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# ON FRIEDRICHS CONSTANT AND HORGAN-PAYNE ANGLE FOR LBB CONDITION

Monique Dauge, Christine Bernardi, Martin Costabel,  
and Vivette Girault

**Abstract.** In dimension 2, the Horgan-Payne angle serves to construct a lower bound for the inf-sup constant of the divergence arising in the so-called LBB condition. This lower bound is equivalent to an upper bound for the Friedrichs constant. Explicit upper bounds for the latter constant can be found using a polar parametrization of the boundary. Revisiting carefully the original paper which establishes this strategy, we found out that some proofs need clarification, and some statements, replacement.

*Keywords:* LBB condition, inf-sup constant, Friedrichs constant, Horgan-Payne angle.

*AMS classification:* 30A10, 35Q35.

## §1. The inf-sup constant and some general properties

Here we only consider *bounded connected* open sets  $\Omega$  in  $\mathbb{R}^2$ , the generic point in  $\mathbb{R}^2$  being denoted by  $\mathbf{x} = (x_1, x_2)$ . For such a domain  $\Omega$ , the inf-sup constant of the divergence associated with Dirichlet boundary conditions, also called LBB constant after LADYZHENSKAYA, BABUŠKA [3, 2] and BREZZI [4], is defined as

$$\beta(\Omega) = \inf_{q \in L^2_0(\Omega)} \sup_{\mathbf{v} \in H^1_0(\Omega)^2} \frac{\langle \operatorname{div} \mathbf{v}, q \rangle_\Omega}{|\mathbf{v}|_{1,\Omega} \|q\|_{0,\Omega}}. \quad (1)$$

Here

- $L^2_0(\Omega)$  stands for the space of square integrable scalar functions  $q$  with zero mean value in  $\Omega$  endowed with its natural norm  $\|\cdot\|_{0,\Omega}$  and natural scalar product  $\langle \cdot, \cdot \rangle_\Omega$ ,
- $H^1_0(\Omega)^2$  is the standard Sobolev space of vector functions  $\mathbf{v} = (v_1, v_2)$  with square integrable gradients and zero traces on the boundary, endowed with its natural semi-norm

$$|\mathbf{v}|_{1,\Omega} = \left( \sum_{k=1}^2 \sum_{j=1}^2 \|\partial_{x_j} v_k\|_{0,\Omega}^2 \right)^{1/2}.$$

Since  $\Omega$  is bounded, by virtue of the Poincaré inequality, the above semi-norm on  $H^1_0(\Omega)^2$  is equivalent to the usual norm in  $H^1(\Omega)^2$ .

We list some elementary properties of  $\beta(\Omega)$ :

- (a)  $\beta(\Omega) \geq 0$ ,

- (b)  $\beta(\Omega) \leq 1$ , because of the identity  $\|v\|_{1,\Omega}^2 = \|\operatorname{curl} v\|_{0,\Omega}^2 + \|\operatorname{div} v\|_{0,\Omega}^2$  for any  $v \in H_0^1(\Omega)^2$ ,
- (c)  $\beta(\Omega)$  is invariant by translations, dilations, symmetries and rotations by virtue of Piola transform. Thus  $\beta(\Omega)$  only depends on the *shape* of  $\Omega$ .

The constant  $\beta(\Omega)$  is positive for Lipschitz domains (see [8, Chap. 1, Section 2.2], which relies on [12, Chap. 3, Lemme 7.1]), and also for domains with less regular boundary like John domains [1]. In contrast, domains with an external cusp (also called thin peak) satisfy  $\beta(\Omega) = 0$ , see [15, Chap. 15].

Finding calculable lower bounds for  $\beta(\Omega)$  is of great interest, since it is involved in any analysis of the Stokes and Navier-Stokes equations with no-slip boundary conditions. Moreover, discrete inf-sup constants between finite dimensional subspaces of  $H_0^1(\Omega)^2$  and  $L_0^2(\Omega)$  are influenced by both the continuous inf-sup constant  $\beta(\Omega)$  and the type of chosen (mixed) discrete spaces, see [13] and also [6].

In reference [10], HORGAN & PAYNE design an efficient strategy for calculating lower bounds of  $\beta(\Omega)$  in domains  $\Omega$  whose boundary can be described in polar coordinates  $(r, \theta)$  by a relation  $r = f(\theta)$  with a Lipschitz-continuous function  $f$ :

- First, state a relation between  $\beta(\Omega)$  and the Friedrichs constant  $\Gamma(\Omega)$ ,
- Second, find bounds for  $\Gamma(\Omega)$  using  $f$  and its first derivative  $f'$ .

In the present paper, we revisit these two steps, with more emphasis on the second one.

## §2. The Friedrichs constant

In dimension 2, the coordinates  $(x_1, x_2)$  are identified with the complex number  $x_1 + ix_2$ . Two real valued functions  $h$  and  $g$  are said to be harmonic conjugate if they are the real and imaginary parts of a holomorphic function  $h + ig$ . The functions  $h$  and  $g$  are harmonic conjugate if and only if they satisfy the relations

$$\Delta h = 0, \quad \Delta g = 0, \quad \text{and} \quad \operatorname{grad} h = \operatorname{curl} g \quad \text{in} \quad \Omega.$$

Let  $\mathfrak{F}(\Omega)$  denote the space of complex valued  $L^2(\Omega)$  holomorphic functions and let  $\mathfrak{F}_0(\Omega)$  be its subspace of functions with mean value 0.

