left(Z-\mathbb{E}(Z) \geq D_{n}(1-x)\right) \leq x^{\left(1-x^{2}\right) D_{n}^{2} / \sigma_{n}^{2}} \tag{1.8}
\end{equation*}
$$

This extension is also suitable for large values of the deviation when $\sigma_{n}^{2} \sim D_{n}^{2} / n$. However, as shown by the converse inequality (1.7), (1.8) has be improved when $\sigma_{n} \gg n^{-1 / 2} D_{n}$. We
give a first result in this direction in Section 3. The results of Sections 2 and 3 are proved in Section 4. To conclude this paper, we give a more general inequality in Section 5. This inequality, based on partitions of the set of diameters, provides better numerical estimates than the results of Section 3 for intermediate values of the deviation.
2. The case $\Delta_{1}=\Delta_{2}=\cdots=\Delta_{n}$

In this section we assume that $\Delta_{1}=\Delta_{2}=\cdots=\Delta_{n}=1$. Then (1.2) yields

$$
\begin{equation*}
\mathbb{P}(Z-\mathbb{E}(Z) \geq n x) \leq \exp (-n \varphi(x)) \text { with } \varphi(x)=\varphi_{0}(x)=2 x^{2} \tag{2.1}
\end{equation*}
$$

From Theorem A in Rio (2001), (2.1) still holds true with the greater function $\varphi(x)=$ $\varphi_{1}(x)=2((1+x) \log (1+x)+(1-x) \log (1-x))$ (here log is the Neper logarithm). Nevertheless $\varphi_{1}(1)=4 \log 2<\infty$, so that this result still is suboptimal for $x=1$. In Theorem 2.1 below, we give a better large deviations rate function.

Theorem 2.1. For any positive $t$,
(a) $\quad n^{-1} \log \mathbb{E}(\exp (t Z-t \mathbb{E}(Z))) \leq(t-\log t-1)+t\left(e^{t}-1\right)^{-1}+\log \left(1-e^{-t}\right):=\ell(t)$.

Let then $\psi_{1}(x)=2 x^{2}+\left(4 x^{4} / 9\right)$ and $\psi_{2}(x)=\left(x^{2}-2 x\right) \log (1-x)$, with the convention that $\psi_{2}(1)=+\infty$. For any $x$ in $[0,1]$,

$$
\begin{equation*}
\mathbb{P}(Z-\mathbb{E}(Z) \geq n x) \leq \exp \left(-n \max \left(\psi_{1}(x), \psi_{2}(x)\right)\right) \leq(1-x)^{n\left(2 x-x^{2}\right)} \tag{b}
\end{equation*}
$$

Remark 2.1. Let $\ell^{*}$ denotes the Young transform of $\ell$. (b) follows from the fact that

$$
\begin{equation*}
\ell^{*}(x) \geq \max \left(\psi_{1}(x), \psi_{2}(x)\right) \tag{2.2}
\end{equation*}
$$

Note that $\psi_{2}(x)>2 x^{2}$. Consequently, the two upper bounds in Theorem 1.1(b) improve McDiarmid's result. Furthermore it can be proven that $\max \left(\psi_{1}(x), \psi_{2}(x)\right)>\varphi_{1}(x)$ for any $x$ in $] 0,1$. Consequently (b) also improves Theorem A in Rio (2001).
By (1.7) and $(2.2),-\log (1-x)+(1-x)^{2} \log (1-x) \leq \ell^{*}(x) \leq-\log (1-x)$. It follows that $\lim _{x \uparrow 1} \ell^{*}(x)+\log (1-x)=0$, which gives the asymptotics of $l^{*}$ as $x \uparrow 1$.

Remark 2.2. The expansion of $\ell$ at point 0 is $\ell(t)=\frac{1}{8} t^{2}-\frac{1}{576} t^{4}+O\left(t^{5}\right)$. It follows that $\ell^{*}(x)=2 x^{2}+(4 / 9) x^{4}+O\left(x^{5}\right)$ as $x$ tends to 0 . Hence $\psi_{1}$ is the exact expansion of $\ell^{*}$ at order 4.

## 3. The general case : moderate and large deviations

Here we assume that the diameters $\Delta_{i}$ do not satisfy $\Delta_{1}=\Delta_{2}=\cdots=\Delta_{n}$. Let us introduce the quantities below, which will be used to state our bounds:

$$
\begin{equation*}
A_{n}=\left(D_{n} / n\right), B_{n}=n^{-1 / 2} \sigma_{n} \text { and } G_{n}=\left(\Delta_{1} \Delta_{2} \ldots \Delta_{n}\right)^{1 / n} \tag{3.1}
\end{equation*}
$$

Then $G_{n}<A_{n}<B_{n}$. Our first result is an extension of Theorem 3.1 below gives an extension of Theorem 2.1, which preserves the variance factor $\sigma_{n}^{2}$. This result is suitable for moderate deviations. Here $\ell$ denotes the function already defined in Theorem 2.1(a) and $\ell^{*}$ is the Young transform of $\ell$.

