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# HOMOGENIZATION OF THE GINZBURG-LANDAU EQUATION IN A DOMAIN WITH OSCILLATING BOUNDARY

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**ABSTRACT :** We study the asymptotic behaviour, as  $h$  tends to  $+\infty$ , of the nonlinear system:

$$\begin{cases} -\Delta u_h - u_h + |u_h|^2 u_h = f & \text{in } \Omega_h, \\ Du_h \cdot \nu = 0 & \text{on } \partial\Omega_h, \\ u_h : \Omega_h \rightarrow \mathbb{R}^2, \end{cases}$$

in a varying domain  $\Omega_h$  in  $\mathbb{R}^2$ . The boundary  $\partial\Omega_h$  contains an oscillating part like a comb with fine teeth periodically distributed in the first direction  $0x_1$  with period  $h^{-1}$  and thickness  $\lambda h^{-1}$ ,  $0 < \lambda < 1$ .

We identify the limit problem where the operator  $-\Delta$  is reduced to  $-\frac{\partial^2}{\partial x_2^2}$  in the domain corresponding to the oscillating boundary.

**AMS (MOS) Subject Classification.** 35B27, 35Q55, 58E50.

## 1. INTRODUCTION

In this paper we consider the Ginzburg-Landau equation ( $GL_h$ ):

$$\begin{cases} -\Delta u_h - u_h + |u_h|^2 u_h = f & \text{in } \Omega_h, \\ u_h : \Omega_h \longrightarrow \mathbb{R}^2, \end{cases}$$

with homogeneous Neumann boundary condition, in a varying domain  $\Omega_h$  in  $\mathbb{R}^2$ . We are interested in a class of domains  $\Omega_h$  which have the shape of a comb with fine teeth periodically distributed in the first direction  $0x_1$  with period  $\frac{1}{h}$  and thickness  $\frac{\lambda}{h}$ ,  $0 < \lambda < 1$  (see Fig. 1). The goal is to study the asymptotic behaviour of such problem when  $h$  tends to  $+\infty$  (see Theorem 2.1).

For general references about homogenization, we refer to [2], [3], [4], [13], [24] and [25]. In the scalar case, for this kind of domains with crenelated part of the boundary, the limit problem has been studied for the Laplace operator in [9], [10], [15] and for quasilinear operator, more generally

for a monotone operator in [12] and [7]. For reinforcement problems by a layer with oscillating thickness see [11], for problems related to the asymptotic behaviour of thin cylinders see [19] and [20].

An extensive study of Ginzburg-Landau equations is developed by Bethuel-Brezis and Hélein in [5] and [6]. The limit behaviour of the Ginzburg-Landau equation in a perforated domain in  $\mathbb{R}^3$  with holes along a plane is studied in [17].

In order to identify the limit problem of  $(GL_h)$ , as  $h$  tends to  $+\infty$ , the main steps are to establish a uniform estimate of  $u_h$  in  $(L^\infty(\Omega_h))^2$  and to obtain an extension of  $u_h$  on a fixed domain; this is the object of Section 3 and 4. Then, in Section 5, we find the limit problem where the operator  $-\Delta$  is reduced to  $-\frac{\partial^2}{\partial x_2^2}$  in the domain corresponding to the oscillating boundary. The main difficulty is to pass to the limit in the nonlinear term of the Ginzburg-Landau equation (see Remark 5.1).

For sake of completeness, we give also the limit behaviour of the previous Ginzburg-Landau equation with homogeneous Dirichlet boundary condition (for scalar problems with Dirichlet boundary condition in a domain with oscillating boundary see [7], [9], [10], [14],[18], [21], [22] and [23]).

## 2. STATEMENT OF THE PROBLEM AND MAIN RESULT

Let  $a, b_1, b_2, \alpha$  be in  $]0, +\infty[$  such that  $0 < \alpha < \frac{a}{2}$  and let us introduce the following domains in  $\mathbb{R}^2$  (see Fig.1):

$$(2.1) \quad \left\{ \begin{array}{l} \Omega = ]0, a[ \times ]-b_1, b_2[, \\ \Omega^- = ]0, a[ \times ]-b_1, 0[, \quad \Omega^+ = ]0, a[ \times ]0, b_2[, \\ \Sigma = ]0, a[ \times \{0\}, \\ \Omega_h = \Omega^- \cup \left( \bigcup_{k=0}^{h-1} \left( \frac{1}{h} \alpha, a - \alpha \left[ + \frac{ak}{h} \right) \times [0, b_2[ \right) \quad h \in \mathbb{N}, \\ \Omega_h^+ = \Omega^+ \cap \Omega_h \quad h \in \mathbb{N}. \end{array} \right.$$

In the sequel,  $x = (x_1, x_2)$  denotes the generic point of  $\mathbb{R}^2$ ,  $\chi_A$  the characteristic function of a subset  $A$  of  $\Omega$  and  $\tilde{v}$  the zero-extension to  $\Omega$  of any (vector) function  $v$  defined on a subset of  $\Omega$ .

