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Hamiltonian structure and stability analysis of a reduced four-field model for plasmas in the presence of a strong guide field

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Abstract. We analyze the collisionless and inviscid limit of a reduced fluid model for tokamak plasma dynamics. We show that the model under consideration possesses a noncanonical Hamiltonian structure with four infinite families of Casimir invariants. Sufficient conditions for energy stability are derived and formulated in terms of requirements on the current, electron pressure, and parallel ion velocity gradients, as well as on the ion temperature. In particular, the stability condition requires a gradient of the parallel flow in the presence of ion temperature and a negative upper bound on the current gradient, in the presence of pressure gradients and/or parallel velocity gradients. Examples of stable configurations are derived.

1. Introduction

Reduced fluid models play an important role in the modeling of tokamak plasmas. Although they cannot describe important phenomena such as wave-particle interaction, their advantage, in terms of computational efficiency, when compared to kinetic models, make them a valuable tool for the investigation of phenomena such as turbulence, magnetic reconnection, or formation and propagation of coherent structures. Derivation of reduced fluid models from a parent model is typically carried out at the level of the equations of motion, operating with truncations and averages. Because the parent model ultimately has to go back to the Hamiltonian system describing the interaction between charged particles and electromagnetic fields, it is expected that the resulting reduced model possesses a non-trivial Hamiltonian core, plus dissipative terms, obtained as the result of the truncations and averages [1].

Identifying such a structure in the reduced model has several advantages. For instance, it gives access to an unambiguous definition of conserved energy (the Hamiltonian in the non-dissipative limit), which is necessary for correctly evaluating energy transfers in turbulent plasmas. Moreover, from the Hamiltonian structure, further non-trivial conservation laws can be systematically derived and stability methods can be applied [2].

In this spirit, we investigate the, initially supposed, dissipationless limit of a four-field fluid model for neoclassical tearing modes [3], which we obtain by omitting from the original model the terms proportional to the collisional and viscous coefficients. In such limit, of course, the

model does not account for tearing modes and does not possess all the sink and forcing terms, due, for instance, to neoclassical mechanisms or externally driven current, and whose influence on the reconnection process was investigated in Refs. [3, 4]. Nevertheless, for the above reasons, it remains an interesting model to investigate. Making use of the Hamiltonian structure, we obtain also criteria for plasma stability in the dissipationless limit.

First, we show that the model possesses a noncanonical Hamiltonian structure, characterized by four infinite families of Casimir invariants. Subsequently, we apply the so-called Energy-Casimir method (see, e.g., Refs. [5, 6, 7, 8, 9, 10, 2]) to determine sufficient conditions for the energy stability of equilibria obtained from a variational principle and formulate the results in terms of conditions on pressure, parallel flow, current density gradients, and ion temperature. Finally, we provide examples of stable equilibria.

2. Model equations and Hamiltonian structure

Considering a Cartesian coordinate system (x, y, z), the dissipationless limit of the four-field model derived in Ref. [3], can be written as:

$$\frac{\partial \psi}{\partial t} + [\phi, \psi] = d_i[\exp p, \psi], \tag{1}$$

$$\frac{\partial U}{\partial t} + [\phi, U] = -[\nabla^2 \psi, \psi], \tag{2}$$

$$\frac{\partial v_{i\|}}{\partial t} + [\phi, v_{i\|}] = (1 + \tau_T)[\psi, \exp p], \tag{3}$$

$$\frac{\partial p}{\partial t} + [\phi, p] = \frac{5}{3}[\psi, v_{i\|}] + \frac{5}{3} d_i[\psi, \nabla^2 \psi], \tag{4}$$

where ψ is the poloidal flux function of a magnetic field $\mathbf{B} = \nabla\psi \times \hat{z} + \hat{z}$, ϕ is the electrostatic potential, $U = \nabla^2\phi$ the $\mathbf{E} \times \mathbf{B}$ vorticity, p the logarithm of the electron pressure, $v_{i\|}$ the ion parallel velocity, whereas the constants d_i and τ_T indicate the ion skin depth and the ratio between the ion and the electron temperature, respectively. The fields ψ, ϕ, p, and $v_{i\|}$ are assumed to be translationally invariant along z and the bracket $[\,,]$ is defined by $[f, g] \equiv \hat{z} \cdot (\nabla f \times \nabla g)$, for generic fields f and g.