**Definition 1.** The *Friedrichs constant* (named after [7]) denoted by  $\Gamma(\Omega)$ , is the smallest constant  $\Gamma \in \mathbb{R} \cup \{\infty\}$  such that for all  $h + ig \in \mathfrak{F}_0(\Omega)$

$$\|h\|_{L^2(\Omega)}^2 \leq \Gamma \|g\|_{L^2(\Omega)}^2.$$

**Theorem 1** ([10], [5]). *Let  $\Omega$  be any bounded connected domain in  $\mathbb{R}^2$ . The LBB constant  $\beta(\Omega)$  is positive if and only if  $\Gamma(\Omega)$  is finite and*

$$\Gamma(\Omega) + 1 = \frac{1}{\beta(\Omega)^2}.$$

This relation between  $\beta(\Omega)$  and  $\Gamma(\Omega)$  was proved in [10] under additional regularity properties on the domain. A new proof is provided in [5], in which no regularity assumption is needed.

### §3. An upper bound for the Friedrichs constant

Let  $\Omega$  be *strictly star-shaped*, which means that there is an open ball  $B \subset \Omega$  such that any segment with one end in  $B$  and the other in  $\Omega$ , is contained in  $\Omega$ . Let  $O$  be the center of  $B$  and  $(r, \theta)$  be polar coordinates centered at  $O$ . Let  $\theta \mapsto r = f(\theta)$  be the polar parametrization of the boundary  $\partial\Omega$ , defined on the torus  $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ .

**Lemma 2** ([11, Lemma 1.1.8]). *Let  $\Omega$  be a bounded strictly star-shaped domain, and  $f$  be a polar parametrization of its boundary as described above. Then  $f$  belongs to  $W^{1,\infty}(\mathbb{T})$ .*

Since  $\Gamma(\Omega)$  is invariant by dilation, we may assume without restriction that

$$\boxed{\max_{\theta \in \mathbb{T}} f(\theta) = 1} \quad (2)$$

Following the approach in [10], we are prompted to introduce the following notation.

*Notation 3.* Under condition (2), let  $P = P(\alpha, \theta)$  be the function defined on  $\mathbb{R}_+ \times \mathbb{T}$  as

$$P(\alpha, \theta) = \frac{1}{\alpha f(\theta)^2} \left( 1 + \frac{f'(\theta)^2}{f(\theta)^2 - \alpha f(\theta)^4} \right). \quad (3)$$

Let  $M(\Omega)$  and  $m(\Omega)$  be the following two positive numbers

$$M(\Omega) = \inf_{\alpha \in (0,1)} \left\{ \sup_{\theta \in \mathbb{T}} P(\alpha, \theta) \right\} \quad \text{and} \quad m(\Omega) = \sup_{\theta \in \mathbb{T}} \left\{ \inf_{\alpha \in (0, \frac{1}{f(\theta)^2})} P(\alpha, \theta) \right\}. \quad (4)$$

*Remark 1.* Let us choose  $\theta \in \mathbb{T}$ . Calculating the second derivative of the function  $P_\theta : \alpha \mapsto P(\alpha, \theta)$  defined on the interval  $(0, \frac{1}{f(\theta)^2})$ , we find that  $P_\theta$  is strictly convex. The function  $P_\theta$  tends to  $+\infty$  as  $\alpha \rightarrow 0$ , and if  $f'(\theta) \neq 0$ , as  $\alpha \rightarrow \frac{1}{f(\theta)^2}$ . In any case, there exists a unique  $\alpha(\theta)$  in  $(0, \frac{1}{f(\theta)^2}]$  such that

$$P(\alpha(\theta), \theta) = \inf_{\alpha \in (0, \frac{1}{f(\theta)^2})} P(\alpha, \theta).$$

So,

$$m(\Omega) = \sup_{\theta \in \mathbb{T}} P(\alpha(\theta), \theta). \quad (5)$$

Since, in particular, for all  $\alpha \in (0, 1)$  and  $\theta \in \mathbb{T}$ ,  $P(\alpha(\theta), \theta) \leq P(\alpha, \theta)$ , we find that

$$M(\Omega) \geq m(\Omega). \quad (6)$$

The quantity  $m(\Omega)$  is the original bound introduced by Horgan-Payne in [10] and  $M(\Omega)$  is our modified Horgan-Payne like bound.

**Theorem 4** (Estimate (6.24) in [10]). *Let  $\Omega$  be a bounded strictly star-shaped domain. Its Friedrichs constant satisfies the bound*

$$\Gamma(\Omega) \leq M(\Omega). \quad (7)$$

*Proof.* We assume for simplicity that the origin  $O$  of polar coordinates coincides with the origin  $\mathbf{0}$  of Cartesian coordinates. Let  $g \in \mathcal{D}(\overline{\Omega})$  be an harmonic function and let  $h \in \mathcal{D}(\overline{\Omega})$  be its harmonic conjugate such that  $h(\mathbf{0}) = 0$ . If we bound the  $L^2(\Omega)$  norm of  $h$ , we bound a fortiori the  $L^2(\Omega)$  norm of  $h - \frac{1}{|\Omega|} \int_{\Omega} h$  which is the harmonic conjugate of  $g$  in  $L^2_c(\Omega)$ , hence with minimal  $L^2(\Omega)$  norm. The extension of the estimate to all pairs of harmonic conjugate functions in  $L^2(\Omega)$  follows from a density argument.