We recall that

$$(2.2) \quad \chi_{\Omega_h^+} \rightharpoonup \theta = \frac{a - 2\alpha}{a} \text{ weakly } \star \text{ in } L^\infty(\Omega^+)$$

and

$$(2.3) \quad \chi_{\Omega_h \cap \Sigma} \rightharpoonup \theta \text{ weakly } \star \text{ in } L^\infty(\Sigma)$$

as  $h$  diverges.

The aim of this paper is to study the asymptotic behaviour, as  $h$  tends to  $+\infty$ , of the following homogeneous Neumann problem:

$$(2.4) \quad \left\{ \begin{array}{l} -\Delta u_h - u_h + |u_h|^2 u_h = f \text{ in } \Omega_h, \\ Du_h \cdot \nu = 0 \text{ on } \partial\Omega_h, \end{array} \right.$$

where  $f = (f_1, f_2)$  is a given function in  $(L^2(\Omega))^2$  and  $\nu$  denotes the exterior unit normal to  $\Omega_h$ .

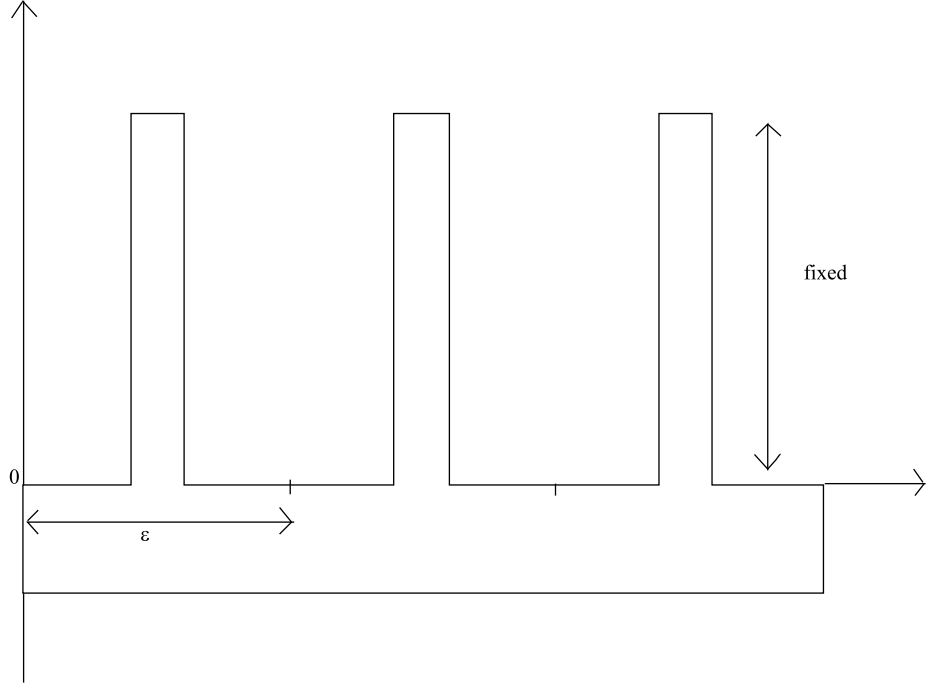


Figure 1: the middle surface of our three-dimensional plate.

Problem (2.4) admits a weak solution  $u_h \in (H^1(\Omega_h))^2$ . In fact, it is easy to see that a minimizing sequence of the following problem:

$$(2.5) \quad \inf \left\{ \int_{\Omega_h} (|Dv|^2 + \frac{1}{2}(1 - |v|^2)^2 - 2fv) dx : v \in (H^1(\Omega_h))^2 \right\}$$

is bounded in  $(L^4(\Omega_h))^2$  and  $(H^1(\Omega_h))^2$  and then the infimum in (2.5) is achieved by  $u_h$  satisfying the following variational equation :

$$(2.6) \quad \begin{cases} \int_{\Omega_h} (Du_h Dv - u_h v + |u_h|^2 u_h v) dx = \int_{\Omega_h} f v dx \quad \forall v \in (H^1(\Omega_h))^2, \\ u_h = (u_h^{(1)}, u_h^{(2)}) \in (H^1(\Omega_h))^2. \end{cases}$$

The main result of this paper is given by the following theorem:

**Theorem 2.1.** *For every  $h$  in  $\mathbb{N}$ , let  $u_h$  be a solution of Problem (2.4) with  $f$  in  $(H^1(\Omega))^2 \cap (L^\infty(\Omega))^2$  and let  $\theta$  be defined by (2.2).*

*Then, for every  $h$  in  $\mathbb{N}$ , there exists a linear extension - operator  $P_h \in \mathcal{L} \left( (H^1(\Omega_h^+))^2, (H^1(\Omega^+))^2 \right)$ , a strictly increasing sequence of positive integer numbers  $\{h_k\}_{k \in \mathbb{N}}$  and  $u$  in  $(H^1(\Omega))^2 \cap (L^\infty(\Omega))^2$  (depending possibly on the selected subsequence) such that*

$$(2.7) \quad \begin{cases} P_{h_k} u_{h_k} \rightharpoonup u \text{ weakly in } (H^1(\Omega^+))^2, \\ u_{h_k} \rightharpoonup u \text{ weakly in } (H^1(\Omega^-))^2, \end{cases}$$