In Eqs. (1)-(4), lengths and time are normalized with respect to a characteristic length scale and to the Alfvén time, respectively. Eq. (1) represents an ideal Ohm's law, (2) is the plasma vorticity equation, (3) determines the evolution of the parallel ion motion, whereas (4) is an adiabatic equation of state for the electron pressure.

In what follows, we shall assume that we work on a finite domain with periodic functions, or in function spaces with enough decay at infinity that ensures that all boundary terms appearing in integration by parts vanish. We shall also assume, that in these function spaces, for any smooth functions f, g, h, we have the identity $\int d^2x[f, g]h = \int d^2x f[g, h]$ (i.e., the boundary terms appearing in the integration by parts used to establish this formula, vanish).

We address now the question of whether the system (1)-(4) is Hamiltonian. First, we recall that a dynamical system describing the evolution of a, possibly infinite, set of dynamical variables $(\chi^1, \chi^2, \ldots)$, is Hamiltonian if it can be written in the form

$$\frac{\partial \chi^i}{\partial t} = \{\chi^i, H\}, \qquad i = 1, 2, \cdots, \tag{5}$$

w here $H(\chi^1, \chi^2, \cdots)$ is the Hamiltonian (the total energy of the system) and $\{\,,\}$ is a Poisson bracket, in general noncanonical (which is hence bilinear, antisymmetric, and satisfies the Leibniz and Jacobi identities).

In checking for the existence of a Hamiltonian structure of (1)-(4), great help comes from Ref. [11], where a Hamiltonian model (Eqs. (35)-(38) of Ref. [11]) very similar to (1)-(4) is

considered. Indeed, the only differences between the two models consist in the presence of ion temperature and total pressure in (1)-(4), whereas [11], considers only pressure fluctuations but accounts also for magnetic field curvature and additional z-dependence of all variables. Thus, taking advantage of the results of Ref. [11], one sees by direct calculation that the functional

$$H = \int d^2x \left(\frac{|\nabla\psi|^2}{2} + \frac{|\nabla\phi|^2}{2} + \frac{v_{i\|}^2}{2(1+\tau_T)} + \frac{\exp p}{(5/3)} \right) \tag{6}$$

$$= \int d^2x \left(\frac{|\nabla\psi|^2}{2} + \frac{|\nabla\nabla^{-2}U|^2}{2} + \frac{D^2}{2d_i^2(1+\tau_T)} + \frac{(1+\tau_T)\psi^2}{2d_i^2} - \frac{D\psi}{d_i^2} + \frac{3}{5}\exp p \right), \tag{7}$$

where $D \equiv (1+\tau_T)\psi + d_i v_{i\|}$, is conserved by the dynamics generated by (1)-(4) and is, therefore, a good candidate for Hamiltonian (total energy) of the system. The first two terms of (6) can obviously be interpreted as poloidal magnetic and kinetic energy. The fourth term, is not of quadratic type, unlike in Ref. [11], which reflects the fact that the pressure has not been expanded about a constant equilibrium. This term represents internal energy, but only of the electron fluid. The third term, on the other hand, is analogous to the parallel kinetic energy term of the model of Ref. [12], with the same unfamiliar coefficient $1/(1+\tau_T)$.

Concerning the bracket, direct calculation shows that the operation

$$\begin{aligned} \{F,G\} = \int d^2x \Bigg(\psi \Bigg(& [F_\psi, G_U] + [F_U, G_\psi] + \frac{5}{3} d_i([F_\psi, G_p] + [F_p, G_\psi]) - \\ & \frac{5}{3}(1+\tau_T)([F_{v_{i\|}}, G_p] + [F_p, G_{v_{i\|}}]) \Bigg) + U[F_U, G_U] + \\ p \Bigg(& [F_p, G_U] + [F_U, G_p] + \frac{5}{3} d_i [F_p, G_p] \Bigg) + v_{i\|} \left([F_{v_{i\|}}, G_U] + [F_U, G_{v_{i\|}}] \right) \Bigg), \end{aligned} \tag{8}$$

when combined with (6), according to (5) and using $(\psi, U, v_{i\|}, p)$ as dynamical variables, gives the equations (1)-(4). In (8) the subscripts indicate functional derivatives. The bracket (8) indeed satisfies the properties of a Poisson bracket and is of the same form as the Poisson bracket of Ref. [11]. Analogously to most of the brackets of reduced fluid models for tokamaks, this bracket is of Lie-Poisson type (see, e.g., Refs. [2, 5]).