Since  $h + ig$  is holomorphic, its square is holomorphic too and we deduce that the function  $H := h^2 - g^2$  is harmonic conjugate of  $G := 2gh$ . Hence equation  $\text{grad } H = \text{curl } G$  leads to the relation in polar coordinates

$$\partial_{\rho} \tilde{H} = \frac{1}{\rho} \partial_{\theta} \tilde{G}$$

where  $\tilde{H}(r, \theta) = H(\mathbf{x})$  and  $\tilde{G}(r, \theta) = G(\mathbf{x})$  for  $\mathbf{x} = (r \cos \theta, r \sin \theta)$ . Thus for any  $\theta \in \mathbb{T}$  and  $r \in (0, f(\theta))$  we have

$$\tilde{H}(r, \theta) - H(\mathbf{0}) = \int_0^r \partial_{\rho} \tilde{H}(\rho, \theta) d\rho = \int_0^r \frac{1}{\rho} \partial_{\theta} \tilde{G}(\rho, \theta) d\rho.$$

We divide by  $f(\theta)^2$  and integrate for  $\theta \in \mathbb{T}$  and  $r \in (0, f(\theta))$ :

$$\begin{aligned} \int_{\mathbb{T}} \int_0^{f(\theta)} \frac{\tilde{H}(r, \theta) - H(\mathbf{0})}{f(\theta)^2} r dr d\theta &= \int_{\mathbb{T}} \int_0^{f(\theta)} \frac{1}{f(\theta)^2} \left\{ \int_0^r \frac{1}{\rho} \partial_{\theta} \tilde{G}(\rho, \theta) d\rho \right\} r dr d\theta \\ &= \int_{\mathbb{T}} \int_0^{f(\theta)} \frac{1}{f(\theta)^2} \frac{1}{\rho} \partial_{\theta} \tilde{G}(\rho, \theta) \left\{ \int_{\rho}^{f(\theta)} r dr \right\} d\rho d\theta \\ &= \frac{1}{2} \int_{\mathbb{T}} \int_0^{f(\theta)} \frac{f(\theta)^2 - \rho^2}{\rho^2 f(\theta)^2} \partial_{\theta} \tilde{G}(\rho, \theta) \rho d\rho d\theta. \end{aligned}$$

Since the function  $f(\theta)^2 - \rho^2$  is 0 on the boundary, integration by parts yields

$$\int_{\mathbb{T}} \int_0^{f(\theta)} \frac{\tilde{H}(r, \theta) - H(\mathbf{0})}{f(\theta)^2} r dr d\theta = - \int_{\mathbb{T}} \int_0^{f(\theta)} \frac{f'(\theta)}{f(\theta)^3} \tilde{G}(\rho, \theta) \rho d\rho d\theta.$$

We set for any  $\theta \in \mathbb{T}$

$$t(\theta) = \frac{f'(\theta)}{f(\theta)}.$$

Coming back to  $h$  and  $g$  we find:

$$\int_{\Omega} \frac{h(\mathbf{x})^2}{f(\theta)^2} d\mathbf{x} = \int_{\Omega} \frac{g(\mathbf{x})^2 - g(\mathbf{0})^2}{f(\theta)^2} d\mathbf{x} - 2 \int_{\Omega} \frac{t(\theta)h(\mathbf{x})g(\mathbf{x})}{f(\theta)^2} d\mathbf{x}. \quad (8)$$

In order to take the best advantage of the previous identity we introduce a parameter

$$\alpha \in (0, 1)$$

and write for any  $\theta \in \mathbb{T}$  (here we use condition (2) which ensures that  $1 - \alpha f(\theta)^2 > 0$ )

$$2|t(\theta)h(\mathbf{x})g(\mathbf{x})| \leq \{1 - \alpha f(\theta)^2\}h(\mathbf{x})^2 + \frac{t(\theta)^2}{1 - \alpha f(\theta)^2}g(\mathbf{x})^2$$

and deduce from (8) that (note that the same  $\alpha$  is used for all  $\theta$ )

$$\alpha \int_{\Omega} h(\mathbf{x})^2 \, d\mathbf{x} \leq \int_{\Omega} \frac{g(\mathbf{x})^2}{f(\theta)^2} + \frac{t(\theta)^2}{1 - \alpha f(\theta)^2} \frac{g(\mathbf{x})^2}{f(\theta)^2} \, d\mathbf{x}.$$

Thus, for any  $\alpha \in (0, 1)$

$$\int_{\Omega} h(\mathbf{x})^2 \, d\mathbf{x} \leq \sup_{\theta \in \mathbb{T}} \left\{ \frac{1}{\alpha f(\theta)^2} \left( 1 + \frac{t(\theta)^2}{1 - \alpha f(\theta)^2} \right) \right\} \int_{\Omega} g(\mathbf{x})^2 \, d\mathbf{x}.$$

Optimizing on  $\alpha \in (0, 1)$  and coming back to the definition of  $t$  and  $P$ , we find

$$\int_{\Omega} h(\mathbf{x})^2 \, d\mathbf{x} \leq \inf_{\alpha \in (0,1)} \left\{ \sup_{\theta \in \mathbb{T}} P(\alpha, \theta) \right\} \int_{\Omega} g(\mathbf{x})^2 \, d\mathbf{x},$$

which is nothing else than  $\|h\|_{0,\Omega}^2 \leq M(\Omega) \|g\|_{0,\Omega}^2$ , whence the theorem.  $\square$

*Remark 2.* The proof above is due to Horgan and Payne in [10, § 6]. Unfortunately, instead of simply concluding that  $M(\Omega)$  is an upper bound for  $\Gamma(\Omega)$ , they try to show that  $M(\Omega)$  coincides with  $m(\Omega)$  and this part of their argument is flawed. In the rest of our paper we discuss cases where equality or non-equality holds between these two quantities.