*as  $k$  tends to  $+\infty$  and  $u$  is a solution of the following problem:*

$$(2.8) \quad \begin{cases} -\frac{\partial^2 u}{\partial x_2^2} - u + |u|^2 u = f \text{ in } \Omega^+, \\ -\Delta u - u + |u|^2 u = f \text{ in } \Omega^-, \\ \theta \frac{\partial u^+}{\partial x_2} = \frac{\partial u^-}{\partial x_2} \text{ on } \Sigma, \\ \frac{\partial u}{\partial x_2} = 0 \text{ on } ]0, a[ \times \{b_2\}, \\ Du \cdot \nu = 0 \text{ on } \partial\Omega^- \setminus \Sigma. \end{cases}$$

Moreover the energies converge in the sense that

$$(2.9) \quad \begin{aligned} & \lim_{k \rightarrow +\infty} \int_{\Omega_{h_k}} (|Du_{h_k}|^2 - |u_{h_k}|^2 + |u_{h_k}|^4) dx \\ & = \theta \int_{\Omega^+} \left( \left| \frac{\partial u}{\partial x_2} \right|^2 - |u|^2 + |u|^4 \right) dx + \int_{\Omega^-} (|Du|^2 - |u|^2 + |u|^4) dx. \end{aligned}$$

The variational formulation of Problem (2.8) is given by

$$(2.10) \quad \begin{cases} \theta \int_{\Omega^+} \frac{\partial u}{\partial x_2} \frac{\partial v}{\partial x_2} - uv + |u|^2 uv \, dx + \int_{\Omega^-} DuDv - uv + |u|^2 uv \, dx = \\ = \theta \int_{\Omega^+} fv \, dx + \int_{\Omega^-} fv \, dx \quad \forall v \in (H^1(\Omega))^2, \\ u \in (H^1(\Omega))^2 \cap (L^\infty(\Omega))^2. \end{cases}$$

**Remark 2.2.** If Problem (2.10) admits a unique solution, then convergences (2.7) and (2.9) hold true for the whole sequence.

**Remark 2.3.** If in Problem (2.4), instead of the Neumann boundary condition, we assume the Dirichlet boundary condition  $u_h = 0$  on  $\partial\Omega_h$ , it is easy to show that every sequence of zero-extension to  $\Omega$  of solutions of the Dirichlet problem admits a subsequence which strongly converges in  $(H_0^1(\Omega))^2$  to a solution of the following problem:

$$\begin{cases} u = 0 \text{ a.e. in } \Omega^+, \\ -\Delta u - u + |u|^2 u = f \text{ in } \Omega^-, \\ u = 0 \text{ on } \partial\Omega^-. \end{cases}$$

Moreover the convergence of the energies holds.

### 3. A PRIORI $L^\infty$ ESTIMATE

In this section, we establish an a priori norm-estimate for the solution  $u_h$  of problem (2.4).

Let  $A$  be a bounded open set of  $\mathbb{R}^n$ . We shall denote by  $C_b(A)$  the Banach space defined by  $C_b(A) = \{v \in C(A) : v \text{ is bounded}\}$  provided with the  $L^\infty$ - norm.

The goal of this section is to prove the following result:

**Proposition 3.1.** *For every  $h$  in  $\mathbb{N}$ , let  $u_h$  be a solution of Problem (2.4) with  $f$  in  $(L^\infty(\Omega))^2$ . Then  $u_h$  is in  $(C_b(\Omega_h))^2$  and there exists a constant  $c$  (independent of  $h$ ) such that*

$$\|u_h\|_{(L^\infty(\Omega_h))^2} \leq c \quad \forall h \in \mathbb{N}.$$

To this aim, we begin by giving some preliminary results:

**Lemma 3.2.** *Let  $u_h, h$  in  $\mathbb{N}$ , be a solution of Problem (2.4) with  $f$  in  $(L^2(\Omega))^2$ . Then,  $u_h$  is in  $(C_b(\Omega_h))^2$ .*

**Proof.** Let us fix  $h$  in  $N$  and let us set

$$f_h = u_h - |u_h|^2 u_h + f \text{ a.e. in } \Omega_h.$$

Since  $u_h$  belongs to  $(H^1(\Omega_h))^2$  and  $H^1(\Omega_h)$  is embedded into  $L^p(\Omega_h)$  for every  $p$  in  $[1, +\infty[$ , it turns out that  $f_h$  belongs to  $(L^2(\Omega_h))^2$ . Moreover, from (2.4) it follows that  $u_h$  is a solution of

$$\begin{cases} -\Delta u_h = f_h \text{ in } \Omega_h, \\ Du_h \cdot \nu = 0 \text{ on } \partial\Omega_h. \end{cases}$$

Consequently there exists  $s$  in  $\left] \frac{3}{2}, \frac{5}{3} \right[$  such that  $u_h$  belongs to  $(H^s(\Omega_h))^2$  (see [16], Lemma 5.1 and Theorem 5.2). Finally, the thesis follows from the embeddings of  $H^s(\Omega_h)$ , with  $s > 1$ , into  $C_b(\Omega_h)$  (see [1], 7.57).  $\square$

We recall the following well-known classical variational inequality and we give the proof for the reader's convenience.