As it is typical for noncanonical Poisson brackets, (8) possesses Casimir invariants, i.e., elements of the center of the Poisson bracket (8), which, in this specific case are the infinite families of functionals:

$$C_1 = \int d^2x\, \mathcal{S}(\psi), \quad C_2 = \int d^2x\, p\, \mathcal{C}(\psi), \quad C_3 = \int d^2x\, \mathcal{K}(D), \quad C_4 = \int d^2x \left(U - \frac{3}{5d_i} p \right) \mathcal{F}(D), \tag{9}$$

where $\mathcal{F}$, $\mathcal{C}$, $\mathcal{S}$, and $\mathcal{K}$ are arbitrary smooth functions of a real variable.

Casimir functions corresponding to the integral of an arbitrary function of field variables, such as C_1 and C_3, are associated with Lagrangian invariants. The conservation of the Casimir functions C_1 obviously reflects the fact that the magnetic field is frozen into the poloidal flow determined by the $\mathbf{E} \times \mathbf{B}$ and diamagnetic velocities. The conservation of C_3, on the other hand, is related to the parallel momentum conservation when electron inertia is neglected and ion pressure is assumed to be proportional to the electron pressure. The Casimir functions C_2 and C_4 are remnants of cross-helicities of the parent model from which the system (1)-(4) is derived [11]. We remark that the addition of electron inertia into the model changes the nature of the Casimir functions by replacing a family of Casimir functions of cross-helicity type, with one associated with a Lagrangian invariant [13].

3. Equilibria and stability

One advantage of the Hamiltonian formulation, is that it makes it possible to apply the Energy-Casimir method in order to find formal stability conditions for equilibria of the system. In particular, the method can provide conditions for those equilibria which are extremals of the *free energy functional* F, obtained as a linear combination of the Hamiltonian with the Casimir functions. Then, according to the Energy- Casimir method, if $\delta^2 F$, evaluated at an equilibrium of interest, has a definite sign, the equilibrium is formally energy stable, which implies linear stability (for nonlinear stability in infinite dimensional systems, additional convexity arguments have to be provided [5, 9]). In the context of plasma equilibria, this method has been adopted in a number of cases (see for instance Refs. [14, 15, 16, 9, 17, 18]).

Equilibria. From (1)–(4), one obtains that equilibrium solutions have to satisfy

$$p = \ln\left(\frac{1}{d_i}(\mathcal{G}(D) - \mathcal{H}(\psi))\right), \tag{10}$$

$$\nabla^{-2} U = \mathcal{G}(D), \tag{11}$$

$$\left[\mathcal{G}(D), \nabla^2 \mathcal{G}(D)\right] = \left[\psi, \nabla^2 \psi\right], \tag{12}$$

$$\left[\mathcal{G}(D), \ln\left(\frac{1}{d_i}(\mathcal{G}(D) - \mathcal{H}(\psi))\right)\right] = \frac{5}{3d_i}\left[\psi, D\right] + \frac{5d_i}{3}\left[\psi, \nabla^2\psi\right], \tag{13}$$

with $\mathcal{G}, \mathcal{H}$ given arbitrary functions. Eq. (10) indicates that, at equilibrium, the pressure must be a function of the magnetic flux and of the modified ion canonical momentum D. Note that (10) implies also that we have to assume that $\mathcal{G}(D) - \mathcal{H}(\psi) > 0$. The relation (11), on the other hand, tells us that the equilibrium stream function must be a function of the modified ion canonical momentum D. Eqs. (12)-(13), finally, have to be considered as a system to be solved with respect to ψ and D, for given $\mathcal{G}(D)$ and $\mathcal{H}(\psi)$.

Two interesting examples of classes of equilibrium solutions are the following

Case 1: $\mathcal{G}(D) = \psi$. This case corresponds to the so called Alfvénic solutions, in which, at equilibrium, the $\mathbf{E} \times \mathbf{B}$ speed equals the Alfvén speed based on the poloidal magnetic field. Eq. (10) then implies that the equilibrium pressure is a function of ψ, i.e., it is a magnetic flux function. If $\mathcal{G}(D) = \psi$, a sufficient condition for (13) to be satisfied is $\nabla^2\psi + \frac{1}{d_i^2} D = \Psi(\psi)$ for some function Ψ. This corresponds to an equilibrium in which the electron parallel velocity is also a flux function.

