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Stefano Morra

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Iwasawa modules with extra structures and p -modular representations of GL_2

Stefano Morra

ABSTRACT

Let F be a finite extension of \mathbf{Q}_p . We associate, to certain smooth p -modular representations π of $\mathbf{GL}_2(F)$, a module $\mathfrak{S}(\pi)$ on the mod p Iwasawa of the standard Iwahori subgroup I of $\mathbf{GL}_2(F)$. When F is unramified, we obtain a module on a suitable formally smooth \mathbf{F}_q -algebra, endowed with an action of \mathcal{O}_F^\times (the units in the ring of integers of F) and an \mathcal{O}_F^\times equivariant, Frobenius semilinear endomorphism which turns out to be p -étale. We study the torsion properties of such module, as well as its Iwahori-radical filtration.

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1. Introduction

The p -modular Langlands program. Let F be a p -adic field, \mathcal{O}_F its ring of integers and k_F the residue field. The p -adic Langlands program has the ambition to establish a dictionary between n -dimensional p -adic Galois representation of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/F)$ and certain p -adic Banach space representations of $\mathrm{GL}_n(F)$. Such correspondence is expected to be compatible with mod p -reduction of coefficients and to be realized in the cohomology of suitable Shimura varieties.

This correspondence is nowadays well understood for the particular case of $\mathbf{GL}_2(\mathbf{Q}_p)$. The first breakthrough was Breuil's classification of the p -modular supercuspidal representations of $\mathbf{GL}_2(\mathbf{Q}_p)$

(cf. [Bre]) and the intertwinings between them (yielding a natural parametrization of their isomorphism classes by means of irreducible 2-dimensional Galois representations). The second breakthrough was the realization, by Colmez, of a *functor* from smooth admissible p -modular representations of $\mathbf{GL}_2(\mathbf{Q}_p)$ to Fontaine's (φ, Γ) -modules ([Col], §VI).

The last years have experienced an extensive research of a p -modular correspondence for \mathbf{GL}_2 over finite extensions of \mathbf{Q}_p . The most striking phenomenon is the proliferation of supercuspidal representations (as showed by the work of Breuil and Paskunas [BP] and Hu [Hu]), which does not seem to find any justification on the Galois side (the irreducible Galois representations of $\text{Gal}(\overline{\mathbf{Q}}_p/F)$ are well known since the work of Serre [Se72]). Moreover, by the recent work of Schraen (cf. [Sch]), supercuspidal representations are not of finite presentations if F is a quadratic extension of \mathbf{Q}_p .

Although many problems in the category of smooth p -modular representations of $\mathbf{GL}_2(F)$ are extremely delicate, the investigations of the last years showed that their approach by Iwasawa theoretical methods can be extremely fruitful. It is the case for the faithfulness of the action of the group algebra associated to $\begin{bmatrix} 1 & \mathcal{O}_F \\ 0 & 1 \end{bmatrix}$ on irreducible supercuspidal representations (cf. [HMS]) or their presentation properties as G -modules (cf. [Sch]). Indeed, even Colmez's Montreal functor can be seen as principle to associate an Iwasawa module to a smooth admissible p -modular representation of $\mathbf{GL}_2(\mathbf{Q}_p)$.

The aim of this paper is to develop this approach, describing a way to associate to a universal p -modular representation of \mathbf{GL}_2 an Iwasawa module over a power series ring of characteristic p , endowed with commuting semilinear actions of \mathcal{O}_F^\times and a Frobenius morphism \mathcal{F} , and study some of its properties when F is unramified.

It turns out that such module is (almost always) torsion free over the Iwasawa algebra of $\begin{bmatrix} 1 & 0 \\ p\mathcal{O}_F & 1 \end{bmatrix}$ but, at the same time, its quotients by certain non-zero sub-modules have dense torsion.

Description of the main results. We give now a more precise account of the results contained in this paper. All representations are smooth, over k -linear spaces, where k is a (sufficiently large) finite extension of k_F . By the classical results of Barthel and Livné [BL94] a *supersingular* representation π of $\mathbf{GL}_2(F)$ is (up to twist) an admissible quotient of an explicit universal representation $\pi(\sigma, 0)$. The latter is defined to be the cokernel of a suitable Hecke operator, acting on the compact induction $\text{ind}_{\mathbf{GL}_2(\mathcal{O}_F)F^\times}^{\mathbf{GL}_2(F)}(\sigma)$ where σ is an irreducible smooth representation of $\mathbf{GL}_2(\mathcal{O}_F)F^\times$ with trivial action of the uniformizer $\varpi \in F^\times$ (*Serre weight*).

From the results of [Mo1] the representation theoretic properties of $\pi(\sigma, 0)$ are controlled by an explicit sub- I -representation $R_{\infty,0}^- \oplus R_{\infty,1}^-$ of $\pi(\sigma, 0)$, where I is the standard Iwahori subgroup of $\mathbf{GL}_2(\mathcal{O}_F)$:

THEOREM 1.1 ([Mo1], Theorems 1.1 and 1.2). *There is a canonical $\mathbf{GL}_2(\mathcal{O}_F)F^\times$ -isomorphism $\pi(\sigma, 0)|_{\mathbf{GL}_2(\mathcal{O}_F)F^\times} \cong R_{\infty,0} \oplus R_{\infty,1}$ where the representations $R_{\infty,0}, R_{\infty,1}$ fit in the following exact sequences of $k[\mathbf{GL}_2(\mathcal{O}_F)]$ -modules:*

$$\begin{aligned} 0 \rightarrow V_1 \rightarrow \text{ind}_I^{\mathbf{GL}_2(\mathcal{O}_F)}(R_{\infty,0}^-) \rightarrow R_{\infty,0} \rightarrow 0 \\ 0 \rightarrow V_2 \rightarrow \text{ind}_I^{\mathbf{GL}_2(\mathcal{O}_F)}(R_{\infty,1}^-) \rightarrow R_{\infty,1} \rightarrow 0 \end{aligned}$$

for suitable explicit subquotients V_1, V_2 of a finite parabolic induction from a smooth character of the Iwahori subgroup, depending on σ .

Moreover the I -restriction of $R_{\infty,0}, R_{\infty,1}$ fit in the exact sequences

$$\begin{aligned} 0 \rightarrow W_1 \rightarrow (R_{\infty,1}^-)^s \oplus R_{\infty,0}^- \rightarrow R_{\infty,0}|_I \rightarrow 0 \\ 0 \rightarrow W_2 \rightarrow (R_{\infty,0}^-)^s \oplus R_{\infty,1}^- \rightarrow R_{\infty,1}|_I \rightarrow 0 \end{aligned}$$

where W_1, W_2 are appropriate 1-dimensional $k[I]$ -modules and the action of the element $\begin{bmatrix} 0 & 1 \\ \varpi & 0 \end{bmatrix}$ on the universal representation $\pi(\sigma, 0)$ induces the $k[I]$ -equivariant involution

$$\begin{aligned} (R_{\infty,1}^-)^s \oplus R_{\infty,0}^- &\xrightarrow{\sim} \left((R_{\infty,0}^-)^s \oplus R_{\infty,1}^- \right)^s \\ (v_1, v_2) &\longmapsto (v_2, v_1) \end{aligned}$$

which restricts to an isomorphism $W_1 \xrightarrow{\sim} W_2^s$.

Here, the notation $(*)^s$ means that we are considering the action of I on $*$ obtained by conjugation by the element $\begin{bmatrix} 0 & 1 \\ \varpi & 0 \end{bmatrix}$ (which normalizes I). Furthermore, the representations $R_{\infty,0}^-, R_{\infty,1}^-$ admit an explicit construction, by ‘‘cutting out’’ the positive part of the tree associated to $\mathbf{GL}_2(F)$ (see [Mo1], §3 for more details).

The universal Iwasawa module and its torsion properties. Let $\mathfrak{S}_{\infty}^0, \mathfrak{S}_{\infty}^1$ be the Pontryagin duals of $R_{\infty,0}^-, R_{\infty,1}^-$ respectively. They are profinite modules over the Iwasawa algebra $k[[I]]$ of I . By restriction, they can equivalently be seen as modules for the Iwasawa algebra A associated to the p -adic analytic group $U_0^- \stackrel{\text{def}}{=} \begin{bmatrix} 1 & 0 \\ \varpi \mathcal{O}_F & 1 \end{bmatrix}$, endowed with continuous actions of the groups

$$\Gamma \stackrel{\text{def}}{=} \begin{bmatrix} 1 & 0 \\ 0 & 1 + \varpi \mathcal{O}_F \end{bmatrix}, \quad U_0^+ \stackrel{\text{def}}{=} \begin{bmatrix} 1 & \mathcal{O}_F \\ 0 & 1 \end{bmatrix}, \quad \text{and } H \stackrel{\text{def}}{=} \left\{ \begin{bmatrix} [a] & 0 \\ 0 & [d] \end{bmatrix}, a, d \in k_F^\times \right\}$$

(here, $[\cdot] : k_F^\times \rightarrow \mathcal{O}_F^\times$ is the Teichmüller character). Since Γ, H normalize U_0^- one sees that their actions on $\mathfrak{S}_{\infty}^0, \mathfrak{S}_{\infty}^1$ are *semilinear*.

The first result of this paper is a precise description of $\mathfrak{S}_{\infty}^\bullet$ (for $\bullet \in \{0, 1\}$), as a limit of a projective system of finite dimensional A -modules endowed with continuous actions of Γ, H, U_0^+ (i.e. finite dimensional $k[[I]]$ -modules). From now on, we assume F to be *unramified* over \mathbf{Q}_p and we write $\underline{r} \in \{0, \dots, p-1\}^f$ for the f -tuple parametrizing the isomorphism class of $\sigma|_{\mathrm{SL}_2(k_F)}$.

THEOREM 1.2 (Proposition 3.5). *Let $\bullet \in \{0, 1\}$. The Iwasawa module $\mathfrak{S}_{\infty}^\bullet$ is obtained as the limit of a projective system of finite length Iwasawa modules $\{\mathfrak{S}_{n+1}^\bullet\}_{n \in 2\mathbf{N}_{-1+\bullet}}$ where, for all $n \in 2\mathbf{N}_{-1+\bullet}$, $n \geq 2$, the transition morphism $\mathfrak{S}_{n+1}^\bullet \rightarrow \mathfrak{S}_{n-1}^\bullet$ fit into the following commutative diagram of Iwasawa modules:*

$$\begin{array}{ccc} \mathfrak{S}_{n-1}^\bullet \hookrightarrow A/\langle X_i^{p^{n-2}(r_{i+n-2+1})}, i = 0, \dots, f-1 \rangle & & \downarrow 1 \\ & \downarrow & \downarrow \\ & A/\langle X_i^{p^{n-1}(r_{i+n-1+1})}, i = 0, \dots, f-1 \rangle & \prod_{i=0}^{f-1} X_i^{p^{n-2}(p(r_{i+n-1+1})-(r_{i+n-2+1}))} \\ & \uparrow \text{proj}_{n+1} & \\ \mathfrak{S}_{n+1}^\bullet \hookrightarrow A/\langle X_i^{p^n(r_{i+n+1})}, i = 0, \dots, f-1 \rangle & & \\ \uparrow & & \uparrow \\ \ker(\text{proj}_{n+1}) & \xlongequal{\quad} & \ker(\text{proj}_{n+1}) \end{array} \tag{1}$$

where the left vertical complex is exact and proj_{n+1} denotes the natural projection.

We precise briefly the content of Theorem 1.2. The Iwasawa algebra A can be seen, by the Iwahori and Mackey decomposition, as a $k[[I]]$ -module. We recall that A is a complete local regular k algebra and we determine (Lemma 3.2) a regular system of parameters $X_0, \dots, X_{f-1} \in A$, which give rise to a system of eigenvectors for the action of H on the tangent space of A . All the morphisms in the diagram (1) are $k[[I]]$ -linear and it is shown (§3) that the ideals of A of the form $\langle X_i^{p^n(r_{i+n}+1)}, i = 0, \dots, f-1 \rangle_A$ (where the indices $i+n$ appearing in r_{i+n} are understood to be element in $\mathbf{Z}/f\mathbf{Z}$) are stable under the actions of Γ, H, U_0^+ .

Moreover, we can describe precisely the monomorphisms

$$\mathfrak{S}_{n+1}^\bullet \hookrightarrow A / \langle X_i^{p^n(r_{i+n}+1)}, i = 0, \dots, f-1 \rangle,$$

deducing an explicit family of A -generators $\mathcal{G}_{n+1}^\bullet$ for $\mathfrak{S}_{n+1}^\bullet$.

The families $\mathcal{G}_{n+1}^\bullet$ are compatible with the transition maps (the image of $\mathcal{G}_{n+1}^\bullet$ via the transition epimorphism $\mathfrak{S}_{n+1}^\bullet \twoheadrightarrow \mathfrak{S}_{n-1}^\bullet$ is $\mathcal{G}_{n-1}^\bullet$, for all $n \in 2N+1+\bullet$), yielding a set $\mathcal{G}_\infty^\bullet$ of topological A -generators for $\mathfrak{S}_\infty^\bullet$ which is finite if and only if $F = \mathbf{Q}_p$ (in which case is a one-point set). In other words we have an A -linear (and H -equivariant) continuous epimorphism

$$\prod_{e \in \mathcal{G}_\infty^\bullet} A \cdot e \twoheadrightarrow \mathfrak{S}_\infty^\bullet \quad (2)$$

and the next step is to investigate the torsion properties of $\mathfrak{S}_\infty^\bullet$:

THEOREM 1.3 (Propositions 7.6, 7.7, 7.9). *Let $\bullet \in \{0, 1\}$.*

If either f is odd or f is even and $\underline{r} \neq (\dots, 0, p-1, 0, p-1, \dots)$ the module $\mathfrak{S}_\infty^\bullet$ is torsion free over A and contains a dense A -sub-module of rank one over $\text{Frac}(A)$.

If f is even and $\underline{r} = (\dots, 0, p-1, 0, p-1, \dots)$ then the torsion sub-module of $\mathfrak{S}_\infty^\bullet$ is dense in $\mathfrak{S}_\infty^\bullet$.

Finally, if $x \in \mathfrak{S}_\infty^\bullet \setminus \{0\}$ is in the image of $\bigoplus_{e \in \mathcal{G}_\infty^\bullet} A \cdot e \rightarrow \mathfrak{S}_\infty^\bullet$ and M_x is the A -sub-module of $\mathfrak{S}_\infty^\bullet$ generated by x , then the torsion sub-module of $\mathfrak{S}_\infty^\bullet/M_x$ is dense in $\mathfrak{S}_\infty^\bullet/M_x$.

We recall that for an A -module M we define its *rank* over $\text{Frac}(A)$ as the dimension of the vector space $M \otimes \text{Frac}(A)$. We remark that it is not clear, a priori, that the module $\mathfrak{S}_\infty^\bullet$ is of rank one over $\text{Frac}(A)$ (and this should indeed be *false*, [Sch1]).

Even if $\mathfrak{S}_\infty^\bullet$ is not of finite type over A (unless $F = \mathbf{Q}_p$) it is possible to determine an Iwasawa sub-module of finite co-length, which is finitely generated over an appropriate skew power series ring. More precisely, A is endowed with a Frobenius endomorphism $\mathcal{F} : A \rightarrow A$, which is k -linear, Γ, H -equivariant and characterized by the condition $\mathcal{F}(X_i) = X_{i-1}^p$.

We set $\mathfrak{S}_\infty^{\geq 1} \stackrel{\text{def}}{=} \ker(\mathfrak{S}_\infty^0 \twoheadrightarrow \mathfrak{S}_0^0)$ and, similarly, $\mathfrak{S}_\infty^{\geq 2} \stackrel{\text{def}}{=} \ker(\mathfrak{S}_\infty^1 \twoheadrightarrow \mathfrak{S}_1^1)$; the result is then the following:

THEOREM 1.4 (Proposition 5.12, 5.13). *The module $\mathfrak{S}_\infty^{\geq 1} \oplus \mathfrak{S}_\infty^{\geq 2}$ is an Iwasawa sub-module of $\mathfrak{S}_\infty^0 \oplus \mathfrak{S}_\infty^1$ of finite co-length endowed with an \mathcal{F} -semilinear, Γ, H -equivariant endomorphism \mathcal{F}_∞ .*

The topological A -linearization of \mathcal{F}_∞

$$A \widehat{\otimes}_{\mathcal{F}, A} (\mathfrak{S}_\infty^{\geq 1} \oplus \mathfrak{S}_\infty^{\geq 2}) \xrightarrow{id \otimes \mathcal{F}_\infty} \mathfrak{S}_\infty^{\geq 1} \oplus \mathfrak{S}_\infty^{\geq 2}$$

has an image of finite co-length.

Finally, $\mathfrak{S}_\infty^{\geq 1} \oplus \mathfrak{S}_\infty^{\geq 2}$ (resp. $\mathfrak{S}_\infty^{\geq \bullet+1}$) admits a finite family of generators as a module over the skew power series ring $A[[\mathcal{F}]]$ (resp. $A[[\mathcal{F}^2]]$).

The family of $A[[\mathcal{F}]]$ -generators of the module $\mathfrak{S}_\infty^{\geq 1} \oplus \mathfrak{S}_\infty^{\geq 2}$ can be explicitly described as a lift of the A -generators of the subquotient $\ker(\mathfrak{S}_3^1 \rightarrow \mathfrak{S}_1^1)$ (cf. Corollary 5.13 and Proposition 5.12). If the f -tuple \underline{r} verifies $r_{i-1} - r_i < p - 1$ for all i such family has cardinality f (otherwise, the cardinality is strictly smaller, cf. Corollary 5.13 and Definition 3.6, where we give a detailed description of the A generators of the subquotients of the Iwasawa module).

We refer the reader to the paper [Ven], §2 for the definitions and basic properties of the skew power series ring $A[[\mathcal{F}]]$.

The case $F = \mathbf{Q}_p$. If $F = \mathbf{Q}_p$ we have a precise Galois theoretic description of the Iwasawa module $\mathfrak{S}_\infty^{\geq 1} \oplus \mathfrak{S}_\infty^{\geq 2}$ in terms of Wach modules.

Fix an embedding $k \hookrightarrow \overline{\mathbf{F}}_p$ and let $\omega_2 : G_{\mathbf{Q}_{p^2}} \rightarrow k$ be a choice for the Serre fundamental character of niveau 2, where $G_{\mathbf{Q}_{p^2}}$ is the absolute Galois group of the quadratic unramified extension \mathbf{Q}_{p^2} of \mathbf{Q}_p . For $0 \leq r \leq p-1$ we write $\mathrm{ind}(\omega_2^{r+1})$ for the unique (absolutely) irreducible $G_{\mathbf{Q}_p}$ -representation whose restriction on the inertia $I_{\mathbf{Q}_p}$ is described by $\omega_2^{r+1} \oplus \omega_2^{p(r+1)}$ and whose determinant is ω^{r+1} (where ω is the mod p -cyclotomic character).

Under the p -modular Langlands correspondence for $\mathbf{GL}_2(\mathbf{Q}_p)$, defined in [Bre], Definition 4.2.4 and realized in full generality by Colmez ([Col]), the Galois representation $\mathrm{ind}(\omega_2^{r+1})$ is associated to the supersingular representation $\pi(\sigma_r, 0)$ (see §1.1 for the precise definition of the Serre weight σ_r).

In section 7.3 we verify that the \mathcal{F}_∞ -module $\mathfrak{S}_\infty^{\geq 1} \oplus \mathfrak{S}_\infty^{\geq 2}$ associated to $\pi(\sigma_r, 0)$ is compatible with the p -modular Langlands correspondence for $\mathbf{GL}_2(\mathbf{Q}_p)$. Indeed, the explicit description of the elements in $\mathcal{G}_\infty^\bullet$ lets us control in detail the \mathcal{F}_∞ -action on $\mathfrak{S}_\infty^{\geq 1} \oplus \mathfrak{S}_\infty^{\geq 2}$.

It is therefore easy to compare such \mathcal{F}_∞ -action with the Frobenius action on the Wach modules associated to crystalline Galois representations and we get:

PROPOSITION 1.5 (Proposition 7.11). *Let $0 \leq r \leq p-1$, and write $\chi_{(0,1)}$ for the crystalline character of $G_{\mathbf{Q}_{p^2}}$ such that $\chi_{(0,1)}(p) = 1$ and with labelled Hodge-Tate weights $-(0, 1)$ (for a choice of an embedding $\mathbf{Q}_{p^2} \hookrightarrow \overline{\mathbf{Q}}_p$). Define the crystalline representation $V_{r+1} \stackrel{\mathrm{def}}{=} \mathrm{ind}_{G_{\mathbf{Q}_{p^2}}}^{G_{\mathbf{Q}_p}} \chi_{(0,1)}^{r+1}$.*

Then we have an isomorphism of φ -modules

$$\mathfrak{S}_\infty^{\geq 1} \oplus \mathfrak{S}_\infty^{\geq 2} \xrightarrow{\sim} \mathbf{N}(\overline{V}_{r+1})$$

where $\mathbf{N}(\overline{V}_{r+1})$ is the mod- p reduction of the Wach module associated to V_{r+1} and $\mathfrak{S}_\infty^{\geq 1} \oplus \mathfrak{S}_\infty^{\geq 2}$ is the Iwasawa module of Theorem 1.4 (endowed with the φ action defined by \mathcal{F}_∞).

We recall that the mod p -reduction of V_{r+1} is the dual of $\mathrm{ind}(\omega_2^{r+1})$, in particular the statement of Proposition 1.5 is consistent with the p -modular Langlands correspondence for $\mathbf{GL}_2(\mathbf{Q}_p)$.

Radical filtration on the universal Iwasawa module. We finally focus our interest on the radical filtration for the module $\mathfrak{S}_\infty^\bullet$. Indeed, even if we have the surjection (2) we point out that none of the composite morphisms $A \rightarrow \mathfrak{S}_\infty^\bullet$ (which are almost always monomorphisms) are equivariant for the extra actions of Γ , U_0^+ .

Our results give a new, much simplified proof of the main theorems of [Mo2] describing the socle filtration for the representations $R_{\infty, \bullet}^-$, avoiding almost completely manipulations on Witt vectors.

The first result in this direction is the proof that the A -radical filtration on A is stable by the actions of Γ , U_0^+

PROPOSITION 1.6 (Corollary 4.5). *For any $k \in \mathbf{N}$ the k -th power \mathfrak{m}^k of the maximal ideal \mathfrak{m} of A is endowed with a continuous action of Γ , U_0^+ , which is trivial on the quotient*

$$\mathfrak{m}^k / \mathfrak{m}^{k+(p-1)}.$$

In particular, the $k[[I]]$ -radical filtration on A (seen as a $k[[I]]$ -profinite module) is described by the A -radical filtration.

Since $\mathfrak{S}_\infty^\bullet|_A$ is free of rank one if $F = \mathbf{Q}_p$, Proposition 1.6 together with Theorem 1.1 give another proof of [Bre], Théorème 3.2.4 :

THEOREM 1.7 ([Bre], Théorème 3.2.4, Corollaire 4.1.4). *Assume $F = \mathbf{Q}_p$ and write $I(1)$ for the pro- p Sylow subgroup of the Iwahori I . For any $r \in \{0, \dots, p-1\}$ we have*

$$\dim (\pi(r, 0)^{I(1)}) = 2.$$

The action of Γ being by k -algebra endomorphisms, the main difficulty to deduce Proposition 1.6 consists in the control of the U_0^+ -action; this is done by a delicate induction argument (Proposition 4.4). The statement of Proposition 1.6 is expected to be false as soon as F ramifies over \mathbf{Q}_p : the $k[[I]]$ -radical filtration on A does not coincide with its A -radical filtration.

Similarly as we did in the paper [Mo2], the next step is to control the action of $k[[I]]$ on the graded pieces $\mathcal{K}er_{n+1} \stackrel{\text{def}}{=} \ker (\mathfrak{S}_{n+1}^\bullet \rightarrow \mathfrak{S}_{n-1}^\bullet)$. The result is the following:

PROPOSITION 1.8 (Proposition 6.1). *Let $n \geq 2$. For any $k \geq 0$ the A -sub-module $\mathfrak{m}^k \mathcal{K}er_{n+1}$ is endowed with a discrete action of Γ , U_0^+ , which is trivial on the quotient*

$$\mathfrak{m}^k \mathcal{K}er_{n+1} / \mathfrak{m}^{k+(p-1)} \mathcal{K}er_{n+1}.$$

In particular, the $k[[I]]$ -radical filtration on $\mathcal{K}er_{n+1}$ (seen as a $k[[I]]$ -profinite module) is described by its A -radical filtration.

We can state a similar but less precise result for $\mathcal{K}er_2$, depending on some weak regularity conditions on the Serre weight σ . This discrepancy reflects a similar phenomenon on the representation theoretic side, for a certain, “small”, sub I -representation of $R_{\infty,0}^-$ (cf. section 5.1.2 and the representation $(R_1/R_0)^+$ in [Mo2]).

As for Proposition 1.6, the main difficulty in Proposition 1.8 is the control of the action of U_0^+ (again we need an inductive argument) and we use in a crucial way some of the properties of the Frobenius \mathcal{F} on A .

The last step in order to recover the $k[[I]]$ -radical filtration on $\mathfrak{S}_\infty^\bullet$ consists in an appropriate “gluing” of the filtrations obtained by Proposition 1.8 on the subquotients $\{\mathcal{K}er_{n+1}\}_{n \in 2\mathbf{N}+1+\bullet}$. The argument is now mainly formal (as happened for the representation theoretic approach), based on general estimates on the A -radical filtration for $\mathfrak{S}_{n+1}^\bullet$.

THEOREM 1.9 (Proposition 7.1). *Let $\bullet \in \{0, 1\}$ and, for $k \in \mathbf{N}$, write \mathcal{J}_k for the closure of $\mathfrak{m}^k \mathfrak{S}_\infty^\bullet$ in $\mathfrak{S}_\infty^\bullet$. Assume for simplicity that $|r_{i_1} - r_{i_2}| < p-1$ for all $i_1, i_2 \in \{0, \dots, p-1\}$.*

Then the A -linear filtration $\{\mathcal{J}_k\}_{k \in \mathbf{N}}$ coincides with the $k[[I]]$ -radical filtration on $\mathfrak{S}_\infty^\bullet$.