## §4. The Horgan-Payne angle

STOYAN in [14] propose an interesting geometrical interpretation of the lower bound on  $\beta(\Omega)$  under the condition that  $m(\Omega)$  is an upper bound for  $\Gamma(\Omega)$ .

*Notation 5.* For  $\theta \in \mathbb{T}$ , let  $\mathbf{x}$  be the point  $(f(\theta) \cos \theta, f(\theta) \sin \theta)$  in  $\partial\Omega$ , let  $\gamma(\theta) \in [0, \frac{\pi}{2})$  denote the (non-oriented) angle between the line  $[\mathbf{0}, \mathbf{x}]$  and the outward normal vector to  $\partial\Omega$  at  $\mathbf{x}$ . We set

$$\gamma(\Omega) = \sup_{\theta \in \mathbb{T}} \gamma(\theta) \quad \text{and} \quad \omega(\Omega) = \frac{\pi}{2} - \gamma(\Omega). \quad (9)$$

The angle  $\omega(\Omega)$  is referred as the Horgan-Payne angle in [14].

**Lemma 6.** *We have the identities*

$$m(\Omega) = \frac{1 + \sin \gamma(\Omega)}{1 - \sin \gamma(\Omega)} \quad \text{and} \quad \frac{1}{\sqrt{m(\Omega) + 1}} = \sin \frac{\omega(\Omega)}{2}. \quad (10)$$

*Proof.* Let us recall the formulas

$$\cos \gamma(\theta) = \frac{f(\theta)}{\sqrt{f(\theta)^2 + f'(\theta)^2}}, \quad \sin \gamma(\theta) = \frac{f'(\theta)}{\sqrt{f(\theta)^2 + f'(\theta)^2}}, \quad \tan \gamma(\theta) = \frac{f'(\theta)}{f(\theta)}.$$

Hence we have

$$P(\alpha, \theta) = \frac{1}{\alpha f(\theta)^2} \left( 1 + \frac{\tan^2 \gamma(\theta)}{1 - \alpha f(\theta)^2} \right). \quad (11)$$

Let  $\theta$  be chosen. To determine the value  $\alpha(\theta)$  which realizes the minimum of  $P(\alpha, \theta)$  for  $\alpha \in (0, 1/f(\theta)^2]$ , cf. (5), we calculate

$$\partial_\alpha P(\alpha, \theta) = -\frac{1}{\alpha^2 f(\theta)^2} \left( 1 + \frac{\tan^2 \gamma(\theta)}{1 - \alpha f(\theta)^2} \right) + \frac{1}{\alpha f(\theta)^2} \frac{\tan^2 \gamma(\theta) f(\theta)^2}{(1 - \alpha f(\theta)^2)^2}.$$

Setting  $\zeta = \alpha f(\theta)^2$ , we see that  $\partial_\alpha P(\alpha, \theta) = 0$  if and only if

$$\zeta^2 - 2(1 + \tan^2 \gamma(\theta))\zeta + 1 + \tan^2 \gamma(\theta) = 0. \quad (12)$$

We look for  $\zeta \in (0, 1]$ . The convenient root of equation (12) is

$$\begin{aligned} \alpha(\theta) f(\theta)^2 = \zeta &= 1 + \tan^2 \gamma(\theta) - \tan \gamma(\theta) \sqrt{1 + \tan^2 \gamma(\theta)} \\ &= \frac{1}{1 + \sin \gamma(\theta)}. \end{aligned} \quad (13)$$

Hence we find

$$P(\alpha(\theta), \theta) = \frac{1 + \sin \gamma(\theta)}{1 - \sin \gamma(\theta)}, \quad (14)$$

whose supremum is attained for the supremum  $\gamma(\Omega)$  of  $\gamma(\theta)$ , whence the first formula in (10). The second formula is obtained using  $\sin \frac{\omega(\Omega)}{2} = \sin(\frac{\pi}{4} - \frac{\gamma(\Omega)}{2}) = \frac{1}{\sqrt{2}}(\cos \frac{\gamma(\Omega)}{2} - \sin \frac{\gamma(\Omega)}{2})$ .  $\square$

As a straightforward consequence of Theorem 1 and Lemma 6, we obtain the following.

**Corollary 7.** *For any domain such that  $\Gamma(\Omega) \leq m(\Omega)$ , the inf-sup constant  $\beta(\Omega)$  satisfies*

$$\beta(\Omega) \geq \sin \frac{\omega(\Omega)}{2}. \quad (15)$$

*Remark 3.* The estimate (15) is stated in [14] for any strictly star-shaped domain. The reality is that (15) is true if and only if  $\Gamma(\Omega) \leq m(\Omega)$ . The latter estimate is true for some categories of domains as we will see in the next section. We will also exhibit domains for which  $m(\Omega)$  is distinct from  $M(\Omega)$ . In [5] it is proved that, in fact, there exists strictly star-shaped domains such that  $\Gamma(\Omega) > m(\Omega)$  (equivalently,  $\beta(\Omega) < \sin \frac{\omega(\Omega)}{2}$ ).

## §5. Examples

In this section, we consider some particular shapes of domains, namely ellipses, polygons, and limaçons.