Theorem 3.1. For any positive $t$,
(a)

$$
\log \mathbb{E}(\exp (t Z-t \mathbb{E}(Z))) \leq\left(D_{n} / \sigma_{n}\right)^{2} \ell\left(\sigma_{n}^{2} x / D_{n}\right)
$$

Consequently, for any $x$ in $[0,1]$,

$$
\begin{equation*}
\mathbb{P}\left(Z-\mathbb{E}(Z) \geq D_{n} x\right) \leq \exp \left(-\left(D_{n} / \sigma_{n}\right)^{2} \ell^{*}(x)\right) \tag{b}
\end{equation*}
$$

Contrary to the McDiarmid inequality, the upper bound in Theorem 3.1(b) converges to 0 as $x$ tends to 1 . Now, by the Cauchy-Schwarz inequality, $\left(D_{n} / \sigma_{n}\right)^{2} \leq n$ in the general case. However, in some cases $\left(D_{n} / \sigma_{n}\right)^{2}=o(n)$ as $n$ tends to $\infty$. In that case Theorem 3.2 below provides better results for large values of $x$. In order to state this result we need to introduce a second rate function. This is done in Proposition 3.1. below.

Proposition 3.1. Let $\eta(t)=\ell(t)-(t-\log t-1)$ and let $t_{0} \simeq 1.5936$ be the solution of the equation $1-e^{-t}=t / 2$. Then $\eta$ is concave and increasing on $\left.] 0, t_{0}\right]$ and decreasing on $\left[t_{0}, \infty\left[\right.\right.$. Define $\eta_{c}$ by $\eta_{c}(t)=\eta(t)$ for $t \leq t_{0}$ and $\eta_{c}(t)=\eta\left(t_{0}\right)$ for $t \geq t_{0}$. Let $\ell_{c}$ be defined by $\ell_{c}(t)=(t-\log t-1)+\eta_{c}(t)$. Then $\ell_{c}$ is a convex continuously differentiable and increasing function on $\mathbb{R}+$, and
(a) $\ell_{c}^{*}(x)=\ell^{*}(x)$ for $x \leq x_{0}=1-t_{0}^{-1}$ and $\ell_{c}^{*}(x)=-\eta\left(t_{0}\right)-\log (1-x)$ for $x \geq x_{0}$.

The numerical value of $\eta\left(t_{0}\right)$ is $\eta\left(t_{0}\right) \simeq 0.17924$ and $x_{0} \simeq 0.3725$. Furthermore

$$
\begin{equation*}
\ell_{c}(t) \leq t^{2} / 8 \text { for any } t>0 \text { and } \ell_{c}^{*}(x) \geq 2 x^{2} \text { for any } x>0 \tag{b}
\end{equation*}
$$

We now state our second result

Theorem 3.2. For any positive $t$,
(a)

$$
n^{-1} \log \mathbb{E}(\exp (t Z-t \mathbb{E}(Z))) \leq \log \left(A_{n} / G_{n}\right)+\ell_{c}\left(A_{n} t\right)
$$

Consequently, for any $x$ in $[0,1]$,

$$
\begin{equation*}
\mathbb{P}\left(Z-\mathbb{E}(Z) \geq D_{n} x\right) \leq \exp \left(n \log \left(A_{n} / G_{n}\right)-n \ell_{c}^{*}(x)\right) \tag{b}
\end{equation*}
$$

Remark 3.1. Since the maximum value of $\eta$ is $\eta\left(t_{0}\right), \ell_{c}(t) \leq t-\log t-1+\eta\left(t_{0}\right)$ for any positive $t$. Hence, for any $x$ in $[0,1]$,

$$
\begin{equation*}
\ell_{c}^{*}(x) \leq \eta\left(t_{0}\right)-\log (1-x) \leq \log (56 / 67)-\log (1-x) \tag{3.2}
\end{equation*}
$$

It follows that, for any positive $y$,

$$
\begin{equation*}
\mathbb{P}\left(Z-\mathbb{E}(Z) \geq D_{n}-(56 / 67) n y\right) \leq y^{n} /\left(\Delta_{1} \Delta_{2} \ldots \Delta_{n}\right) \tag{3.3}
\end{equation*}
$$

The factor $1 /\left(\Delta_{1} \Delta_{2} \ldots \Delta_{n}\right)$ appearing in (3.3) cannot be removed, as shown by (1.7), For sake of completeness, we give here the proof of (1.7). let $\Delta_{1} \geq \Delta_{2} \geq \cdots \geq \Delta_{n}$ be positive reals and $y$ be any positive real in $\left[0, \Delta_{n}\right]$. Let $b_{1}, b_{2}, \ldots, b_{n}$ be independent random variables such that $b_{k}$ has the Bernoulli law $b\left(y / \Delta_{k}\right)$. Set $T_{n}=\Delta_{1} b_{1}+\Delta_{2} b_{2}+\cdots+\Delta_{n} b_{n}$. Then $\mathbb{P}\left(T_{n}-\mathbb{E}\left(T_{n}\right) \geq D_{n}-n y\right)=y^{n} /\left(\Delta_{1} \Delta_{2} \ldots \Delta_{n}\right)$.

Example 3.1. Take $n=100, \Delta_{1}=49$ and $\Delta_{k}=1$ for $k \geq 2$. Then $\sigma_{n}=50, D_{n}=148$ and $A_{n}=1.48$. Let $p=\mathbb{P}(Z-\mathbb{E}(Z) \geq 75)$. The McDiarmid inequality (1.2) applied with $x=3 / 2$ yields $p \leq e^{-9 / 2} \simeq 1.110^{-2}$ and (1.3) yields $p \leq 7.710^{-3}$. Theorem 3.1(b) yields $p \leq 8.610^{-3}$ and Theorem 3.2(b) yields $p \leq 2.710^{-8}$.

To conclude this section, we give an inequality, which is a byproduct of the proofs of Theorems 3.1 and 3.2. In some cases this inequality provides better estimates.