We remark that we can find an explicit sub-module $\mathfrak{S}_\infty^{\geq 3} \stackrel{\text{def}}{=} \ker (\mathfrak{S}_\infty^0 \rightarrow \mathfrak{S}_2^0)$ for which the result of Theorem 1.9 can be strengthened, without assumptions on the Serre weight σ :

THEOREM 1.7'. *For $k \in \mathbf{N}$, write \mathcal{J}_k for the closure of $\mathfrak{m}^k \mathfrak{S}_\infty^{\geq 3}$ in $\mathfrak{S}_\infty^{\geq 3}$ (resp. for the closure of $\mathfrak{m}^k \mathfrak{S}_\infty^1$ in \mathfrak{S}_∞^1).*

The A -sub-module \mathcal{I}_k is endowed with a continuous action of Γ , U_0^+ , which is trivial on the quotient

$$\mathcal{I}_k / \mathcal{I}_{k+(p-1)}.$$

In particular, we are able to describe the isotypical components of for $\mathrm{cosoc}_{k[[I]]}(\mathfrak{S}_\infty^\bullet)$, the $k[[I]]$ -cosocle of $\mathfrak{S}_\infty^\bullet$. If χ is an irreducible $k[[I]]$ -module, we write $V(\chi)$ to denote the χ -isotypical component of the $k[[I]]$ -cosocle of $\mathfrak{S}_\infty^\bullet$ and the result is the following:

COROLLARY 1.10 (Corollary 7.3). *Assume that either f is odd or f is even and $\underline{r} \neq (\dots, 0, p-1, 0, p-1, \dots)$. Then*

$$\begin{aligned} \mathrm{cosoc}_{k[[I]]}(\mathfrak{S}_\infty^0) &= V(\chi_{-\underline{r}}) \oplus \bigoplus_{i=0}^{f-1} V(\chi_{\underline{r}} \det^{-\underline{r}} \mathbf{a}^{-p^i(r_i+1)}) \\ \mathrm{cosoc}_{k[[I]]}(\mathfrak{S}_\infty^1) &= V(\chi_{\underline{r}} \det^{-\underline{r}}) \oplus \bigoplus_{i=0}^{f-1} V(\chi_{\underline{r}} \det^{-\underline{r}} \mathbf{a}^{-p^i(r_i+1)}) \end{aligned}$$

where

$$\begin{aligned} \dim(V(\chi_{-\underline{r}})) &= \dim(V(\chi_{\underline{r}} \det^{-\underline{r}})) = 1, \\ \dim(V(\chi_{\underline{r}} \det^{-\underline{r}} \mathbf{a}^{-p^i(r_i+1)})) &= \begin{cases} \infty & \text{for all } i \in \{0, \dots, f-1\} \text{ if } F \neq \mathbf{Q}_p \\ 0 & \text{for all } i \in \{0, \dots, f-1\} \text{ if } F = \mathbf{Q}_p. \end{cases} \end{aligned}$$

We have a similar result for the special case where f is even and $\underline{r} = (\dots, 0, p-1, 0, p-1, \dots)$ (cf. Corollary 7.3). Here, $\chi_{\underline{r}}$, \mathbf{a} are the smooth characters of I characterized by

$$\chi_{\underline{r}} \left(\begin{bmatrix} [a] & 0 \\ 0 & [d] \end{bmatrix} \right) = a^{\sum_{i=0}^{f-1} p^i r_i}, \quad \mathbf{a} \left(\begin{bmatrix} [a] & 0 \\ 0 & [d] \end{bmatrix} \right) = ad^{-1}.$$

Organisation of the paper. The paper is organized as follow.

In section 2 we recall the structure theorems for universal p -modular representations of \mathbf{GL}_2 (Theorem 2.1), describing the construction of the representations $R_{\infty,0}^-, R_{\infty,1}^-$ as it appears in [Mo1], §3.

We subsequently dualize these constructions in §3. After recalling the main formal properties of Pontryagin duality for compact p -adic analytic groups, we determine the dual of a Serre weight (§3.2), thanks to an appropriate choice of a regular system of parameters for the Iwasawa algebra A . The description of the universal modules $\mathfrak{S}_\infty^\bullet$ follows finally from the construction of the Hecke operator T (3.3). The main result is Proposition 3.5, where we give a precise account of $\mathfrak{S}_\infty^\bullet$ as a $k[[I]]$ -module.

Section 4 is devoted to the investigation of the A -radical filtration on A with respect to the extra action of the groups Γ , U_0^+ and the main result is Corollary 4.5.

In §5 we study the Frobenius \mathcal{F} on A and its relations with the universal modules $\mathfrak{S}_\infty^\bullet$. After its formal definition (as a morphism between certain Iwasawa modules, §5.1) and its first properties, we recall the constructions of [Ven] on the skew power series ring $A[[\mathcal{F}]]$ (§5.1.1). We subsequently deduce, in §5.2, the behavior of \mathcal{F} with respect to certain modules (associated to the projective system defining $\mathfrak{S}_\infty^\bullet$) and we conclude (section 5.3) with the construction of a Frobenius, with a p -étale action, on an appropriate sub-module of $\mathfrak{S}_\infty^\bullet$ of finite co-length. Moreover, we show that such sub-module is of finite type over the skew power series ring $A[[\mathcal{F}]]$.

Section 6 is concerned with the $k[[I]]$ -radical filtration for certain subquotients Ker_{n+1} of $\mathfrak{S}_\infty^\bullet$. The techniques are similar to those of §4 and the new ingredient (in order to control the action of

U_0^+) is the crucial use of the properties of the Frobenius. We remark that the behavior of \mathcal{Ker}_{n+1} is different for $n = 1$ and $n \geq 2$. The main result is Proposition 6.1.

Finally, the results of §6 and §4 are used in section 7.1 in order to recover the $k[[I]]$ -radical filtration on $\mathfrak{S}_\infty^\bullet$ (7.1). In section 7.2 we conclude describing the torsion properties of the universal module $\mathfrak{S}_\infty^\bullet$.

The paper ends (§8) with a brief comment on the parallel constructions for the principal and special series representations for $\mathbf{GL}_2(F)$, where all the results are much simplified.

1.1 Notation

Let p be an odd prime. We consider a p adic field F , with ring of integers \mathcal{O}_F , uniformizer ϖ and residue field k_F . We assume that $[k_F : \mathbf{F}_p] = f$ is finite. We write $val : F \rightarrow \mathbf{Z}$ for the valuation on F , normalized by $val(\varpi) = 1$, $x \mapsto \bar{x}$ for the reduction morphism $\mathcal{O}_F \rightarrow k_F$ and $\bar{x} \mapsto [\bar{x}]$ for the Teichmüller lift $k_F^\times \rightarrow \mathcal{O}_F^\times$ (we set $[0] \stackrel{\text{def}}{=} 0$).

Consider the general linear group \mathbf{GL}_2 . We fix the maximal torus \mathbf{T} of diagonal matrices and the unipotent radical \mathbf{U} of upper unipotent matrices, so that $\mathbf{B} \stackrel{\text{def}}{=} \mathbf{T} \ltimes \mathbf{U}$ is the Borel subgroup of upper triangular matrices. We write \mathcal{T} for the Bruhat-Tits tree associated to $\mathbf{GL}_2(F)$ (cf. [Ser77]) and we consider the hyperspecial maximal compact subgroup $K \stackrel{\text{def}}{=} \mathbf{GL}_2(\mathcal{O}_F)$.

The following subgroups of K will play an important role in this article:

$$U_0^- \stackrel{\text{def}}{=} \begin{bmatrix} 1 & 0 \\ \varpi \mathcal{O}_F & 1 \end{bmatrix}, \quad \Gamma \stackrel{\text{def}}{=} \begin{bmatrix} 1 & 0 \\ 0 & 1 + \varpi \mathcal{O}_F \end{bmatrix}, \quad U_0^+ \stackrel{\text{def}}{=} \mathbf{U}(\mathcal{O}_F).$$

The natural reduction map $\mathbf{T}(\mathcal{O}_F) \rightarrow \mathbf{T}(k_F)$ has a section (induced by the Teichmüller lift) and we define H to be its image.

For notational convenience, we introduce the following objects

$$\omega \stackrel{\text{def}}{=} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathbf{GL}_2(F), \quad \alpha \stackrel{\text{def}}{=} \begin{bmatrix} 0 & 1 \\ \varpi & 0 \end{bmatrix} \in \mathbf{GL}_2(F), \quad K_0(\varpi) \stackrel{\text{def}}{=} \text{red}^{\leftarrow}(\mathbf{B}(k_F))$$

(where $\text{red} : K \rightarrow \mathbf{GL}_2(k_F)$ is the reduction morphism).

Let E be a p -adic field, with ring of integers \mathcal{O} and finite residue field k (the ‘‘coefficient field’’). Up to enlarging E , we can assume that $\text{Card}(\text{Hom}_{\mathbf{F}_p}(k_F, k)) = [k_F : \mathbf{F}_p]$.

A representation σ of a subgroup H_1 of $\mathbf{GL}_2(\mathbf{Q}_p)$ is always understood to be smooth with coefficients in k . If $h \in H_1$ we will sometimes write $\sigma(h)$ to denote the k -linear automorphism induced by the action of h on the underlying vector space of σ . We denote by $(\sigma)^{H_1}$ the linear space of H_1 invariant vectors of σ and by $(\sigma)_{H_1}$ the linear space of H_1 co-invariant vectors.

Let $H_2 \leq H_1$ be compact open subgroups of K . For a smooth representation σ of H_2 we write $\text{ind}_{H_2}^{H_1} \sigma$ to denote the (compact) induction of σ from H_2 to H_1 . If $v \in \sigma$ and $h \in H_1$ we write $[h, v]$ for the unique element of $\text{ind}_{H_2}^{H_1} \sigma$ supported in $H_2 h^{-1}$ and sending h to v . We deduce in particular the following equalities:

$$h' \cdot [h, v] = [h' h, v], \quad [h k, v] = [h, \sigma(k)v] \tag{3}$$

for any $h' \in H_1, k \in H_2$.

If $Z \cong F^\times$ is the center of $\mathbf{GL}_2(F)$ and σ is a representation of KZ we will similarly write $\text{ind}_{KZ}^{\mathbf{GL}_2(F)} \sigma$ for the subspace of the full induction $\text{Ind}_{KZ}^{\mathbf{GL}_2(F)} \sigma$ consisting of functions which are compactly supported modulo the center Z (cf. [Bre], §2.3). For $g \in \mathbf{GL}_2(F), v \in \sigma$ we use the same notation $[g, v]$ for the element of $\text{ind}_{KZ}^{\mathbf{GL}_2(F)} \sigma$ having support in $KZ g^{-1}$ and sending g to v ; the element $[g, v]$ verifies similar compatibility relations as in (3).

A *Serre weight* is an absolutely irreducible representation of K . Up to isomorphism they are of the form

$$\bigotimes_{\tau \in \mathrm{Gal}(k_F/\mathbf{F}_p)} (\det^{t_\tau} \otimes_{k_F} \mathrm{Sym}^{r_\tau} k_F^2) \otimes_{k_{F,\tau}} k \quad (4)$$

where $r_\tau, t_\tau \in \{0, \dots, p-1\}$ for all $\tau \in \mathrm{Gal}(k_F/\mathbf{F}_p)$ and $t_\tau < p-1$ for at least one τ . This gives a bijective parametrization of isomorphism classes of Serre weights by $2f$ -tuples of integers $r_\tau, t_\tau \in \{0, \dots, p-1\}$ such that $t_\tau < p-1$ for some τ . The Serre weight characterized by $t_\tau = 0, r_\tau = p-1$ for all $\tau \in \mathrm{Gal}(k_F/\mathbf{F}_p)$ will be referred as the *Steinberg weight* and denoted by \overline{St} .

Recall that the K representations $\mathrm{Sym}^{r_\tau} k_F^2$ can be identified with $k_F[X, Y]_{r_\tau}^h$, the homogeneous component of degree r_τ of the monoidal algebra $k_F[X, Y]$. In this case, the action of K is described by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot X^{r_\tau-i} Y^i \stackrel{\mathrm{def}}{=} (\overline{a}X + \overline{c}Y)^{r_\tau-i} (\overline{b}X + \overline{d}Y)^i$$

for any $0 \leq i \leq r_\tau$ and $\overline{\cdot} : \mathcal{O}_F \rightarrow k_F$ is the reduction modulo ϖ .

We fix once for all a field homomorphism $k_F \hookrightarrow k$. The results of this paper do not depend on this choice.

Up to twist by a power of \det , a Serre weight has now the more concrete expression

$$\sigma_{\underline{r}} \cong \bigotimes_{i=0}^{f-1} (\mathrm{Sym}^{r_i} k^2)^{\mathrm{Frob}^i}$$

where $\underline{r} = (r_0, \dots, r_{f-1}) \in \{0, \dots, p-1\}^f$ and $(\mathrm{Sym}^{r_i} k^2)^{\mathrm{Frob}^i}$ is the representation of K obtained from $\mathrm{Sym}^{r_i} k^2$ via the homomorphism $\mathbf{GL}_2(k_F) \rightarrow \mathbf{GL}_2(k_F)$ induced by the i -th Frobenius $x \mapsto x^{p^i}$ on k_F .

We will usually extend the action of K on a Serre weight to the group KZ , by imposing the scalar matrix $\varpi \in Z$ to act trivially.

Let G be a compact p -adic analytic group (cf. [DDSMS], §8.4). It is a profinite topological group, with an open pro- p subgroup of finite rank.

The Iwasawa algebra $k[[G]]$ associated to G is the limit of the group algebras associated to the finite quotients of G :

$$\Omega_G \stackrel{\mathrm{def}}{=} \varprojlim_U k[G/U]$$

where the limit is taken over the open normal subgroups U of G (cf. [AB] for the main properties of Iwasawa algebras). If G is pro- p , the associated Iwasawa algebra is a local noetherian regular domain, whose maximal ideal \mathfrak{m} is the augmentation ideal:

$$\mathfrak{m} = \ker(\Omega_G \twoheadrightarrow k) = \langle x - 1, x \in G \rangle_{\Omega_G}$$

(notice that the abstract ideal on the RHS is automatically closed since Ω_G is noetherian and compact). In this case the Krull dimension of the associated graded ring $\mathrm{gr}(\Omega_G)$ equals the dimension of the group G . If moreover G is a finitely generated free abelian pro- p -group then $\dim(G)$ is the Krull dimension of Ω_G .

A module M over the Iwasawa algebra Ω_G will always be understood to be a profinite left Ω_G -module (i.e. an inverse system of finite left Ω_G -modules). If M, N are profinite left and right Ω_G modules respectively, their *completed tensor product* is the profinite k -module defined by

$$M \widehat{\otimes}_{\Omega_G} N \stackrel{\mathrm{def}}{=} \varprojlim_{M', N'} M/M' \otimes_{\Omega_G} N/N'$$

where the projective system is taken over the open Ω_G -sub-modules M', N' of M, N respectively. We refer the reader to [RZ], §5.5 or [Wil], §7.7 for the basic properties of completed tensor product of profinite modules.

A k -valued character χ of the torus $\mathbf{T}(k_F)$ will be considered, by inflation, as a smooth character of any subgroup of $K_0(\varpi)$. We will write χ^s to denote the conjugate character of χ , defined by

$$\chi^s(t) \stackrel{\text{def}}{=} \chi(\omega t \omega)$$

for any $t \in \mathbf{T}(k_F)$.

Similarly, if τ is any representation of $K_0(\varpi)$, we will write τ^s to denote the conjugate representation, defined by

$$\tau^s(h) = \tau(\alpha h \alpha)$$

for any $h \in K_0(\varpi)$.

If $\underline{r} = (r_0, \dots, r_{f-1}) \in \{0, \dots, p-1\}^f$ is an f -tuple we define the characters of $\mathbf{T}(k_F)$:

$$\chi_{\underline{r}} \left(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \right) \stackrel{\text{def}}{=} a^{\sum_{i=0}^{f-1} p^i r_i}, \quad \mathbf{a} \left(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \right) \stackrel{\text{def}}{=} ad^{-1}.$$

If τ is a semisimple representation of $K_0(\varpi)$ we will write $V_\tau(\chi)$ (or simply $V(\chi)$ if the representation τ is clear from the context) for the χ -isotypical component of τ ; thus

$$\tau = \bigoplus_{\chi \in X^*(\mathbf{T}(k_F))} V_\tau(\chi).$$

Let \mathcal{C} be an abelian category and write \mathcal{C}^{ss} for the full subcategory consisting of semisimple objects; if $X \in \mathcal{C}$ we can consider the functor

$$\begin{aligned} \mathcal{C}^{ss} &\longrightarrow \mathcal{S}ets \\ Y &\longmapsto \text{Hom}_{\mathcal{C}}(X, Y). \end{aligned}$$

If the functor is representable, by a couple $X \rightarrow Q$, we define the *radical* $\text{Rad}(X)$ of X to be the kernel

$$\text{Rad}(X) \stackrel{\text{def}}{=} \ker(X \rightarrow Q).$$

If R is a ring which is semisimple modulo its Jacobson ideal J and \mathcal{C} is a full subcategory of the category of left- R modules, then the radical of an object in \mathcal{C} always exists and we have

$$\text{Rad}(M) = J \cdot M$$

for any $M \in \mathcal{C}$. In particular, for any object $M \in \mathcal{C}$ we can define, by induction, the *radical filtration* $\{\text{Rad}^n(M)\}_{n \in \mathbf{N}}$ by $\text{Rad}^0(M) \stackrel{\text{def}}{=} M$ and $\text{Rad}^n(M) \stackrel{\text{def}}{=} J \cdot \text{Rad}^{n-1}(M)$ for $n \geq 1$.

The dual notion of the radical filtration is the *socle filtration*.

We recall some conventions on the multi-index notations. We write $\underline{\alpha} \stackrel{\text{def}}{=} (\alpha_0, \dots, \alpha_{f-1})$ to denote an f -tuple $\underline{\alpha} \in \mathbf{N}^f$ and if $\underline{\alpha}, \underline{\beta}$ are f -tuples we define

- i) $\underline{\alpha} \geq \underline{\beta}$ if and only if $\alpha_s \geq \beta_s$ for all $s \in \{0, \dots, f-1\}$;
- ii) $\underline{\alpha} \pm \underline{\beta} \stackrel{\text{def}}{=} (\alpha_0 \pm \beta_0, \dots, \alpha_{f-1} \pm \beta_{f-1})$ (where the difference $\underline{\alpha} - \underline{\beta}$ is defined only if $\underline{\alpha} \geq \underline{\beta}$).

The *length* of an f -tuple $\underline{\alpha}$ is defined as $|\underline{\alpha}| \stackrel{\text{def}}{=} \sum_{s=0}^{f-1} \alpha_s$ and, for $s \in \{0, \dots, f-1\}$ we define the element $e_s \stackrel{\text{def}}{=} (0, \dots, 0, 1, 0, \dots, 0)$ where the only non-zero coordinate appears at position s .

If $A = k[[X_0, \dots, X_{f-1}]]$ and $\underline{\alpha} \in \mathbf{N}^f$ is an f -tuple we write

$$\underline{X}^{\underline{\alpha}} \stackrel{\text{def}}{=} \prod_{s=0}^{f-1} X_s^{\alpha_s}.$$

Finally, we recall that if S is any set, and $s_1, s_2 \in S$ the Kronecker delta $\delta_{(s_1, s_2)}$ is defined by

$$\delta_{(s_1, s_2)} \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } s_1 \neq s_2 \\ 1 & \text{if } s_1 = s_2. \end{cases}$$

2. Reminders on the universal representation for \mathbf{GL}_2

We recall here the definition of the universal representation for \mathbf{GL}_2 , quickly specializing its construction in terms of certain amalgamated sums of finite inductions. The main upshot is Theorem 2.1, which shows that in order to control the universal representation it is sufficient to consider a suitable sub-representation of the Iwahori subgroup of K . The reader is invited to refer to [Mo1], §2.1 and §3.1 for the omitted details.

We fix an f -tuple $\underline{r} \in \{0, \dots, p-1\}^f$ and write $\sigma = \sigma_{\underline{r}}$ for the associated Serre weight described in (4). In particular, the highest weight space of σ affords the character $\chi_{\underline{r}}$. We recall ([BL94], [Her1]) that the Hecke algebra $\mathcal{H}_{KZ}(\sigma)$ is commutative and isomorphic to the monoidal algebra of \mathbf{N} on k :

$$\mathcal{H}_{KZ}(\sigma) \xrightarrow{\sim} k[T].$$

The Hecke operator T is supported on the double coset $K\alpha KZ$ and completely determined as a suitable linear projection on σ (cf. [Her1], Theorem 1.2); it admits an explicit description in terms of the Bruhat-Tits tree of $\mathbf{GL}_2(F)$ (cf. [Bre], §2.5).

The universal representation $\pi(\sigma, 0)$ for \mathbf{GL}_2 is then defined¹ by the exact sequence

$$0 \rightarrow \mathrm{ind}_{KZ}^G \sigma \xrightarrow{T} \mathrm{ind}_{KZ}^G \sigma \rightarrow \pi(\sigma, 0) \rightarrow 0.$$

Using the Makey decomposition for the KZ -restriction, we are able to describe $\pi(\sigma, 0)|_{KZ}$ as a compact induction from an explicit $K_0(\varpi)$ -representation, as we will outline in the following lines.

Let $n \in \mathbf{N}$. We consider the anti-dominant co-weight $\lambda_n \in X(\mathbf{T})_*$ characterized by

$$\lambda_n(\varpi) = \begin{bmatrix} 1 & 0 \\ 0 & \varpi^n \end{bmatrix}$$

and we introduce the subgroup

$$K_0(\varpi^n) \stackrel{\text{def}}{=} (\lambda_n(\varpi)K\lambda_n(\varpi^{-1})) \cap K = \left\{ \begin{bmatrix} a & b \\ \varpi^n c & d \end{bmatrix} \in K \right\}.$$

The element $\begin{bmatrix} 0 & 1 \\ \varpi^n & 0 \end{bmatrix}$ normalizes $K_0(\varpi^n)$ and we define the $K_0(\varpi^n)$ -representation $\sigma^{(n)}$ as

the $K_0(\varpi^n)$ restriction of σ endowed with the twisted action of $K_0(\varpi^n)$ by the element $\begin{bmatrix} 0 & 1 \\ \varpi^n & 0 \end{bmatrix}$.

Explicitly,

$$\sigma^{(n)} \left(\begin{bmatrix} a & b \\ p^n c & d \end{bmatrix} \right) \cdot \underline{X}^{\underline{r}-\underline{j}} \underline{Y}^{\underline{j}} \stackrel{\text{def}}{=} \sigma \left(\begin{bmatrix} d & c \\ p^n b & a \end{bmatrix} \right) \underline{X}^{\underline{r}-\underline{j}} \underline{Y}^{\underline{j}}.$$

¹In the current literature the universal representation is written $\pi(\sigma, 0, 1)$. We decided to write $\pi(\sigma, 0)$ in order to lighten the notations.

Finally, for $n \geq 1$ we write

$$R_n^-(\sigma) \stackrel{\text{def}}{=} \text{ind}_{K_0(\varpi^n)}^{K_0(\varpi)}(\sigma^{(n)}), \quad R_0^- \stackrel{\text{def}}{=} \text{cosoc}_{K_0(\varpi)}(\sigma^{(1)}).$$

For notational convenience, we will write \underline{Y}^r for a linear basis of R_0^- . If the Serre weight σ is clear from the context, we write R_n^- instead of $R_n^-(\sigma)$.

The interest of the representations R_n^- is that they realize the Mackey decomposition for $\text{ind}_{KZ}^G \sigma$:

$$(\text{ind}_{KZ}^G \sigma)|_{KZ} \xrightarrow{\sim} \sigma^{(0)} \oplus \bigoplus_{n \geq 1} \text{ind}_{K_0(\varpi)}^K(R_n^-).$$

The interpretation in terms of the tree of \mathbf{GL}_2 is clear: the $k[K_0(\varpi)]$ -module R_n^- maps isomorphically onto the space of elements of $\text{ind}_{KZ}^G \sigma$ having support on the double coset $K_0(\varpi)\lambda_n(\varpi)KZ$. In particular, if σ is the trivial weight, a linear basis for R_n^- is parametrized by the vertices of \mathcal{T} , belonging to the negative part of the tree and lying at distance n from the central vertex.

The Hecke morphism T induces, by transport of structure, a family of $K_0(\varpi)$ -equivariant morphisms $\{(T_n)^{\text{neg}}\}_{n \geq 1}$ defined on the $k[K_0(\varpi)]$ -modules R_n^- : for $n \geq 2$ we have $(T_n)^{\text{neg}} \stackrel{\text{def}}{=} T|_{R_n^-}$ and, for $n = 1$, we define

$$(T_1)^{\text{neg}} : R_1^- \xrightarrow{T|_{R_1^-}} R_2^- \oplus \sigma^{(0)} \rightarrow R_2^- \oplus R_0^-$$

(notice that $\text{soc}_{K_0(\varpi)}(\sigma^{(0)}) \cong \text{cosoc}_{K_0(\varpi)}(R_1^-)$).

More expressively, one shows (cf. [Mo1], §2.1) that for any $n \geq 1$ the Hecke operator $(T_n)^{\text{neg}}$ admits a decomposition $(T_n)^{\text{neg}} = T_n^+ \oplus T_n^-$ where² the morphisms $T_n^\pm : R_n^- \rightarrow R_{n \pm 1}^-$ are obtained by compact induction (from $K_0(\varpi^n)$ to K) from the following morphisms:

$$t_n^+ : \sigma^{(n)} \hookrightarrow \text{ind}_{K_0(\varpi^{n+1})}^{K_0(\varpi^n)} \sigma^{(n+1)}$$

$$\underline{X}^{r-j} \underline{Y}^j \mapsto \sum_{\lambda_n \in k_F} (-\lambda_n)^j \begin{bmatrix} 1 & 0 \\ \varpi^n [\lambda_n] & 1 \end{bmatrix} [1, \underline{X}^r];$$

$$t_{n+1}^- : \text{ind}_{K_0(\varpi^{n+1})}^{K_0(\varpi^n)} \sigma^{(n+1)} \rightarrow \sigma^{(n)}$$

$$[1, \underline{X}^{r-j} \underline{Y}^j] \mapsto \delta_{j,r} \underline{Y}^r$$

and, for $n = 0$, we have the natural epimorphism

$$T_1^- : R_1^- \rightarrow R_0^-$$

$$\underline{X}^{r-j} \underline{Y}^j \mapsto \delta_{j,r} \underline{Y}^r$$

(this shows that T_n^+ are monomorphisms and T_n^- epimorphisms for all $n \geq 1$).