Case 2: $\nabla^2 \mathcal{G}(D) = \Xi(D)$ and $\nabla^2\psi = \Upsilon(\psi)$, for some functions Ξ and Υ. In this case, the equilibrium vorticity is a function of D, whereas the current density $J = -\nabla^2\psi$ is a magnetic flux function. Equation (13) then implies either

$$a.) \quad [\psi, D] = 0, \qquad \text{or} \qquad b.) \quad \frac{\mathcal{G}'(D)\mathcal{H}'(\psi)}{\mathcal{G}(D) - \mathcal{H}(\psi)} = \frac{5}{3d_i}.$$

Case *a.*) is solved if we assume that $D = \mathcal{D}(\psi)$ for some function $\mathcal{D}$. Case *b.*) can also be solved assuming the same condition, but, in addition to that, it also requires a further relation between $\mathcal{G}$ and $\mathcal{H}$.

A special case (not applicable in the situation *b*) is obtained by requesting $\mathcal{G}'(D) = \Xi(D) = 0$, which refers to a case with no equilibrium $\mathbf{E} \times \mathbf{B}$ flow and with a current density being a magnetic flux function. The equilibrium magnetic field is then amenable to a number of solutions depending on the choice of Υ. A remarkable example is $\Upsilon(\psi) = \exp(2\psi)$, leading to the Liouville equation, which includes classical solutions such as the Harris sheet, the Kelvin-Stuart "cat's eye", and the Gold-Hoyle equilibrium (see, e.g., Ref. [19, 20]). The choice of $\Upsilon(\psi)$ linear in ψ, on the other hand, can give modon-like solutions [21, 22].

First variation. Following the Energy-Casimir method, we look for equilibria that are critical points of the free energy functional

$$F = H + \sum_i C_i,$$

where C_i are the Casimir functionals given in (9). In our case the free energy functional F is

$$F(\psi, U, D, p) = \int d^2x \left[\frac{1}{2}\left(|\nabla\psi|^2 - U\nabla^{-2}U + \frac{1}{d_i^2(1+\tau_T)}D^2 + \frac{1+\tau_T}{d_i^2}\psi^2 + \right.\right.$$
$$\left.\left. -\frac{2}{d_i^2}D\psi + \frac{6}{5}\exp p\right) + \mathcal{S}(\psi) + p\mathcal{C}(\psi) + \mathcal{K}(D) + \left(U - \frac{3}{5d_i}p\right)\mathcal{F}(D)\right]. \tag{14}$$

Setting the first variation

$$\delta F = \int d^2x \left[\left(-\nabla^2\psi + \frac{1+\tau_T}{d_i^2}\psi - \frac{1}{d_i^2}D + \mathcal{S}'(\psi) + p\mathcal{C}'(\psi)\right)\delta\psi \right.$$
$$+ \left(\frac{1}{d_i^2(1+\tau_T)}D - \frac{1}{d_i^2}\psi + \mathcal{K}'(D) + \left(U - \frac{3}{5d_i}p\right)\mathcal{F}'(D)\right)\delta D$$
$$\left. + \left(\frac{3}{5}\exp p + \mathcal{C}(\psi) - \frac{3}{5d_i}\mathcal{F}(D)\right)\delta p + \left(-\nabla^{-2}U + \mathcal{F}(D)\right)\delta U\right]$$

of F equal to zero, leads to the following set of equations:

$$\mathcal{F}(D) - d_i \exp p = \frac{5}{3}d_i\mathcal{C}(\psi), \tag{15}$$

$$\nabla^{-2}U = \mathcal{F}(D), \tag{16}$$

$$-\nabla^2\psi + \frac{1+\tau_T}{d_i^2}\psi - \frac{D}{d_i^2} + \ln\left(\frac{\mathcal{F}(D)}{d_i} - \frac{5}{3}\mathcal{C}(\psi)\right)\mathcal{C}'(\psi) + \mathcal{S}'(\psi) = 0, \tag{17}$$

$$\frac{1}{d_i^2(1+\tau_T)}D - \frac{\psi}{d_i^2} + \nabla^2\mathcal{F}(D)\mathcal{F}'(D) - \frac{3}{5d_i}\ln\left(\frac{\mathcal{F}(D)}{d_i} - \frac{5}{3}\mathcal{C}(\psi)\right)\mathcal{F}'(D) + \mathcal{K}'(D) = 0, \tag{18}$$

where the prime denotes derivative with respect to the argument of the function.