Theorem 3.3. For $k$ in $[1, n]$, let the quantities $\sigma_{k}, D_{k}, A_{k}$ and $G_{k}$ be defined as in (1.1), (1.4) and (3.1). Set $D_{0}=0$ and $\sigma_{0}=0$. Then, for any $k$ in $[0, n]$ and any positive $u$,

$$
\mathbb{P}\left(Z-\mathbb{E}(Z) \geq\left(D_{n}-D_{k}\right) \ell^{*-1}\left(\frac{\left(\sigma_{n}^{2}-\sigma_{k}^{2}\right) u}{\left(D_{n}-D_{k}\right)^{2}}\right)+D_{k} \ell_{c}^{*-1}\left(\log \left(\frac{A_{k}}{G_{k}}\right)+\frac{u}{k}\right)\right) \leq e^{-u}
$$

where $\ell^{*-1}$ and $\ell_{c}^{*-1}$ denote the inverse functions of $\ell^{*}$ and $\ell_{c}^{*}$. Furthemore, for $k=1$, the function $\ell_{c}^{*-1}$ may be replaced by $\ell^{*-1}$ in the above inequality.

Remark 3.2. For $k=n$, Theorem 3.3 is just an other formulation of Theorem 3.2. For $k=0$, Theorem 3.3 is an other formulation of Theorem 3.1.

We now give a "ready to be used" inequality". From $(2.2), \ell^{*-1}(y) \leq \psi_{1}^{-1}(y)$. Moreover

$$
\begin{equation*}
\psi_{1}^{-1}(y)=\sqrt{y}(1+\sqrt{1+(4 / 9) y})^{-1 / 2} \tag{3.4}
\end{equation*}
$$

Now, from (3.2), $\ell_{c}^{*-1}(y) \leq 1-\frac{56}{67} e^{-y}$. Hence Theorem 3.3 yields

$$
\begin{equation*}
\mathbb{P}\left(Z-\mathbb{E}(Z) \geq\left(D_{n}-D_{k}\right) \psi_{1}^{-1}\left(\frac{\left(\sigma_{n}^{2}-\sigma_{k}^{2}\right) u}{\left(D_{n}-D_{k}\right)^{2}}\right)+D_{k}-\frac{56}{67} k G_{k} e^{-u / k}\right) \leq e^{-u} \tag{3.5}
\end{equation*}
$$

Example 3.1 (continued). Let $p=e^{-9 / 2}$. Denote by $Q$ the quantile function of $Z-\mathbb{E}(Z)$, which is the inverse of the tail function. The McDiarmid inequality yields $Q(p) \leq 75$. By Theorem 3.3 applied with $k=1$ (the optimal choice), $Q(p) \leq 63.35$. Next $Q(p) \leq 73.05$ by Theorem 3.1 and $Q(p) \leq 64.93$ by Theorem 3.2.

## 4. Proofs of the results of Sections 2 and 3 .

We start by proving an upper bound on the Laplace transform of $Z$ which implies Theorem 2.1(a) in the case $\Delta_{1}=\Delta_{2}=\ldots=\Delta_{n}$.

Lemma 4.1. Let $\ell$ be the function already defined in Theorem 2.1(a). Then, for any positive $t, \log \mathbb{E}(\exp (t Z-t \mathbb{E}(Z))) \leq \ell\left(\Delta_{1} t\right)+\ell\left(\Delta_{2} t\right)+\cdots+\ell\left(\Delta_{n} t\right):=L(t)$.

Proof of Lemma 4.1. Let us briefly recall the martingale decomposition of $Z$. Let $\mathcal{F}_{0}=\{\emptyset, \Omega\}$ and $\mathcal{F}_{k}=\sigma\left(X_{1}, \ldots, X_{n}\right)$. Set $Z_{k}=\mathbb{E}\left(Z \mid \mathcal{F}_{k}\right)$. Then $Z=Z_{n}$ and $Z_{0}=\mathbb{E}(Z)$. Furthermore $\left(Z_{k}\right)_{k}$ is a martingale sequence adapted to the above filtration. Now, set $Y_{k}=Z_{k}-Z_{k-1}$. Define the $\mathcal{F}_{k-1}$-measurable random variable $W_{k-1}$ by

$$
\begin{equation*}
W_{k-1}=\mathbb{E}\left(\inf _{x \in E_{k}} f\left(X_{1}, \ldots X_{k-1}, x, X_{k+1}, \ldots, X_{n}\right) \mid \mathcal{F}_{k-1}\right)-Z_{k-1} \tag{4.1}
\end{equation*}
$$

By (1.1),

$$
\begin{equation*}
W_{k-1} \leq Y_{k} \leq W_{k-1}+\Delta_{k} \tag{4.2}
\end{equation*}
$$

From this inequality and the convexity of the exponential function,

$$
\Delta_{k} e^{t Y_{k}} \leq\left(Y_{k}-W_{k-1}\right) e^{t\left(W_{k-1}+\Delta_{k}\right)}+\left(\Delta_{k}+W_{k-1}-Y_{k}\right) e^{t W_{k-1}}
$$

Hence, using the martingale property

$$
\begin{equation*}
\Delta_{k} \mathbb{E}\left(e^{t Y_{k}} \mid \mathcal{F}_{k-1}\right) \leq-W_{k-1} e^{t\left(W_{k-1}+\Delta_{k}\right)}+\left(\Delta_{k}+W_{k-1}\right) e^{t W_{k-1}} \tag{4.3}
\end{equation*}
$$