**Lemma 3.3.** *Let  $A$  be an open subset of  $\mathbb{R}^n$  satisfying the segment property,  $\gamma$  a positive constant and  $v$  a function in  $H^1(A)$  such that*

$$\int_A (DvD\varphi + \gamma v\varphi) dx \geq 0 \quad \forall \varphi \in C^1(\mathbb{R}^n) \text{ with } \varphi \geq 0.$$

*Then, it results that*

$$v \geq 0 \text{ a.e. in } A.$$

**Proof.** By the assumptions it follows that

$$(3.1) \quad \int_A (DvD\varphi + \gamma v\varphi) dx \geq 0 \quad \forall \varphi \in H^1(A) \text{ with } \varphi \geq 0 \text{ a.e. in } A.$$

By choosing  $\varphi = v^- = -\min\{v, 0\}$  in (3.1), it results

$$-\int_A (|Dv^-|^2 + \gamma |v^-|^2) dx \geq 0.$$

This inequality provides

$$v^- = 0 \text{ a.e. in } A$$

and consequently

$$v \geq 0 \text{ a.e. in } A.$$

$\square$

Now we can prove the main result of this section.

**Proof of Proposition 3.1.** Let us fix  $h$  in  $\mathbb{N}$ .

Lemma 3.2 provides that

$$(3.2) \quad |u_h|^2 \in H^1(\Omega_h)$$

and consequently

$$(3.3) \quad \int_{\Omega_h} D(|u_h|^2)D\varphi \, dx = 2 \int_{\Omega_h} u_h Du_h D\varphi \, dx \quad \forall \varphi \in C^1(\mathbb{R}^2).$$

On the other hand, by choosing  $v = \varphi u_h$  in (2.6) with  $\varphi$  in  $C^1(\mathbb{R}^2)$ , it results

$$(3.4) \quad \begin{aligned} & \int_{\Omega_h} |Du_h|^2 \varphi \, dx + \int_{\Omega_h} u_h Du_h D\varphi \, dx = \\ & = \int_{\Omega_h} (|u_h|^2(1 - |u_h|^2)\varphi + fu_h\varphi) \, dx \quad \forall \varphi \in C^1(\mathbb{R}^2). \end{aligned}$$

By combining (3.3) with (3.4) it follows that

$$(3.5) \quad \begin{aligned} & \int_{\Omega_h} D(|u_h|^2)D\varphi \, dx = \\ & = 2 \int_{\Omega_h} (-|Du_h|^2 + |u_h|^2(1 - |u_h|^2) + fu_h)\varphi \, dx \quad \forall \varphi \in C^1(\mathbb{R}^2). \end{aligned}$$

If we set

$$(3.6) \quad v_h = 1 - |u_h|^2 \quad \text{a.e. in } \Omega_h,$$

equation (3.5) provides that

$$\int_{\Omega_h} Dv_h D\varphi \, dx = 2 \int_{\Omega_h} (|Du_h|^2 - v_h(1 - v_h) - fu_h)\varphi \, dx \quad \forall \varphi \in C^1(\mathbb{R}^2)$$

and consequently

$$(3.7) \quad \begin{aligned} & \int_{\Omega_h} (Dv_h D\varphi + 2v_h\varphi) \, dx = 2 \int_{\Omega_h} (|Du_h|^2 + |v_h|^2 - fu_h)\varphi \, dx \geq \\ & \geq -2 \int_{\Omega_h} fu_h\varphi \, dx \quad \forall \varphi \in C^1(\mathbb{R}^2) \text{ with } \varphi \geq 0. \end{aligned}$$

Now let us fix  $\eta$  in  $]0, \frac{1}{2}[$ .

By applying the Young inequality it results

$$u_h f = u_h^{(1)} f_1 + u_h^{(2)} f_2 \leq \eta |u_h|^2 + \frac{1}{\eta} |f|^2 = \eta(1 - v_h) + \frac{1}{\eta} |f|^2 \quad \text{a.e. in } \Omega_h$$

and consequently

$$(3.8) \quad -2u_h f \geq -2\eta + 2\eta v_h - \frac{2}{\eta} \|f\|_{(L^\infty(\Omega))^2}^2 \quad \text{a.e. in } \Omega_h.$$

By combining (3.7) with (3.8) we obtain

$$(3.9) \quad \begin{aligned} & \int_{\Omega_h} (Dv_h D\varphi + 2(1 - \eta)v_h\varphi) \, dx \geq \\ & \geq -2 \left( \eta + \frac{\|f\|_{(L^\infty(\Omega))^2}^2}{\eta} \right) \int_{\Omega_h} \varphi \, dx \quad \forall \varphi \in C^1(\mathbb{R}^2) \text{ with } \varphi \geq 0. \end{aligned}$$