### 5.1. Disks and ellipses

The equation of an ellipse can always be written in suitable Cartesian coordinates as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

with positive coefficients  $a \leq b$ . The constant  $\Gamma(\Omega)$  is analytically known, cf. [7], namely

$$\Gamma(\Omega) = \frac{b^2}{a^2} \quad \text{and} \quad \beta(\Omega) = \frac{a}{\sqrt{a^2 + b^2}}. \quad (16)$$

In polar coordinates, the parametrization of the ellipse is

$$f(\theta) = ab \left( b^2 \cos^2 \theta + a^2 \sin^2 \theta \right)^{-1/2}. \quad (17)$$

i) Let us calculate  $m(\Omega)$ . We have

$$\tan \gamma(\theta) = \frac{f'(\theta)}{f(\theta)} = \frac{\sin \theta \cos \theta (b^2 - a^2)}{b^2 \cos^2 \theta + a^2 \sin^2 \theta} = \frac{\tan \theta (b^2 - a^2)}{b^2 + a^2 \tan^2 \theta}. \quad (18)$$

The maximal value  $\tan \gamma(\Omega)$  of  $\tan \gamma(\theta)$  is obtained for

$$\tan \theta = \frac{b}{a},$$

hence

$$\tan \gamma(\Omega) = \frac{b^2 - a^2}{2ab}$$

from which we deduce

$$\sin \gamma(\Omega) = \frac{b^2 - a^2}{b^2 + a^2}.$$

Formula (10) then yields

$$m(\Omega) = \frac{b^2}{a^2}.$$

ii) Let us calculate  $M(\Omega)$ . In order to comply with the condition  $\max_{\theta \in \mathbb{T}} f(\theta) = 1$ , we set  $\tilde{a} = b/a$  and  $\tilde{b} = 1$ , and consider  $f$  given by (17) with  $a, b$  replaced by  $\tilde{a}, \tilde{b}$ . We use formula (11) for  $P$  to write:

$$P(\alpha, \theta) = \frac{1}{\alpha} \frac{(1 + \tan^2 \gamma(\theta)) f(\theta)^{-2} - \alpha}{1 - \alpha f(\theta)^2}.$$

From (17) and (18) we deduce

$$(1 + \tan^2 \gamma(\theta)) f(\theta)^{-2} = \tilde{a}^{-2}.$$

Therefore

$$P(\alpha, \theta) = \frac{1}{\alpha} \frac{\tilde{a}^{-2} - \alpha}{1 - \alpha f(\theta)^2}.$$

For each  $\alpha \in (0, 1)$ , the supremum in  $\theta$  of  $P(\alpha, \theta)$  is attained for  $f(\theta)$  minimum, i.e. in  $\theta = 0$  for which  $f(\theta) = \tilde{a}$ . We deduce

$$\sup_{\theta \in \mathbb{T}} P(\alpha, \theta) = \frac{1}{\alpha} \frac{\tilde{a}^{-2} - \alpha}{1 - \alpha \tilde{a}^2} = \frac{1}{\alpha} \frac{1}{\tilde{a}^2} = \frac{1}{\alpha} \frac{b^2}{a^2},$$



hence, taking the infimum over  $\alpha \in (0, 1)$ :

$$M(\Omega) = \frac{b^2}{a^2}.$$

Comparing with (16), we finally obtain

$$m(\Omega) = M(\Omega) = \frac{b^2}{a^2} = \Gamma(\Omega). \quad (19)$$

In particular, if  $\Omega$  is a disk

$$m(\Omega) = M(\Omega) = \Gamma(\Omega) = 1. \quad (20)$$

## 5.2. Star-shaped polygons

A polygon  $\Omega$  is characterized by the fact that its boundary is a finite union of segments. Let us first investigate the behavior of the function  $P$  along a segment.

For ease of computation, we consider a segment  $I$  lying on a vertical line of equation  $x_1 = d$  with  $d > 0$ . Note that  $d$  is the distance of this line to the origin. Normals to  $I$  are horizontal. We find

$$f(\theta) = \frac{d}{\cos \theta} \quad \text{and} \quad \gamma(\theta) = \theta. \quad (21)$$

Hence, under the global condition  $\max_{\theta \in \mathbb{T}} f(\theta) = 1$ , the contribution to the function  $P$  of such a segment is — here we use formula (11),

$$\begin{aligned} P(\alpha, \theta) &= \frac{\cos^2 \theta}{\alpha d^2} \frac{1 - \alpha d^2}{\cos^2 \theta - \alpha d^2} \\ &= \frac{1}{\alpha d^2} \frac{1 - \alpha d^2}{1 - \alpha f(\theta)^2}. \end{aligned} \quad (22)$$

For any  $\alpha \in (0, 1)$ , the maximal value of  $P$  is attained for  $f(\theta)$  maximal, i.e., at an end of the segment  $I$ , and this end is the most distant from the origin. That is why we introduce:

*Notation 8.* For any side  $I_j$ ,  $j = 1, \dots, J$ , of a polygon  $\Omega$ , we define its radius  $r_j$  as the distance between the origin and its most distant endpoint  $E_j$ . Denoting by  $\tilde{I}_j$  the line containing  $I_j$ , we define  $d_j$  as the distance of  $\tilde{I}_j$  to the origin.