The Hecke operators T_n^\pm can be used to construct a family of amalgamated sums, in the following way. We define $R_0^- \oplus_{R_1^-} R_2^-$ as the push out:

$$\begin{array}{ccc} R_1^- & \xrightarrow{T_1^+} & R_2^- \\ \downarrow -T_1^- & & \downarrow pr_2 \\ R_0^- & \xrightarrow{\quad} & R_0^- \oplus_{R_1^-} R_2^- \end{array}$$

²According to [Mo1], the morphisms T_n^\pm should be written as $(T_n^\pm)^{\text{neg}}$. We decided to use here the lighter notation T_n^\pm .

and, assuming we have inductively constructed $pr_{n-1} : R_{n-1}^- \rightarrow R_0^- \oplus_{R_1^-} \cdots \oplus_{R_{n-2}^-} R_{n-1}^-$ (where $n \geq 3$ is odd), we define the amalgamated sum $R_0^- \oplus_{R_1^-} \cdots \oplus_{R_n^-} R_{n+1}^-$ by the following co-cartesian diagram:

$$\begin{array}{ccc} R_n^- & \xrightarrow{T_n^+} & R_{n+1}^- \\ \downarrow -pr_{n-1} \circ T_n^- & & \downarrow pr_{n+1} \\ R_0^- \oplus_{R_1^-} R_2^- \oplus_{R_3^-} \cdots \oplus_{R_{n-2}^-} R_{n-1}^- & \xrightarrow{\quad} & R_0^- \oplus_{R_1^-} R_2^- \oplus_{R_3^-} \cdots \oplus_{R_n^-} R_{n+1}^- \end{array}$$

The amalgamated sums $R_0^- \oplus_{R_1^-} \cdots \oplus_{R_n^-} R_{n+1}^-$ (where n is odd) form, in an evident manner, an inductive system and we define

$$R_{\infty,0}^- \stackrel{\text{def}}{=} \varinjlim_{n \in 2\mathbf{N}+1} R_0^- \oplus_{R_1^-} \cdots \oplus_{R_n^-} R_{n+1}^-.$$

We can repeat the previous construction for n even, defining an inductive system of $K_0(\varpi)$ -representations $R_1^- \oplus_{R_2^-} \cdots \oplus_{R_n^-} R_{n+1}^-$ and we write

$$R_{\infty,1}^- \stackrel{\text{def}}{=} \varinjlim_{n \in 2\mathbf{N}+2} R_1^- \oplus_{R_2^-} \cdots \oplus_{R_n^-} R_{n+1}^-.$$

The relation between the representations $R_{\infty,\bullet}^-$ and the universal representation $\pi(\sigma, 0)$ is described by the following

THEOREM 2.1 ([Mo1], Theorem 1.1). *Let $\sigma = \sigma_{\underline{r}}$ be a Serre weight. The KZ restriction of the universal representation $\pi(\sigma, 0)$ decomposes as $\pi(\sigma, 0)|_{KZ} = R_{\infty,0}^- \oplus R_{\infty,1}^-$ and we have short exact sequences of K -representations*

$$\begin{aligned} 0 &\rightarrow \text{Rad}(\chi_{\underline{r}}) \rightarrow \text{ind}_{K_0(\varpi)}^K(R_{\infty,0}^-) \rightarrow R_{\infty,0}^- \rightarrow 0 \\ 0 &\rightarrow \text{Soc}(\chi_{\underline{r}}^s) \rightarrow \text{ind}_{K_0(\varpi)}^K(R_{\infty,-1}^-) \rightarrow R_{\infty,-1}^- \rightarrow 0 \end{aligned}$$

where $\text{Rad}(\chi_{\underline{r}})$, $\text{Soc}(\chi_{\underline{r}}^s)$ are defined by

$$\begin{aligned} \text{Rad}(\chi_{\underline{r}}) &\stackrel{\text{def}}{=} \overline{St}, & \text{Soc}(\chi_{\underline{r}}^s) &\stackrel{\text{def}}{=} 1 & \text{if } \underline{r} = \underline{0} \\ \text{Rad}(\chi_{\underline{r}}) &\stackrel{\text{def}}{=} 1, & \text{Soc}(\chi_{\underline{r}}^s) &\stackrel{\text{def}}{=} \overline{St} & \text{if } \underline{r} = \underline{p-1} \\ \text{Rad}(\chi_{\underline{r}}) &\stackrel{\text{def}}{=} \text{Rad}\left(\text{ind}_{K_0(\varpi)}^K \chi_{\underline{r}}\right), & \text{Soc}(\chi_{\underline{r}}^s) &\stackrel{\text{def}}{=} \text{Soc}\left(\text{ind}_{K_0(\varpi)}^K \chi_{\underline{r}}^s\right) & \text{otherwise.} \end{aligned}$$

Proof. Omissis. This is Corollary 3.4 in [Mo1]. \square

We remark that the representations $R_{\infty,0}^-$ $R_{\infty,0}^+$ let us control the N -action on $\pi(\sigma, 0)$ as well. Indeed, if we define $R_{\infty,\bullet}^+$ to be the $K_0(\varpi)$ -representation deduced from $R_{\infty,1-\bullet}^-$ by conjugation by α , we can endow the $K_0(\varpi)$ -representation

$$R_{\infty,0}^- \oplus R_{\infty,0}^+ \oplus R_{\infty,1}^- \oplus R_{\infty,1}^+$$

with an action of N , by making α act by the involution $\alpha \cdot (v_0^-, v_0^+, v_1^-, v_1^+) \stackrel{\text{def}}{=} (v_1^+, v_1^-, v_0^+, v_0^-)$ for $v_{\bullet}^* \in R_{\infty,\bullet}^*$, where $\bullet \in \{0, 1\}$, $*$ $\in \{+, -\}$.

Then one can show (cf. [Mo1], Propositions 3.6 and 3.7) that we have an N -equivariant exact sequence

$$0 \rightarrow \langle v_0 \rangle_k \oplus \langle v_1 \rangle_k \rightarrow R_{\infty,0}^- \oplus R_{\infty,0}^+ \oplus R_{\infty,1}^- \oplus R_{\infty,1}^+ \rightarrow \pi(\sigma, 0)|_N \rightarrow 0$$

where $v_\bullet \in R_{\infty, \bullet}^- \oplus R_{\infty, \bullet}^+$ is an appropriate nonzero, $K_1(\varpi)$ -fixed vector, affording the character χ_r if $\bullet = 0$ (resp. χ_r^s if $\bullet = 1$) and such that $\alpha \cdot v_\bullet = v_{1-\bullet}$ (indeed, the image of v_\bullet in $R_{\infty, \bullet}^-$ is described by the socle $\text{soc}_{K_0(\varpi)}(R_\bullet^-)$).

3. Dual translation

The first step in order to control the representations $R_{\infty, 0}^-, R_{\infty, 1}^-$ consists in a precise knowledge of their Pontryagin duals $\mathfrak{S}_\infty^0, \mathfrak{S}_\infty^1$. We start by recalling some well-known results about the duality between smooth k representations of compact p -adic analytic groups and profinite modules (§3.1) and we specialize the construction to the group U_0^- . In particular, we determine a family of H -eigenvectors for the tangent space of $k[[U_0^-]]$, which lets us easily deduce the dual of a Serre weight (3.2). The description of $\mathfrak{S}_\infty^0, \mathfrak{S}_\infty^1$ follows then by a formal construction, which is detailed in section 3.3.

We fix throughout this section a Serre weight $\sigma = \sigma_r$. In particular, the highest weight space of σ affords the $K_0(p)$ -character χ_r .

3.1 Review of Pontryagin duality

The aim of this section is to give a precise survey of the main formal properties of Pontryagin duality for compact p -adic analytic groups. The subject is classical and we invite the reader to refer to the work of Emerton [Eme], §2.2 or Ribes-Zaleskii [RZ], §5.1 for more details.

Let A be a complete, Noetherian local \mathcal{O}_F -algebra with finite residue field and let G be a compact p -adic analytic group (cf. [DDSMS], §8.4).

The category $\text{Mod}_G^{\text{sm}}(A)$ of smooth, A -linear G -representations is defined as the category of locally Artinian A -modules endowed with the discrete topology and a continuous action of G . On the other hand, we have the category $\text{Mod}_G^{\text{pro}}(A)$ of profinite $A[[G]]$ -modules.

We recall the following result

THEOREM 3.1 (Pontryagin Duality). *For any compact-open subgroup K of G we have an involutive anti-equivalence of categories*

$$\begin{aligned} \text{Mod}_K^{\text{sm}}(A) &\xrightarrow{\sim} \text{Mod}_G^{\text{pro}}(K) \\ V &\longrightarrow V^\vee. \end{aligned}$$

Moreover, if $K_1 \leq K_2$ are two compact open subgroups of G and $V \in \text{Mod}_{K_1}^{\text{sm}}(A)$ then

$$(\text{ind}_{K_1}^{K_2} V)^\vee = A[[K_2]] \widehat{\otimes}_{A[[K_1]]} (V)^\vee.$$

We content ourselves to recall that the dual of $V \in \text{Mod}_K^{\text{sm}}(A)$ is defined as $V^\vee \stackrel{\text{def}}{=} \text{Hom}_{\mathcal{O}_F}(V, F/\mathcal{O}_F)$, the latter endowed with the compact-open topology (hence the topology of the simple convergence as the \mathcal{O}_F -modules $V, F/\mathcal{O}_F$ are endowed with the discrete topology) and the action of K given by $(g \cdot f)(v) \stackrel{\text{def}}{=} f(g^{-1}v)$ for any $g \in K, v \in V, f \in V^\vee$.

Conversely, if $M \in \text{Mod}_K^{\text{pro}}(A)$ one consider the topological dual $M^\vee \stackrel{\text{def}}{=} \text{Hom}_{\mathcal{O}_F}^{\mathcal{C}^0}(M, F/\mathcal{O}_F)$, endowed with the discrete topology and the (continuous) contragradient action of G .

If $K_1 \leq K_2$ are two closed subgroups of G , it is clear that the duality commutes with the restriction functor $\text{Mod}_{K_2}^*(A) \rightarrow \text{Mod}_{K_1}^*(A)$, for $*$ in $\{\text{sm}, \text{pro}\}$. Therefore, if K_2 is compact and K_1 is open, the induction $\text{ind}_{K_1}^{K_2} \circ$ is right adjoint to the restriction (Frobenius reciprocity), and hence we have a natural isomorphism of functors

$$(\text{ind}_{K_1}^{K_2} \circ)^\vee \cong A[[K_1]] \widehat{\otimes}_{A[[K_2]]} (\circ)^\vee.$$

More generally, for a family $\{K_n\}_{n \in \mathbf{N}}$ of compact open subgroups of G such that $K_\infty = \bigcap_{n \in \mathbf{N}} K_n$ is closed and $V \in \text{Mod}_G^{\text{sm}}(A)$ we have $\text{ind}_{K_\infty}^G V|_{K_\infty} = \lim_{\substack{\longrightarrow \\ n}} \text{ind}_{K_n}^G V|_{K_n}$ and hence

$$\left(\text{ind}_{K_\infty}^G V|_{K_\infty} \right)^\vee = \lim_{\substack{\longleftarrow \\ n}} \left(\text{ind}_{K_n}^G V|_{K_n} \right)^\vee = \lim_{\substack{\longleftarrow \\ n}} A[[G]] \widehat{\otimes}_{A[[K_n]]} (V|_{K_n})^\vee = A[[G]] \widehat{\otimes}_{A[[K_\infty]]} (V|_{K_\infty})^\vee$$

(the last equality clearly holds if G is discrete, and one passes to the inverse limit over the open compact normal subgroups of G , cf. [RZ], Theorem 6.10.8).

We deduce, using the continuity of the restriction functor and the Mackey decomposition, that for a closed subgroup U of G we have an isomorphism of profinite $A[[U]]$ -modules

$$\left(A[[G]] \widehat{\otimes}_{A[[K_\infty]]} (V|_{K_\infty})^\vee \right) |_{A[[U]]} \cong \prod_{e \in U \backslash G / K_\infty} A[[U]] \widehat{\otimes}_{A[[eK_\infty e^{-1} \cap U]]} (e(V|_{K_\infty})^\vee) |_{A[[eK_\infty e^{-1} \cap U]]} \quad (5)$$

We can now specialize the previous construction to our situation.

Let $n \in \mathbf{N}$. By the Iwasawa decomposition and Theorem 3.1 we can introduce

$$S_{n+1}(\sigma) \stackrel{\text{def}}{=} (R_{n+1}^- |_{U_0^-})^\vee = k[[U_0^-]] \otimes_{k[[\varpi^n U_0^-]]} (\sigma^{(n+1)})^\vee.$$

It is a pseudo-compact module over $k[[U_0^-]]$ (it is moreover of finite length over k , since $\varpi^n U_0^-$ is open in U_0^- and σ is finite dimensional). We will simply write S_{n+1} instead of $S_{n+1}(\sigma)$ if the weight σ is clear from the context. Since R_{n+1}^- is endowed with a smooth action of Γ, H, U_0^+ we deduce that S_{n+1} is endowed, by functoriality, with \mathcal{O}_F -linear, continuous actions of the subgroups H, Γ, U_0^+ (obviously, the same holds true if we replace S_{n+1} with any other object in $\text{Mod}_{K_0(p^\infty)}^{\text{sm}}(k)$).

We will say that S_{n+1} is an *Iwasawa module with extra structure* or, simply, an Iwasawa module. We will call it a $k[[U_0^-]]$ -module when we consider S_{n+1} as a module over the algebra $k[[U_0^-]]$ via the forgetful functor.

For ease of notation we set $A \stackrel{\text{def}}{=} k[[U_0^-]]$. We notice that, by (5), A is itself an Iwasawa module, since

$$(k[[K_0(p)]] \widehat{\otimes}_{k[[K_0(p^\infty)]]} 1) |_{k[[U_0^-]]} \cong k[[U_0^-]] \quad (6)$$

where we have defined $K_0(p^\infty) \stackrel{\text{def}}{=} \bigcap_{n \in \mathbf{N}} K_0(p^n)$.

As Γ, H normalize U_0^- it is easy to see that their induced actions on A (via (6) and the contra-gradient action on the LHS) coincides with the action induced on A by Γ, H -conjugation:

$$\gamma \cdot \begin{bmatrix} 1 & 0 \\ \varpi x & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \varpi \gamma x & 1 \end{bmatrix}$$

for any $\gamma \in \Gamma, H, x \in \mathcal{O}_F$.

In particular, Γ, H act on A by local k -algebra endomorphism. Similarly, the action of Γ, H on S_{n+1} are *semilinear* (notice that Γ acts trivially on σ).

The finite torus H acts semi-simply on the tangent space of A and we are able to determine, *in the unramified case*, a regular system of parameters for the maximal ideal \mathfrak{m} of A , formed by H -eigenvectors.

LEMMA 3.2. For $i \in \{0, \dots, f-1\}$ define the following elements of A :

$$X_i \stackrel{\text{def}}{=} \sum_{\lambda \in k_F^\times} \lambda^{-p^i} \begin{bmatrix} 1 & 0 \\ p[\lambda] & 1 \end{bmatrix} \in A.$$

The family $\{X_0, \dots, X_{f-1}\}$ is a regular system of parameters for the maximal ideal \mathfrak{m} in A and H acts on X_i by the character \mathfrak{a}^{-p^i}

Proof. We have to show that the elements X_i form a linear basis for the tangent space of A . This is equivalent to ask that the discrete A module

$$k[[U_0^-/pU_0^-]] \cong k[U_0^-/pU_0^-] \cong k[U_0^-] \otimes_{k[pU_0^-]} 1$$

admits the images of the elements X_i as a linear basis for the first graded piece in its radical filtration.

We can now apply [Mo2], Proposition 4.4 (with $m = n = 1$), noticing that X_i is nothing that the element $F_{\underline{p-1-e_i}}^{(1)}$ in the notation of *loc. cit.*

The statement about the action of H is an easy check. □

For $n \geq 1$ consider the natural injection $K_0(p^{n+1}) \hookrightarrow K_0(p)$. It induces a monomorphism of Iwasawa algebras, hence a morphism of Iwasawa modules

$$k[[K_0(p^{n+1})]] \widehat{\otimes}_{k[[K_0(p^\infty)]]} 1 \rightarrow k[[K_0(p)]] \widehat{\otimes}_{k[[K_0(p^\infty)]]} 1. \quad (7)$$

If we restrict this morphism to $k[[p^n U_0^-]]$ we deduce, from (5), a monomorphism of Iwasawa algebras

$$k[[p^n U_0^-]] \hookrightarrow k[[U_0^-]]$$

induced from the inclusion $p^n U_0^- \hookrightarrow U_0^-$.

Define the following elements of $k[[p^n U_0^-]]$:

$$X_{i-n}^{p^n} \stackrel{\text{def}}{=} \sum_{\lambda \in k_F^\times} \lambda^{-p^i} \begin{bmatrix} 1 & 0 \\ p^{n+1}[\lambda] & 1 \end{bmatrix} \in k[[p^n U_0^-]]$$

where the indices $i - n$ appearing in $X_{i-n}^{p^n}$ are understood to be elements of $\mathbf{Z}/f\mathbf{Z}$.

As in the proof of Lemma 3.2 one sees that the elements $X_{i-n}^{p^n}$, for $i = 0, \dots, f-1$, form a regular system of parameters for the maximal ideal of $k[[p^n U_0^-]]$, and moreover each $X_{i-n}^{p^n}$ is an H -eigenvector, of associated eigencharacter \mathfrak{a}^{-p^i} .

We deduce that the morphism of k -algebras

$$\begin{aligned} k[[p^n U_0^-]] &\hookrightarrow k[[U_0^-]] \\ X_{i-n}^{p^n} &\mapsto X_{i-n}^{p^n}, \end{aligned} \quad (8)$$

is the natural morphism induced from (7) by restriction to $k[[p^n U_0^-]]$ (i.e. is compatible with the extra structures on both sides of (8))

3.2 The dual of a Serre weight

We realize here the dual of the Serre weight $\sigma = \sigma_{\underline{r}}$ as a suitable explicit quotient of the Iwasawa module A . We will use in a crucial way the H -eigenvectors decomposition of the tangent space of A given in Lemma 3.2. Thus, from now until the end of the paper, **we will assume that F is unramified**.

PROPOSITION 3.3. *Let $\sigma = \sigma_{\underline{r}}$ be a Serre weight and let $\chi_{\underline{r}}$ be the character of $\mathbf{T}(k_F)$ associated to a highest weight eigenvector $v_{\underline{r}} \in \sigma$. Let $n \geq 1$.*

If $\dim_k(\sigma) \neq q$ then the following Hom spaces are 1-dimensional,

$$\text{Hom}_{K_0(p^n)}(\sigma^{(n)}, \text{ind}_{K_0(p^{n+1})}^{K_0(p^n)} \chi_{\underline{r}}^s) = \langle \phi_n \rangle_k; \quad \text{Hom}_{K_0(p^n)}(\text{ind}_{K_0(p^{n+1})}^{K_0(p^n)} \chi_{\underline{r}}, \sigma^{(n)}) = \langle \psi_n \rangle_k$$

and ϕ_n (resp. ψ_n) is a monomorphism (resp. epimorphism).

If $\dim_k(\sigma) = q$ the following Hom spaces are 2 dimensional

$$\mathrm{Hom}_{K_0(p^n)}(\sigma^{(n)}, \mathrm{ind}_{K_0(p^{n+1})}^{K_0(p^n)} \chi_r^s) = \langle \phi_n, \tilde{\phi}_n \rangle_k; \quad \mathrm{Hom}_{K_0(p^n)}(\mathrm{ind}_{K_0(p^{n+1})}^{K_0(p^n)} \chi_r, \sigma^{(n)}) = \langle \psi_n, \tilde{\psi}_n \rangle_k$$

and ϕ_n, ψ_n are isomorphisms, while $\tilde{\phi}_n, \tilde{\psi}_n$ have one dimensional image.

Moreover we have exact sequences of $k[[p^{n-1}U_0^-]]$ -modules with extra structure

$$0 \rightarrow \langle X_{i-n+1}^{p^{n-1}(r_i+1)}, i = 0, \dots, f-1 \rangle \rightarrow k[p^{n-1}U_0^-/p^nU_0^-] \otimes_k (\chi_r \det^{-r}) \rightarrow (\sigma^{(n)})^\vee \rightarrow 0 \quad (9)$$

$$0 \rightarrow (\sigma^{(n)})^\vee \rightarrow k[p^{n-1}U_0^-/p^nU_0^-] \otimes_k (\chi_r^s \det^{-r}) \rightarrow (k[p^{n-1}U_0^-/p^nU_0^-] \otimes_k (\chi_r^s \det^{-r})) / \langle \prod_{i=0}^{f-1} X_{i-n+1}^{p^{n-1}(p-1-r_i)} \rangle \rightarrow 0. \quad (10)$$

Proof. We start from the the exact sequence (9). Notice that the action of $K_0(p^n)$ on v_r in the twisted module $\sigma^{(n)}$ is given by χ_r^s , whose dual is the one dimensional Iwasawa module, with trivial actions of $\Gamma, U_0^+, k[[p^{n-1}U_0^-]]$, and the action of H given by the character $(\chi_r^s)^{-1} = \chi_r \det^{-r}$.

We see that the $K_0(p^{n+1})$ -restriction of $\sigma^{(n)}$ is described by

$$\sigma^{(n)}|_{K_0(p^{n+1})} = \bigoplus_{0 \leq j \leq r} \chi_r^s \alpha^j.$$

If $\dim(\sigma) \neq q$, the H -character of σ are *all distinct* so that the Hom space

$$\mathrm{Hom}_{K_0(p^{n+1})}(\sigma^{(n)}, \chi_r^s) \cong \mathrm{Hom}_{K_0(p^n)}(\sigma^{(n)}, \mathrm{ind}_{K_0(p^{n+1})}^{K_0(p^n)} \chi_r^s)$$

is one dimensional.

Moreover a k -generator ϕ_n is an *injective* morphism: the space of $p^{n-1}U_0^-$ -invariants of $\sigma^{(n)}$ is generated by v_r , and, by construction, $(\phi_n(v_r))(1)$ is a linear generator of χ_r^s in particular it is non-zero. Therefore $(\ker(\phi_n))^{p^{n-1}U_0^-} = 0$, and the claim follows as $p^{n-1}U_0^-$ is a pro- p group.

If $\dim(\sigma) = q$ then the lowest weight vector $v^{(x)}$ and the highest weight vector v_r are the only H -eigenvectors of $\sigma^{(n)}$ affording the character χ_r^s . We deduce two linearly independent morphisms

$$\phi_n, \tilde{\phi}_n \in \mathrm{Hom}_{K_0(p^n)}(\sigma^{(n)}, \mathrm{ind}_{K_0(p^{n+1})}^{K_0(p^n)} \chi_r^s)$$

characterized by $(\phi_n(v_r))(1) = e$ and $(\tilde{\phi}_n(v^{(x)}))(1) = e$ for a linear generator e of χ_r^s . As above, we see that ϕ_n is a monomorphism (hence an isomorphism for dimension reasons) and that $\mathrm{soc}(\sigma^{(n)})$ is a subspace of $\ker(\tilde{\phi}_n)$. As the characters of the socle filtration for $\mathrm{ind}_{K_0(p^{n+1})}^{K_0(p^n)} \chi_r^s$ are all distinct except for the those appearing in the socle and the cosocle we deduce that $\tilde{\phi}_n$ has to factor via $\mathrm{cosoc}(\sigma^{(n)})$ into a non-zero morphism.

Passing to duals we deduce an epimorphism of Iwasawa modules with extra-structures

$$k[p^{n-1}U_0^-/p^nU_0^-] \otimes_k (\chi_r \det^{-r}) \xrightarrow{\phi^\vee} (\sigma^{(n)})^\vee \rightarrow 0.$$

which is an isomorphism if $\dim(\sigma) = q$.

Assume $\dim(\sigma) \neq q$. By counting dimensions, the equality

$$\ker(\phi^\vee) = \langle X_{i-n+1}^{p^{n-1}(r_i+1)}, i = 0, \dots, f-1 \rangle$$

is established once we show that $X_{i-n+1}^{p^{n-1}(r_i+1)} \in \ker(\phi^\vee)$ for any $i = 0, \dots, f-1$.

This is immediate, since the H eigencharacter of $(\sigma^{(n)})^\vee$ (which are all distinct) are described

by

$$(\sigma^{(n)})^\vee|_H = \bigoplus_{0 \leq j \leq r} (\chi_r \det^{-r}) \alpha^{-j}$$

while H acts on $X_{i-n+1}^{p^{n-1}(r_i+1)}$ by $(\chi_r \det^{-r}) \alpha^{-(r_i+1)e_i}$.

The proof of the existence of the natural exact sequence (10) is similar and left to the reader, noticing that $\text{cosoc}((\sigma^{(n)})^\vee) \cong (\text{soc}(\sigma^{(n)}))^\vee = (\chi_r^s)^\vee$ and that the H -eigencharacter of $\prod_{i=0}^{f-1} X_{i-n+1}^{p^{n-1}(p-1-r_i)}$ is $(\chi_r^s)^\vee$. \square

3.3 The dual of the universal module

In this section we complete the dictionary between the representations $R_{\infty, \bullet}^-$ and the corresponding Pontryagin dual $\mathfrak{S}_\infty^\bullet$. We first describe the dual of the Hecke morphisms T_n^\pm in terms of Iwasawa modules, hence the realization of the pseudo-compact Iwasawa module $\mathfrak{S}_\infty^\bullet$ as a limit of certain explicit subquotients of A .

Let $n \geq 2$. Recall (§2) that the Hecke morphism T_{n-1}^+ is obtained as the induction, from $K_0(p^{n-1})$ to $K_0(p)$, of the $K_0(p^{n-1})$ -equivariant morphism

$$t_{n-1}^+ : \sigma^{(n-1)} \rightarrow \text{ind}_{K_0(p^n)}^{K_0(p^{n-1})} \sigma^{(n)}.$$

Let $K_1(p^n)$ be the maximal pro- p subgroup of $K_0(p^n)$. Since it is normal in $K_0(p^{n-1})$ and it acts trivially on $\sigma^{(n-1)}$ we deduce a factorization

$$\begin{array}{ccc} \sigma^{(n-1)} & \longrightarrow & \text{ind}_{K_0(p^n)}^{K_0(p^{n-1})} \sigma^{(n)} \\ & \searrow \phi_{n-1} & \uparrow \\ & & \text{ind}_{K_0(p^n)}^{K_0(p^{n-1})} \chi_r^s \end{array} \quad (11)$$

where the vertical arrow is induced from the isomorphism $\chi_r^s \xrightarrow{\sim} (\sigma^{(n)})^{K_1(p^n)}$.