If we set

$$c = -2 \left( \eta + \frac{\|f\|_{(L^\infty(\Omega))^2}^2}{\eta} \right)$$

and

$$(3.10) \quad w_h = v_h - c \text{ a.e. in } \Omega_h,$$

since  $0 < \eta < \frac{1}{2}$  and  $c < 0$ , from (3.9) it follows that

$$(3.11) \quad \begin{aligned} & \int_{\Omega_h} (Dw_h D\varphi + 2(1-\eta)w_h\varphi) dx \geq \\ & \geq c(2\eta-1) \int_{\Omega_h} \varphi dx \geq 0 \quad \forall \varphi \in C^1(\mathbb{R}^2) \text{ with } \varphi \geq 0. \end{aligned}$$

Now observe that, by virtue of (3.10), (3.6) and (3.2),  $w_h$  belongs to  $H^1(\Omega_h)$ . Consequently, by virtue Lemma 3.3, from (3.11) we deduce that

$$(3.12) \quad w_h \geq 0 \text{ a.e. in } \Omega_h.$$

By recalling the definitions (3.10) and (3.6), the last inequality provides

$$|u_h|^2 \leq 1 - c \text{ a.e. in } \Omega_h.$$

Since  $c$  does not depend on  $h$ , the thesis holds.  $\square$

**Corollary 3.4.** *For every  $h$  in  $\mathbb{N}$ , let  $u_h$  be a solution of Problem (2.4) with  $f$  in  $(L^\infty(\Omega))^2$ . Then, there exists a constant  $c$  (independent of  $h$ ) such that*

$$\|u_h\|_{(H^1(\Omega_h))^2} \leq c \quad \forall h \in \mathbb{N}.$$

**Proof.** By choosing  $v = u_h$  in (2.6) and by using Hölder's inequality, it results

$$\begin{aligned} & \int_{\Omega_h} |Du_h|^2 dx \leq \int_{\Omega_h} (|Du_h|^2 + |u_h|^4) dx = \int_{\Omega_h} f u_h dx + \int_{\Omega_h} |u_h|^2 dx \leq \\ & \leq \|f\|_{(L^2(\Omega))^2} \|u_h\|_{(L^2(\Omega_h))^2} + \|u_h\|_{(L^2(\Omega_h))^2}^2 \quad \forall h \in \mathbb{N} \end{aligned}$$

from which, by virtue of Proposition 3.1, the thesis follows.  $\square$

**Remark 3.5.** For every  $h$  in  $\mathbb{N}$ , let  $u_h$  be a solution of Problem (2.4). By assuming  $f$  only in  $(L^2(\Omega))^2$  (and without making use of Proposition 3.1), it is easy to prove the existence of a constant  $c$  (independent of  $h$ ) such that

$$\begin{aligned} & \|u_h\|_{(L^4(\Omega_h))^2} \leq c \quad \forall h \in \mathbb{N}. \\ & \|u_h |u_h|^2\|_{L^{\frac{4}{3}}(\Omega_h)} \leq c \quad \forall h \in \mathbb{N}. \\ & \|u_h\|_{(H^1(\Omega_h))^2} \leq c \quad \forall h \in \mathbb{N}. \end{aligned}$$



#### 4. EXTENSION RESULT

This section is devoted to prove the following result:

**Proposition 4.1.** *For every  $h$  in  $\mathbb{N}$ , let  $u_h$  be a solution of Problem (2.4) with  $f$  in  $(L^\infty(\Omega))^2 \cap (H^1(\Omega))^2$ . Then, for every  $h$  in  $\mathbb{N}$ , there exists a linear extension-operator  $P_h \in \mathcal{L} \left( (H^1(\Omega_h^+))^2, (H^1(\Omega^+))^2 \right)$  such that*

$$(4.1) \quad \|P_h u_h\|_{(H^1(\Omega^+))^2} \leq c \quad \forall h \in \mathbb{N},$$

where  $c$  is a constant independent of  $h$ .

In this section, if  $\phi(x_1, x_2)$  is a real function defined on  $\Omega_h$ , we assume  $\phi$  extended on  $T_h = \bigcup_{k \in \mathbb{Z}} ((ka, 0) + \Omega_h)$  in the following way: first we extend  $\phi$  on  $(-a, 0) + \Omega_h$  by reflection, then we extend this function on  $T_h$  by  $2a$ -periodicity in the variable  $x_1$ . Moreover, we define

$$(4.2) \quad \tau_h \phi : (x_1, x_2) \in T_h \longrightarrow \phi(x_1, x_2) - \phi\left(x_1 + \frac{a}{h}, x_2\right).$$

We recall the following extension result proved in [10] Lemma 2.2:

**Lemma 4.2.** [10] *Let  $\tau_h$ ,  $h$  in  $\mathbb{N}$ , be defined by (4.2). Then, for every  $h$  in  $\mathbb{N}$ , there exists a linear extension-operator  $Q_h \in \mathcal{L} (H^1(\Omega_h^+), H^1(\Omega^+))$  such that*

$$\|Q_h \phi\|_{H^1(\Omega^+)}^2 \leq c \left( \|\phi\|_{H^1(\Omega_h^+)}^2 + \|\tau_h \phi\|_{H^1(\Omega_h^+)}^2 + h^2 \|\tau_h \phi\|_{L^2(\Omega_h^+)}^2 \right) \quad \forall \phi \in H^1(\Omega_h^+) \quad \forall h \in \mathbb{N},$$

where  $c$  is a constant independent of  $\phi$  and  $h$ .