The normalization (2) here takes the form  $\max_j r_j = 1$ . From the previous computation (22) we find the formula

$$M(\Omega) = \inf_{\alpha \in (0,1)} \max_{j=1}^J \frac{1}{\alpha d_j^2} \frac{1 - \alpha d_j^2}{1 - \alpha r_j^2} \quad (23)$$

$$= \inf_{\alpha \in (0,1)} \max_{j=1}^J \frac{1}{\alpha r_j^2} \frac{r_j^2 d_j^{-2} - \alpha r_j^2}{1 - \alpha r_j^2}. \quad (24)$$

In order to find a similar formula for the quantity  $m(\Omega)$ , we are going to use (5) and we go back to expression (14) which can be written in function of  $\cos \gamma(\theta)$  instead of  $\sin \gamma(\theta)$ :

$$P(\alpha(\theta), \theta) = \left( \frac{1}{\cos \gamma(\theta)} + \sqrt{\frac{1}{\cos^2 \gamma(\theta)} - 1} \right)^2. \quad (25)$$

For the angles  $\theta$  corresponding to the segment  $I_j$ , the supremum of  $P(\alpha(\theta), \theta)$  is attained for  $\cos \gamma(\theta)$  minimum, i.e. for  $\cos \gamma(\theta) = \frac{d_j}{r_j}$ . Therefore formula (25) yields

$$m(\Omega) = \max_{j=1}^J \left( \frac{r_j}{d_j} + \sqrt{\frac{r_j^2}{d_j^2} - 1} \right)^2. \quad (26)$$

The maximum is attained when  $r_j/d_j$  is maximal.

**Proposition 9.** *Let  $\Omega$  be a polygon, with  $d_j$  and  $r_j$  the distances in Notation 8. We have*

- i) *If all  $r_j$  are equal, then  $M(\Omega) = m(\Omega)$ .*
- ii) *If all  $d_j$  are equal, then  $M(\Omega) = m(\Omega)$ .*
- iii) *If the largest value of  $r_j/d_j$  is attained for two different indices  $j$  and  $k$  and if  $r_j \neq r_k$ , then  $M(\Omega) > m(\Omega)$ .*

*Proof.* i) If all  $r_j$  are equal, the normalization  $\max_j r_j = 1$  yields that  $r_j = 1$ . Formula (24) then gives that

$$M(\Omega) = \inf_{\alpha \in (0,1)} \frac{1}{\alpha} \frac{d_{\min}^{-2} - \alpha}{1 - \alpha}$$

where  $d_{\min}$  is the minimum value of the  $d_j$ . The optimization with respect to  $\alpha$  provides the optimal value

$$\alpha_0 = \frac{1}{d_{\min}^2} - \sqrt{\frac{1}{d_{\min}^4} - \frac{1}{d_{\min}^2}} \in (0, 1)$$

for  $\alpha$ , hence the infimum

$$M(\Omega) = \left( \frac{1}{d_{\min}} + \sqrt{\frac{1}{d_{\min}^2} - 1} \right)^2$$

which coincides with  $m(\Omega)$  given by (26) since  $r_j/d_j$  is maximal for  $1/d_{\min}$ .

ii) If all  $d_j$  are equal, Formula (23) gives that

$$M(\Omega) = \inf_{\alpha \in (0,1)} \frac{1}{\alpha d^2} \frac{1 - \alpha d^2}{1 - \alpha r_{\max}^2}$$

where  $d$  is the common value of the  $d_j$  and  $r_{\max}$  the maximum value of the  $r_j$ . Due to normalization  $\max_j r_j = 1$ , this formula becomes

$$M(\Omega) = \inf_{\alpha \in (0,1)} \frac{1}{\alpha d^2} \frac{1 - \alpha d^2}{1 - \alpha} = \inf_{\alpha \in (0,1)} \frac{1}{\alpha} \frac{d^{-2} - \alpha}{1 - \alpha}.$$

As in the previous case we find

$$M(\Omega) = \left( \frac{1}{d} + \frac{\sqrt{1 - d^2}}{d} \right)^2,$$

which coincides with  $m(\Omega)$  given by (26) since  $r_j/d_j$  is maximal for  $1/d$ .

iii) For  $\ell \in \{j, k\}$ , let  $\theta_\ell$  be the angle  $\theta$  corresponding to the end  $E_\ell$ . We have

$$M(\Omega) \geq \min_{\alpha \in (0,1)} \max_{\theta \in \{\theta_j, \theta_k\}} P(\alpha, \theta).$$

And let  $\alpha_m$  be the value of  $\alpha \in (0, 1]$  minimizing  $\max_{\theta \in \{\theta_j, \theta_k\}} P(\alpha, \theta)$ . We have

$$M(\Omega) \geq \max\{P(\alpha_m, \theta_j), P(\alpha_m, \theta_k)\}.$$

Now, still for  $\ell \in \{j, k\}$ , let  $\alpha_\ell$  be the value of  $\alpha \in (0, r_\ell^{-2})$  minimizing  $P(\alpha, \theta_\ell)$ . Since  $r_j/d_j = r_k/d_k$  maximizes the quotients  $r_i/d_i$ , we have by (26)

$$\begin{aligned} m(\Omega) &= \left( \frac{r_j}{d_j} + \sqrt{\frac{r_j^2}{d_j^2} - 1} \right)^2 = \left( \frac{r_k}{d_k} + \sqrt{\frac{r_k^2}{d_k^2} - 1} \right)^2 \\ &= P(\alpha_j, \theta_j) = P(\alpha_k, \theta_k). \end{aligned}$$

By (13),  $\alpha_j$  and  $\alpha_k$  satisfy

$$\alpha_\ell r_\ell^2 = \frac{1}{1 + \sin \gamma(\theta_\ell)}, \quad \ell = j, k.$$

But  $\sin \gamma(\theta_j) = \sin \gamma(\theta_k)$  because  $\cos \gamma(\theta_\ell) = r_\ell/d_\ell$ . Hence, since  $r_j \neq r_k$ , we have  $\alpha_j \neq \alpha_k$ , therefore  $\alpha_m$  cannot coincide with  $\alpha_j$  and  $\alpha_k$  at the same time. So, since the functions  $\alpha \mapsto P(\alpha, \theta)$  are strictly convex in the interval  $(0, f(\theta)^{-2})$ , we deduce

$$M(\Omega) \geq \max\{P(\alpha_m, \theta_j), P(\alpha_m, \theta_k)\} > P(\alpha_j, \theta_j) = P(\alpha_k, \theta_k) = m(\Omega),$$

and conclude that  $M(\Omega) > m(\Omega)$  as announced in the proposition.  $\square$

Here are examples for the three situations *i) – iii)* investigated in Proposition 9.