Set then

$$
\begin{equation*}
\gamma(r, t)=\log \left(1+r\left(e^{t}-1\right)\right)-t r \text { and } r_{k-1}=-\left(W_{k-1} / \Delta_{k}\right) \tag{4.4}
\end{equation*}
$$

Since $\left(Z_{k}\right)$ is a martingale sequence, $\mathbb{E}\left(Y_{k} \mid \mathcal{F}_{k-1}\right)=0$. Hence, from (4.2), $W_{k-1} \leq 0$ and $0 \leq W_{k-1}+\Delta_{k}$. Consequently $r_{k-1}$ belongs to [0,1]. Furthermore, by (4.3),

$$
\log \mathbb{E}\left(e^{t Y_{k}} \mid \mathcal{F}_{k-1}\right) \leq \gamma\left(r_{k-1}, \Delta_{k} t\right)
$$

Define now

$$
\begin{equation*}
\ell(s)=\sup _{r \in[0,1]} \gamma(r, s)=\sup _{r \in] 0,1[ }\left(\log \left(1+r\left(e^{t}-1\right)\right)-t r\right) \tag{4.5}
\end{equation*}
$$

From the above inequality

$$
\begin{equation*}
\log \mathbb{E}\left(e^{t Y_{k}} \mid \mathcal{F}_{k-1}\right) \leq \ell\left(\Delta_{k} t\right) \quad \text { almost surely }, \tag{4.6}
\end{equation*}
$$

which implies Lemma 4.1 for the function $\ell$ defined in (4.5). It remains to prove that $\ell$ is equal to the function already defined in Theorem 2.1(a). Now

$$
\frac{\partial \gamma}{\partial r}(r, t)=\frac{e^{t}-t-1-r t\left(e^{t}-1\right)}{1+r\left(e^{t}-1\right)}
$$

and consequently the function $\gamma(r, t)$ has an unique maximum with respect to $r$ in the interval $[0,1]$. This maximum is obtained for $r=r_{t}=\left(e^{t}-t-1\right) /\left(t\left(e^{t}-1\right)\right)$, whence

$$
\ell(t)=\log \left(\left(e^{t}-1\right) / t\right)-1+t /\left(e^{t}-1\right)=(t-\log t-1)+t\left(e^{t}-1\right)^{-1}+\log \left(1-e^{-t}\right) .
$$

We now prove (2.2), and therefore Theorem 2.1(b). The first step is to compare the functions $\psi_{1}$ and $\psi_{2}$.

Lemma 4.2. There exists an unique real $x_{0}$ in [0.6670, 0.6675] such that $\psi_{1}(x) \geq \psi(x)$ for any $x \leq x_{0}$ and $\psi_{1}(x)<\psi_{2}(x)$ for $x>x_{0}$.

Proof of Lemma 4.2. For any $x<1, \psi_{2}(x)=2 x^{2}+\left(x^{4} / 6\right)+\sum_{k>4} a_{k} x^{k}$ with $a_{k}=(k-3) /\left(k^{2}-3 k+2\right)$. Define now $f$ by $f(x)=x^{-4}(\psi(x)-\varphi(x))$. Then

$$
f(x):=x^{-4}(\psi(x)-\varphi(x))=(-5 / 18)+\sum_{k>4} a_{k} x^{k-4},
$$

which implies that $f$ is increasing on $[0,1]$. Lemma 4.2 follows then from the facts that $f(0.6670)<0$ and $f(0.6675)>0$.

The second step is to prove that $\ell^{*}(x) \geq \psi_{1}(x)$ for any $x$ in $[0,1]$.
Lemma 4.3. $\quad \ell^{*}(x) \geq \psi_{1}(x)$ for any $x$ in $[0,1]$.
Proof of Lemma 4.3. From (4.5), for any positive $x$,

$$
\begin{equation*}
\ell^{*}(x)=\inf _{r \in] 0,1[ } \gamma_{r}^{*}(x) \text { with } \gamma_{r}^{*}(x)=\sup _{t>0}(t x-\gamma(r, x)) . \tag{4.6}
\end{equation*}
$$

Let the function $h$ be defined by $h(x)=(1+x) \log (1+x)-x$ for $x>-1, h(-1)=1$ and $h(x)=+\infty$ for $x<-1$. Since $h(0)=h^{\prime}(0)=0$ and $h^{\prime \prime}(x)=1 /(1+x)$, the function $h$ is convex and nonnegative. Next

$$
\begin{equation*}
\gamma_{r}^{*}(x)=r h(x / r)+(1-r) h(-x /(1-r)):=h_{r}(x) . \tag{4.7}
\end{equation*}
$$

Consequently $\gamma_{r}^{*}(x)=+\infty$ for $r>1-x$. Furthermore the above function is continuous with respect to $r$ for $r$ in $[0,1-x]$, so that we may take the infimum over $] 0,1-x[$ in (4.6). Our way to prove Lemma 4.3 is consider $h_{r}$ as a function of $u_{x}=2(r+x)-1$. Clearly

$$
h_{r}^{\prime \prime}(x)=(x+r)^{-1}+(1-x-r)^{-1}=4 /\left(1-u_{x}^{2}\right) \geq 4\left(1+u_{x}^{2}\right) .
$$

Now $h_{r}(0)=h_{r}^{\prime}(0)=0$. Consequently, by the Taylor integral formula,

$$
\begin{equation*}
h_{r}(x)=\int_{0}^{x}(x-y) h_{r}^{\prime \prime}(y) d y \geq 4 \int_{0}^{x}\left(1+u_{y}^{2}\right)(x-y) d y \tag{4.8}
\end{equation*}
$$

Some elementary calculations show that

$$
\begin{equation*}
4 \int_{0}^{x}\left(1+4 u_{y}^{2}\right)(x-y) d y=2 x^{2}+\frac{4}{9} x^{4}+2 x^{2}(2 r-1+(2 x / 3))^{2} \tag{4.9}
\end{equation*}
$$

Both (4.6), (4.7), (4.8) and (4.9) then imply Lemma 4.3.
To complete the proof of Theorem 2.1(b), it remains to prove that $\ell^{*}(x) \geq \psi_{1}(x)$ for $x \geq x_{0}$.