If $\dim(\sigma) < q$, the diagonal arrow is deduced to be equal to ϕ_{n-1} (up to a scalar) by Lemma 3.3; an easy check on the definition of t_{n-1}^+ shows that this is the case also for $\dim(\sigma) = q$.

Dualizing (11) and using Lemma 3.3 we obtain (we omit the twist by $(\chi_r^s)^\vee$ not to overload the notations)

$$\begin{array}{ccc} (\sigma^{(n-1)})^\vee & \longleftarrow & k[[p^{n-2}U_0^-]] \otimes k[[p^{n-1}U_0^-]] / \langle X_{i-n+1}^{p^{n-1}(r_i+1)}, i = 0, \dots, f-1 \rangle \\ & \swarrow & \downarrow \\ & & k[[p^{n-2}U_0^-]] \otimes (\chi_r^s)^\vee \end{array}$$

where the vertical arrow is induced by base change from the projection $(\sigma^{(n)})^\vee \rightarrow (\chi_r^s)^\vee$ and the tensor product is over $k[[p^{n-1}U_0^-]]$. We conclude from (8) that $(T_{n-1}^+)^\vee$ is the natural projection for any $n \geq 2$.

We turn our attention to $(T_n^-)^\vee$, for $n \geq 1$. Again, the Hecke morphism T_n^- is obtained as the induction, from $K_0(p^n)$ to $K_0(p)$, of the $K_0(p^n)$ -equivariant morphism

$$t_n^- : \text{ind}_{K_0(p^{n+1})}^{K_0(p^n)} \sigma^{(n+1)} \rightarrow \sigma^{(n)}.$$

Since $K_1(p^{n+1})$ (the maximal pro- p subgroup of $K_0(p^{n+1})$) acts trivially on $\sigma^{(n)}$ and is normal in

$K_0(p^n)$ we deduce a factorization

$$\begin{array}{ccc} \text{ind}_{K_0(p^{n+1})}^{K_0(p^n)} \sigma^{(n+1)} & \twoheadrightarrow & \sigma^{(n)} \\ \downarrow & \nearrow \psi_n & \\ \text{ind}_{K_0(p^{n+1})}^{K_0(p^n)} \chi_{\underline{r}} & & \end{array} \quad (12)$$

where the vertical arrow is induced from the isomorphism $(\sigma^{(n+1)})_{K_1(p^n)} \xrightarrow{\sim} \chi_{\underline{r}}$.

If $\dim(\sigma) < q$, the diagonal arrow is deduced to be equal to ψ_n (up to a scalar) by Lemma 3.3; another easy check on the definition of T_n^- shows that this is the case also for $\dim(\sigma) = q$.

Dualizing (12) and using Lemma 3.3 we obtain (again omitting the twist by $(\chi_{\underline{r}}^s)^\vee$ in the first line not to overload the notations)

$$\begin{array}{ccc} k[[p^{n-1}U_0^-]] \otimes k[[p^n U_0^-]] / \langle X_{i-n}^{p^n(r_i+1)}, i = 0, \dots, f-1 \rangle & \longleftarrow & k[[p^{n-1}U_0^-]] / \langle X_{i-n+1}^{p^{n-1}(r_i+1)}, i = 0, \dots, f-1 \rangle \\ \uparrow & & \swarrow \\ k[[p^{n-1}U_0^-]] \otimes (\chi_{\underline{r}})^\vee & & \end{array}$$

where the vertical arrow is induced by base change from the injection

$$\begin{aligned} (\chi_{\underline{r}})^\vee &\hookrightarrow k[[p^n U_0^-]] \otimes_k (\chi_{\underline{r}}^s)^\vee / \langle X_{i-n}^{p^n(r_i+1)}, i = 0, \dots, f-1 \rangle \\ 1 &\mapsto \prod_{i=0}^{f-1} X_{i-n}^{p^n r_i}, \end{aligned}$$

the diagonal arrow is deduced Lemma 3.3 and the tensor product in the RHS is over $k[[p^n U_0^-]]$.

We obtain, using (8),

PROPOSITION 3.4. *Let $n \geq 1$.*

The dual morphism of $T_n^+ : R_n^- \otimes \chi_{-\underline{r}}^s \rightarrow R_{n+1}^- \otimes \chi_{-\underline{r}}^s$ is described by the natural surjection:

$$A / \langle X_i^{p^n(r_{i+n+1})}, i = 0, \dots, f-1 \rangle \twoheadrightarrow A / \langle X_i^{p^{n-1}(r_{i+n-1+1})}, i = 0, \dots, f-1 \rangle.$$

The dual of the Hecke morphism $T_{n+1}^- : R_{n+1}^- \otimes \chi_{-\underline{r}}^s \rightarrow R_n^- \otimes \chi_{-\underline{r}}^s$ is described by the monomorphism:

$$\begin{aligned} A / \langle X_i^{p^{n-1}(r_{i+n-1+1})}, i = 0, \dots, f-1 \rangle &\hookrightarrow A / \langle X_i^{p^n(r_{i+n+1})}, i = 0, \dots, f-1 \rangle \\ 1 &\mapsto \prod_{i=0}^{f-1} X_i^{p^{n-1}(p(r_{i+n+1})-(r_{i+n-1+1}))}. \end{aligned}$$

Finally $T_1^- : R_1 \rightarrow R_0$ is dualized to the natural morphism

$$\begin{aligned} (\chi_{\underline{r}})^\vee &\hookrightarrow A \otimes_k (\chi_{\underline{r}}^s)^\vee / \langle X_i^{(r_i+1)}, i = 0, \dots, f-1 \rangle \\ 1 &\mapsto \prod_{i=0}^{f-1} X_i^{r_i} \end{aligned}$$

Proof. This is deduced from the previous discussion for $n \geq 1$. The case $n = 1$ is immediate. \square

We can now describe precisely the Pontryagin dual $\mathfrak{S}_\infty^\bullet$ of $R_{\infty, \bullet}^-$ as a projective limit of certain explicit Iwasawa modules of finite length. More precisely

PROPOSITION 3.5. Let $\sigma = \sigma_{\underline{r}}$ be a Serre weight. For $\bullet \in \{0, 1\}$ there is an inductive system of Iwasawa modules

$$\cdots \rightarrow S_{\bullet} \times_{S_{\bullet+1}} \cdots \times_{S_n} S_{n+1} \rightarrow S_{\bullet} \times_{S_{\bullet+1}} \cdots \times_{S_{n-2}} S_{n-1} \rightarrow \cdots \rightarrow S_{\bullet}$$

such that, for all $n \geq 1$, the transition morphisms fit into a commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{Ker}_{n+1} & \longrightarrow & \cdots \times_{S_n} S_{n+1} & \longrightarrow & \cdots \times_{S_{n-2}} S_{n-1} \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{Ker}_{n+1} & \longrightarrow & S_{n+1} & \xrightarrow{(T_n^+)^{\vee}} & S_n \longrightarrow 0 \end{array} \quad (13)$$

$\downarrow S_{n-1}$
 $\downarrow (T_n^-)^{\vee}$

and where $\mathcal{Ker}_{n+1} = \langle X_{i-n+1}^{p^{n-1}(r_i+1)}, i = 0, \dots, f-1 \rangle \leq S_{n+1}$. In particular, for $n \geq 1$ the fibered product $S_{\bullet} \times_{S_{\bullet+1}} \cdots \times_{S_n} S_{n+1}$ is realized as the sub-module of S_{n+1} generated by the elements:

$$e_{2(j+1)+\bullet, i} \stackrel{\text{def}}{=} X_i^{p^{2j+\bullet}(r_{i+2j+\bullet+1})} \prod_{l=0}^{f-1} X_l^{\sum_{s=2j+1+\bullet}^{n-1} (r_{l+s+1}) p^s (-1)^{s+\bullet}}$$

for $i = 0, \dots, f-1$, $j = 0, \dots, \frac{n-1-\bullet}{2}$ and

- a) if $\bullet = 0$, the element $e_0 \stackrel{\text{def}}{=} \prod_{l=0}^{f-1} X_l^{-1 + \sum_{s=0}^{n-1} (r_{l+s+1}) p^s (-1)^s}$;
- b) if $\bullet = 1$ the element $e_1 \stackrel{\text{def}}{=} \prod_{l=0}^{f-1} X_l^{\sum_{s=0}^{n-1} (r_{l+s+1}) p^s (-1)^{s+1}}$.

Note that, *a priori*, it is not obvious that sub-module of S_{n+1} generated by the elements listed in Proposition 3.5 is stable under the action of Γ , U_0^+ .

Proof. The fibered products $S_{\bullet} \times_{S_{\bullet+1}} \cdots \times_{S_n} S_{n+1}$ are defined inductively through the morphisms $(T_n^{\pm})^{\vee}$ in the evident way: if we assume the injection $S_{\bullet} \times_{S_{\bullet+1}} \cdots \times_{S_{n-2}} S_{n-1} \hookrightarrow S_{n-1}$ being constructed, we define $S_{\bullet} \times_{S_{\bullet+1}} \cdots \times_{S_n} S_{n+1}$ through the following cartesian diagram

$$\begin{array}{ccc} S_{\bullet} \times_{S_{\bullet+1}} \cdots \times_{S_n} S_{n+1} & \xrightarrow{\cdots} & S_{\bullet} \times_{S_{\bullet+1}} \cdots \times_{S_{n-1}} S_{n-1} \\ \downarrow \text{dotted} & & \downarrow \\ S_{n+1} & \xrightarrow{(T_n^+)^{\vee}} & S_n \end{array}$$

$\downarrow S_{n-1}$
 $\downarrow (T_n^-)^{\vee}$

where the upper (resp. left) dotted arrow is an epimorphism (resp. monomorphism) by base change.

This shows that $S_{\bullet} \times_{S_{\bullet+1}} \cdots \times_{S_n} S_{n+1}$ is the Pontryagin dual of $R_{\bullet} \oplus_{R_{\bullet+1}} \cdots \oplus_{R_n} R_{n+1}$.

The commutative diagram (13) is obtained by the universal property of the fibered product $S_{\bullet} \times_{S_{\bullet+1}} \cdots \times_{S_n} S_{n+1}$; an easy diagram chase shows that the upper row is indeed an exact sequence of modules.

The realization of $S_{\bullet} \times_{S_{\bullet+1}} \cdots \times_{S_n} S_{n+1}$ as an explicit sub-module of S_{n+1} is an elementary induction, using the definition of the morphisms $(T_n^-)^{\vee}$ and noticing that $\mathcal{Ker}_{n+1} = \langle X_i^{p^{n-1}(r_{i+n-1+1})}, i = 0, \dots, f-1 \rangle$ for $n \geq 1$. \square

We can now introduce the universal Iwasawa module associated to the representation $\pi(\sigma, 0)$:

DEFINITION 3.6. Let $n \in 2\mathbf{N} + 1 + \bullet$, where $\bullet \in \{0, 1\}$. We define the Iwasawa modules

$$\mathfrak{S}_{n+1}^\bullet \stackrel{\text{def}}{=} S_\bullet \times_{S_{\bullet+1}} \cdots \times_{S_n} S_{n+1}, \quad \mathfrak{S}_\infty^\bullet \stackrel{\text{def}}{=} \varprojlim_{n \in 2\mathbf{N} + 1 + \bullet} \mathfrak{S}_{n+1}^\bullet.$$

Similarly, if $m \in 2\mathbf{N} + 1 + \bullet$, we define

$$\mathfrak{S}_{n+1}^{\geq m} \stackrel{\text{def}}{=} \ker \left(\mathfrak{S}_{n+1}^\bullet \rightarrow \mathfrak{S}_{m-1}^\bullet \right), \quad \mathfrak{S}_\infty^{\geq m} \stackrel{\text{def}}{=} \ker \left(\mathfrak{S}_\infty^\bullet \rightarrow \mathfrak{S}_{m-1}^\bullet \right).$$

By Proposition 3.5 we can deduce an explicit family of A generators (resp. topological generators) for the modules \mathfrak{S}_{n+1} , $\mathfrak{S}_{n+1}^{\geq m}$ (resp. \mathfrak{S}_∞ , $\mathfrak{S}_\infty^{\geq m}$).

Indeed, by the construction of the elements $e_{2(j+1)+\bullet, i}$ we see that

$$e_{2(j+1)+\bullet, i} = 0 \quad \text{if and only if} \quad (r_{i+2j+\bullet}, r_{i+2j+\bullet+1}) = (p-1, 0)$$

and the sets

$$\begin{aligned} \mathcal{G}_{n+1}^\bullet &\stackrel{\text{def}}{=} \{e_{2(j+1)+\bullet, i}, e_\bullet, \text{ for } 0 \leq j \leq \frac{n-1-\bullet}{2}, i \in \{0, \dots, f-1\} \text{ s.t. } (r_{i+2j+\bullet}, r_{i+2j+\bullet+1}) \neq (p-1, 0)\} \\ \mathcal{G}_{n+1}^{\geq m} &\stackrel{\text{def}}{=} \{e_{2(j+1)+\bullet, i}, \text{ for } \frac{m-1-\bullet}{2} \leq j \leq \frac{n-1-\bullet}{2}, i \in \{0, \dots, f-1\} \text{ s.t. } (r_{i+2j+\bullet}, r_{i+2j+\bullet+1}) \neq (p-1, 0)\} \end{aligned}$$

give a family of A -generators for the modules \mathfrak{S}_{n+1} , $\mathfrak{S}_{n+1}^{\geq m}$ (where $m \in 2\mathbf{N} + 1 + \bullet$).

Moreover, for any $0 \leq j \leq \frac{n-1-\bullet}{2}$ the families $\mathcal{G}_{2(j+1)+\bullet}^\bullet$, $\mathcal{G}_\bullet \stackrel{\text{def}}{=} \{e_\bullet\}$ (resp. $\mathcal{G}_{2(j+1)+\bullet}^{\geq m}$) are compatible with the transition morphisms

$$\begin{aligned} \mathfrak{S}_{n+1}^\bullet &\rightarrow \mathfrak{S}_{2(j+1)+\bullet}^\bullet \\ (\text{resp. } \mathfrak{S}_{n+1}^{\geq m} &\rightarrow \mathfrak{S}_{2(j+1)+\bullet}^{\geq m}). \end{aligned}$$

Therefore, we deduce a set of *topological* A -generators for the modules $\mathfrak{S}_\infty^\bullet$, $\mathfrak{S}_\infty^{\geq m}$:

$$\begin{aligned} \mathcal{G}_\infty^\bullet &\stackrel{\text{def}}{=} \{e_{2(j+1)+\bullet, i}, e_\bullet, \text{ for } j \in \mathbf{N}, i \in \{0, \dots, f-1\} \text{ s.t. } (r_{i+2j+\bullet}, r_{i+2j+\bullet+1}) \neq (p-1, 0)\} \\ \mathcal{G}_\infty^{\geq m} &\stackrel{\text{def}}{=} \{e_{2(j+1)+\bullet, i}, \text{ for } j \geq \frac{m-1-\bullet}{2}, i \in \{0, \dots, f-1\} \text{ s.t. } (r_{i+2j+\bullet}, r_{i+2j+\bullet+1}) \neq (p-1, 0)\} \end{aligned}$$

such that, for any $n \in 2\mathbf{N} + 1 + \bullet$, $i \in \{0, \dots, f-1\}$, we have $pr_{n+1}(e_{n+1, i}) = e_{n+1, i} \in \mathfrak{S}_{n+1}^\bullet$ (where $pr_{n+1} : \mathfrak{S}_\infty^\bullet \rightarrow \mathfrak{S}_{n+1}^\bullet$ is the canonical projection).

4. A filtration on monogenic Iwasawa modules

The first step in order to control the universal modules $\mathfrak{S}_\infty^\bullet$ lies in a precise knowledge of the Γ, U_0^+ -stable sub-modules of S_{n+1} .

This is a delicate problem: the action of Γ on the regular system of parameters X_i in A is rather subtle and our first approach is to work by successive approximation by appropriate p -th powers of the maximal ideal \mathfrak{m} of A . An explicit control of the U_0^+ -action is expected to be even more involved.

In this section, we want to show that the filtration by the maximal ideal \mathfrak{m} of A is stable by the extra action of Γ and U_0^+ . More precisely, we see that the action of Γ, U_0^+ is *trivial* on a graded piece $\mathfrak{m}^s/\mathfrak{m}^{s+i}$ if $0 \leq i \leq p-1$. Note that a non zero element of \mathfrak{m} can *not* be fixed by Γ , by [HMS], Théorème 1.1.

Recall (6) that A has a structure of an Iwasawa module via the isomorphism

$$\left(k[[K_0(p)]] \widehat{\otimes}_{k[[\mathbf{B}(\mathcal{O}_F) \cap K_0(p)]]} 1 \right) |_A \xrightarrow{\sim} A.$$

Moreover, A is endowed with a valuation

$$A \xrightarrow{\text{ord}} \mathbf{N} \cup \{\infty\}$$

$$\sum_{\underline{j}} \kappa_{\underline{j}} \underline{X}^{\underline{j}} \mapsto \min\{|\underline{j}|, \underline{j} \text{ s.t. } \kappa_{\underline{j}} \neq 0\}.$$

We are going to show that ord is compatible with the action of Γ , U_0^+ and, even more precisely,

PROPOSITION 4.1. *Let $g \in \Gamma$, U_0^+ and $P(\underline{X}) \in A$. Then*

$$\text{ord}((g-1) \cdot P(\underline{X})) \geq \text{ord}(P(\underline{X})) + (p-1).$$

The proof will occupy the rest of this section.

Recall that A is endowed with a Frobenius homomorphism ϕ defined by $\phi(X_i) = X_i^p$. For $k \in \mathbf{N}$ we write $(\phi^k(\mathfrak{m}))$ for the ideal of A generated by the image of \mathfrak{m} via the k -th composite of the Frobenius ϕ^k . Notice that any element $P(\underline{X})$ in $(\phi^k(\mathfrak{m}))$ verifies

$$\text{ord}(P(\underline{X})) \geq p^k.$$

The following obvious result will be used constantly:

LEMMA 4.2. *Let $n \in \mathbf{N}$ and $z \in \mathcal{O}_F$. In the Iwasawa algebra A we have:*

$$\begin{bmatrix} 1 & 0 \\ pz & 1 \end{bmatrix} \in 1 + \phi^{\text{val}(z)}(\mathfrak{m}).$$

Proof. Writing $z = p^{\text{val}(z)}z_0$ we have

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ pz & 1 \end{bmatrix} &= \left(\begin{bmatrix} 1 & 0 \\ pz_0 & 1 \end{bmatrix} \right)^{p^{\text{val}(z)}} \\ &= \left(\begin{bmatrix} 1 & 0 \\ pz_0 & 1 \end{bmatrix} - 1 \right)^{p^{\text{val}(z)}} + 1 \end{aligned}$$

and the result follows since the maximal ideal of A is the augmentation ideal. \square

We start from the action of Γ :

LEMMA 4.3. *Let $\gamma = 1 + px \in 1 + p\mathcal{O}_F \cong \Gamma$. Then, for any $i = 0, \dots, f-1$ we have*

$$\gamma \cdot X_i = X_i + (\phi^{\text{val}(x)+1}(\mathfrak{m})).$$

In particular, if $\underline{j} \in \mathbf{N}^f$ we have

$$\gamma \cdot \underline{X}^{\underline{j}} = \underline{X}^{\underline{j}} + \sum_{\underline{i} \geq 0} \underline{X}^{\underline{j}-\underline{i}} (\phi^{\text{val}(x)+1}(\mathfrak{m}))^{|\underline{i}|}.$$

Proof. We have, writing $z \stackrel{\text{def}}{=} p[\lambda]x$

$$\begin{aligned} \gamma \cdot X_i &= \sum_{\lambda \in k_F^\times} \lambda^{-p^i} \begin{bmatrix} 1 & 0 \\ p[\lambda] + p^2[\lambda]x & 1 \end{bmatrix} \\ &= \sum_{\lambda \in k_F^\times} \lambda^{-p^i} \begin{bmatrix} 1 & 0 \\ p[\lambda] & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ pz & 1 \end{bmatrix} \\ &= \sum_{\lambda \in k_F^\times} \lambda^{-p^i} \begin{bmatrix} 1 & 0 \\ p[\lambda] & 1 \end{bmatrix} (1 + \phi^{\text{val}(x)+1}(\mathfrak{m})), \end{aligned}$$

where the last equality follows from Lemma 4.2.

The second statement is then clear, as the elements of Γ act by k -algebra endomorphisms. \square

The action of U_0^+ is more complicate to clarify: as the group U_0^+ does not normalize U_0^- , its does not act by ring endomorphisms on A . Notice that for $x \in \mathcal{O}_F$ and $j \geq 1$ the action of $\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$ on the Iwasawa algebra $k[[p^{j-1}U_0^-]]$ is obtained, by linearity, from the following continuous maps

$$\begin{aligned} \mathcal{O}_F &\xrightarrow{\delta_{x,j}} \mathcal{O}_F \\ z &\longmapsto \sum_{k=0}^{\infty} (-p^j x)^k z^{k+1} \end{aligned}$$

in the sense that

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ p^j z & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ p^j \delta_{x,j}(z) & 1 \end{bmatrix} \begin{bmatrix} 1 + p^j x z & 0 \\ 0 & 1 - p^j x \delta_{x,j}(z) \end{bmatrix} \begin{bmatrix} 1 & x(1 + p^j x z)^{-1} \\ 0 & 1 \end{bmatrix} \quad (14)$$

In order to shorten the notations, we will write $\delta_x \stackrel{\text{def}}{=} \delta_{x,1}$ when $j = 1$.

PROPOSITION 4.4. *Let $x \in \mathcal{O}_F$ and $\underline{X}^j \in A$. Then we have*

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \cdot \underline{X}^j = \underline{X}^j + \sum_{i \neq 0} \underline{X}^{j-i} (\phi^{val(x)+1}(\mathfrak{m}))^{|i|}.$$

Proof. We recall (6) that the action of U_0^+ on A is induced by the A -linear isomorphism

$$\begin{aligned} A &\xrightarrow{\sim} (k[[K_0(p)]] \widehat{\otimes}_k [\mathbf{B}(\mathcal{O}_F) \cap K_0(p)] 1) |_A \\ \underline{X}^j &\longmapsto \underline{X}^j \otimes 1. \end{aligned}$$

Notice that left hand side is *just a module* over $k[[K_0(p)]]$.

We are going to prove the result by induction on the length of \underline{j} .

For $\underline{j} = e_i$ we deduce from (14):

$$\begin{aligned} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \cdot (X_i \otimes 1) &= \sum_{\lambda \in k_F^\times} \lambda^{-p^i} \begin{bmatrix} 1 & 0 \\ p\delta_x([\lambda]) & 1 \end{bmatrix} \begin{bmatrix} 1 + px[\lambda] & x \\ 0 & 1 - px\delta_x([\lambda]) \end{bmatrix} \otimes 1 \\ &= \sum_{\lambda \in k_F^\times} \lambda^{-p^i} \begin{bmatrix} 1 & 0 \\ p[\lambda] & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ p(\delta_x([\lambda]) - [\lambda]) & 1 \end{bmatrix} \otimes 1 \\ &= (X_i + (\phi^{val(x)+1}(\mathfrak{m}))) \otimes 1 \end{aligned}$$

where the last equality is obtained from Lemma 4.2 noticing that $val(\delta_x([\lambda]) - [\lambda]) \geq val(x) + 1$.

We treat now the inductive step.

Let $i \in \{0, \dots, f-1\}$ be such that $j_i > 0$. We then have, from (14)

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \cdot (\underline{X}^{\underline{j}} \otimes 1) = \left(\sum_{\lambda \in k_F^\times} \lambda^{-p^i} \begin{bmatrix} 1 & 0 \\ p\delta_x([\lambda]) & 1 \end{bmatrix} \begin{bmatrix} 1 + px[\lambda] & x \\ 0 & 1 - px\delta_x([\lambda]) \end{bmatrix} \underline{X}^{j-e_i} \right) \otimes 1. \quad (15)$$

The inductive hypothesis let us write

$$\begin{bmatrix} 1 & x(1 + px[\lambda])^{-1} \\ 0 & 1 \end{bmatrix} \underline{X}^{j-e_i} \otimes 1 = \left(\underline{X}^{j-e_i} + \sum_{l \neq 0} \underline{X}^{j-e_i-l} (\phi^{val(x)+1}(\mathfrak{m}))^{|l|} \right) \otimes 1$$

and, by Lemma 4.3 we deduce

$$\begin{bmatrix} 1 + px[\lambda] & 0 \\ 0 & 1 - px\delta_x([\lambda]) \end{bmatrix} \cdot \left(\underline{X}^{j-e_i-l}(\phi^{val(x)+1}(\mathbf{m}))^{|\underline{l}|} \right) = \sum_{l' \geq 0} \underline{X}^{j-e_i-l-l'}(\phi^{val(x)+1}(\mathbf{m}))^{|\underline{l}+l'|}.$$

Thus,

$$\begin{bmatrix} 1 + px[\lambda] & x \\ 0 & 1 - px\delta_x([\lambda]) \end{bmatrix} \cdot (\underline{X}^{j-e_i} \otimes 1) = \left(\underline{X}^{j-e_i} + \sum_{l \geq 0} \underline{X}^{j-e_i-l}(\phi^{val(x)+1}(\mathbf{m}))^{|\underline{l}|} \right) \otimes 1 \quad (16)$$

and we deduce from (15) and (16)

$$\begin{aligned} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \cdot \underline{X}^j &= \sum_{\lambda \in k_F^\times} \lambda^{-p^i} \begin{bmatrix} 1 & 0 \\ p[\lambda] & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ p(\delta_x([\lambda]) - [\lambda]) & 1 \end{bmatrix} \left(\underline{X}^{j-e_i} + \sum_{l \geq 0} \underline{X}^{j-e_i-l}(\phi^{val(x)+1}(\mathbf{m}))^{|\underline{l}|} \right) \\ &= ((\phi^{val(x)+1}(\mathbf{m})) + X_i) \left(\underline{X}^{j-e_i} + \sum_{l \geq 0} \underline{X}^{j-e_i-l}(\phi^{val(x)+1}(\mathbf{m}))^{|\underline{l}|} \right) \\ &= \underline{X}^j + \sum_{i \geq 0} \underline{X}^{j-i}(\phi^{val(x)+1}(\mathbf{m}))^{|\underline{i}|}. \end{aligned}$$

This concludes the inductive step and the proof is complete. \square

As a corollary, we get

COROLLARY 4.5. *For any $k \geq 0$ define the following subspace of A :*

$$\mathcal{I}_k \stackrel{\text{def}}{=} \text{ord}^{k-}(\{n \in \mathbf{N} \cup \{\infty\}, n \geq k\}).$$

Then \mathcal{I}_k is a A -ideal, stable by the action of Γ , U_0^+ and H . Moreover the action of Γ , U_0^+ is trivial on the quotient

$$\mathcal{I}_k / \mathcal{I}_{k+(p-1)}.$$

Proof. It follows from Lemma 4.3 and Proposition 4.4. \square

As we have, for all $n \geq 1$, an epimorphism of Iwasawa modules $A \twoheadrightarrow S_{n+1}(\sigma)$ we deduce

COROLLARY 4.6. *Let $n, k \geq 1$. Let σ be a Serre weight and write $\overline{\mathcal{I}}_k$ for the image of the Iwasawa sub-module \mathcal{I}_k via the projection $A \twoheadrightarrow S_{n+1}(\sigma)$. The filtration $\{\overline{\mathcal{I}}_k\}_k$ is stable by the action of Γ , U_0^+ and H . Moreover the action of Γ , U_0^+ is trivial on the quotient*

$$\overline{\mathcal{I}}_k / \overline{\mathcal{I}}_{k+(p-1)}.$$

Proof. Omissis. \square

5. The twisted Frobenius

In this section we construct a “twisted” Frobenius morphism between the graded pieces $\mathcal{K}er_{n+1}$ of the natural filtration on $\mathfrak{S}_\infty^\bullet$. This morphism turns out to be Γ , H -equivariant and it is obtained from the twisted Frobenius \mathcal{F} on A (the latter induced from conjugation by the element $\alpha\omega$).