**Example 1.** In each of the examples below, the center  $\mathbf{0}$  of polar and Cartesian coordinates is chosen at the *barycenter* of the domain.

- i) If  $\Omega$  is a regular polygon or a rectangle, then all  $r_j$  are equal, thus  $M(\Omega) = m(\Omega)$ .
- ii) If  $\Omega$  is a triangle or a rhombus, then all  $d_j$  are equal, thus  $M(\Omega) = m(\Omega)$ .
- iii) See Figure 1: For this hexagonal domain, the quotients  $r_j/d_j$  are all equal to  $\sqrt{2}$ , but  $r_1 = 1$  and  $r_2 = 1/\sqrt{2}$  (with  $\theta_1 = 0$  and  $\theta_2 = \frac{\pi}{4}$ ). Therefore  $m(\Omega) < M(\Omega)$ .

### 5.3. Limaçons

Limaçons of Pascal (named after Etienne Pascal, father of Blaise Pascal) are curves defined in polar coordinates by a formula of the type

$$f_\varepsilon(\theta) = a(1 + \varepsilon \cos \theta), \quad a > 0, \quad \varepsilon > 0. \quad (27)$$

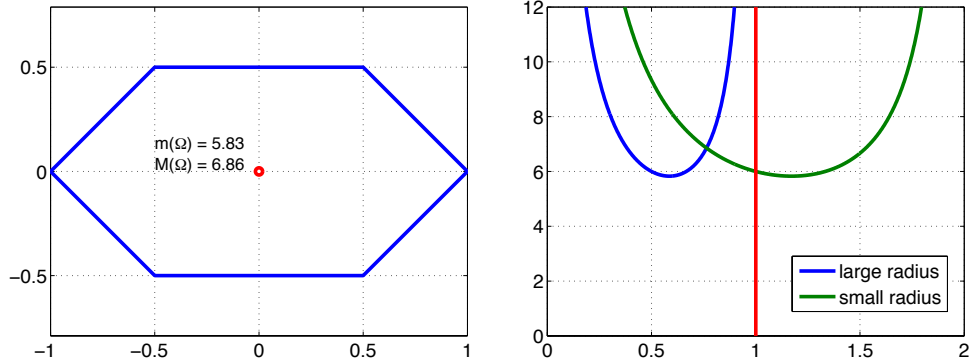


Figure 1: Example where  $M(\Omega) > m(\Omega)$ . Domain  $\Omega$  with center of coordinates, left. Plot of  $\alpha \mapsto P(\alpha, \theta_j)$  for  $\alpha \in (0, \frac{1}{f(\theta_j)^2})$ ,  $j = 1, 2$ , right.

Such a curve is simple if  $\varepsilon$  is less than 1, and so defines the boundary of a domain  $\Omega_\varepsilon$ . In [10] the case of such limaçons is considered. Here the constant  $\Gamma(\Omega_\varepsilon)$  is analytically known, [9, 10], which provides an explicit formula for  $\beta(\Omega_\varepsilon)$  via Theorem 1

$$\Gamma(\Omega_\varepsilon) = \frac{2 + \varepsilon^2}{2 - \varepsilon^2} \quad \text{and} \quad \beta(\Omega_\varepsilon) = \frac{\sqrt{2 - \varepsilon^2}}{2}. \quad (28)$$

This example can serve as a benchmark for bounds  $m$  and  $M$ . We have computed by a Matlab program the two constants  $m(\Omega_\varepsilon)$  and  $M(\Omega_\varepsilon)$ . It happens that as soon as  $\varepsilon$  is not zero, i.e.,  $\Omega_\varepsilon$  is not a circle, these two constants are distinct, see the top two curves in Figure 2. The other curves are explained below.

Here comes the question of the choice of polar coordinates defining  $m(\Omega)$  and  $M(\Omega)$ . For limaçons, the first choice is to consider the polar coordinates in which the domain is defined by (27). But, considering that  $\Omega_\varepsilon$  intersects the horizontal axis between the points  $-a + a\varepsilon$  and  $a + a\varepsilon$ , choosing new polar coordinates  $(r', \theta')$  centered at  $O' = (a\varepsilon, 0)$  appears more judicious. A new equation

$$r' = f'_\varepsilon(\theta')$$

is associated with  $\Omega_\varepsilon$ , leading to new quantities

$$m'(\Omega_\varepsilon) \quad \text{and} \quad M'(\Omega_\varepsilon).$$

In fact these new quantities are very different from the old ones. We have observed that  $m'(\Omega_\varepsilon)$  and  $M'(\Omega_\varepsilon)$  do coincide, and are much smaller than  $m(\Omega_\varepsilon)$  and  $M(\Omega_\varepsilon)$ , see Figure 2. Moreover, the asymptotic behavior of  $m'(\Omega_\varepsilon) = M'(\Omega_\varepsilon)$  as  $\varepsilon \rightarrow 0$  is very good. Indeed, using formula (28) allows us to compute the difference  $m'(\Omega_\varepsilon) - \Gamma(\Omega_\varepsilon)$ . We have found numerical evidence, see Figure 3, for the asymptotic behavior

$$m'(\Omega_\varepsilon) - \Gamma(\Omega_\varepsilon) = M'(\Omega_\varepsilon) - \Gamma(\Omega_\varepsilon) = O(\varepsilon^3).$$