Different choices of the free functions $\mathcal{S}$, $\mathcal{C}$, $\mathcal{F}$, and $\mathcal{K}$, of course lead to different equilibria.

Given equilibrium solutions of (10)-(13), on the other hand, are extremals of F, if the free functions $\mathcal{F}$, $\mathcal{C}$, $\mathcal{S}$, and $\mathcal{K}$ satisfy the appropriate constraints. Indeed, by comparing (11) with (16) and (10) with (15) one immediately sees that the functions $\mathcal{F}$ and $\mathcal{C}$ are determined by $\mathcal{F} = \mathcal{G}$ and $\mathcal{C} = (3/5d_i)\mathcal{H}$. The case with no $\mathbf{E}\times\mathbf{B}$ flow, then corresponds to $\mathcal{F} = 0$. The relations determining $\mathcal{S}$ and $\mathcal{K}$, on the other hand, are less straightforward. For the solutions of Case 2, for instance, they are determined by

$$\mathcal{S}'(\psi) = \Upsilon(\psi) - \frac{1+\tau_T}{d_i^2}\psi + \frac{\mathcal{D}(\psi)}{d_i^2} - \frac{3}{5d_i}\ln\left(\frac{\mathcal{G}(\mathcal{D}(\psi)) - \mathcal{H}(\psi)}{d_i}\right)\mathcal{H}'(\psi),$$
$$\mathcal{K}'(D) = \frac{\mathcal{D}^{-1}(D)}{d_i^2} - \frac{1}{d_i^2(1+\tau_T)}D - \Xi(D)\mathcal{G}'(D) + \frac{3}{5d_i}\ln\left(\frac{\mathcal{G}(D) - \mathcal{H}(\mathcal{D}^{-1}(D))}{d_i}\right)\mathcal{G}'(D),$$

where we assumed $\mathcal{D}$ to be invertible.

We notice also that, in the absence of $\mathbf{E}\times\mathbf{B}$ flow, the choices $\mathcal{K}(D) =$ constant and $\mathcal{C}(\psi) =$ constant, correspond to no parallel flow and no pressure gradient at equilibrium, respectively. Indeed, in that limit, one has $v_{i\|}(D) = -d_i(1+\tau_T)\mathcal{K}'(D)$ and $p(\psi) = \ln((5/3)|\mathcal{C}(\psi)|)$ at equilibrium.

Second variation The expression for the second variation of F for our model, after some manipulation, can be written as

$$\delta^2 F = \int d^2x \left[|\nabla\delta\psi|^2 + |\nabla\delta\phi|^2 + \left(\frac{\tau_T}{d_i^2} + p\mathcal{C}''(\psi) + \mathcal{S}''(\psi) - \mathcal{C}'(\psi)\right)|\delta\psi|^2 + \frac{|\delta\psi - \delta D|^2}{d_i^2} \right.$$
$$+ \left(-\frac{\tau_T}{d_i^2(1+\tau_T)} + U\mathcal{F}''(D) - \frac{3}{5d_i}p\mathcal{F}''(D) + \mathcal{K}''(D) - \left(1 + \frac{3}{5d_i}\right)\mathcal{F}'(D)\right)|\delta D|^2 - \mathcal{F}'(D)|\delta U|^2 +$$
$$\left. + \left(\frac{3}{5}\exp p - \frac{3}{5d_i}\mathcal{F}'(D) - \mathcal{C}'(\psi)\right)|\delta p|^2 + \mathcal{F}'(D)|\delta D + \delta U|^2 + \frac{3}{5d_i}\mathcal{F}'(D)|\delta p - \delta D|^2 + \mathcal{C}'(\psi)|\delta\psi + \delta p|^2 \right].$$

We consider now a generic equilibrium $(\psi, U, D, p) = (\psi_{eq}, U_{eq}, D_{eq}, p_{eq})$, solution of (15)-(18). A sufficient condition for $\delta^2 F$, evaluated at such equilibrium, to be positive, is given by:

$$\mathcal{F} = 0, \tag{19}$$
$$-\frac{\tau_T}{d_i^2(1+\tau_T)} + \mathcal{K}''(D_{eq}) \geq 0, \tag{20}$$
$$\frac{\tau_T}{d_i^2} + p_{eq}\mathcal{C}''(\psi_{eq}) + \mathcal{S}''(\psi_{eq}) - \mathcal{C}'(\psi_{eq}) \geq 0, \tag{21}$$
$$\frac{3}{5}\exp p_{eq} - \mathcal{C}'(\psi_{eq}) \geq 0, \tag{22}$$
$$\mathcal{C}'(\psi_{eq}) \geq 0. \tag{23}$$