The main properties of the twisted Frobenius are listed in Proposition 5.5 and 5.8: roughly speaking, this morphism lets us translate information from $S_2(\sigma)$, where computations are still accessible, to higher dimensional quotients $S_{n+1}(\sigma)$, where things get considerably more complicated.

We subsequently pursue the investigation on the twisted Frobenius (§5.3). We determine an explicit sub-module of $\mathfrak{S}_\infty^0 \oplus \mathfrak{S}_\infty^1$ of finite colength, endowed with an \mathcal{F} -semilinear, Γ , H -equivariant

endomorphism, which turns out to be p -étale. The main result is summarized in Proposition 5.12. As a corollary, we deduce that such sub-module is of finite type over the skew polynomial ring $A[[\mathcal{F}]]$ (Corollary 5.13).

We remark that some of the statements of §5.1, 5.2, which refer to the $k[[K_0(p^{n+1})]]$ -modules $(\sigma^{(n+1)})^\vee$, hold in greater generality for any $k[[K_0(p^{n+1})]]$ -module. Nevertheless we believe that the specialized statements of Lemmas 5.2 and 5.7 are more expressive for the subsequent applications to the universal module $\mathfrak{S}_\infty^\bullet$.

5.1 Analysis for the trivial weight

For $l \geq j \geq 1$ we write

$$\mathbf{S}_l^{(j)} \stackrel{\text{def}}{=} ((\text{ind}_{K_0(p^l)}^{K_0(p^j)} 1)|_{U_0^-})^\vee.$$

The aim of this paragraph is to construct a suitable Frobenius morphism between Iwasawa modules of the form $\mathbf{S}_l^{(j)}$.

Let $j \geq 1$ and define the following parahoric subgroup of G

$$\tilde{K}_0(p^{j+1}) \stackrel{\text{def}}{=} \left\{ \begin{bmatrix} a & p^{-1}b \\ p^{j+1}c & d \end{bmatrix}, a, b, c, d \in \mathcal{O}_F \right\}.$$

Conjugation by $\begin{bmatrix} p^{-1} & 0 \\ 0 & 1 \end{bmatrix}$ gives an isomorphism of profinite groups

$$\begin{aligned} c_j : K_0(p^j) &\xrightarrow{\sim} \tilde{K}_0(p^{j+1}) \\ k &\longmapsto \begin{bmatrix} p^{-1} & 0 \\ 0 & 1 \end{bmatrix} k \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

hence an isomorphism of Iwasawa algebras

$$\mathcal{F}_j^0 : k[[K_0(p^j)]] \xrightarrow{\sim} k_F[[\tilde{K}_0(p^{j+1})]]$$

and, for any $l \geq j$, an isomorphism of Iwasawa modules

$$\begin{array}{ccc} k[[K_0(p^j)]] \times k[[K_0(p^j)]] \widehat{\otimes}_{k[[K_0(p^l)]]} 1 & \longrightarrow & k[[K_0(p^j)]] \widehat{\otimes}_{k[[K_0(p^l)]]} 1 \\ \downarrow \wr & & \downarrow \wr \\ k[[\tilde{K}_0(p^{j+1})]] \times k[[\tilde{K}_0(p^{j+1})]] \widehat{\otimes}_{k[[\tilde{K}_0(p^{l+1})]]} 1 & \longrightarrow & k[[\tilde{K}_0(p^{j+1})]] \widehat{\otimes}_{k[[\tilde{K}_0(p^{l+1})]]} 1. \end{array}$$

Moreover for any $j \geq 2$ we have that $\overline{\mathbf{B}}(p^j) \stackrel{\text{def}}{=} \overline{\mathbf{B}}(\mathcal{O}_F) \cap K_0(p^j)$ is naturally a subgroup of $K_0(p^j) \cap \tilde{K}_0(p^j)$ and we obtain the following commutative diagram of Iwasawa modules over $k[[\overline{\mathbf{B}}(p^j)]] \xrightarrow{\sim} k[[\overline{\mathbf{B}}(p^{j+1})]]$:

$$\begin{array}{ccc} (k[[K_0(p^j)]] \widehat{\otimes}_{k[[K_0(p^l)]]} 1)|_{k[[\overline{\mathbf{B}}(p^j)]]} & \xrightarrow{\sim} & (k_F[[\tilde{K}_0(p^{j+1})]] \widehat{\otimes}_{k[[\tilde{K}_0(p^{l+1})]]} 1)|_{k[[\overline{\mathbf{B}}(p^{j+1})]]} & (17) \\ & & \downarrow \wr & \\ & & (k[[K_0(p^{j+1})]] \widehat{\otimes}_{k[[K_0(p^{l+1})]]} 1)|_{k[[\overline{\mathbf{B}}(p^{j+1})]]} & \\ & \searrow \mathcal{F}_{j,l}^0 & \downarrow & \\ & & (k[[K_0(p^j)]] \widehat{\otimes}_{k[[K_0(p^{l+1})]]} 1)|_{k[[\overline{\mathbf{B}}(p^{j+1})]]} & \end{array}$$

DEFINITION 5.1. Let $l \geq j \geq 1$. We define the morphism

$$\mathcal{F} = \mathcal{F}_{j,l} : \mathbf{S}_l^{(j)} \hookrightarrow \mathbf{S}_{l+1}^{(j)}$$

as the $k[[p^{j-1}U_0^-]]$ -restriction of the morphism $\mathcal{F}_{j,l}^0$ defined by diagram (17).

As the morphism \mathcal{F} is induced by conjugation by an element of the torus $\mathbf{T}(F)$ we can easily deduce its behavior with respect to the $\mathbf{T}(\mathcal{O}_F)$ -action.

Recall that the $k[[p^{j-1}U_0^-]]$ -modules $\mathbf{S}_l^{(j)}$, $\mathbf{S}_{l+1}^{(j)}$ are endowed with a natural structure of k -algebras.

LEMMA 5.2. The morphism \mathcal{F} respects the natural k -algebra structures on $\mathbf{S}_l^{(j)}$, $\mathbf{S}_{l+1}^{(j)}$. It is injective, with image $\mathbf{S}_{l+1}^{(j+1)}$ and it is described explicitly by

$$\begin{aligned} \mathcal{F} : \mathbf{S}_l^{(j)} &\longrightarrow \mathbf{S}_{l+1}^{(j)} \\ X_i^{p^{j-1}} &\longmapsto X_{i-1}^{p^j}. \end{aligned}$$

Moreover it is H and Γ -equivariant.

Proof. From the diagram (17) we obtain, by restriction to $k[[p^{j-1}U_0^-]]$, an isomorphism of Iwasawa modules over $k[[p^{j-1}U_0^-]] \xrightarrow{\sim} k[[p^jU_0^-]]$:

$$\mathbf{S}_l^{(j)}|_{k[[p^{j-1}U_0^-]]} \xrightarrow{\sim} \mathbf{S}_{l+1}^{(j+1)}|_{k[[p^jU_0^-]]}.$$

This shows that \mathcal{F} respects the k -algebra structures, and it induces an isomorphism on $\mathbf{S}_{l+1}^{(j)}$.

The explicit description of \mathcal{F} in terms of the parameters $X_i^{p^{j-1}} \in k[[K_0(p^j)]]$ is a straightforward computation.

Finally, notice that the elements of H , Γ , as well as the element $\begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}$, belong to the torus $\mathbf{T}(F)$. As the torus is commutative and the actions of H , Γ and \mathcal{F} are induced by conjugation on $p^{j-1}U_0^-$, the H , Γ equivariance of the twisted Frobenius is clear. □

We recall that U_0^+ acts on $\mathbf{S}_l^{(j)}$ by k -linear morphisms which are not, a priori, compatible with the multiplicative structure on $\mathbf{S}_l^{(j+1)}$.

Nevertheless we have

LEMMA 5.3. The action of U_0^+ on $\mathbf{S}_l^{(j)}$ is trivial if and only if $l \leq 2j$.

Proof. Recall that for $z \in \mathcal{O}_F$ the elements

$$\begin{bmatrix} 1 & 0 \\ p^j z & 1 \end{bmatrix} \otimes 1$$

give a family of linear generators for the Iwasawa module $k[[K_0(p^j)]] \widehat{\otimes}_{k[[K_0(p^l)]]} 1$. If $x \in \mathcal{O}_F$ we have

$$p^j \delta_{x,j}(z) = p^j z + p^{2j} z'$$

for a suitable $z' \in \mathcal{O}_F$ with $val(z') = val(xz^2)$ and the statement is then clear from the equality (14). □

We deduce

COROLLARY 5.4. *Let $m \geq l \geq j \geq 1$. The $(m-l)$ -composite of \mathcal{F} :*

$$\mathbf{S}_l^{(j)} \xrightarrow{\mathcal{F}^{m-l}} \mathbf{S}_m^{(j)}$$

factors through the U_0^+ invariants of $\mathbf{S}_m^{(j)}$ if and only if $0 \leq 2(j-l) + m$.

Proof. It suffices to remark that \mathcal{F}^{m-l} induces an isomorphism of $\mathbf{S}_l^{(j)}$ onto the sub-algebra $\mathbf{S}_m^{(j+m-l)}$ of $\mathbf{S}_m^{(j)}$ and use Lemma 5.3. \square

5.1.1 *The skew power series ring $A[[\mathcal{F}]]$.* The evident, similar constructions of the previous section, with $K_0(p^l)$ replaced by $K_0(p^\infty)$, give us a Frobenius endomorphism on A :

PROPOSITION 5.5. *We have a Γ , H -equivariant monomorphism of local k -algebras $\mathcal{F} : A \rightarrow A$ described by*

$$\begin{aligned} \mathcal{F} : A &\longrightarrow A \\ X_i &\longmapsto X_{i-1}^p. \end{aligned}$$

In particular we have a decomposition

$$A \cong \bigoplus_{0 \leq i \leq p-1} \mathcal{F}(A) \underline{X}^i$$

and \mathcal{F} is a flat endomorphism of A .

Proof. Omissis. \square

We recall (cf. [Ven], §2) that we can consider the skew power series ring $A[[\mathcal{F}]]$, whose elements are formal power series $\sum_{i=0}^{\infty} a_i \mathcal{F}^i$ with $a_i \in A$ and multiplication law induced by

$$\mathcal{F} \cdot a \stackrel{\text{def}}{=} \mathcal{F}(a) \mathcal{F}$$

for any $a \in A$. It is a local ring, endowed with a structure of complete, T_2 topological ring, a basis of open neighborhood of 0 being described by

$$\mathcal{B}_k \stackrel{\text{def}}{=} \prod_{i=0}^{k-1} \mathfrak{m}^k \mathcal{F}^i \times \prod_{i=k}^{\infty} A \mathcal{F}^i$$

for $k \in \mathbf{N}$. In particular, the skew polynomial ring $A[\mathcal{F}]$ is a dense sub-ring of $A[[\mathcal{F}]]$.

We introduce the following notion (cf. [Fon], §B 1.3)

DEFINITION 5.6. *An \mathcal{F} -semilinear morphism φ of profinite A -modules $D_1 \xrightarrow{\varphi} D_2$ is p -étale if the image of the natural map*

$$A \widehat{\otimes}_{A, \mathcal{F}} D_1 \xrightarrow{id \otimes \varphi} D_2$$

has finite colength.

5.2 Analysis for a general Serre weight

In this section we are going to use the results of §5.1 in order to deduce Proposition 5.8. One again we will crucially make use of Proposition 3.3.

Let σ be a Serre weight. Up to a twist by a smooth character of $K_0(p)$ we can assume that the highest weight space affords the trivial character of $K_0(p)$. We have

LEMMA 5.7. *Let $l \geq j \geq 1$. There exists a unique morphism $\mathcal{F}_\sigma : S_l^{(j)}(\sigma) \rightarrow S_{l+1}^{(j)}(\sigma)$ of k -algebras such that the following diagram commutes*

$$\begin{array}{ccc} \mathbf{S}_{l+1}^{(j)} & \xrightarrow{\mathcal{F}} & \mathbf{S}_{l+2}^{(j)} \\ \downarrow & & \downarrow \\ S_l^{(j)}(\sigma) & \xrightarrow{\mathcal{F}_\sigma} & S_{l+1}^{(j)}(\sigma), \end{array} \quad (18)$$

where the vertical arrows are induced by the morphisms $(\phi_l)^\vee$ and $(\phi_{l+1})^\vee$ of Proposition 3.3.

Proof. It suffices to use the explicit definition of \mathcal{F} and recall that the kernel of the vertical arrow on the RHS (resp. on the LHS) is the ideal generated by the elements $X_{i-l}^{p^l(r_i+1)}$ (resp. $X_{i-l+1}^{p^{l-1}(r_i+1)}$) for $i = 0, \dots, f-1$. \square

As $\mathbf{S}_{l+1}^{(j)} \rightarrow S_l^{(j)}(\sigma)$ is a morphism of Iwasawa modules we get

PROPOSITION 5.8. *Let $l \geq j \geq 1$. We have a monomorphism of k -algebras*

$$\begin{aligned} \mathcal{F}_\sigma : S_l^{(j)}(\sigma) &\longrightarrow S_{l+1}^{(j)}(\sigma) \\ X_i^{p^{j-1}} &\longmapsto X_{i-1}^{p^j} \end{aligned}$$

verifying the following properties :

- i) *The morphism \mathcal{F}_σ is Γ and H -equivariant.*
- ii) *The morphism \mathcal{F}_σ admits the factorization*

$$\begin{array}{ccc} S_n^{(j)}(\sigma) & \xrightarrow{\mathcal{F}_\sigma} & S_{n+1}^{(j)}(\sigma) \\ & \searrow \cong & \uparrow \\ & & S_{n+1}^{(j+1)}(\sigma). \end{array}$$

- iii) *Let $m \geq l$. If $0 \leq 2(j-l) + m - 1$ then the composite morphism*

$$S_l^{(j)}(\sigma) \xrightarrow{\mathcal{F}^{m-l}} S_m^{(j)}(\sigma)$$

factors through the U_0^+ -invariants of $S_m^{(j)}(\sigma)$.

Proof. Part i) and ii) follow from the properties of the morphism \mathcal{F} (Lemma 5.2) and from Lemma 5.7, recalling that the vertical arrows in the diagram (18) are morphisms of Iwasawa modules.

Property iii) follows from Corollary 5.4 using the epimorphism $\mathbf{S}_{m+1}^j \rightarrow S_m^{(j)}(\sigma)$ (which is U_0^+ -equivariant). \square

5.3 The twisted Frobenius on the universal Iwasawa module

The aim of this section is to construct, starting from the twisted Frobenius \mathcal{F}_σ of Proposition 5.8, a Frobenius morphism \mathcal{F} on a suitable Iwasawa sub-module $\mathfrak{S}_\infty^{\geq 1} \oplus \mathfrak{S}_\infty^{\geq 2}$ of $\mathfrak{S}_\infty^0 \oplus \mathfrak{S}_\infty^1$. Such sub-module is of finite co-length and the action of \mathcal{F} is p -étale. Moreover, $\mathfrak{S}_\infty^{\geq 1} \oplus \mathfrak{S}_\infty^{\geq 2}$ is of finite rank on the skew power series ring $A[[\mathcal{F}]]$. Throughout this section $\sigma = \sigma_\tau$ is a fixed Serre weight.

We start from the following

LEMMA 5.9. For $n \geq 3$ we have commutative diagrams of k -linear spaces:

$$\begin{array}{ccc} S_{n-1} & \xrightarrow{\mathcal{F}_\sigma} & S_n \\ \uparrow & & \uparrow \\ S_{n-2} & \xrightarrow{\mathcal{F}_\sigma} & S_{n-1} \end{array} \qquad \begin{array}{ccc} S_{n-1} & \xrightarrow{\mathcal{F}_\sigma} & S_n \\ \downarrow & & \downarrow \\ S_{n-2} & \xrightarrow{\mathcal{F}_\sigma} & S_{n-1} \end{array}$$

where the horizontal arrows are the monomorphisms of k -algebras of Proposition 5.8 (with $j = 1$) and the vertical arrows are the morphisms of Iwasawa modules defined in Proposition 3.4.

Proof. The commutativity of the diagrams can be checked directly, using the definition of the morphisms in terms of the regular parameters X_i (noticing that \mathcal{F}_σ is a morphism of k -algebras). The details are left to the reader. \square

Since the Iwasawa modules $\mathfrak{S}_{n+1}^\bullet$ admit an explicit family of A -generators, we easily see that $\mathcal{F}_\sigma : S_{n+1} \rightarrow S_{n+2}$ induces a morphism between appropriate sub-modules of \mathfrak{S}_{n+1} and \mathfrak{S}_{n+2} .

PROPOSITION 5.10. Let $n \in 2\mathbb{N} + 1$. The morphism $\mathcal{F}_\sigma : S_{n+1} \hookrightarrow S_{n+2}$ verifies

$$\mathcal{F}_\sigma(e_{2(j+1),i}) = e_{2(j+1)+1,i-1} \in \mathcal{G}_{n+2}^{\geq 2}$$

for all $e_{2(j+1),i} \in \mathcal{G}_{n+1}^{\geq 1}$ (cf. Definition 3.6 et seq.).

Similarly, for $m \in 2\mathbb{N} + 2$, the morphism $\mathcal{F}_\sigma : S_{m+1} \hookrightarrow S_{m+2}$ verifies

$$\mathcal{F}_\sigma(e_{2(j+1)+1,i}) = e_{2(j+2),i-1} \in \mathcal{G}_{m+2}^{\geq 1}$$

for all $e_{2(j+1)+1,i} \in \mathcal{G}_{m+1}^{\geq 2}$

In particular, we have the following commutative diagrams

$$\begin{array}{ccc} S_{n+1} & \xrightarrow{\mathcal{F}_\sigma} & S_{n+2} \\ \uparrow & & \uparrow \\ \mathfrak{S}_{n+1}^{\geq 1} & \xrightarrow{\mathcal{F}} & \mathfrak{S}_{n+2}^{\geq 2} \end{array} \qquad \begin{array}{ccc} S_{m+1} & \xrightarrow{\mathcal{F}_\sigma} & S_{m+2} \\ \uparrow & & \uparrow \\ \mathfrak{S}_{m+1}^{\geq 2} & \xrightarrow{\mathcal{F}} & \mathfrak{S}_{m+2}^{\geq 1} \end{array}$$

and the morphisms

$$\mathfrak{S}_{n+1}^{\geq 1} \xrightarrow{\mathcal{F}} \mathfrak{S}_{n+2}^{\geq 2}, \qquad \mathfrak{S}_{m+1}^{\geq 2} \xrightarrow{\mathcal{F}} \mathfrak{S}_{m+2}^{\geq 1}$$

are Γ , H equivariant, \mathcal{F} -semilinear and p -étale over A .

Proof. The first part of the statement follows from an elementary computation on the elements $e_{2(j+1),i} \in \mathcal{G}_{n+1}^{\geq 1}$ (resp. $e_{2(j+1)+1,i} \in \mathcal{G}_{m+1}^{\geq 2}$).

We deduce the factorization of the morphism $S_{n+1} \xrightarrow{\mathcal{F}_\sigma} S_{n+2}$ (resp. $S_{m+1} \hookrightarrow S_{m+2}$), as the module $\mathfrak{S}_{n+2}^{\geq 2}$ (resp. $\mathfrak{S}_{m+2}^{\geq 1}$) is generated, over A , by the elements $e_{2(j+1)+1,i} \in \mathcal{G}_{n+2}^{\geq 2}$ (resp. by the elements $e_{2(j+1),i} \in \mathcal{G}_{m+2}^{\geq 1}$).

The induced morphisms on the Iwasawa modules are clearly \mathcal{F} -semilinear and Γ , H equivariant. Their p -étale nature follows again by noticing that the A -generators of $\mathfrak{S}_{n+2}^{\geq 2}$ (resp. $\mathfrak{S}_{m+2}^{\geq 3}$) are the elements $e_{2(j+1)+1,i} \in \mathcal{G}_{n+2}^{\geq 2}$ (resp. the elements $e_{2(j+1),i} \in \mathcal{G}_{m+2}^{\geq 3}$) and the cokernel of $\mathfrak{S}_{m+2}^{\geq 3} \hookrightarrow \mathfrak{S}_{m+2}^{\geq 1}$ is the finite A -module \mathcal{Ker}_2 . \square

We are now left to prove that the morphisms of Proposition 5.10 are compatible with the transition maps of the projective system defining the universal modules $\mathfrak{S}_\infty^{\geq 1}$, $\mathfrak{S}_\infty^{\geq 2}$.

which is p -étale and verifies

$$\mathcal{F}(e_{2(j+1)+1,i}) = e_{2(j+2)+1,i-1} \in \mathcal{G}_\infty^{\geq 1}$$

for all $e_{2(j+1),i} \in \mathcal{G}_\infty^{\geq 2}$.

Proof. The assertions follow from Proposition 5.10 and the compatibility with the transition morphisms given by Proposition 5.11. For the p -étale property of the second morphism we just remark that, from the proof of Proposition 5.10, we have an exact sequence

$$A \otimes_{\mathcal{F},A} \mathfrak{S}_{m+1}^{\geq 2} \rightarrow \mathfrak{S}_{m+2}^{\geq 1} \rightarrow \mathcal{Ker}_2 \rightarrow 0$$

for all $m \in 2\mathbf{N} + 2$, and by passing to the limit we get a complex

$$A \widehat{\otimes}_{\mathcal{F},A} \mathfrak{S}_\infty^{\geq 2} \rightarrow \mathfrak{S}_\infty^{\geq 1} \rightarrow \mathcal{Ker}_2 \rightarrow 0$$

which is exact (the transition morphisms in the projective system are all epi). \square

In particular, we deduce a finiteness property for the modules $\mathfrak{S}_\infty^{\geq 1}$, $\mathfrak{S}_\infty^{\geq 2}$ on the twisted polynomial algebra $A[[\mathcal{F}^2]]$:

COROLLARY 5.13. *For $\bullet \in \{0, 1\}$ we have a $A[[\mathcal{F}^2]]$ -equivariant surjection*

$$\bigoplus_{i=0}^{f-1} A[[\mathcal{F}^2]]e_{2+\bullet,i} \longrightarrow \mathfrak{S}_\infty^{\geq \bullet+1}$$

$$e_{2+\bullet,i} \mapsto e_{2+\bullet,i}.$$

Proof. To ease notation, we consider the case where $\bullet = 0$. It is clear by proposition 5.12 that for all $l \in \mathbf{N}$ we have a semilinear morphism

$$\mathcal{F}^{2l} : \mathfrak{S}_\infty^{\geq 1} \hookrightarrow \mathfrak{S}_\infty^{\geq 1}$$

which verifies

$$\mathcal{F}^{2l}(e_{2(j+1)+1,i}) = e_{2(j+l+1),i-2l}$$

for all $j \in \mathbf{N}$, $i \in \{0, \dots, f-1\}$. We deduce that the natural morphism

$$\bigoplus_{i=0}^{f-1} A[\mathcal{F}^2]e_{2+\bullet,i} \longrightarrow \mathfrak{S}_\infty^{\geq 1}$$

$$e_{2+\bullet,i} \mapsto e_{2+\bullet,i}.$$

is $A[\mathcal{F}^2]$ -linear, continuous and with dense image. Since the completion $A[[\mathcal{F}^2]]$ is compact and $\mathfrak{S}_\infty^{\geq \bullet+1}$ is separated, the statement follows. \square

6. A filtration on the ideals \mathcal{Ker}_{n+1}

We recall that, for $\bullet \in \{0, 1\}$, the universal module $\mathfrak{S}_\infty^\bullet$ is a pseudo-compact module over A , with a separated filtration of open neighborhood of 0 whose graded pieces are isomorphic to \mathcal{Ker}_{n+1} for $n \in 2\mathbf{N} + 1 + \bullet$. For $n \geq 1$ the description of the graded pieces \mathcal{Ker}_{n+1} in terms of the regular parameters X_i is deduced by Proposition 3.3:

$$\mathcal{Ker}_{n+1} = \langle e_{n+1,i}, i \in \{0, \dots, f-1\} \rangle_A$$

where $e_{n+1,i} = \mathcal{F}^{n-1}(X_{i+n-1}^{r_{i+n-1}+1})$.

As we did for the monogenic modules $S_{n+1}(\sigma)$, we endow the module \mathcal{Ker}_{n+1} with a natural filtration $I_{k,n+1}$, which turns out to be stable for the extra structures defined on \mathcal{Ker}_{n+1} .

The H , Γ -stability of $I_{k,n+1}$ follows easily, as these groups act by algebra homomorphisms on $S_{n+1}(\sigma)$. The action of U_0^+ is, again, more delicate : we use in a crucial way the fact that the generating family of $\mathcal{K}er_{n+1}$ lies *in the image* of the twisted Frobenius \mathcal{F}^{n-1} . In particular we find some differences in the behavior of the filtration $I_{k,n+1}$ and $I_{k,2}$ when $n \geq 2$.