Proposition 3.1 allows us to adapt the proof of Lemma 2.4 in [10] to our nonlinear case and obtain the following estimate result:

**Lemma 4.3.** *For every  $h$  in  $\mathbb{N}$ , let  $\tau_h$  be defined by (4.2) and  $u_h$ , be a solution of Problem (2.4) with  $f$  in  $(L^\infty(\Omega))^2 \cap (H^1(\Omega))^2$ . Then there exists a constant  $c$  (independent of  $h$ ) such that*

$$\|\tau_h u_h^{(i)}\|_{H^1(\Omega_h^+)} \leq c \frac{1}{h} \quad \forall h \in \mathbb{N} \quad \forall i \in \{1, 2\}.$$

**Proof.** Let us fix  $i$  in  $\{1, 2\}$ .

For every  $h$  in  $\mathbb{N}$  let us define

$$f_h^{(i)} = 2u_h^{(i)} - |u_h|^2 u_h^{(i)} + f_i \quad \text{a.e. in } \Omega_h.$$

Let us observe that  $f_h^{(i)}$  is in  $H^1(\Omega_h)$  since  $f$  is in  $(H^1(\Omega))^2$  and  $u_h$  is in  $(H^1(\Omega_h))^2 \cap (L^\infty(\Omega_h))^2$  by virtue of Lemma 3.2. Moreover, by using Proposition 3.1 and Corollary 3.4 we obtain the existence of a constant  $\bar{c}$ , independent of  $h$ , such that

$$(4.3) \quad \left\| \frac{\partial f_h^{(i)}}{\partial x_1} \right\|_{L^2(\Omega_h)} \leq \bar{c} \quad \forall h \in \mathbb{N}.$$

Since  $u_h^{(i)}$  is the solution of

$$\begin{cases} -\Delta u_h^{(i)} + u_h^{(i)} = f_h^{(i)} & \text{in } \Omega_h, \\ Du_h^{(i)} \cdot \nu = 0 & \text{on } \partial\Omega_h, \end{cases}$$

it turn out that  $\tau_h u_h^{(i)}$  is the solution of

$$(4.4) \quad \begin{cases} -\Delta \tau_h u_h^{(i)} + \tau_h u_h^{(i)} = \tau_h f_h^{(i)} & \text{in } \bigcup_{k=-1,0} ((ak, 0) + \Omega_h), \\ \frac{\partial (\tau_h u_h^{(i)})}{\partial \nu} = 0 & \text{on } \partial \left( \bigcup_{k=-1,0} ((ak, 0) + \Omega_h) \right) - (\{-a, a\} \times ] - b_1, 0[), \\ \tau_h u_h^{(i)}(\cdot, x_2) \text{ is } 2a\text{-periodic in } x_1 & \forall x_2 \in ] - b_1, 0[. \end{cases}$$

By choosing  $\tau_h u_h^{(i)}$  as test function in (4.4) and making use of Prop. IX.3 in [8] and (4.3), it results

$$\begin{aligned} \left\| \tau_h u_h^{(i)} \right\|_{H^1(\Omega_h)} &\leq \left\| \tau_h u_h^{(i)} \right\|_{H^1 \left( \bigcup_{k=-1,0} ((ak, 0) + \Omega_h) \right)} \leq \\ &\leq \left\| \tau_h f_h^{(i)} \right\|_{L^2 \left( \bigcup_{k=-1,0} ((ak, 0) + \Omega_h) \right)} \leq \frac{1}{h} \left\| \frac{\partial f_h^{(i)}}{\partial x_1} \right\|_{L^2 \left( \bigcup_{k=-2}^1 ((ak, 0) + \Omega_h) \right)} \leq \\ &\leq \frac{4}{h} \left\| \frac{\partial f_h^{(i)}}{\partial x_1} \right\|_{L^2(\Omega_h)} \leq \frac{4\bar{c}}{h} \quad \forall h \in \mathbb{N} \end{aligned}$$

and the thesis holds.  $\square$

**Proof of Proposition 4.1.** By choosing  $\phi = u_h^{(i)}$ ,  $i = 1, 2$ , in Lemma 4.2 and making use of Lemma 4.3 and Corollary 3.4 we obtain

$$(4.5) \quad \left\| Q_h u_h^{(i)} \right\|_{H^1(\Omega^+)} \leq c \quad \forall h \in \mathbb{N} \quad \forall i \in \{1, 2\},$$

where  $c$  is a constant independent of  $h$ . The estimate (4.1) follows from (4.5), by setting

$$P_h = (Q_h, Q_h) \quad \forall h \in \mathbb{N}.$$

$\square$

## 5. PROOF OF THEOREM 2.1.

By virtue of Proposition 3.1, Corollary 3.4 and Proposition 4.1 there exist a strictly increasing sequence of positive integer numbers  $\{h_k\}_{k \in \mathbb{N}}$ ,  $u^+$  in  $(H^1(\Omega^+))^2$ ,  $u^-$  in  $(H^1(\Omega^-))^2 \cap (L^\infty(\Omega^-))^2$ ,  $u^*$  in  $(L^\infty(\Omega^+))^2$ ,  $z$  in  $(L^\infty(\Omega))^2$  and  $d, e$  in  $(L^2(\Omega^+))^2$  such that