Condition (19) means that we are restricting to the case of no $\mathbf{E}\times\mathbf{B}$ flow. The above conditions could then apply to equilibria of Case 2. We remark that it is well known (see, e.g. Refs. [23, 2]), that an equilibrium flow can lead to the presence of negative energy modes, which can imply indefiniteness of $\delta^2 F$. We observe also that our stability conditions are sufficient but not necessary. Therefore, there could exist, in general, spectrally stable equilibria of the system, that do not satisfy our stability conditions.

For $\mathcal{F} = 0$, from (15) one has $p_{eq} = \ln((5/3)|\mathcal{C}(\psi_{eq})|)$ and the condition $\mathcal{C}(\psi_{eq}) < 0$ is required in order for the equilibrium relation (15) to be valid. Conditions (22) and (23) can then be written as

$$-1 \leq p'_{eq}(\psi_{eq}) \leq 0, \tag{24}$$

which implies a restriction on the equilibrium pressure gradient.
Because $v_{i\|}(D_{eq}) = -d_i(1+\tau_T)\mathcal{K}'(D_{eq})$, the presence of a parallel equilibrium sheared flow is necessary in order to satisfy condition (20). This is not required for negligible ion temperature. On the other hand, this condition can be reformulated as $(dv_{i\|}/dD)(D_{eq}) \leq -\tau_T/d_i$, which directly shows the role of the parallel velocity shear. Finally, inequality (21) can be interpreted as a condition on the current density gradient. Indeed, because for the equilibria of Case 2 one has $D_{eq} = \mathcal{D}(\psi_{eq})$, it follows that the equilibrium current density can be written as

$$J_{eq}(\psi_{eq}) = -\Upsilon(\psi_{eq}) = -\frac{1+\tau_T}{d_i^2}\psi_{eq} + \frac{\mathcal{D}(\psi_{eq})}{d_i^2} + \frac{3}{5}p_{eq}(\psi_{eq})p'_{eq}(\psi_{eq})\exp p_{eq}(\psi_{eq}) - \mathcal{S}'(\psi_{eq}), \tag{25}$$

and the stability condition (21) can be expressed as

$$J'_{eq}(\psi_{eq}) \leq \frac{\mathcal{D}'(\psi_{eq}) - 1}{d_i^2} + \frac{3}{5}p'_{eq}(\psi_{eq})(p'_{eq}(\psi_{eq}) + 1)\exp p_{eq}(\psi_{eq}), \tag{26}$$

which relates the current with pressure and velocity gradients. In the absence of parallel ion flow and ion temperature (corresponding to $\mathcal{K}'(D) = 0$ and $\mathcal{D}(\psi) = \psi$) , this condition becomes

$$J'_{eq}(\psi_{eq}) \leq \frac{3}{5}p'_{eq}(\psi_{eq})(p'_{eq}(\psi_{eq}) + 1)\exp p_{eq}(\psi_{eq}). \tag{27}$$

Note that, because of (24), the right-hand side of (27) is negative (except for $p'_{eq} = 0$ or $p'_{eq} = -1$), so that formal stability is attained if the current gradient is negative and sufficiently steep. An analogous situation occurs for flat pressure equilibrium profiles, with ion parallel flow and temperature satisfying (20). Indeed, in this case the stability condition (26) reduces to