We are strongly convinced that for a supersingular sub-module S_π of $\mathfrak{S}_\infty^\bullet$ (i.e. a sub-module induced by a supersingular quotient π of $\pi(\sigma, 0)$) the graded pieces of the induced filtration on S_π are generated by elements which are *not all* U_0^+ -invariant (at least eventually).

Throughout this section $\sigma = \sigma_r$ is a fixed Serre weight as in §3.

For any $n \geq 1$ we have an epimorphism of A -modules

$$\bigoplus_{i=0}^{f-1} A \cdot e_{n+1,i} \longrightarrow \mathcal{K}er_{n+1} \tag{19}$$

$$e_{n+1,i} \longmapsto \mathcal{F}^{n-1}(X_{i+n-1}^{r_{i+n-1}+1})$$

which is H -equivariant if we make H act by the character $\mathfrak{a}^{-p^i(r_i+1)}\chi_{-r}^s$ on $e_{n+1,i-n+1}$.

Notice that for any $i \in \{0, \dots, f-1\}$ we have

$$\ker(A \cdot e_{n+1,i} \rightarrow \mathcal{K}er_{n+1}) = \langle X_i^{p^n(r_{i+n+1})-p^{n-1}(r_{i+n-1}+1)}, X_s^{p^n(r_s+1)} \text{ for } s \neq i \rangle$$

so that

$$\{e_{n+1,i}, \text{ for } i \in \{0, \dots, f-1\} \text{ s.t. } (r_{i+(n-1)}, r_{i+n}) \neq (p-1, 0)\}$$

is a minimal set of A -generators for $\mathcal{K}er_{n+1}$.

The Iwasawa module $\bigoplus_{i=0}^{f-1} A \cdot e_{n+1,i}$ is endowed with the valuation ord_{n+1} of the infimum

$$ord_{n+1}\left(\sum_{i=0}^{f-1} P_i(\underline{X})e_{(n+1),i}\right) \stackrel{\text{def}}{=} \min\{ord(P_i(\underline{X})), i = 0, \dots, f-1\}$$

hence with a filtration $\{I_{k,n+1}^0\}_k$. By Proposition 4.1 the filtration $\{I_{k,n+1}^0\}_k$ is Γ , U_0^+ -stable (with these groups acting trivially on the quotient of two layers at distance $p-1$).

Let $\{I_{k,n+1}\}_k$ be the filtration on $\mathcal{K}er_{n+1}$ induced by the morphism (19). As the latter is *not* Γ , U_0^+ -equivariant, there is no reason for which $\{I_{k,n+1}\}_k$ should be a filtration of *Iwasawa modules* on $\mathcal{K}er_{n+1}$.

Define the integers attached to σ :

$$h'_i \stackrel{\text{def}}{=} \min\{r_{i-1} - r_i, p-2\}$$

$$h' \stackrel{\text{def}}{=} \max\{h'_i, \text{ for } i \in \{0, \dots, f-1\}\}$$

$$h \stackrel{\text{def}}{=} \max\{|r_{i_1} - r_{i_2}|, \text{ for } i_1, i_2 \in \{0, \dots, f-1\}\}.$$

The result is the following:

PROPOSITION 6.1. *Let $n \geq 1$ and consider the induced filtration $\{I_{k,n+1}\}_{k \in \mathbf{N}}$ on $\mathcal{K}er_{n+1}$. Then $\{I_{k,n+1}\}_{k \in \mathbf{N}}$ defines the radical filtration for the A -module $\mathcal{K}er_{n+1}$.*

The filtration is stable with respect to the action of H , Γ , U_0^+ and for any $k \in \mathbf{N}$, $n \geq 2$ the action of Γ , U_0^+ is trivial on the quotients

$$I_{k,2}/I_{k+(p-1-h),2} \quad \text{and} \quad I_{k,n+1}/I_{k+(p-1),n+1}.$$

In particular, for $n \geq 2$ the filtration $\{I_{k,n+1}\}_{k \in \mathbf{N}}$ defines the radical filtration for $\mathcal{K}er_{n+1}$ as an Iwasawa module on $k[[K_0(p)]]$ and, if $n = 1$ and $h < p-1$, the same result is true for the filtration $\{I_{k,2}\}_{k \in \mathbf{N}}$ on $\mathcal{K}er_2$.

Finally, the action of Γ is trivial on the quotient

$$I_{k,2}/I_{k+(p-1-h'),2}.$$

As the morphism (19) is A -linear and H -equivariant it is clear that the filtration $\{I_{k,n+1}\}_{k \in \mathbf{N}}$ is H -stable and defines the A -radical filtration on $\mathcal{K}er_{n+1}$.

The rest of this section is devoted to the proof of the Γ and U_0^+ -stability of $\{I_{k,n+1}\}_{k \in \mathbf{N}}$; the techniques will be similar to those introduced in section §4, using now in a crucial way the properties of the twisted Frobenius \mathcal{F} . Indeed, as the A -generators of $\mathcal{K}er_{n+1}$ lie in the image of the twisted Frobenius it suffices to investigate the Γ , U_0^+ action on the Iwasawa module $S_2(\sigma)$ (where the computations are still accessible) to get the control of the filtration $\{I_{k,n+1}\}_{k \in \mathbf{N}}$ for a general n .

We start with the following lemma

LEMMA 6.2. *Let $x \in \mathcal{O}_F^\times$ and $i \in \{0, \dots, f-1\}$. In the module $S_2(\sigma)$ we have the following equalities:*

a) *We have*

$$\begin{bmatrix} 1 & 0 \\ 0 & 1+px \end{bmatrix} \cdot X_i^{r_i+1} = X_i^{r_i+1} + X_{i-1}^{r_{i-1}+1} \mathfrak{m}^{(p-1)-h'_i};$$

in particular if $(r_{i-1}, r_i) = (p-1, 0)$ we have $\gamma \cdot X_i^{r_i+1} = X_i^{r_i+1}$ for all $\gamma \in \Gamma$.

b) *We have*

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \cdot X_i^{r_i+1} = X_i^{r_i+1} + \sum_s X_s^{r_s+1} \mathfrak{m}^{(p-1)-h}.$$

Proof. Up to twist by the $K_0(p)$ -character χ_{-x} we can assume that the highest weight space of σ affords the trivial character.

By Lemma 4.3 and Proposition 4.4 we have, for $g = \begin{bmatrix} 1 & 0 \\ 0 & 1+px \end{bmatrix}$ or $g = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$, the equality

$$g \cdot X_i^{r_i+1} = X_i^{r_i+1} + \left(\sum_{l=1}^{r_i+1} c_{x,l} X_i^{r_i+1-l} (\phi(\mathfrak{m}))^l \right)$$

for appropriate scalars $c_{x,l} \in k$.

We fix $l \in \{1, \dots, r_i+1\}$ and a monomial $X_i^{r_i+1-l} \underline{X}^{pj+\underline{t}}$ appearing, with nonzero coefficient, in the term $X_i^{r_i+1-l} (\phi(\mathfrak{m}))^l$. Notice that we have $|\underline{j}| \geq l \geq 1$. As $X_i^{p^2} \equiv 0$ in $S_2(\sigma)$, we can further assume that $\underline{t}, \underline{j} \in \{0, \dots, p-1\}^f$, up to increase the length $|\underline{j}|$ of \underline{j} .

Proof of a): Write $\gamma \stackrel{\text{def}}{=} \begin{bmatrix} 1 & 0 \\ 0 & 1+px \end{bmatrix}$.

As the actions of H and Γ commutes, we deduce that $\gamma \cdot X_i^{r_i+1}$ is an H -eigenvector, whose associated eigencharacter is $\mathfrak{a}^{-p^i(r_i+1)}$.

Thus, $\underline{X}^{pj+\underline{t}} X_i^{r_i+1}$ must be an H -eigenvector, of associated eigencharacter $\mathfrak{a}^{-p^i(r_i+1)}$.

The associated eigencharacter of $\underline{X}^{pj+\underline{t}} X_i^{r_i+1-l}$ is described by the following f -tuple:

$$\begin{aligned} & (p-1-j_{f-1}-t_0, p-1-j_0-t_1, \dots \\ & \dots, p-1-j_{i-2}-t_{i-1}, p-1-j_{i-1}-t_i - (r_i+1-l), p-1-j_i-t_{i+1}, \dots \\ & \dots, p-1-j_{f-2}-t_{f-1}) \end{aligned}$$

which is well defined up to an element of $(p-1)\mathbf{Z}$. Comparing with the f -tuple associated to $\mathfrak{a}^{-p^i(r_i+1)}$ and recalling that $0 \leq j_s, t_s \leq p-1$ for all $s \in \{0, \dots, p-1\}$ we deduce one of the following possibilities:

- 1) we have $j_{s-1} + t_s = 0$ for all $s \neq i$ and $j_{i-1} + t_i = l$;
- 2) we have $j_{s-1} + t_s = p - 1$ for all $s \neq i$ and $j_{i-1} + t_i = l + (p - 1)$,
- 3) we have $j_{s-1} + t_s = 2(p - 1)$ for all $s \neq i$ and $j_{i-1} + t_i = l + 2(p - 1)$

However, as $l \geq 1$ and $j_{i-1}, t_i \leq p - 1$ the third possibility can not occur.

If 1) holds true, then $t_s = 0 = j_{s-1}$ for all $s \neq i$ and, since $|j| \geq l$, $j_{i-1} = l$, $t_i = 0$. Thus

$$\underline{X}^{pj+t} X_i^{r_i+1-l} = X_{i-1}^{pl} X_i^{r_i+1-l} = X_{i-1}^{r_{i-1}+1} X_i^{r_i+1-l} X_{i-1}^{p-(r_{i-1}+1)}$$

and the required result follows.

Assume 2) holds true. Then we get $t_s = p - 1 - j_{s-1} + \delta_{s,i} \cdot l$ and therefore

$$\underline{X}^{pj+t} X_i^{r_i+1-l} = X_i^{r_i+1-l} \underline{X}^{pj} \underline{X}^{p-1-j[1]} X_i^l$$

(here $j[1]$ is the shifted f -tuple $(j_{f-1}, j_0, \dots, j_{f-2})$). Since $l \geq 1$ and $t_i + j_{i-1} = (p - 1) + l$ we have $j_{i-1} > 0$ so that we can write

$$\underline{X}^{pj} = X_{i-1}^{r_{i-1}+1} \underline{X}^{pj-e_{i-1}(r_{i-1}+1)}$$

and again

$$X_{i-1}^{r_{i-1}+1} \underline{X}^{pj-e_{i-1}(r_{i-1}+1)} \underline{X}^{p-1-j[1]} X_i^{r_i+1} \in X_{i-1}^{r_{i-1}+1} \mathfrak{m}^{(p-1)-(r_{i-1}-r_i)}.$$

Finally if $(r_{i-1}, r_i) = (p - 1, 0)$ then $X_{i-1}^p = X_{i-1}^{p(r_i+1)} \equiv 0$ by definition of $S_2(\sigma)$ so that

$$\gamma \cdot X_i^{r_i+1} = X_i^{r_i+1}.$$

This proves part a) of the Lemma.

Proof of b): As $|j| \geq l \geq 1$ certainly it exists an index s such that $j_s \geq 1$. We can therefore write

$$\underline{X}^{pj+t} X_i^{r_i+1-l} = X_i^{r_i+1-l} X_s^{r_s+1} \underline{X}^{pj-(r_s+1)e_s+t}$$

and we notice that

$$(r_i + 1 - l) + p|j| - (r_s + 1) \geq (p - 1)l + (r_i - r_s) \geq (p - 1) - h.$$

The statement follows. \square

We can now use the properties of the twisted Frobenius to deduce, from Lemma 6.2, the behavior of the filtration $\{I_{k,n+1}\}_k$ with respect to the Γ, U_0^+ -action.

LEMMA 6.3. *Let $\underline{X}^j \in A$ be a monomial verifying $\text{ord}(\underline{X}^j) \geq k$ and let $\gamma \in \Gamma$.*

For any $i \in \{0, \dots, f - 1\}$ we have the following equality in $S_{n+1}(\sigma)$:

$$\gamma \cdot (\underline{X}^j e_{n+1,i}) = (\underline{X}^j + \sum_{i \neq 0} \underline{X}^{j-i} (\phi(\mathfrak{m}))^{|\underline{i}|}) (e_{n+1,i} + e_{n+1,i-1} \mathcal{F}^{n-1} (\mathfrak{m}^{p-1-h'_{i+n-1}})). \quad (20)$$

In particular if $n \geq 2$ the action of Γ is trivial on the quotients

$$I_{k,n+1}/I_{k+(p-1),n+1}; \quad I_{k,2}/I_{k+(p-1-h'),2}.$$

Proof. Recall that $e_{n+1,i} = \mathcal{F}^{n-1}(X_{i+n-1}^{r_{i+n-1}+1})$. As Γ acts by k -algebra homomorphisms, and \mathcal{F} is Γ equivariant, the equality (20) follows from Lemma 4.3 and Lemma 6.2.

Notice moreover that a monomial issued from $(\gamma - 1) \cdot (\underline{X}^j e_{n+1,i})$ has one of the following forms

$$\begin{aligned} e_{n+1,i} \underline{X}^{j-i} (\phi(\mathfrak{m}))^{|\underline{i}|} &\in e_{n+1,i} \mathfrak{m}^{|\underline{j}|+(p-1)|\underline{i}|} \\ e_{n+1,i-1} \underline{X}^j \mathcal{F}^{n-1} (\mathfrak{m}^{p-1-h'_{i+n-1}}) &\in e_{n+1,i-1} \mathfrak{m}^{|\underline{j}|+p^{n-1}(p-1-h'_{i+n-1})} \\ e_{n+1,i-1} \underline{X}^{j-i} (\phi(\mathfrak{m}))^{|\underline{i}|} \mathcal{F}^{n-1} (\mathfrak{m}^{p-1-h'_{i+n-1}}) &\in e_{n+1,i-1} \mathfrak{m}^{|\underline{j}|+(p-1)|\underline{i}|+p^{n-1}(p-1-h'_{i+n-1})}. \end{aligned}$$

As $|j| > 0$ and $p - 1 - h'_i > 0$ for all $i = 0, \dots, f - 1$, this implies that the action of Γ is trivial on $I_{k,n}/I_{k+(p-1),n}$ if $n \geq 2$. \square

We turn our attention to the action of the upper unipotent radical. We recall that U_0^+ acts *just* by k -linear morphisms: in order to avoid this difficulty, we are forced to use an inductive argument as in Proposition 4.4.

We have

LEMMA 6.4. *Let $n \geq 2$ (resp. $n = 1$) and $x \in \mathcal{O}_F$. For any $i \in \{0, \dots, f - 1\}$ we have the following equality in $S_{n+1}(\sigma)$:*

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \cdot (\underline{X}^j e_{n+1,i}) = (\underline{X}^j + \sum_{i \geq 0} \underline{X}^{j-i} (\phi(\mathbf{m}))^{|i|}) (e_{n+1,i} + \sum_s e_{n+1,s} \mathcal{F}^{n-1}(\mathbf{m}^{p-1-h'})) \quad (21)$$

$$\text{(resp. } \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \cdot (\underline{X}^j e_{2,i}) = (\underline{X}^j + \sum_{i \geq 0} \underline{X}^{j-i} (\phi(\mathbf{m}))^{|i|}) (e_{2,i} + \sum_s (e_{2,s} \mathbf{m}^{p-1-h})) \text{ if } n = 1).$$

In particular if $n \geq 2$ the action of U_0^+ is trivial on the quotients

$$I_{k,n+1}/I_{k+(p-1),n+1}; \quad I_{k,2}/I_{k+(p-1-h),2}.$$

Proof. The proof is completely analogous to the proof of Proposition 4.4, using now the extra data coming from Lemma 6.2 and Proposition 5.8.

It suffices to treat the case $\text{val}(x) = 1$; we use the notations in the proof of Proposition 4.4.

Recall $e_{n+1,i} = \mathcal{F}^{n-1}(X_{i+n-1}^{r_i+n-1+1})$. In particular, by Lemma 6.2 and Proposition 5.8-iii) we can write

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} e_{n+1,i} = \begin{cases} e_{n+1,i} & \text{if } n \geq 2 \\ e_{2,i} + \sum_s e_{2,s} \mathbf{m}^{(p-1)-h} & \text{if } n = 1. \end{cases}$$

This implies the statement if $|j| = 0$.

We consider now the inductive step. Let $s \in \{0, \dots, f - 1\}$ be such that $j_s > 0$ so that we have

$$\begin{aligned} & \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \cdot (\underline{X}^j e_{n+1,i} \otimes 1) = \\ & = \sum_{\lambda \in k_F^\times} \lambda^{-p^s} \begin{bmatrix} 1 & 0 \\ p\delta_x([\lambda]) & 1 \end{bmatrix} \begin{bmatrix} 1 + px[\lambda] & x \\ 0 & 1 - px\delta_x([\lambda]) \end{bmatrix} \underline{X}^{j-e_s} e_{n+1,i} \otimes 1. \end{aligned}$$

The inductive hypothesis lets us write (for $n \geq 2$)

$$\begin{aligned} & \begin{bmatrix} 1 & x(1 + px[\lambda])^{-1} \\ 0 & 1 \end{bmatrix} \underline{X}^{j-e_s} e_{n+1,i} \otimes 1 = \\ & \left((\underline{X}^{j-e_s} + \sum_{l \geq 0} \underline{X}^{j-e_s-l} (\phi(\mathbf{m}))^{|l|}) (e_{n+1,i} + \sum_s e_{n+1,s} \mathcal{F}^{n-1}(\mathbf{m}^{p-1-h'})) \right) \otimes 1 \end{aligned}$$

so that a simple manipulation (exactly as in Proposition 4.4) using the multiplicativity of Γ , Lemma 4.3 and the Γ equivariance of \mathcal{F} gives

$$\begin{aligned} & \begin{bmatrix} 1 + px[\lambda] & x \\ 0 & 1 - px\delta_x([\lambda]) \end{bmatrix} \cdot \underline{X}^{j-e_s} e_{n+1,i} \otimes 1 = \\ & \left((\underline{X}^{j-e_s} + \sum_{l \geq 0} \underline{X}^{j-e_s-l} (\phi(\mathbf{m}))^{|l|}) (e_{n+1,i} + \sum_s e_{n+1,s} \mathcal{F}^{n-1}(\mathbf{m}^{p-1-h'})) \right) \otimes 1. \end{aligned}$$

Thus we have

$$\begin{aligned}
 & \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \cdot \underline{X}^j e_{n+1,i} = \\
 & = \sum_{\lambda \in k_F^\times} \lambda^{-p^s} \begin{bmatrix} 1 & 0 \\ p[\lambda] & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ p(\delta_x([\lambda]) - [\lambda]) & 1 \end{bmatrix} \left((\underline{X}^{j-e_s} + \sum_{l \geq 0} \underline{X}^{j-e_s-l} (\phi(\mathbf{m}))^{[l]}) \right) \\
 & \qquad \qquad \qquad \cdot (e_{n+1,i} + \sum_s e_{n+1,s} \mathcal{F}^{n-1}(\mathfrak{m}^{p-1-h'})) \\
 & = ((\phi(\mathbf{m}) + X_s) \left((\underline{X}^{j-e_s} + \sum_{l \geq 0} \underline{X}^{j-e_s-l} (\phi(\mathbf{m}))^{[l]}) (e_{n+1,i} + \sum_s e_{n+1,s} \mathcal{F}^{n-1}(\mathfrak{m}^{p-1-h'})) \right)
 \end{aligned}$$

and the equality (21) follows. The case $n = 1$ is completely analogous.

The assertion on the U_0^+ -action on the quotients $I_{k,n}/I_{k+(p-1),n}$ (for $n \geq 2$) and $I_{k,1}/I_{k+(p-1-h),1}$ is deduced in the same manner as for Lemma 6.3. \square

7. The universal Iwasawa module

We are finally in the condition to analyze some of the properties of the universal Iwasawa module $\mathfrak{S}_\infty^\bullet$. We first focus on the Iwahori radical filtration (§7.1). The main result is Proposition 7.1, where we show that, under some mild hypotheses on the Serre weight σ , the A -radical filtration of $\mathfrak{S}_\infty^\bullet$ coincides with the $k[[K_0(p)]]$ -radical filtration. In Corollary 7.3 we deduce the isotypical components of the cosocle of $\mathfrak{S}_\infty^\bullet$: we have a 2-dimensional isotypical space, together with some other infinite dimensional spaces as soon as $F \neq \mathbf{Q}_p$.

In section 7.2 we study some torsion properties, over A , for the universal module $\mathfrak{S}_\infty^\bullet$. Under mild conditions on the Serre weight, such module is torsion free over A and indeed it contains a dense sub-module of rank one over $\text{Frac}(A)$ (Proposition 7.9).

We fix a Serre weight $\sigma = \sigma_r$ as in section §6. We introduce the following regularity conditions on the weight σ :

- (I1) There exists an index $i \in \{0, \dots, f-1\}$ such that $r_i + 1 > \sum_{s=0}^{f-1} r_s$;
- (I2) there exist indices $i_1, i_2 \in \{0, \dots, f-1\}$ such that $|r_{i_1} - r_{i_2}| = p-1$.

This conditions affect slightly the behavior of the natural filtration on the universal module associated to σ (cf. Proposition 7.1).

Recall that, by Proposition 3.5, we have an A -linear morphism with dense image

$$\mathcal{M}_\infty^\bullet \stackrel{\text{def}}{=} \left(\bigoplus_{e_{2(j+1)+\bullet}, i \in \mathcal{G}_\infty^\bullet} A \cdot e_{2(j+1)+\bullet, i} \right) \oplus A \cdot e_\bullet \xrightarrow{\Psi_\infty} \mathfrak{S}_\infty^\bullet$$

as well as a family of compatible commutative diagrams

$$\begin{array}{ccc}
 \mathcal{M}_\infty^\bullet & \xrightarrow{\Psi_\infty} & \mathfrak{S}_\infty^\bullet \\
 \downarrow & & \downarrow \text{pr}_{n+1} \\
 \mathcal{M}_{n+1}^\bullet & \xrightarrow{\Psi_{n+1}} & \mathfrak{S}_{n+1}^\bullet
 \end{array} \tag{22}$$

where $\mathcal{M}_{n+1}^\bullet \stackrel{\text{def}}{=} \left(\bigoplus_{\substack{e_{2(j+1)+\bullet}, i \in \mathcal{G}_\infty^\bullet \\ 0 \leq j \leq \frac{n-1-\bullet}{2}}} A \cdot e_{2(j+1)+\bullet, i} \right) \oplus A \cdot e_\bullet$.

We make H act by $\mathfrak{a}^{-p^{i+2j+\bullet}(r_{i+2j+\bullet}+1)}(\chi_r^s)^\vee$ on $e_{2(j+1)+\bullet,i}$ and by $(\chi_r)^\vee$ on e_0 (resp. by $(\chi_r^s)^\vee$ on e_1); in this way the morphisms Ψ_∞, Ψ_{n+1} become H -equivariant.

7.1 Filtration on the Fibered products

We endow the free Iwasawa module $\mathcal{M}_\infty^\bullet$ with the valuation of the minimum ord_∞ .

This induces a filtration $\{\mathcal{J}_k^\bullet\}_k$ on $\mathcal{M}_\infty^\bullet$ by the condition

$$\mathcal{J}_k^\bullet \stackrel{\text{def}}{=} \left\{ \sum_{i,j} P_{i,j}(\underline{X}) e_{2(j+1)+\bullet,i} + P_\bullet(\underline{X}) e_\bullet \in \mathcal{M}_\infty^\bullet, \min\{ord(P_{i,j}(\underline{X})), ord(P_\bullet(\underline{X}))\} \geq k \right\}.$$

By Proposition 4.5 the filtration $\{\mathcal{J}_k^\bullet\}_k$ is Γ, U_0^+ -stable and it coincides with the A -radical filtration on $\mathcal{M}_\infty^\bullet$.

We define in the analogous, evident way the filtration $\{\mathcal{J}_{k,n+1}^\bullet\}_k$ on the module $\mathcal{M}_{n+1}^\bullet$ for $n \geq 1$.

The natural filtration $\{\mathcal{J}_k^\bullet\}_k$ on $\mathfrak{S}_\infty^\bullet$ is therefore given by

$$\mathcal{J}_k^\bullet \stackrel{\text{def}}{=} \overline{\Psi_\infty(\mathcal{J}_k^\bullet)};$$

If we define $\mathcal{J}_{k,n+1}^\bullet \stackrel{\text{def}}{=} \Psi_{n+1}(\mathcal{J}_{k,n+1}^\bullet)$ the commutative diagram (22) let us write more expressively

$$\mathcal{J}_k^\bullet = \bigcap_{n \in 2\mathbf{N}+1+\bullet} pr_{n+1}^{\leftarrow}(\mathcal{J}_{k,n+1}^\bullet). \quad (23)$$

As the morphisms Ψ_∞, Ψ_{n+1} are *not* Γ, U_0^+ -equivariant there is no reason, *a priori*, that the filtration on $\mathfrak{S}_\infty^\bullet, \mathfrak{S}_{n+1}^\bullet$ should be Γ, U_0^+ stable. As we did in §4 and §6 we show that such filtrations are indeed well behaved.

We remark that, contrary to what happens for the modules A and $\mathcal{K}er_{n+1}$, the properties of the filtration $\{\mathcal{J}_k^\bullet\}_k$ follow (almost) directly from Corollary 4.6 and Proposition 6.1, using a formal argument on the valuation ord_{n+1} on the Iwasawa modules $S_{n+1}(\sigma)$.

7.1.1 The even case. In order to get an optimal result for the behavior of the filtrations $\mathcal{J}_k^0, \mathcal{J}_{k,n+1}^0$ we need to slightly refine their construction. This is due to the phenomenology on the Iwasawa module \mathfrak{S}_2^0 (cf. Lemma 6.2). We assume in the rest of this paragraph that $n \geq 3$ is odd.

Write $\mathfrak{S}_*^{\geq 3} \stackrel{\text{def}}{=} \ker(\mathfrak{S}_*^0 \rightarrow \mathfrak{S}_2^0)$ for $* \in \{n+1, \infty\}$ and set $\mathcal{M}_*^{\geq 3}$ for the complement of \mathcal{M}_2^0 in \mathcal{M}_*^\bullet .