$$(5.1) \quad P_{h_k} u_{h_k} \rightharpoonup u^+ \quad \text{weakly in } (H^1(\Omega^+))^2 \\ \text{and strongly in } (L^p(\Omega^+))^2 \quad \forall p \in [1, +\infty[,$$

$$(5.2) \quad u_{h_k} \rightharpoonup u^- \quad \text{weakly in } (H^1(\Omega^-))^2 \text{ and weakly } \star \text{ in } (L^\infty(\Omega^-))^2,$$

$$(5.3) \quad \widetilde{u_{h_k}} \rightharpoonup \theta u^* \quad \text{weakly } \star \text{ in } (L^\infty(\Omega^+))^2,$$

$$(5.4) \quad \widetilde{u_{h_k}} |\widetilde{u_{h_k}}|^2 \rightharpoonup z \quad \text{weakly } \star \text{ in } (L^\infty(\Omega))^2,$$

$$(5.5) \quad \frac{\partial \widetilde{u_{h_k}}}{\partial x_1} \rightharpoonup d \quad \text{weakly in } (L^2(\Omega^+))^2,$$

$$(5.6) \quad \frac{\partial \widetilde{u_{h_k}}}{\partial x_2} \rightharpoonup e \quad \text{weakly in } (L^2(\Omega^+))^2,$$

as  $k$  diverges.

Since

$$\chi_{\Omega_{h_k}^+ \cap \Sigma} P_{h_k} u_{h_k} = \chi_{\Omega_{h_k} \cap \Sigma} u_{h_k} \quad \text{a.e. in } \Sigma \quad \forall k \in \mathbb{N},$$

from (2.3), (5.1) and (5.2) it follows that

$$u^+ = u^- \quad \text{a.e. in } \Sigma$$

and consequently

$$(5.7) \quad u = \begin{cases} u^+ & \text{a.e. in } \Omega^+ \\ u^- & \text{a.e. in } \Omega^- \end{cases} \in (H^1(\Omega))^2.$$

On the other hand, since

$$\chi_{\Omega_{h_k}^+} P_{h_k} u_{h_k} = \widetilde{u_{h_k}} \quad \text{a.e. in } \Omega^+ \quad \forall k \in \mathbb{N},$$

from (2.2), (5.1) and (5.3) it follows that

$$(5.8) \quad u^+ = u^* \quad \text{a.e. in } \Omega^+$$

and consequently

$$(5.9) \quad u \in (L^\infty(\Omega))^2,$$

where  $u$  is defined in (5.7).

By combining (5.1) and (5.2) with (5.7) and (5.9) we obtain (2.7).

Now let us prove that

$$(5.10) \quad z = \begin{cases} \theta|u|^2u & \text{a.e. in } \Omega^+, \\ |u|^2u & \text{a.e. in } \Omega^-. \end{cases}$$

To this purpose, first let us verify that

$$(5.11) \quad |P_{h_k}u_{h_k}|^2 \rightarrow |u|^2 \text{ strongly in } L^2(\Omega^+),$$

as  $k$  diverges. In fact, Hölder's inequality provides that

$$\begin{aligned} & \| |P_{h_k}u_{h_k}|^2 - |u|^2 \|_{L^2(\Omega^+)}^2 \leq 2 \int_{\Omega^+} |P_{h_k}u_{h_k} - u|^2 (|P_{h_k}u_{h_k}|^2 + |u|^2) dx \leq \\ & \leq 2 \|P_{h_k}u_{h_k} - u\|_{(L^4(\Omega^+))^2}^2 \left( \|P_{h_k}u_{h_k}\|_{(L^4(\Omega^+))^2}^2 + \|u\|_{(L^4(\Omega^+))^2}^2 \right) \quad \forall k \in \mathbb{N}, \end{aligned}$$

from which, by virtue of (5.1) and definition (5.7), convergence (5.11) follows.

Since

$$\widetilde{u_{h_k}|u_{h_k}|^2} = \widetilde{u_{h_k}}|P_{h_k}u_{h_k}|^2 \text{ a.e. in } \Omega^+ \quad \forall k \in \mathbb{N},$$

by making use of (5.4), (5.3), (5.8), (5.7) and (5.11), we obtain  $z = \theta|u|^2u$  a.e. in  $\Omega^+$ . The identification of  $z$  on  $\Omega^-$  is similar.

By following arguments identical to those used in [12] Proposition 2.2, it is easy to prove that

$$(5.12) \quad e = \theta \frac{\partial u}{\partial x_2} \text{ a.e. in } \Omega^+.$$

Arguing as in [10], let us prove that

$$(5.13) \quad d = 0 \text{ a.e. in } \Omega^+.$$

Let  $\{w_h\}_{h \in \mathbb{N}}$  be a sequence in  $H^1(\Omega^+) \cap L^\infty(\Omega^+)$  such that

$$(5.14) \quad w_h \rightarrow x_1 \text{ strongly in } L^\infty(\Omega^+)$$

as  $h$  diverges and

$$Dw_h = 0 \text{ a.e. in } \Omega_h^+ \quad \forall h \in \mathbb{N}.$$

The existence of such sequence is proved in [10] (see also [12], Lemma 4.3).