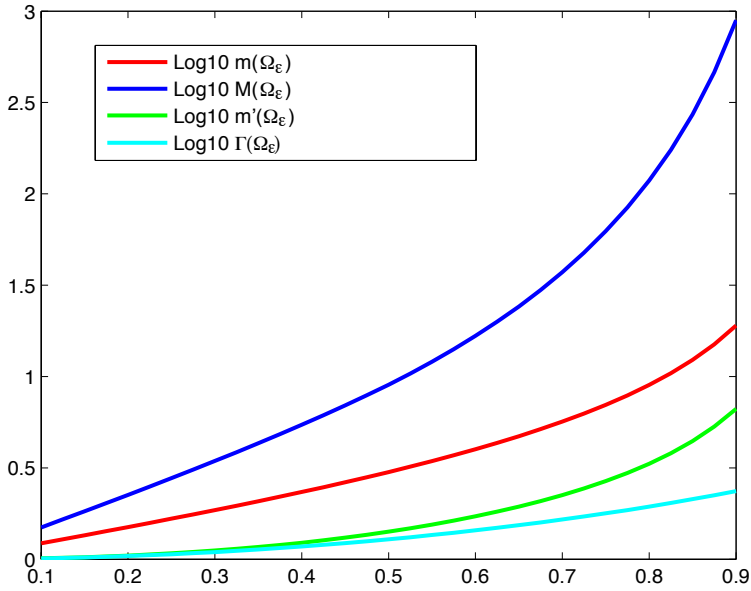


Figure 2: Plot of  $\varepsilon \mapsto \log_{10} \{m(\Omega_\varepsilon), M(\Omega_\varepsilon), m'(\Omega_\varepsilon) = M'(\Omega_\varepsilon), \Gamma(\Omega_\varepsilon)\}$  for  $\varepsilon = 0.1$  to  $0.9$ .

## 5.4. Decentered disks

We end this section by a curiosity which sheds some light on the discrepancy between  $m(\Omega)$  and  $M(\Omega)$  and their dependency on the center of polar coordinates. Let  $\Omega$  be a disk. We have seen in (20) that we have the optimal values  $m(\Omega) = M(\Omega) = \Gamma(\Omega) = 1$ .

Now, we consider decentered disks, moving off the center of polar coordinates by a relative amount  $\delta$  with respect to the radius of the disk. We can assume that the new center lies on the horizontal axis. This defines new versions of the constants, denoted  $m[\delta](\Omega)$  and  $M[\delta](\Omega)$ . It is not very hard to prove the following

(i) The maximal value of the angle  $\gamma$  occurs for  $\theta_0 = \frac{\pi}{2}$ , so  $\sin \gamma = \delta$ . Hence, cf (10),

$$m[\delta](\Omega) = \frac{1 + \delta}{1 - \delta}.$$

This value is the same for the limaçon (27) choosing  $\varepsilon = \delta$ , see [10, (6.34)].

(ii) For any  $\alpha \in (0, 1)$ , the max in  $\theta$  of  $P(\alpha, \theta)$  is attained for  $\theta = \pi$ , then the inf in  $\alpha$  corresponds to  $P(1, \pi)$ . Hence

$$M[\delta](\Omega) = \frac{1}{f(\pi)^2} = \left(\frac{1 + \delta}{1 - \delta}\right)^2 = m[\delta](\Omega)^2 > m[\delta](\Omega).$$

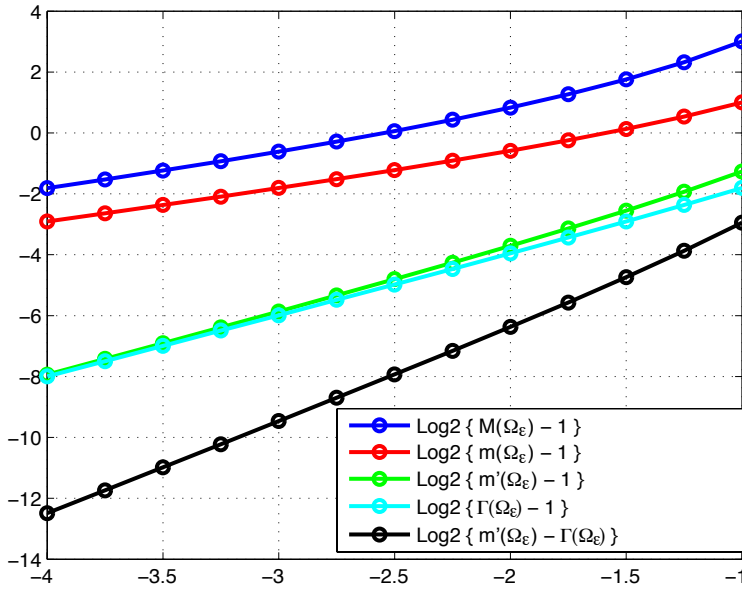


Figure 3: Plot of  $\log_2 \varepsilon \mapsto \log_2 \{M(\Omega_\varepsilon) - 1, m(\Omega_\varepsilon) - 1, m'(\Omega_\varepsilon) - 1, \Gamma(\Omega_\varepsilon) - 1, m'(\Omega_\varepsilon) - \Gamma(\Omega_\varepsilon)\}$  for  $\varepsilon = 0.0625$  to  $0.5$ .

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