Lemma 4.4. $\quad \ell^{*}(x) \geq \psi_{1}(x)$ for any $x \geq 2 / 3$.
Proof of Lemma 4.4. Let $t_{x}=1 /(1-x)$. By definition, $\ell^{*}(x) \geq x t_{x}-\ell\left(t_{x}\right)$. Define the function $\eta$ by

$$
\begin{equation*}
\eta(t)=\ell(t)-(t-\log t-1)=t\left(e^{t}-1\right)^{-1}+\log \left(1-e^{-t}\right) . \tag{4.10}
\end{equation*}
$$

With this definition, $x t_{x}-\ell\left(t_{x}\right)=\log t_{x}-\eta\left(t_{x}\right)$. Now, $t_{x} \geq 3$ for any $x \geq(2 / 3)$. Since $\psi_{2}(x)=\log t_{x}-t_{x}^{-2} \log t_{x}$, the proof of Lemma 4.3 will be complete if we prove that

$$
\begin{equation*}
t^{2} \eta(t) \leq \log t \text { for any } t \geq 3 \tag{4.11}
\end{equation*}
$$

Since $\log 3 \geq 1$, it is enough to prove that $t^{2} \eta(t) \leq 1$. Now, by concavity of the logarithm,

$$
t^{2} \eta(t) \leq t^{2}\left(t\left(e^{t}-1\right)^{-1}-e^{-t}\right)=\left(t^{2}+\left(t^{3}-t^{2}\right) e^{t}\right) /\left(e^{2 t}-e^{t}\right)
$$

Hence the inequality $t^{2} \eta(t) \leq 1$ holds true if $\delta(t):=\left(e^{t}+t^{2}-t^{3}-1\right) e^{t}-t^{2} \geq 0$ for $t \geq 3$. Let $\beta(t):=e^{t}+t^{2}-t^{3}-1 . \beta$ is strictly convex on $[3, \infty[$ and has an unique infimum at $t_{0} \simeq 3.1699$. Now $\beta\left(t_{0}\right) \simeq 1.00137>1$, whence $\delta(t)>e^{t}-t^{2}>0$ for $t \geq 3$. Hence $t^{2} \eta(t) \leq 1$ for $t \geq 3$, which implies (4.11). Consequently Lemma 4.3 holds true. Inequality (2.2) follows then from both Lemmas 4.2, 4.3 and 4.4. Theorem 2.1(b) Follows.

Proof of Theorem 3.1. The proof of Theorem 3.1 is based on the convexity property below.

Lemma 4.5. The function $\ell^{\prime}$ is concave on $\mathbb{R}^{+}$.
Proof of Lemma 4.5. Set $v=1 /\left(e^{t}-1\right)$. Then $\ell(t)=v t-\log v-\log t-1$. Since $v^{\prime}=-v(1+v)$,

$$
\ell^{\prime}=1+2 v-t v-t v^{2}-(1 / t), \ell^{\prime \prime}=-3\left(v+v^{2}\right)+t v(1+v)(1+2 v)+\left(1 / t^{2}\right)
$$

and

$$
-\ell^{\prime \prime \prime}=\left(2 / t^{3}\right)-4 v(1+v)(1+2 v)+t v(1+v)(1+6 v(1+v)) .
$$

Let $f(t):=-\ell^{\prime \prime \prime}(t) /\left(t v^{2}(1+v)^{2}\right)$. We prove that $f \geq 0$. Since $2 v(1+v)(\cosh t-1)=1$, the function $f$ can be decomposed as follows:
$f(t)=f_{1}(t)+f_{2}(t)$ with $f_{1}(t)=8 t^{-4}(\cosh t-1)^{2}$ and $f_{2}(t)=2 \cosh t+4-8(\sinh t / t)$.
Now $f_{1}$ and $f_{2}$ are analytic. First $f_{2}(t)=-2-\left(t^{2} / 3\right)+\sum_{k \geq 2} a_{k} t^{2 k}$, for positive coefficients $a_{k}$. More precisely $a_{k}=2(2 k-3) /(2 k+1)$ !. Consequently $f_{2}(t) \geq-2-\left(t^{2} / 3\right)$. And second $2(\cosh t-1) \geq t^{2}\left(1+t^{2} / 12\right)$, whence

$$
f_{1}(t)+f_{2}(t) \geq 2\left(1+t^{2} / 12\right)^{2}-2-\left(t^{2} / 3\right) \geq\left(t^{4} / 72\right)>0 .
$$

Hence $f(t)>0$ for any positive $t$, which ensures that $\ell^{\prime}$ is concave.