From (2.6) it follows that

$$(5.15) \quad \begin{aligned} & \int_{\Omega} (\widetilde{Du_{h_k}}Dv - \widetilde{u_{h_k}}v + |\widetilde{u_{h_k}}|^2\widetilde{u_{h_k}}v) dx = \\ & = \int_{\Omega} \chi_{\Omega_{h_k}} f v dx \quad \forall v \in (H^1(\Omega))^2 \quad \forall k \in \mathbb{N}. \end{aligned}$$

By choosing  $v = w_{h_k}\varphi$ , with  $\varphi$  in  $(C_0^\infty(\Omega^+))^2$ , as test function in (5.15) it results

$$(5.16) \quad \begin{aligned} & \int_{\Omega^+} (w_{h_k}\widetilde{Du_{h_k}}D\varphi - w_{h_k}\widetilde{u_{h_k}}\varphi + w_{h_k}|\widetilde{u_{h_k}}|^2\widetilde{u_{h_k}}\varphi) dx = \\ & = \int_{\Omega^+} \chi_{\Omega_{h_k}} w_{h_k} f \varphi dx \quad \forall \varphi \in (C_0^\infty(\Omega^+))^2 \quad \forall k \in \mathbb{N}. \end{aligned}$$

By passing to the limit, as  $k$  diverges, in (5.16) and by making use of (5.14), (5.5), (5.6), (5.12), (5.3), (5.8), (5.7), (5.4), (5.10) and (2.2), it results

$$(5.17) \quad \begin{aligned} & \int_{\Omega^+} \left( x_1 d \frac{\partial \varphi}{x_1} + x_1 \theta \frac{\partial u}{x_2} \frac{\partial \varphi}{x_2} - x_1 \theta u \varphi + x_1 \theta |u|^2 u \varphi \right) dx = \\ & = \int_{\Omega^+} \theta x_1 f \varphi dx \quad \forall \varphi \in (C_0^\infty(\Omega^+))^2. \end{aligned}$$

On the other hand, by passing to the limit in (5.15) with  $v = x_1 \varphi$  as test function and by making use of (5.5), (5.6), (5.12), (5.3), (5.8), (5.7), (5.4), (5.10) and (2.2) it results

$$(5.18) \quad \begin{aligned} & \int_{\Omega^+} \left( x_1 d \frac{\partial \varphi}{x_1} + d \varphi + x_1 \theta \frac{\partial u}{x_2} \frac{\partial \varphi}{x_2} - x_1 \theta u \varphi + x_1 \theta |u|^2 u \varphi \right) dx = \\ & = \int_{\Omega^+} \theta x_1 f \varphi dx \quad \forall \varphi \in (C_0^\infty(\Omega^+))^2. \end{aligned}$$

By comparing (5.17) with (5.18), we obtain

$$\int_{\Omega^+} d \varphi dx = 0 \quad \forall \varphi \in (C_0^\infty(\Omega^+))^2,$$

which implies (5.13).

By passing to the limit, as  $k$  diverges, in (5.15) and by making use of (5.2), (5.5), (5.13), (5.6), (5.12), (5.3), (5.8), (5.7), (5.4), (5.10) and (2.2) we obtain that  $u$  is a solution of (2.10).

To prove the convergence of the energies, let us choose  $v = u_h$  as test function in (2.6). Then, by virtue of (2.7) and (2.2), we obtain

$$(5.19) \quad \lim_{k \rightarrow +\infty} \int_{\Omega_{h_k}} (|Du_{h_k}|^2 - |u_{h_k}|^2 + |u_{h_k}|^4) dx = \theta \int_{\Omega^+} f u dx + \int_{\Omega^-} f u dx.$$

On the other hand, by choosing  $v = u$  as test function in (2.10), it results

$$(5.20) \quad \begin{aligned} & \theta \int_{\Omega^+} \left( \left| \frac{\partial u}{\partial x_2} \right|^2 - |u|^2 + |u|^4 \right) dx + \int_{\Omega^-} (|Du|^2 - |u|^2 + |u|^4) dx = \\ & = \theta \int_{\Omega^+} f u dx + \int_{\Omega^-} f u dx. \end{aligned}$$

By comparing (5.19) with (5.20), the convergence of the energies (2.9) holds.  $\square$

**Remark 5.1.** Let us observe that the assumption  $f$  in  $(L^\infty(\Omega))^2$  is used to obtain the uniform estimate in Proposition 3.1. This estimate together with the assumption  $f$  in  $(H^1(\Omega))^2$  allows us to obtain (4.3) in order to prove Lemma 4.3 and consequently Proposition 4.1. Thanks to Proposition 4.1, we have the strong convergence (5.1) in  $(L^p(\Omega^+))^2$  for every  $p$  in  $[1, +\infty[$ , which allows us to pass to the limit in the nonlinear term of the Ginzburg-Landau equation (see (5.11)).

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