$$J'_{eq}(\psi_{eq}) \leq \frac{\mathcal{D}'(\psi_{eq}) - 1}{d_i^2}. \tag{28}$$

Condition (20), on the other hand, can be reformulated as

$$\mathcal{D}^{-1\prime}(D_{eq}) \geq 1. \tag{29}$$

Consequently, for $\psi_{eq} = \mathcal{D}^{-1}(D_{eq})$, one has $\mathcal{D}'(\psi_{eq}) \leq 1$, so, because of (28), also a parallel gradient of the flow imposes a negative upper bound to the current density gradient. Bounds on the current gradient have been shown, by means of dynamical accessible variations [2], to provide stability also in the presence of electron inertia and ion temperature for a collisionless reconnection model [24]. In that case, the bound on the current density gradient depended on the ion temperature and on the electron skin depth. We remark also that, in the limit of a flat pressure profile, with cold ions and no parallel flow, one recovers, from our conditions, the stability condition for reduced magnetohydrodynamics [16], $J'_{eq}(\psi_{eq}) \leq 0$, which reflects the fact that a monotone current density profile provides stability.
To construct a simple concrete example of stable equilibrium one can start from the case $\tau = 0$, and consider an equilibrium of Case 2 with $\mathcal{G} = 0$, $\mathcal{H}(\psi) = -5d_i c/3$, $\Upsilon(\psi) = \lambda^2\psi$, and $\mathcal{D}(\psi) = \psi$ (corresponding to $\mathcal{F} = 0$, $\mathcal{C}(\psi) = -c$, $\mathcal{S}'(\psi) = \lambda^2\psi$ and $\mathcal{K}'(D) = 0$), where $c > 0$ and λ are constant. An equilibrium solution is then given by:

$$U_{eq} = 0, \qquad D_{eq}(\psi_{eq}) = \psi_{eq}, \qquad p_{eq}(\psi_{eq}) = \ln\left(\frac{5}{3}c\right), \qquad \psi_{eq}(x) = \cosh(\lambda x), \tag{30}$$

which describes a one-dimensional magnetic field with a neutral line at $x = 0$. This static equilibrium, with a flat pressure profile and no parallel ion flow, has $J'_{eq} = -\lambda^2$, and thus turns out to be stable, for every λ, according to the criterion (19)-(23) (we observe that such equilibrium can be shown to be also tearing stable according to the Δ' criterion introduced in Ref. [25]).

The inclusion of pressure or parallel velocity profiles that still guarantee stability, can be obtained by choosing, for instance, $p_{eq}(\psi_{eq}) = 1 - \tanh\psi_{eq}$ and $v_{i\|_{eq}}(\psi_{eq}) = \gamma[(1 + \tau_T)/(1 - \gamma d_i)]\psi_{eq}$, with γ constant (corresponding to an equilibrium of Case 2 with $\mathcal{G} = 0$, $\mathcal{H}(\psi) = -d_i\exp(1 - \tanh\psi)$, $\mathcal{D}(\psi) = (1 + \tau_T)/(1 - \gamma d_i)\psi$ and $\Upsilon(\psi) = \lambda^2\psi$). In this way, conditions (22) and (23) are automatically satisfied. Condition (26) reads

$$-\lambda^2 \leq \frac{1}{d_i^2}\frac{\tau_T + d_i\gamma}{1 - d_i\gamma} - \frac{3}{5}\frac{1}{\cosh^2\psi_{eq}}\left(1 - \frac{1}{\cosh^2\psi_{eq}}\right)\exp(1 - \tanh\psi_{eq}). \tag{31}$$

Given that, for this equilibrium, $\psi_{eq}(x) \geq 1$ for all x, condition (31) can be satisfied, if $\lambda^2 \geq (1/d_i^2)((\tau_T + d_i\gamma)/(d_i\gamma - 1)) + (3/5)\mathrm{e}$, which is more restrictive than the condition for cold ions in the absence of pressure and parallel flow. Condition (20), on the other hand, is satisfied if $\gamma \leq -\tau_T/d_i$. In particular, in this stable configuration, the parallel flow turns out to be in the direction opposite to that of the toroidal magnetic field.

4. Summary and outlook

We presented the Hamiltonian structure of a reduced model for plasma dynamics in the presence of a strong guide field, and obtained sufficient stability criteria for its equilibria, making use of the Energy-Casimir method. The stability conditions apply to equilibria with no poloidal flow and can be interpreted as restrictions on the pressure, parallel velocity and current density gradients, thus generalizing previous stability criteria for reduced magnetohydrodynamics. Examples of equilibria with magnetic shear, satisfying the stability conditions, are presented.

As a further remark, we observe that our stability criteria hold also for two-dimensional equilibria, for which traditional spectral stability methods based on the Fourier transform become more difficult to apply. In this respect, we point out that an identification of the relations between the stability conditions derived for this model, with well established stability conditions obtained, for instance, from the classical energy principle, is under progress, as well as an extension of the stability conditions to include poloidal flow, which will be the subject of a forthcoming paper.

A further direction of development would consist of extending the model in such a way that the analysis could be more directly applicable to tokamaks. In particular, we believe that adapting the model to tokamak geometry and reintroducing, in a Hamiltonian framework, the neoclassical effects decribed in the original version of the model, could lead to promising developments.

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