The morphism Ψ_∞ restricts to an A -linear H -equivariant morphism with dense image $\Psi_\infty^{\geq 3} : \mathcal{M}_\infty^{\geq 3} \rightarrow \mathfrak{S}_\infty^{\geq 3}$ (resp. an A -linear H -equivariant epimorphism $\Psi_{n+1}^{\geq 3} : \mathcal{M}_{n+1}^{\geq 3} \rightarrow \mathfrak{S}_{n+1}^{\geq 3}$) and we have the evident compatibility between $\Psi_\infty^{\geq 3}$ and $\Psi_{n+1}^{\geq 3}$ as in (22).

We define in the analogous way the filtrations $\{\mathcal{J}_k^{\geq 3}\}_k, \{\mathcal{J}_{k,n+1}^{\geq 3}\}_k$ on $\mathfrak{S}_\infty^{\geq 3}, \mathfrak{S}_{n+1}^{\geq 3}$, having

$$\mathcal{J}_k^{\geq 3} = \bigcap_{n \in 2\mathbf{N}+3} pr_{n+1}^{\leftarrow}(\mathcal{J}_{k,n+1}^{\geq 3}). \quad (24)$$

Let $\mathcal{J}_k^{\leq 2}$ be the image of $\mathcal{J}_{k,2}^0$ in \mathfrak{S}_∞^0 via $\Psi_\infty|_{\mathcal{M}_2^0}$. As \mathcal{M}_2^0 is finitely generated, it is a closed A -sub-module of \mathfrak{S}_∞^0 .

As $\mathfrak{S}_\infty^0 = \mathfrak{S}_\infty^{\geq 3} \times_{S_3} \mathfrak{S}_2^0$ and since $\mathcal{J}_k^{\geq 3}, \mathcal{J}_k^{\leq 2}$ are closed in \mathfrak{S}_∞^0 we have

$$\mathcal{J}_k^0 = \mathcal{J}_k^{\geq 3} + \mathcal{J}_k^{\leq 2};$$

similarly, $\mathcal{J}_{k,n+1}^0 = \mathcal{J}_{k,n+1}^{\geq 3} + \mathcal{J}_{k,n+1}^{\leq 2}$ (with the obvious definition of $\mathcal{J}_{k,n+1}^{\leq 2}$).

We are now ready to describe the behavior of \mathcal{J}_k with respect to the Γ, U_0^+ actions:

PROPOSITION 7.1. *The filtration $\{\mathcal{F}_k^\bullet\}_k$ is the A -radical filtration on $\mathfrak{S}_\infty^\bullet$. It is H -stable.*

Moreover $\{\mathcal{F}_k^1\}_k$ (resp. $\{\mathcal{F}_k^{\geq 3}\}_k$) is a filtration of Iwasawa sub-modules for \mathfrak{S}_∞^1 (resp. for $\mathfrak{S}_\infty^{\geq 3}$) and the action of Γ, U_0^+ are trivial on the quotients

$$\mathcal{F}_k^1 / \mathcal{F}_{k+(p-1)}^1 \quad (\text{resp. } \mathcal{F}_k^{\geq 3} / \mathcal{F}_{k+(p-1)}^{\geq 3})$$

for all $k \in \mathbf{N}$.

In particular $\{\mathcal{F}_k^1\}_k$ (resp. $\{\mathcal{F}_k^{\geq 3}\}_k$) is the radical filtration on \mathfrak{S}_∞^1 (resp. for $\mathfrak{S}_\infty^{\geq 3}$) as Iwasawa modules.

If $\bullet = 0$, the filtration $\{\mathcal{F}_k^0\}_k$ is Γ and U_0^+ stable.

If the f -tuple \underline{r} does not verify (I1) the action of Γ (resp. and U_0^+) is trivial on the quotients

$$\mathcal{F}_k^0 / (\mathcal{F}_{k+(p-1)}^{\geq 3} + \mathcal{F}_{k+(p-1)-h'}^{\leq 2}) \quad (\text{resp. } \mathcal{F}_k^0 / (\mathcal{F}_{k+(p-1)}^{\geq 3} + \mathcal{F}_{k+(p-1)-h}^{\leq 2})).$$

If the f -tuple \underline{r} verifies (I1), say $r_i + 1 > \sum_s r_s$ then the action of Γ, U_0^+ is trivial on the quotient

$$\mathcal{F}_k^0 / (\mathcal{F}_{k+(p-1)}^{\geq 3} + \mathcal{F}_{k+(p-1)-(r_i-1)}^{\leq 2}).$$

In particular, if either \underline{r} verifies (I1) or does not verifies neither (I1) nor (I2), the filtration $\{\mathcal{F}_k^0\}_k$ is the radical filtration on \mathfrak{S}_∞^0 as an Iwasawa module.

As the action of Γ, U_0^+ is continuous and the projection maps pr_{n+1} are Γ, U_0^+ equivariant we deduce from the expressions (23), (24) that it is enough to prove Proposition 7.1 for any arbitrary finite level $\mathfrak{S}_{n+1}^\bullet$.

We first consider the case $n = 1$.

LEMMA 7.2. *Consider the filtration $\mathcal{F}_{k,2}^0$ on \mathfrak{S}_2^0 .*

a) *Assume that \underline{r} does not verify (I1). The filtration $\mathcal{F}_{k,2}^0$ is Γ and U_0^+ -stable. Moreover the action of Γ (resp. and U_0^+) is trivial on the quotients*

$$\mathcal{F}_{k,2}^0 / \mathcal{F}_{k+(p-1)-h',2}^0 \quad (\text{resp. } \mathcal{F}_{k,2}^0 / \mathcal{F}_{k+(p-1)-h,2}^0).$$

b) *Assume there exists an index $i \in \{0, \dots, f-1\}$ such that $r_i + 1 > \sum_{s=0}^{f-1} r_s$. Then the action of Γ, U_0^+ is trivial on the quotient*

$$\mathcal{F}_{k,2}^0 / (\mathcal{F}_{k+(p-1)-(r_i-1),2}^0).$$

Proof. First of all, notice that if $\underline{r} = 0$ then $\mathfrak{S}_2^0 = S_2 \cong (\overline{St}^{(1)})^\vee$ which is endowed with the trivial action of Γ and U_0^+ .

We start from a). As $\mathcal{K}er_2$ is a sub-module of S_2 and $\mathcal{F}_{k,2}^0 = I_{k,2}$ for $k \geq 1$ (recall that $I_{k,2}$ are the sub-module for the filtration on $\mathcal{K}er_2$, cf. §6) by Lemma 6.3 and 6.4 it is enough to show that

$$(g-1)\underline{X}^r \in \mathcal{K}er_2 \mathfrak{m}^{p-1}$$

for any $g \in \Gamma, U_0^+$. Writing

$$(g-1)\underline{X}^r = \sum_s X_s^{r_s+1} P_s(\underline{X})$$

for some $P_s(\underline{X}) \in A$ we have, by Proposition 4.6, $ord(X_s^{r_s+1} P_s(\underline{X})) \geq p-1 + |\underline{r}|$; the result follows as \underline{r} does not verify (I1).

We turn now to the proof of part b). In this case we have

$$\mathcal{F}_{k,2}^0 = \langle P(\underline{X}) X_s^{r_s+1-\delta_{s,i}}, \text{ for } P(\underline{X}) \in \mathfrak{m}^k, s \in \{0, \dots, f-1\} \rangle_k.$$

As for Lemma 6.2, we can show that

$$\begin{aligned}\gamma \cdot X_s^{r_s+1-\delta_{s,i}} &= X_s^{r_s+1-\delta_{s,i}} + X_{s-1}^{r_{s-1}+1-\delta_{s-1,i}} \mathfrak{m}^{p-1-\delta_{s-1,i}(r_i-1)} \\ u^+ \cdot X_s^{r_s+1-\delta_{s,i}} &= X_s^{r_s+1-\delta_{s,i}} + \sum_{l=0}^{f-1} X_l^{r_l+1-\delta_{l,i}} \mathfrak{m}^{p-1-\delta_{l,i}(r_i-1)}\end{aligned}$$

for $\gamma \in \Gamma$, $u^+ \in U_0^+$.

We can now prove, using the very same arguments of Proposition 6.3, 6.4, statement b). The details are left to the reader. \square

Proof of Proposition 7.1 in the finite case. Fix $n \geq 2$ and consider the module $\mathfrak{S}_{n+1}^\bullet \subseteq S_{n+1}(\sigma)$. We notice that $\mathcal{I}_{k,n+1}^\bullet$ is the image, inside $\mathfrak{S}_{n+1}^\bullet$, of the A -module $\mathcal{M}_{n+1}^\bullet \otimes \mathfrak{m}^k A$:

$$\mathcal{I}_{k,n+1}^\bullet = \langle e_{2(j+1)+\bullet,i}, e_\bullet, j = 0, \dots, \frac{n-1-\bullet}{2}, i = 0, \dots, f-1 \rangle_{\mathfrak{m}^k A}.$$

Fix a couple $(j_0, i_0) \in \{0, \dots, \frac{n-1-\bullet}{2}\} \times \{0, \dots, f-1\}$, an f -tuple $\underline{l} \in \mathbf{N}^f$ of length $|\underline{l}| = k$ and consider the element

$$\underline{X}^{\underline{l}} e_{2(j_0+1)+\bullet,i_0} \quad (\text{resp. } \underline{X}^{\underline{l}} e_\bullet).$$

As $\ker(\mathfrak{S}_{n+1}^\bullet \rightarrow \mathfrak{S}_{2j_0+\bullet}^\bullet)$ is generated over A by the elements $e_{2(j+1)+\bullet,i}$ for $j_0 \leq j \leq \frac{n-1-\bullet}{2}$, $i = 0, \dots, f-1$ we can write

$$(g-1) \cdot \underline{X}^{\underline{l}} e_{2(j_0+1)+\bullet,i_0} = \sum_{i=0}^{f-1} \sum_{j \geq j_0} P_{j,i}(\underline{X}) e_{2(j+1)+\bullet,i}$$

(resp. $(g-1) \cdot \underline{X}^{\underline{l}} e_\bullet = \sum_{i=0}^{f-1} \sum_{j \geq 0} P_{j,i}(\underline{X}) e_{2(j+1)+\bullet,i} + P_\bullet(\underline{X}) e_\bullet$) and, by Corollary 4.5, we deduce that

$$\text{ord}(P_{j,i}(\underline{X}) e_{2(j+1)+\bullet,i}) \geq k + (p-1) + \text{ord}(e_{2(j_0+1)+\bullet,i_0})$$

for all i and $j \geq j_0$ (resp. $\text{ord}(P_\bullet(\underline{X}) e_\bullet) \geq k + (p-1) + \text{ord}(e_\bullet)$ and $\text{ord}(P_{j,i}(\underline{X}) e_{2(j+1)+\bullet,i}) \geq k + (p-1) + \text{ord}(e_\bullet)$ for all $j \geq 0, i$).

Thanks to Lemma 7.2 and Proposition 6.1 it is enough to prove that

$$\text{ord}(e_{2(j_0+1)+\bullet,i_0}) \geq \text{ord}(e_{2(j+1)+\bullet,i})$$

for any $j > j_0$ and any i (resp. to prove that $\text{ord}(e_\bullet) \geq \text{ord}(e_{2(j+1)+\bullet,i})$ for any $j \geq 1 - \bullet$ and any i).

Recalling the valuation of the elements $e_{2(j+1)+\bullet,i}$, e_\bullet we are left to prove the inequality

$$\sum_{l=0}^{f-1} \sum_{s=2j_0+1+\bullet}^{n-1} (-1)^{s+\bullet} (r_{l+s} + 1) p^s \geq p^{2j+\bullet} (r_{i+2j+\bullet} + 1) + \sum_{l=0}^{f-1} \sum_{s=2j+1+\bullet}^{n-1} (-1)^{s+\bullet} (r_{l+s} + 1) p^s$$

for all $j > j_0$ and all $i = 0, \dots, f-1$ (resp.

$$\sum_{l=0}^{f-1} \left(\sum_{s=0}^{n-1} (-1)^{s+\bullet} (r_{l+s} + 1) p^s - \delta_{\bullet,0} \right) \geq p^{2j+\bullet} (r_{i+2j+\bullet} + 1) + \sum_{l=0}^{f-1} \sum_{s=2j+1+\bullet}^{n-1} (-1)^{s+\bullet} (r_{l+s} + 1) p^s$$

for all $j \geq 1 - \bullet$ and all $i = 0, \dots, f-1$).

By a simple manipulation we are reduced to prove that

$$p^m (r_{i+m} + 1) \leq \sum_{l=0}^{f-1} p^m (r_{l+m} + 1) - p^{m-1} (r_{l+m-1} + 1)$$

for $m \stackrel{\text{def}}{=} 2j + \bullet \geq 1$.

This is trivially true if $f \geq 3$ or $f = 2$ and $(r_{i+m}, r_{i+m+1}) \neq (p-1, 0)$ and we are done as $(r_{i+2j+\bullet}, r_{i+2j+\bullet+1}) \neq (p-1, 0)$ by definition of $\mathcal{G}_{n+1}^\bullet$.

This ends the proof.

An immediate consequence of Proposition 7.1 is the description of the isotypical components appearing in the cosocle of the universal module $\mathfrak{S}_\infty^\bullet$:

COROLLARY 7.3. *Assume that either f is odd or f is even and $\underline{r} \neq (\dots, 0, p-1, 0, p-1, \dots)$. Then*

$$\begin{aligned} \text{cosoc}_{k[[I]]}(\mathfrak{S}_\infty^0) &= V(\chi_{-\underline{r}}) \oplus \bigoplus_{i=0}^{f-1} V(\chi_{\underline{r}} \det^{-\underline{r}} \mathfrak{a}^{-p^i(r_i+1)}) \\ \text{cosoc}_{k[[I]]}(\mathfrak{S}_\infty^1) &= V(\chi_{\underline{r}} \det^{-\underline{r}}) \oplus \bigoplus_{i=0}^{f-1} V(\chi_{\underline{r}} \det^{-\underline{r}} \mathfrak{a}^{-p^i(r_i+1)}) \end{aligned}$$

where

$$\begin{aligned} \dim(V(\chi_{-\underline{r}})) &= \dim(V(\chi_{\underline{r}} \det^{-\underline{r}})) = 1, \\ \dim(V(\chi_{\underline{r}} \det^{-\underline{r}} \mathfrak{a}^{-p^i(r_i+1)})) &= \begin{cases} \infty & \text{for all } i \in \{0, \dots, f-1\} \text{ if } F \neq \mathbf{Q}_p \\ 0 & \text{for all } i \in \{0, \dots, f-1\} \text{ if } F = \mathbf{Q}_p. \end{cases} \end{aligned}$$

If f is even and $\underline{r} = (\dots, 0, p-1, 0, p-1, \dots)$ we have

$$\begin{aligned} \text{cosoc}_{k[[I]]}(\mathfrak{S}_\infty^0) &= V(\chi_{-\underline{r}}) \oplus \bigoplus_{i=0}^{\frac{f}{2}-1} V\left(\chi_{\underline{r}} \det^{-\underline{r}} \mathfrak{a}^{-p^{2i+\delta(r_0, p-1)}(r_{2i+\delta(r_0, p-1)}+1)}\right) \\ \text{cosoc}_{k[[I]]}(\mathfrak{S}_\infty^1) &= V(\chi_{\underline{r}} \det^{-\underline{r}}) \oplus \bigoplus_{i=0}^{\frac{f}{2}-1} V\left(\chi_{\underline{r}} \det^{-\underline{r}} \mathfrak{a}^{-p^{2i+\delta(r_0, 0)}(r_{2i+\delta(r_0, 0)}+1)}\right) \end{aligned}$$

where again $\dim(V(\chi_{-\underline{r}})) = \dim(V(\chi_{\underline{r}} \det^{-\underline{r}})) = 1$ and the remaining isotypical components in the direct sum decomposition are infinite dimensional (resp. zero dimensional) if $F \neq \mathbf{Q}_p$ (resp. $F = \mathbf{Q}_p$).

Proof. We just remark that the combinatoric in the statement for f even and $\underline{r} = (\dots, 0, p-1, 0, p-1, \dots)$ follow from the definition of $\mathcal{G}_\infty^\bullet$ (as $e_{2(j+1)+\bullet, i} \in \mathcal{G}_\infty^\bullet$ if and only if $(r_{i+2j+\bullet}, r_{i+2j+\bullet+1}) \neq (p-1, 0)$).

Moreover, if the f -tuple \underline{r} does not verify (I1) but does verify (I2), we remark that the quotient Q of $\text{cosoc}_A(\mathfrak{S}_\infty^0)$ by the k -subspace generated by the elements $e_{2, i}$ (for $i = 0, \dots, f-1$) is endowed with a trivial action of Γ, U_0^+ . Indeed from the analysis of the action of Γ, U_0^+ on e_0 in the proof of Proposition 7.1 one sees that

$$(g-1) \cdot e_0 \in \mathcal{I}_{1, n+1}^{\geq 3} + \mathcal{I}_{0, n+1}^{\leq 2}$$

for any $g \in \Gamma, U_0^+$ and, by Lemma 7.2, that the image of $(g-1) \cdot e_0$ in \mathfrak{S}_2^0 belongs to $\mathcal{I}_{1, 2}^0$.

Hence, the action of Γ, U_0^+ on $e_0 \in Q$ is trivial, and on the elements $e_{2(j+1), i} \in Q$ (for $j \geq 1, i = 0, \dots, f-1$) as well, thanks to the first part of Proposition 7.1. \square

7.1.2 A note on a generating set for $\mathfrak{S}_\infty^\bullet$. A slight improvement in the definition of the generating set \mathcal{G}^\bullet let us dispose of a morphism with dense image from a suitable complement of $\mathcal{M}_\infty^\bullet$ into $\mathfrak{S}_\infty^\bullet$.

Define the following subsets of $\mathcal{G}_\infty^\bullet$:

a) if $\underline{r} = 0$,

$$\mathcal{G}_\infty^0(\underline{r}) \stackrel{\text{def}}{=} \{e_{2(j+1),i} \in \mathcal{G}_\infty^0, \text{ s.t. } j > 0, e_0\};$$

b) if \underline{r} verifies (I1), say $r_{i_0} + 1 > \sum_s r_s$, define

$$\mathcal{G}_\infty^0(\underline{r}) \stackrel{\text{def}}{=} \{e_{2(j+1),i} \in \mathcal{G}_\infty^0, \text{ s.t. } (j,i) \neq (0, i_0), e_0\}$$

c) if $\underline{r} \neq 0$ and \underline{r} does not verifies (I1) we set $\mathcal{G}_\infty^\bullet(\underline{r}) \stackrel{\text{def}}{=} \mathcal{G}_\infty^\bullet$.

d) for $n \geq 1$ put $\mathcal{G}_{n+1}^\bullet(\underline{r}) \stackrel{\text{def}}{=} \mathcal{G}_\infty^\bullet(\underline{r}) \cap \mathcal{G}_{n+1}^\bullet$.

The generating families $\mathcal{G}_\infty^\bullet(\underline{r})$ is “minimal” in the following sense:

LEMMA 7.4. For any $n \geq 1$ the restriction of Ψ_{n+1}

$$\bigoplus_{e \in \mathcal{G}_{n+1}^\bullet(\underline{r})} A \cdot e \rightarrow \mathfrak{S}_{n+1}^\bullet$$

is surjective. This result fails if we replace $\mathcal{G}_{n+1}^\bullet(\underline{r})$ by any proper subset.

Proof. By the definition of $\mathcal{G}_{n+1}^\bullet(\underline{r})$ and the description of Ker_{n+1} it is immediate to see that

$$\bigoplus_{e \in \mathcal{G}_{n+1}^\bullet(\underline{r}) \setminus \mathcal{G}_{n-1}^\bullet(\underline{r})} A e \rightarrow \text{Ker}_{n+1}$$

is an epimorphism which fails to be surjective if we replace $\mathcal{G}_{n+1}^\bullet(\underline{r}) \setminus \mathcal{G}_{n-1}^\bullet(\underline{r})$ by a proper subset.

By an immediate induction and the commutative diagram

$$\begin{array}{ccc} \bigoplus_{e \in \mathcal{G}_{n+1}^\bullet(\underline{r})} A \cdot e & \longrightarrow & \mathfrak{S}_{n+1}^\bullet \\ \downarrow & & \downarrow \\ \bigoplus_{e \in \mathcal{G}_{n-1}^\bullet(\underline{r})} A \cdot e & \longrightarrow & \mathfrak{S}_{n-1}^\bullet \end{array}$$

we can therefore reduce to study the morphism

$$\bigoplus_{e \in \mathcal{G}_2^\bullet(\underline{r})} A \cdot e \rightarrow \mathfrak{S}_2^0.$$

The result can now be checked by a direct computation and the statement follows. \square

7.2 Torsion properties of the universal module

In this section we analyze the torsion properties of the universal modules $\mathfrak{S}_\infty^\bullet$ as profinite modules over A . It turns out that, under mild conditions on the Serre weight $\sigma = \sigma_{\underline{r}}$, any nonzero monogenous A -sub-module of $\mathfrak{S}_\infty^\bullet$ is free and, at the same time, for any choice of generators $e, e' \in \mathcal{G}_\infty^\bullet$, the natural morphism $A \cdot e \oplus A \cdot e' \rightarrow \mathfrak{S}_\infty^\bullet$ factors through a rank one quotient of $A \cdot e \oplus A \cdot e'$.

We start from the following elementary observation:

LEMMA 7.5. Assume that either f is odd or f is even and $\underline{r} \neq (\dots, 0, p-1, 0, p-1, \dots)$. Then, for any $l \in \{0, \dots, f-1\}$ and any $j \in \mathbf{N}$ we have

$$\lim_{\substack{n \rightarrow \infty \\ n \in 2\mathbf{N}+1+\bullet}} \sum_{s=2(j+1)+1+\bullet}^n (-1)^{s+1+\bullet} p^s (r_{l+s} + 1) = +\infty.$$

Proof. Omissis. \square

The first striking property of the universal module is that, for any choice of generators $e, e' \in \mathcal{G}_\infty^\bullet$, the natural morphism $A \cdot e \oplus A \cdot e' \rightarrow \mathfrak{S}_\infty^\bullet$ has a nonzero kernel.

PROPOSITION 7.6. *Let $\bullet \in \{0, 1\}$ and fix two elements $e, e' \in \mathcal{G}_\infty^\bullet$ with $e \neq e'$.*

The natural morphism

$$\begin{aligned} A \cdot e \oplus A \cdot e' &\rightarrow \mathfrak{S}_\infty^\bullet \\ (P(\underline{X}), P'(\underline{X})) &\mapsto P(\underline{X})e + P'(\underline{X})e' \end{aligned}$$

has a nonzero kernel.

Proof. To ease notation, we consider the case $\bullet = 0$ (ther other is similar).

By the construction of \mathfrak{S}_∞^0 it is enough to prove that for $n \in 2\mathbf{N} + 1$, $n \gg 0$, we have a commutative diagram

$$\begin{array}{ccc} A \cdot e \oplus A \cdot e' & \xrightarrow{\quad} & \mathfrak{S}_{n+3}^0 \\ \parallel & \searrow & \nearrow \\ & Q & \\ \parallel & \downarrow & \\ A \cdot e \oplus A \cdot e' & \xrightarrow{\quad} & \mathfrak{S}_{n+1}^0 \\ & \searrow & \nearrow \\ & Q & \end{array}$$

where Q is an appropriate A -module of rank at most 1.

There exists $n \in 2\mathbf{N} + 1$, $n \gg 0$ such that the maps $Ae \rightarrow \mathfrak{S}_{n+1}^0$, $Ae' \rightarrow \mathfrak{S}_{n+1}^0$ are both non zero. Since $\mathfrak{S}_{n+1}^0 \hookrightarrow S_{n+1}$ and the latter is a quotient of A , we deduce that there exists two *monomials* $P(\underline{X}), P'(\underline{X}) \in A$ such that $P(\underline{X})e \neq 0 \neq P'(\underline{X})e'$ and $P(\underline{X})e + P'(\underline{X})e' = 0$ in \mathfrak{S}_{n+1}^0 . We can therefore set $Q \stackrel{\text{def}}{=} (Ae \oplus Ae') / \langle P(\underline{X})e, P'(\underline{X})e' \rangle_A$.

It is now enough to show that we have $P(\underline{X})e + P'(\underline{X})e' = 0$ in \mathfrak{S}_{n+3}^0 and this is clear since (the image of) e, e' are monomials of S_{n+3} , hence $P(\underline{X})e, P'(\underline{X})e'$ are monomials of S_{n+3} which maps to nonzero elements in S_{n+2} via the natural projection $S_{n+3} \twoheadrightarrow S_{n+2}$ (and, by construction, their image in S_{n+2} belongs to the image of S_{n+1} in S_{n+2}).

This ends the proof. □

On the other hand, under mild conditions on the Serre weight, *any* nonzero monogenous submodule of $\mathfrak{S}_\infty^\bullet$ is *free*:

PROPOSITION 7.7. *Assume that either f is odd or f is even and $\underline{r} \neq (\dots, 0, p-1, 0, p-1, \dots)$.*

Let $x \in \mathfrak{S}_\infty^\bullet$ be a nonzero element. Then the natural map

$$\begin{aligned} A &\rightarrow \mathfrak{S}_\infty^\bullet \\ P(\underline{X}) &\mapsto P(\underline{X}) \cdot x \end{aligned}$$

is injective.

Proof. For $n \in 2\mathbf{N} + 1 + \bullet$, write $pr_{n+1} : \mathfrak{S}_\infty^\bullet \twoheadrightarrow \mathfrak{S}_{n+1}$ for the natural projection.

Assume the statement is false. Then there is an element $P(\underline{X}) \in A \setminus \{0\}$ such that $P(\underline{X}) \cdot x = 0$ in $\mathfrak{S}_\infty^\bullet$.

Since $x \neq 0$ there is $n_0 \in 2\mathbf{N} + 1 + \bullet$ such that $pr_{n_0+1}(x) \neq 0$ in \mathfrak{S}_{n_0+1} . Write

$$m_i \stackrel{\text{def}}{=} \text{ord}_{X_i}(P(\underline{X})), \quad m'_i \stackrel{\text{def}}{=} \text{ord}_{X_i}(pr_{n_0+1}(x))$$

(we consider $pr_{n_0+1}(x)$ as an element of S_{n_0+1} and hence we dispose of a well defined notion of order in X_i for $pr_{n_0+1}(x)$).