We now complete the proof of Theorem 3.1(a). According to Lemma 4.1, we have to prove that

$$
\begin{equation*}
L(t):=\ell\left(\Delta_{1} t\right)+\ell\left(\Delta_{2} t\right)+\cdots+\ell\left(\Delta_{n} t\right) \leq\left(D_{n} / \sigma_{n}\right)^{2} \ell\left(\sigma_{n}^{2} t / D_{n}\right) . \tag{4.12}
\end{equation*}
$$

Now

$$
L(t)=\int_{0}^{t} L^{\prime}(u) d u=\int_{0}^{t}\left(\Delta_{1} \ell^{\prime}\left(\Delta_{1} u\right) \cdots+\Delta_{n} \ell^{\prime}\left(\Delta_{n} u\right)\right) d u
$$

Next, by Lemma 4.5,

$$
\Delta_{1} \ell^{\prime}\left(\Delta_{1} u\right) \cdots+\Delta_{n} \ell^{\prime}\left(\Delta_{n} u\right) \leq D_{n} \ell^{\prime}\left(\sigma_{n}^{2} u / D_{n}\right)
$$

Hence

$$
L(t) \leq \Delta \int_{0}^{t} \ell^{\prime}\left(\sigma^{2} u / \Delta\right) d u=\left(D_{n} / \sigma_{n}\right)^{2} \ell\left(\sigma_{n}^{2} t / D_{n}\right)
$$

Hence (4.12) holds, which implies Theorem 3.1(a). Theorem 3.1(b) follows from the usual Chernoff calculation.

Proof of Proposition 3.1. With the notations of the proof of Lemma 4.5,

$$
\eta^{\prime}=v(2-(1+v) t) \quad \text { and } \eta^{\prime \prime}=v(1+v)(t(1+2 v)-3) .
$$

Therefrom $\eta^{\prime}(t)>0$ if and only if $2>t e^{t} /\left(e^{t}-1\right)$, which holds if and only $t<t_{0}$. Now $\eta^{\prime \prime}(t)<0$ if and only if $t\left(e^{t}+1\right)<3\left(e^{t}-1\right)$. This condition holds if and only if $t \leq t_{1}$, where $t_{1}$ is the unique positive solution of the equation $t=3 \tanh (t / 2)$. Since $t_{1} \simeq 2.5757>2>t_{0}$, the first part of Proposition 3.1 holds true. Now, by definition $\ell_{c}$ is continuous and convex and continuously differentiable on the two intervals $\left[0, t_{0}\right]$ and $\left[t_{0}, \infty\right]$. Since $\eta^{\prime}\left(t_{0}\right)=0$, the left derivative and the right derivative of $\ell_{c}$ at point $t_{0}$ are equal. Hence $\ell_{c}$ is convex and continuously differentiable on $\mathbb{R}^{+}$.

The proof of (a), being immediate, is omitted. To prove (b), we note that, for any $t \leq t_{0}$, $\ell_{c}(t)=\ell(t) \leq t^{2} / 8$, since $\ell^{*}(x) \geq 2 x^{2}$ for any positive $x$. Now $\ell_{c}(t)=t-\log t-1+\eta\left(t_{0}\right)$ for $t \geq t_{0}$, and consequently $(t / 4)-\ell_{c}^{\prime}(t)=(t-2)^{2} /(4 t)$. Hence $t^{2} / 8-\ell_{c}$ is nondecreasing on $\left[t_{0}, \infty\left[\right.\right.$, whence $t^{2} / 8-\ell_{c}(t) \geq\left(t_{0}^{2} / 8\right)-\ell_{c}\left(t_{0}\right)>0$. Hence (b) holds, wich completes the proof of Proposition 3.1.

Proof of Theorem 3.2. By definition, $\eta_{c}$ is concave. Hence

$$
\eta_{c}\left(\Delta_{1} t\right)+\eta_{c}\left(\Delta_{2} t\right)+\cdots+\eta_{c}\left(\Delta_{n} t\right) \leq n \eta_{c}\left(A_{n} t\right) .
$$

Since $\ell \leq \ell_{c}$,

$$
\ell\left(\Delta_{1} t\right)+\ell\left(\Delta_{2} t\right)+\cdots+\ell\left(\Delta_{n} t\right) \leq n\left(A_{n} t-1-\log t\right)-\log \left(\Delta_{1} \ldots \Delta_{n}\right)+n \eta_{c}\left(A_{n} t\right) .
$$

It follows that

$$
\begin{equation*}
\ell\left(\Delta_{1} t\right)+\ell\left(\Delta_{2} t\right)+\cdots+\ell\left(\Delta_{n} t\right) \leq n \log \left(A_{n} / G_{n}\right)+n \ell_{c}\left(A_{n} t\right) \tag{4.13}
\end{equation*}
$$

which, together with Lemma 4.1, implies Theorem 3.2(a). Theorem 3.2(b) follows from the usual Chernoff calculation.

Proof of Theorem 3.3. Here we use the martingale decomposition. With the same notations as in Lemma 4.1, let $S=Z_{k}-Z_{0}$ and $T=Z_{n}-Z_{k}$. Then

$$
Z-\mathbb{E}(Z)=Z_{n}-Z_{0}=S+T
$$

The random variables $Z$ and $T$ have a finite Laplace transform, and, from their martingale decomposition together with (4.6),

$$
\begin{align*}
& L_{S}(t):=\log \mathbb{E}(\exp (t S)) \leq \ell\left(\Delta_{1} t\right)+\ell\left(\Delta_{2} t\right)+\cdots+\ell\left(\Delta_{k} t\right)  \tag{4.14}\\
& L_{T}(t):=\log \mathbb{E}(\exp (t T)) \leq \ell\left(\Delta_{k+1} t\right)+\ell\left(\Delta_{k+2} t\right)+\cdots+\ell\left(\Delta_{n} t\right) \tag{4.15}
\end{align*}
$$