An immediate induction, together with the definition of the transition morphisms $\mathfrak{S}_{n+1}^\bullet \rightarrow \mathfrak{S}_{n-1}^\bullet$ gives

$$\begin{aligned} \text{ord}_{X_i}(pr_{n_0+1}(x)) &= m'_i + \sum_{j=0}^{n-n_0-1} p^{n_0+j} (-1)^{j+1} (r_{i+n_0+j} + 1) = \\ &= m'_i + \sum_{j=n_0}^{n-1} p^j (-1)^{j+\bullet} (r_{i+j} + 1) \end{aligned}$$

for all $n \in 2\mathbf{N} + 1 + \bullet$, $n \geq n_0$. Hence, by Lemma 7.5, we deduce that for any $i \in \{0, \dots, f-1\}$ there exists $n \in 2\mathbf{N} + 1 + \bullet$, $n \gg 0$ such that:

$$m_i + m'_i + \sum_{j=n_0}^{n-1} p^j (-1)^{j+\bullet} (r_{i+j} + 1) < p^n (r_{i+n} + 1); \quad (25)$$

in particular, it exists $n_1 \in 2\mathbf{N} + 1 + \bullet$, $n_1 \gg n_0$ such that (25) holds for all $i \in \{0, \dots, f-1\}$.

We can thus find a suitable lift $y \in A$ of $pr_{n_1+1}(x)$ for the morphism $A \rightarrow S_{n_1+1}$ such that

$$\text{ord}_{X_i}(P(\underline{X})y) < p^{n_1} (r_{i+n_1} + 1)$$

for all $i \in \{0, \dots, f-1\}$ and this means precisely that $P(\underline{X})y$ maps to a nonzero element via $A \rightarrow S_{n_1+1}$, against the hypothesis that $P(\underline{X})pr_{n_1+1}(x) = 0$ in \mathfrak{S}_{n_1+1} . \square

If the f -tuple of integers defining the Serre weight does not verify the assumptions of Lemma 7.5, the result of Proposition 7.7 fails:

PROPOSITION 7.8. *Assume f even and that $\underline{r} = (\dots, 0, p-1, 0, p-1, \dots)$.*

For any $e \in \mathcal{G}_\infty^\bullet$ the natural morphism

$$A \cdot e \rightarrow \mathfrak{S}_\infty^\bullet$$

is not injective.

Proof. Let $e_{2(j+1)+\bullet, i_0} \in \mathcal{G}_\infty^\bullet$. Hence $(r_{i_0+2j+\bullet}, r_{i_0+2j+\bullet+1}) = (0, p-1)$ and, for $h \in \mathbf{N}$ we have

$$r_h = \begin{cases} 0 & \text{if } h \equiv i_0 + \bullet \pmod{2} \\ p-1 & \text{if } h \equiv i_0 + 1 + \bullet \pmod{2}. \end{cases}$$

Moreover for any $n \in 2\mathbf{N} + 1 + \bullet$ we have $X_l^{p^n} \equiv 0$ in S_{n+1} if and only if $l \equiv i_0 + 1 \pmod{2}$.

Let $n_0 \in 2\mathbf{N} + 1 + \bullet$ be such that $pr_{n_0+1}(e_{2(j+1)+\bullet, i_0}) \neq 0$. In particular, if

$$m_l \stackrel{\text{def}}{=} \text{ord}_{X_l}(pr_{n_0+1}(e_{2(j+1)+\bullet, i_0}))$$

we deduce that $p^{n_0} > m_l$ for any l such that $l \equiv i_0 + 1 \pmod{2}$.

If $l \in \{0, \dots, f-1\}$ and $h \in \mathbf{N}$ we have

$$\begin{aligned} \text{ord}_{X_l}(pr_{n_0+1+2h}(e_{2(j+1)+\bullet, i_0})) &= m_l + \sum_{s=0}^{2h-1} p^{n_0+s} (-1)^{s+1} (r_{l+n_0+s} + 1) \\ &= m_l + \begin{cases} 0 & \text{if } l \equiv i_0 \pmod{2} \\ -p^{n_0} + p^{n_0+2h} & \text{if } l \equiv i_0 + 1 \pmod{2} \end{cases} \end{aligned}$$

Since $e_{2(j+1)+\bullet, i_0}$ is a monomial we conclude that, if $l \in \{0, \dots, f-1\}$ verifies $l \equiv i_0 + 1 \pmod{2}$, we have

$$X_l^{p^{n_0}-m_l} \in \text{Ann}_{\mathfrak{S}_{n_0+1}}(e_{2(j+1)+\bullet, i_0})$$

for any $n \in 2\mathbf{N} + 1 + \bullet$ such that $n \geq n_0$. This ends the proof for the case $e = e_{2(j+1)+\bullet, i_0}$.

The proof for the element e_\bullet is analogous and left to the reader. \square

We deduce, from Proposition 7.6, 7.7 and 7.8, the following result on the torsion properties of the Iwasawa module $\mathfrak{S}_\infty^\bullet$:

PROPOSITION 7.9. *Let $x \in \mathfrak{S}_\infty^\bullet$ be a nonzero element, lying in the image of the natural morphism*

$$\bigoplus_{e \in \mathcal{G}_\infty^\bullet} A \cdot e \rightarrow \mathfrak{S}_\infty^\bullet$$

and write $M_x \stackrel{\text{def}}{=} \langle x \rangle_A$ for the sub A -module of $\mathfrak{S}_\infty^\bullet$ generated by the element x .

Then $\mathfrak{S}_\infty^\bullet/M_x$ has a natural structure of profinite A -module and the torsion sub-module $\text{Tor}(\mathfrak{S}_\infty^\bullet/M_x)$ is dense in $\mathfrak{S}_\infty^\bullet/M_x$ for the natural profinite topology.

Moreover, if f is even and $\underline{r} = (\dots, 0, p-1, 0, p-1 \dots)$ then the torsion sub-module $\text{Tor}(\mathfrak{S}_\infty^\bullet)$ is dense in $\mathfrak{S}_\infty^\bullet$ for the natural profinite topology.

Proof. Since A is compact it is clear that M_x is a closed sub-module of $\mathfrak{S}_\infty^\bullet$. By Proposition 7.6 we deduce that the image of the natural morphism

$$\bigoplus_{e \in \mathcal{G}_\infty^\bullet} A \cdot e \rightarrow \mathfrak{S}_\infty^\bullet$$

is a rank one, dense A -sub-module of $\mathfrak{S}_\infty^\bullet$. The result follows.

The statement for the case where f is even and $\underline{r} = (\dots, 0, p-1, 0, p-1 \dots)$ follows in the analogous way using Proposition 7.8. \square

7.3 The case $F = \mathbf{Q}_p$

The aim of this section is to describe explicitly the Iwasawa module $\mathfrak{S}_\infty^{\geq 1} \oplus \mathfrak{S}_\infty^{\geq 2}$ when $F = \mathbf{Q}_p$ in Galois theoretical terms.

Let \mathbf{Q}_{p^2} be the quadratic unramified extension of \mathbf{Q}_p . We fix an embedding $\mathbf{Q}_{p^2} \xhookrightarrow{\iota} E$ and, for $j \in \{0, 1\}$, we write $\tau_j \stackrel{\text{def}}{=} \iota \circ \text{Frob}_{\mathbf{Q}_{p^2}}^j$ where $\text{Frob}_{\mathbf{Q}_{p^2}}$ is the absolute Frobenius on \mathbf{Q}_{p^2} . With this choice, we can define the fundamental Serre character ω_2 of niveau 2 associated to the residual embedding $\mathbf{F}_{p^2} \hookrightarrow k$. For $n \in \{1, \dots, p\}$ we write $\text{ind}(\omega_2^n)$ for the unique (absolutely) irreducible 2-dimensional representation of $G_{\mathbf{Q}_p}$ whose restriction to the inertia subgroup $I_{\mathbf{Q}_p}$ is isomorphic to $\omega_2^n \oplus \omega_2^{pn}$ and whose determinant is ω^n (the mod p -reduction of the cyclotomic character).

If $0 \leq r \leq p-1$ the Galois representation $\text{ind}(\omega_2^{r+1})$ corresponds to the supersingular representation $\pi(\sigma_r, 0)$ and the aim of this section is to show that the \mathcal{F} -module $\mathfrak{S}_\infty^{\geq 1} \oplus \mathfrak{S}_\infty^{\geq 2}$ is isomorphic to the “mod- p Wach module” associated to the dual of $\text{ind}(\omega_2^{r+1})$.

Recall that for $\bullet \in \{0, 1\}$ we have defined (cf. Definition 3.6) the elements $e_{2(j+1)+\bullet, i}$ where $j \in \mathbf{N}$ and $i = 0, \dots, f-1$; As $F = \mathbf{Q}_p$ we will omit the subscript i in what follows. Then $\mathfrak{S}_\infty^\bullet$ is easily seen to be generated over A by the element e_\bullet .

Fix $n \in 2\mathbf{N} + 1$. Using the definition of the elements e_0, e_2 one verifies the following equality inside \mathfrak{S}_{n+1}^0 :

$$e_2 = X \cdot e_0.$$

By the generality of $n \in 2\mathbf{N} + 1$ we deduce that $\mathfrak{S}_\infty^{\geq 1}$ is the submodule of \mathfrak{S}_∞^0 generated (over A) by the element $e_2 = X \cdot e_0$; a completely analogous argument shows that $\mathfrak{S}_\infty^{\geq 2}$ is the submodule of \mathfrak{S}_∞^1 generated (over A) by the element $e_3 = X^{r+1} \cdot e_1$.

We turn our attention to the action of the Frobenius. By Proposition 5.12 we have

$$\begin{aligned}\mathcal{F}(e_2) &= e_3 \\ \mathcal{F}(e_3) &= e_4 = X^{(p-1)(r+1)}e_2.\end{aligned}\tag{26}$$

As usual, the equality $e_4 = X^{(p-1)(r+1)}e_2$ is verified in the quotient $\mathfrak{S}_{n+1}^\bullet$ using the explicit description of the elements e_4, e_2 given in Proposition 3.5, for all $n \in 2\mathbf{N} + 1$ sufficiently large (here it suffices $n \geq 3$).

We leave to the reader the task to verify that H acts on e_2, e_3 by the character $(\chi_r \mathfrak{a})^\vee$ so that, by Proposition 7.1 we deduce the \mathbf{Z}_p^\times action:

$$\gamma e_{2+\bullet} \equiv \bar{\gamma} e_{2+\bullet} + (X^{p-1}) \cdot e_{2+\bullet}$$

for $\bullet \in \{0, 1\}$.

We claim that $\mathfrak{S}_\infty^{\geq 1} \oplus \mathfrak{S}_\infty^{\geq 2}$ is isomorphic, as a Frobenius module, to the mod- p reduction of Wach module associated to $V_{r+1}(0)$, the irreducible crystalline representation of $G_{\mathbf{Q}_p}$ with Hodge-Tate weights $(0, -(r+1))$ and whose trace of Frobenius equals zero. Even if this is obvious to the experts, we decided to include a self contained argument. We invite the reader to refer to [Ber] for the general theory of Wach modules (cf. also [BLZ] or [Dou]).

Let $0 \leq r \leq p-1$ and $N_{r+1}(0)$ be the rank two φ -module over $\mathcal{O}[[X]]$ whose Frobenius action is characterized by

$$[\varphi(n_1), \varphi(n_2)] = [n_1, n_2] \begin{bmatrix} 0 & 1 \\ \mathfrak{q}^{r+1} & 0 \end{bmatrix}\tag{27}$$

where (n_1, n_2) is a $\mathcal{O}[[X]]$ -basis for $N_{r+1}(0)$ and $\mathfrak{q} \stackrel{\text{def}}{=} \frac{(1+X)^{p-1}}{X} \in \mathcal{O}[[X]]$. By the work of [BLZ] (Proposition 3.1.3), there exists a $\mathcal{O}[[X]]$ -semilinear, φ -equivariant \mathbf{Z}_p^\times -action on $N_{r+1}(0)$, which is trivial modulo $XN_{r+1}(0)$; this gives rise to a well defined structure of Wach module on $N_{r+1}(0)$.

The module $N_{r+1}(0)$ is endowed with a filtration (cf. [Ber], Théorème III.4.4)

$$\text{Fil}^j(N_{r+1}(0)) \stackrel{\text{def}}{=} \{x \in N_{r+1}(0), \phi(x) \in \mathfrak{q}^j N_{r+1}(0)\}$$

and one sees that (cf. [BLZ], proof of Proposition 3.2.4)

$$\text{Fil}^j(N_{r+1}(0)/XN_{r+1}(0)) = \begin{cases} \mathcal{O}n_1 \oplus \mathcal{O}n_2 & \text{if } j \leq 0 \\ \mathcal{O}n_1 & \text{if } 1 \leq j \leq r+1 \\ 0 & \text{if } j \geq r+2. \end{cases}\tag{28}$$

By [Ber], Proposition III.4.2 and Corollaire III.4.5 we have an isomorphism of filtered φ -modules over E :

$$E \otimes_{\mathcal{O}} (N_{r+1}(0)/XN_{r+1}(0)) \xrightarrow{\sim} \mathbf{D}_{\text{cris}}(V_{r+1}(0))$$

for an appropriate crystalline representation $V_{r+1}(0)$ with Hodge-Tate weights $\{0, -(r+1)\}$.

LEMMA 7.10. *In the previous hypotheses, we have an isomorphism of crystalline representations $V_{r+1}(0) \cong \text{ind}_{G_{\mathbf{Q}_{p^2}}}^{G_{\mathbf{Q}_p}} \chi_{(0,1)}^{r+1}$ where $\chi_{(0,1)}$ is the crystalline character of $G_{\mathbf{Q}_{p^2}}$ with labelled Hodge-Tate weights $-(0, 1)$ and such that $\chi_{(0,1)}(p) = 1$.*

Proof. By the equations (27) and (28) we have a complete description of the filtered φ -module $E \otimes_{\mathcal{O}} (N_{r+1}(0)/XN_{r+1}(0))$ (notice that $\mathfrak{q}^{r+1}n_2 \equiv p^{r+1}n_2 \pmod{XN_{r+1}(0)}$). The result follows then from Breuil [Bre] Proposition 3.1.2 et 3.1.1.

Alternatively, we can prove the Lemma using the theory of Wach modules, as we outline in the following lines.

One easily sees by (28) that the filtered module $E \otimes_{\mathcal{O}_E} (\mathbf{N}_{r+1}(0)/X\mathbf{N}_{r+1}(0))$ has no nonzero, φ -admissible proper submodules, and hence $V_{r+1}(0)$ is irreducible. Let $T_{r+1}(0)$ be the \mathcal{O}_E -lattice of $V_{r+1}(0)$ corresponding to $\mathbf{N}_{r+1}(0)$ via the equivalence of [Ber] Proposition III.4.2.

By the results of [Dou], §2 we can describe the $G_{\mathbf{Q}_{p^2}}$ -restriction of $T_{r+1}(0)$ in terms of Wach modules over $\mathbf{Z}_{p^2} \otimes_{\mathbf{Z}_p} \mathcal{O}_E[[X]]$. Recall the natural isomorphism of rings

$$\begin{aligned} \mathbf{Z}_{p^2} \otimes_{\mathbf{Z}_p} \mathcal{O}[[X]] &\xrightarrow{\sim} \mathcal{O}[[X]] \oplus \mathcal{O}[[X]] \\ x \otimes P(X) &\longmapsto (\tau_0(x)P(X), \tau_1(x)P(X)). \end{aligned}$$

By [Dou], Proposition 2.5 and 2.6, the Wach module over $\mathbf{Z}_{p^2} \otimes_{\mathbf{Z}_p} \mathcal{O}_E[[X]]$ associated to $T_{r+1}(0)|_{G_{\mathbf{Q}_{p^2}}}$ is obtained by extension of scalars from $\mathbf{N}_{r+1}(0)$. In particular its Frobenius action is defined by:

$$[\varphi(n_1), \varphi(n_2)] = [n_1, n_2] \begin{bmatrix} (0, 0) & (1, 1) \\ (\mathfrak{q}^{r+1}, \mathfrak{q}^{r+1}) & (0, 0) \end{bmatrix}$$

and the matrix equality

$$\begin{bmatrix} (1, 0) & (0, 1) \\ (0, 1) & (1, 0) \end{bmatrix} \begin{bmatrix} (0, 0) & (1, 1) \\ (\mathfrak{q}^{r+1}, \mathfrak{q}^{r+1}) & (0, 0) \end{bmatrix} = \begin{bmatrix} (1, \mathfrak{q}^{r+1}) & (0, 0) \\ (0, 0) & (\mathfrak{q}^{r+1}, 1) \end{bmatrix} \varphi \left(\begin{bmatrix} (1, 0) & (0, 1) \\ (0, 1) & (1, 0) \end{bmatrix} \right)$$

shows that we have an isomorphism of Wach modules

$$\mathbf{N}(T_{r+1}(0)|_{G_{\mathbf{Q}_{p^2}}}) \xrightarrow{\sim} \mathbf{N}_{(0,r+1),1} \oplus \mathbf{N}_{(r+1,0),1}$$

where $\mathbf{N}_{(0,r+1),1}$ (resp. $\mathbf{N}_{(r+1,0),1}$) are the rank one Wach modules over $\mathbf{Z}_{p^2} \otimes_{\mathbf{Z}_p} \mathcal{O}_E[[X]]$ whose Frobenius action is characterized, on appropriate generators η_0, η_1 , by $\varphi(\eta_0) = (1, \mathfrak{q}^{r+1})\eta_0$ (resp. $\varphi(\eta_1) = (\mathfrak{q}^{r+1}, 1)\eta_1$, cf. also [Dou], §3.1.).

By [Dou], Proposition 3.5 *et seq.* we have

$$E \otimes_{\mathcal{O}} \mathbf{N}_{(0,r+1),1} = \mathbf{D}_{\text{cris}}(\chi_{(0,1)}^{r+1}), \quad E \otimes_{\mathcal{O}} \mathbf{N}_{(r+1,0),1} = \mathbf{D}_{\text{cris}}(\chi_{(1,0)}^{r+1})$$

where $\chi_{(1,0)}$ (resp. $\chi_{(0,1)}$) is the crystalline character of labelled Hodge-Tate weights $-(1, 0)$ (resp. $-(0, 1)$) such that $\chi_{(0,1)}(p) = 1 = \chi_{(1,0)}(p)$. It follows that, $V_{r+1}(0)|_{G_{\mathbf{Q}_{p^2}}} \cong \chi_{(1,0)}^{r+1} \oplus \chi_{(0,1)}^{r+1}$ and we deduce

$$V_{r+1}(0) \cong \text{ind}_{G_{\mathbf{Q}_{p^2}}}^{G_{\mathbf{Q}_p}} \chi_{0,1}^{r+1}$$

as $V_{r+1}(0)$ is irreducible. □

The mod- p reduction of the crystalline character $\chi_{(0,1)}$ is deduced from [Dou], Lemma 6.2:

$$\bar{\chi}_{(0,1)} \cong \omega_2^{-1}$$

hence the mod- p reduction of the crystalline representation $V_{r+1}(0)$ is given by

$$\bar{V}_{r+1}(0) \cong (\text{ind}(\omega_2^{r+1}))^*.$$

(notice that $\bar{V}_{r+1}(0)$ is irreducible as $r \leq p-1$), and we define the mod- p Wach module:

$$\mathbf{N}((\text{ind}(\omega_2^{r+1}))^*) \stackrel{\text{def}}{=} \mathbf{N}_{r+1}(0) \otimes_{\mathcal{O}} k.$$

Since $\mathfrak{q}n_1 \otimes 1 = X^{p-1}n_1 \otimes 1$ in $\mathbf{N}_{r+1}(0) \otimes_{\mathcal{O}} k$, by comparing the equations (27) and (26) we deduce:

PROPOSITION 7.11. *Let $F = \mathbf{Q}_p$ and $\sigma = \sigma_r$ for $r \in \{0, \dots, p-1\}$. We have an isomorphism of φ -modules over A :*

$$\mathfrak{S}_{\infty}^{\geq 1} \oplus \mathfrak{S}_{\infty}^{\geq 2} \cong \mathbf{N}((\text{ind}(\omega_2^{(r+1)}))^*).$$

where $\mathfrak{S}_{\infty}^{\geq 1} \oplus \mathfrak{S}_{\infty}^{\geq 2}$ is the Iwasawa module associated to the supersingular representation $\pi(\sigma, 0)$.

8. A note on Principal and Special series

The aim of this section is to extend the previous constructions to the case of tamely ramified principal series. The arguments are now much simpler; we invite the reader to refer to [Mo1], §5 for the omitted details.

Recall that the tamely ramified principal series for $\mathbf{GL}_2(F)$ are described (up to a twist by a smooth character) by the parabolic induction

$$\pi_{\underline{r}, \mu} \stackrel{\text{def}}{=} \mathrm{ind}_{\mathbf{B}(F)}^{\mathbf{GL}_2(F)}(\mathrm{un}_\mu \otimes \omega^{\underline{r}} \mathrm{un}_{\mu^{-1}}) \quad (29)$$

where $\mu \in \bar{k}^\times$, un_μ is the unramified character of F^\times verifying $\mathrm{un}_\mu(\varpi) = \mu$, $\underline{r} \in \{0, \dots, p-1\}^f$ and ω is the mod p cyclotomic character of F^\times .

It is known by the work of Barthel and Livné [BL94] that the principal series (29) is absolutely irreducible if either $|\underline{r}| \notin \{0, q\}$ or $|\underline{r}| \in \{0, q\}$ and $\mu \notin \{1, -1\}$.

On the other hand if $|\underline{r}| \in \{0, q\}$ and $\mu \in \{1, -1\}$ we have a short exact sequence

$$0 \rightarrow 1 \rightarrow \mathrm{ind}_{\mathbf{B}(\mathbf{Q}_p)}^{\mathbf{GL}_2(\mathbf{Q}_p)} 1 \rightarrow \mathrm{St} \rightarrow 0.$$

where St (the Steinberg representation for $\mathbf{GL}_2(F)$) is absolutely irreducible.

Since $\mathbf{B}(F) \backslash \mathbf{GL}_2(F)$ is compact, we have the following K -equivariant isomorphism

$$(\mathrm{ind}_{\mathbf{B}(F)}^{\mathbf{GL}_2(F)}(\mathrm{un}_\mu \otimes \omega^{\underline{r}} \mathrm{un}_{\mu^{-1}}))|_K \cong \mathrm{ind}_{K_0(\varpi^\infty)}^K \chi_{\underline{r}}^s \cong \varinjlim_{n \geq 1} (\mathrm{ind}_{K_0(\varpi^{n+1})}^K \chi_{\underline{r}}^s) \quad (30)$$

where the transition morphisms for the co-limit in the RHS are obtained inducing the natural monomorphisms of $K_0(\varpi^n)$ -representations

$$\chi_{\underline{r}}^s \hookrightarrow \mathrm{ind}_{K_0(\varpi^{n+1})}^{K_0(\varpi^n)} \chi_{\underline{r}}^s \quad (31)$$

(notice that such monomorphism is unique up to scalar by Frobenius reciprocity).

For the tamely ramified principal series $\pi_{\underline{r}, \mu}$ we associate the $K_0(\varpi)$ -sub-representation

$$R_\infty^- \stackrel{\text{def}}{=} \mathrm{ind}_{K_0(\varpi^\infty)}^{K_0(\varpi)} \chi_{\underline{r}}^s.$$

The representation R_∞^- controls the representation theoretic behavior of principal and special series representations for $\mathbf{GL}_2(F)$:

PROPOSITION 8.1. *Let $\pi_{\underline{r}, \mu}$ be a tamely ramified principal series and let R_∞^- be the associated $K_0(\varpi)$ sub-module. We have a K -equivariant isomorphism*

$$\pi_{\underline{r}, \mu}|_K \cong \mathrm{ind}_{K_0(p)}^K R_\infty^-$$

Moreover we have a N -equivariant isomorphism

$$\pi_{\underline{r}, \mu}|_N \cong R_\infty^- \oplus (R_\infty^-)^s$$

where the action of α on the RHS is given by the involution

$$\begin{aligned} R_\infty^- &\longrightarrow (R_\infty^-)^s \\ v &\longmapsto \mu v. \end{aligned}$$

In particular, the Steinberg representation fits in the following exact sequences:

$$\begin{aligned} 0 &\rightarrow 1 \rightarrow \mathrm{ind}_{K_0(\varpi)}^K R_\infty^- \rightarrow \mathrm{St}|_K \rightarrow 0 \\ 0 &\rightarrow 1 \rightarrow R_\infty^- \oplus (R_\infty^-)^s \rightarrow \mathrm{St}|_N \rightarrow 0. \end{aligned}$$

Proof. The assertions on the K -structure of $\pi_{\mathcal{L},\mu}$ follow from the isomorphism (30) and the formal properties of the compact induction functor.

The assertions on the N -structure of $\pi_{\mathcal{L},\mu}$ can be checked directly using the Mackey decomposition

$$(\pi_{\mathcal{L},\mu})|_{K_0(\varpi)} \cong (\text{ind}_{K_0(\varpi^\infty)}^{K_0(\varpi)} \chi_{\mathcal{L}}^s) \oplus (\text{ind}_{K_0(\varpi) \cap \overline{\mathbf{B}}(F)}^{K_0(\varpi)} \chi_{\mathcal{L}}^s) \quad (32)$$

and noticing that α normalizes $K_0(\varpi)$ (hence the $K_0(\varpi)$ -equivariant isomorphism between the direct summands in the RHS of (32), once we endow one of them with the conjugate action of $K_0(\varpi)$).

□

Assume now that F is unramified over \mathbf{Q}_p .

We define \mathfrak{S}_∞ to be the Pontryagin dual of R_∞^- . Since the transition morphisms $\text{ind}_{K_0(p^n)}^{K_0(p)} \chi_{\mathcal{L}}^s \hookrightarrow \text{ind}_{K_0(p^{n+1})}^{K_0(p)} \chi_{\mathcal{L}}^s$ are obtained, by induction, from the monomorphism (31), it is easy to see that

$$\mathfrak{S}_\infty = \lim_{\substack{\longleftarrow \\ n \geq 1}} A / \langle X_i^{p^n}, i = 0, \dots, f-1 \rangle$$

where the morphisms defining the projective system are the natural projection (and respect the $k[[K_0(p)]]$ -module structures).

Hence

PROPOSITION 8.2. *The universal module \mathfrak{S}_∞ associated to a principal or a special series is described by*

$$\mathfrak{S}_\infty \cong k[[K_0(p)]] \widehat{\otimes}_{k[[K_0(p^\infty)]]} 1.$$

Proof. Omissis.

□

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Stefano Morra stefano.morra@utoronto.ca &
stefano.morra@univ-montp2.fr

Université Montpellier 2, Place Eugène Bataillon, Case Courrier 51, 34090 Montpellier, France
Current address: The Fields Institute, 222 College Street, Toronto, Ontario M5T 3J1, Canada