Let $L_{S}^{*}$ and $L_{T}^{*}$ denote the Young transforms of $L_{S}$ and $L_{T}$ respectively. By Lemma 2.1 in Rio (1994), $L_{S+T}^{*-1} \leq L_{S}^{*-1}+L_{T}^{*-1}$. Hence

$$
\begin{equation*}
\mathbb{P}\left(Z-\mathbb{E}(Z)>L_{T}^{*-1}(u)+L_{S}^{*-1}(u)\right) \leq \exp (-u) \tag{4.16}
\end{equation*}
$$

Next, by (4.15) together with (4.12),

$$
L_{T}(t) \leq\left(\sigma_{n}^{2}-\sigma_{k}^{2}\right)^{-1}\left(D_{n}-D_{k}\right)^{2} \ell\left(\left(\sigma_{n}^{2}-\sigma_{k}^{2}\right) t /\left(D_{n}-D_{k}\right)\right),
$$

which ensures that

$$
\begin{equation*}
L_{T}^{*-1}(u) \leq\left(D_{n}-D_{k}\right) \ell^{*-1}\left(\left(\sigma_{n}^{2}-\sigma_{k}^{2}\right) u /\left(D_{n}-D_{k}\right)^{2}\right) . \tag{4.17}
\end{equation*}
$$

Now, for $k=1, L_{S}(t) \leq \ell\left(\Delta_{1} t\right)$, whence $L_{S}^{*-1}(u) \leq \Delta_{1} \ell^{*-1}(u)$. Theorem 3.3 in the case $k=1$ follows from both this inequality, (4.16), (4.17) and the strict monotonicity of $\ell^{*-1}$ on $\mathbb{R}^{+}$(which allows to replace $>$by $\geq$in (4.16)). For $k \geq 2$, from (4.14) and (4.13), $L_{S}(t) \leq k \log \left(A_{k} / G_{k}\right)+k \ell_{c}\left(A_{k} t\right)$. Therefrom

$$
\begin{equation*}
L_{S}^{*-1}(u) \leq D_{k} \ell_{c}^{*-1}\left(\log \left(A_{k} / G_{k}\right)+(u / k)\right) \tag{4.18}
\end{equation*}
$$

Theorem 3.3 follows then from both (4.16), (4.17), (4.18) and the strict monotonicity of $\ell^{*-1}$ and $\ell_{c}^{*-1}$ on $\mathbb{R}^{+}$.

## 5. An inequality involving partitions

In this section we are interested in intermediate values of the deviation $x$. In the sketchy Example 3.1, it appears that the McDiameter diameter is too big for intermediates values of the deviation. In this section, we introduce a method which minimizes the effect of variations of the values of the individual diameters $\Delta_{1}, \Delta_{2}, \ldots, \Delta_{n}$.

Definition 5.1. A family $\mathcal{P}$ of subsets of $\{1,2, \ldots, n\}$ is called partition of $\{1,2, \ldots, n\}$ iff: (i) for any $I$ in $\mathcal{P}, I \neq \emptyset$; (ii) for any $I$ and any $J$ in $\mathcal{P}$, either $I \cap J=\emptyset$ or $I=J$.

We now define the McDiarmid diameter $\sigma^{2}(\mathcal{P})$ and the entropy $H(\mathcal{P})$ of a partition $\mathcal{P}$ as follows. Let $|J|$ denote the cardinality of the finite set $J$. We set

$$
\begin{equation*}
D_{J}=\sum_{j \in J} \Delta_{j}, A_{J}=|J|^{-1} D_{J} \quad \text { and } \quad \sigma^{2}(\mathcal{P})=\sum_{J \in \mathcal{P}}|J| A_{J}^{2} . \tag{5.1}
\end{equation*}
$$

Let the geometric means $G_{J}$ and the entropy be defined by

$$
\begin{equation*}
G_{J}=\left(\prod_{J \in J} A_{j}\right)^{1 /|J|} \quad \text { and } \quad H(\mathcal{P})=\sum_{J \in \mathcal{P}}|J| \log \left(A_{J} / G_{J}\right) \tag{5.2}
\end{equation*}
$$

The so defined quantities satisfy $\sigma^{2}(\mathcal{P}) \leq \sigma_{n}^{2}$ and $H(\mathcal{P}) \geq 0$. Furthermore $H(\mathcal{P})=0$ if and only if $\sigma^{2}(\mathcal{P})=\sigma_{n}^{2}$.

Theorem 5.1. Let the convex and differentiable function $\ell_{0}$ be defined by

$$
\ell_{0}(t)=t^{2} / 8 \text { for } t \in[0,2] \text { and } \ell_{0}(t)=t-\log t-(3 / 2)+\log 2 \text { for } t \geq 4 .
$$

For any positive $t$ and any partition $\mathcal{P}$ of $\{1,2, \ldots, n\}$,
(a)

$$
\log \mathbb{E}(\exp (t Z-t \mathbb{E}(Z))) \leq H(\mathcal{P})+\left(D_{n}^{2} / \sigma^{2}(\mathcal{P})\right) \ell_{0}\left(\sigma^{2}(\mathcal{P}) t / D_{n}\right)
$$

Consequently, for any $x$ in $[0,1]$,

$$
\begin{equation*}
\mathbb{P}\left(Z-\mathbb{E}(Z) \geq D_{n} x\right) \leq \exp \left(H(\mathcal{P})-\left(D_{n}^{2} / \sigma^{2}(\mathcal{P})\right) \ell_{0}^{*}(x)\right) \tag{b}
\end{equation*}
$$

and, for any positive $y$,

$$
\begin{equation*}
\mathbb{P}\left(Z-\mathbb{E}(Z) \geq D_{n} \ell_{0}^{*-1}\left(D_{n}^{-2} \min _{\mathcal{P}} \sigma^{2}(\mathcal{P})(H(\mathcal{P})+y)\right)\right) \leq e^{-y} \tag{c}
\end{equation*}
$$

Remark 5.1. In Theorem 3.1(c), for small values of $y$, the optimal partition has a small entropy and a large diameter, while, for large values of $y$, the optimal partition has a small diameter and a large entropy.

Remark 5.2. The functions $\ell_{0}^{*}$ and $\ell_{0}^{*-1}$ are explicit. More precisely

$$
\begin{gathered}
\ell_{0}^{*}(x)=2 x^{2} \text { for } x \in[0,1 / 2] \text { and } \ell_{0}^{*}(x)=-\log (1-x)+(1 / 2)-\log 2 \text { for } x \in[1 / 2,1], \\
\quad \ell_{0}^{*-1}(y)=\sqrt{y / 2} \text { for } y \in[0,1 / 2] \text { and } \ell_{0}^{*-1}(y)=1-(\sqrt{e} / 2) e^{-y} \text { for } y \geq 1 / 2 .
\end{gathered}
$$

Example 3.1 (continued). Let $Q$ denote the quantile function of $Z-\mathbb{E}(Z)$. For $p=e^{-9 / 2}$, Theorem 3.1(c) applied with $\mathcal{P}=\{[1,13],[14,100]\}$ (the optimal partition) yields $Q(p) \leq 62.18$, which improves the results of Section 3. For small values of $p$, the optimal partition is $\mathcal{P}=\{[1,100]\}$. In this case Theorem 5.1 is less efficient than Theorem 3.2, since $\ell_{0}^{*-1}(y)>\ell_{c}^{*-1}(y)$. For example, let $q=\mathbb{P}(Z-\mathbb{E}(Z) \geq 75)$. Theorem 5.1(b) yields $q \leq 3.010^{-7}$ instead of $q \leq 2.710^{-8}$ with Theorem 3.2. Recall that the McDiarmid inequality yields $q \leq 1.110^{-2}$.

Proof of Theorem 5.1. By Lemma 4.1 together with (4.13),

$$
\log \mathbb{E}(\exp (t Z-t \mathbb{E}(Z))) \leq \sum_{J \in \mathcal{P}} \sum_{j \in J} \ell\left(\Delta_{j} t\right) \leq H(\mathcal{P})+\sum_{J \in \mathcal{P}}|J| \ell_{c}\left(A_{J} t\right)
$$

Now $\ell_{c}(t) \leq \min \left(t^{2} / 8, \eta\left(t_{0}\right)+t-\log t-1\right) \leq \ell_{0}(t)$ for any positive $t$. Hence

$$
\begin{equation*}
\log \mathbb{E}(\exp (t Z-t \mathbb{E}(Z))) \leq H(\mathcal{P})+\sum_{J \in \mathcal{P}}|J| \ell_{0}\left(A_{J} t\right) \tag{5.3}
\end{equation*}
$$

To complete the proof of Theorem 5.1, we proceed exactly as in the proof of Theorem 3.1: since $\ell_{0}(0)=0$,

$$
\begin{equation*}
\sum_{J \in \mathcal{P}}|J| \ell_{0}\left(A_{J} t\right)=\int_{0}^{t}\left(\sum_{J \in \mathcal{P}} D_{J} \ell_{0}^{\prime}\left(A_{J} u\right)\right) d u \tag{5.4}
\end{equation*}
$$

Now $\ell_{0}^{\prime}(t)=t / 4$ for $t \leq 2$ and $\ell_{0}^{\prime}(t)=1-(1 / t)$ for $t \geq 2$, which ensures that $\ell_{0}^{\prime}$ is continuous and increasing. $\ell_{0}^{\prime \prime}(t)=1 / 4$ for $t<2$ and $\ell^{\prime \prime}(0)=t^{-2}$ for $t>2$, which ensures that $\lim _{t \downarrow 2} \ell_{0}^{\prime \prime}(t)=1 / 4$. Hence, by L'Hospital's rule, $\ell_{0}^{\prime}$ is differentiable at point 2 , and $\ell_{0}^{\prime \prime}(2)=4$. Consequently $\ell_{0}^{\prime \prime}$ is continuous and nonincreasing, which ensures that $\ell_{0}^{\prime}$ is concave. It follows that

$$
\sum_{J \in \mathcal{P}} D_{J} \ell_{0}^{\prime}\left(A_{J} u\right) \leq D_{n} \ell_{0}^{\prime}\left(\sigma^{2}(\mathcal{P}) u / D_{n}\right)
$$

Integrating this inequality, we then get that

$$
\begin{equation*}
\int_{0}^{t}\left(\sum_{J \in \mathcal{P}} D_{J} \ell_{0}^{\prime}\left(A_{J} u\right)\right) d u \leq\left(D_{n}^{2} / \sigma^{2}(\mathcal{P})\right) \ell_{0}\left(\sigma^{2}(\mathcal{P}) t / D_{n}\right), \tag{5.5}
\end{equation*}
$$

which, together with (5.3) and (5.4), implies Theorem 5.1(a). Theorem 5.1(b) follows from the usual Chernoff calculation and Theorem 5.1(c) is an immediate consequence of Theorem 5.1(b)

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