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MATHEMATICAL MODELING OF THIN PIEZOELECTRIC PLATES WITH ELECTRIC FIELD GRADIENT

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Piezoelectric materials are widely used in the design of smart structures. It is thus of major technological interest to provide efficient modelings of such structures. In the case of thin piezoelectric plates, classical studies generally lead to two different models. These two models can be linked to the distinction between sensors and actuators. Here, we extend these results to the second order piezoelectricity, that is to say piezoelectricity with electric field gradient. We recently showed in [1] that *three* different models have to be taken into account, which broadens the scope of the sensors and actuators field. Second order piezoelectricity being compatible with isotropy (see the introduction below), we also propose a systematic study of the impact of crystalline symmetries on our models and show that a striking effect named ‘structural switch-off’ appears for some specific crystal classes. This paper aims at presenting these results in a simplified but accurate way.

Keywords: *Asymptotic analysis, piezoelectric plates, electric field gradient*

1 Introduction

In the 1960’s the study of unexplained aspects of piezoelectricity led Mindlin [2] to extend the classical Voigt’s theory [3] in Toupin’s formulation [4] by assuming that the stored energy function not only depends on the strain tensor and polarization vector but also on the polarization gradient tensor. What motivated Mindlin to study the effects of the polarization gradient was the capacitance of very thin dielectric films. Experiments showed that the capacitance of this kind of film is systematically smaller than the classical prediction. Moreover, performing experimental tests, Mead [5] showed that piezoelectric effect can also appear in centrosymmetric crystals, which is in contradiction with classical Voigt’s theory. And, indeed, the Mindlin’s theory of elastic dielectrics with polarization gradient as introduced in [2] accommodates the observed and experimentally measured phenomena, such as electromechanical interactions in centrosymmetric materials, capacitance of thin dielectric films, surface energy of polarization, deformation and optical activity in quartz (see for example [6], [7] and references quoted therein). In this paper we choose to adopt an alternative to the Mindlin’s formulation by introducing the electric field gradient, as in [8] for example. Because such gradient theories can describe size effects that are important in small-scale problems, it seems unavoidable to use them to deepen our

understanding of smart structures, the wide majority of them being thin. Here we present our results of the mathematical modeling of second order piezoelectric plates. The ground of our method is to view the thickness of the plate as a small parameter denoted by ϵ . Then, we study the behavior of the solution of the genuine electromechanical problem (7) as ϵ tends to 0. We have shown in [1] that depending on the type of electrical loading, three different models (indexed by $p=1, 2$ or 3) appear at the limit. This result extends our previous study in [9] and tends to enlighten that gradient theory broadens the understanding of sensors and actuators. For two of the obtained models (corresponding to $p=2$ and $p=3$) we are able to express the constitutive law of the plate as a Schur complement of the second order piezoelectric tensor (see (2) and (11)). It is important to emphasize that the expression of the constitutive laws in (11) is valid for any symmetry class, which means that we do not make any simplifying assumption dealing with the crystal symmetry of the material constituting the plate. Concerning the remaining model (corresponding to $p=1$), we are not able to explicitly derive the constitutive law in the limit equations. Therefore, as in the case of the first order piezoelectric rods treated in [10], it seems very likely to us that non-local terms appear in this delicate situation. Finally, we study the influence of the crystal symmetries on the constitutive laws of our two

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reducible models (corresponding to $p=2$ and $p=3$) and show that even for the second order piezoelectricity, an electromechanical switch-off may appear in the structure if the plate is designed with specific materials.

2 Setting the problem

We will denote displacement fields by the letters u , v and w while the electric potentials will be denoted by ϕ , ψ and φ . Classically, the tensor of small strains is written $e(u) \in S^3$ where S^N indicates the set of all $N \times N$ real and symmetric matrices. Used as indexes, letters i and j take their values in $\{1, 2, 3\}$ while α and β take their values in $\{1, 2\}$. We recall that $2e_{ij}(u) = \partial_i u_j + \partial_j u_i$ where the symbol ∂_i refers to the partial derivative with respect to the i -th coordinate. The gradient of an electric potential φ will be denoted by $\nabla\varphi \in R^3$ and its bigradient by $\nabla^2\varphi \in S^3$ where $\nabla_{ij}^2\varphi = \partial_{ij}^2\varphi = \partial_i\partial_j\varphi$. Given an electromechanical state (u, φ) we therefore have $(e(u), \nabla\phi, \nabla^2\phi) \in H$, with

$$H = S^3 \times R^3 \times S^3. \quad (1)$$

An element of H will therefore be represented by a triplet, and for the sake of simplicity, the classical symbol ' \cdot ' will stand for the scalar product in H , S^3 and R^3 . The set of all linear mappings from a space V into a space W is denoted by $L(V, W)$ and, if $V = W$, we simply write $L(V)$. In the sequel, for all domain D of R^N , $H_\Gamma^1(D)$ refers to the subset of the Sobolev space $H^1(D)$ whose elements vanish on Γ , included in the boundary ∂D of D .

The reference configuration of a linearly piezoelectric thin plate is the closure in R^3 of the set $\Omega^\varepsilon = \omega \times (-\varepsilon, \varepsilon)$ whose outward unit normal is n^ε . Here, ε is a small positive number and ω a bounded domain of R^2 with a Lipschitz boundary $\partial\omega$. Let $(\Gamma_{mD}^\varepsilon, \Gamma_{mN}^\varepsilon)$, $(\Gamma_{eD,\beta}^\varepsilon, \Gamma_{eN,\beta}^\varepsilon)_{\beta=1,2}$ be three suitable partitions of $\partial\Omega^\varepsilon$ with Γ_{mD}^ε and $\Gamma_{eD,\beta}^\varepsilon$ of strictly positive surface measures. The plate is, on one hand, clamped along Γ_{mD}^ε and the electric potential φ^ε satisfies $\varphi^\varepsilon = \varphi_0^\varepsilon$ on $\Gamma_{eD,1}^\varepsilon$ and $\partial_n \varphi^\varepsilon = \partial_n \varphi_0^\varepsilon$ on $\Gamma_{eD,2}^\varepsilon$, where the symbol ∂_n refers to the normal derivative along the boundary of Ω^ε and φ_0^ε is a smooth enough given field defined in Ω^ε . On the other hand, the plate is subjected to body forces f^ε and electric loading q^ε in Ω^ε . Actually, q^ε vanishes, the material being an insulator, anyway our results are also valid for q^ε different from 0. Moreover, the plate is subjected to surface forces F^ε and electric loading q_s^ε on Γ_{mN}^ε and $\Gamma_{eN,1}^\varepsilon$, respectively. It is also necessary to define 'body' and 'surface' electric dipoles densities, respectively denoted by d^ε , d_s^ε and defined in Ω^ε and on $\Gamma_{eN,2}^\varepsilon$. Finally, we assume that $\Gamma_{mD}^\varepsilon = \gamma_0 \times (-\varepsilon, \varepsilon)$, with $\gamma_0 \subset \partial\omega$.

We now define the operator

$$M^\varepsilon = \begin{pmatrix} a^\varepsilon & -b^\varepsilon & -\alpha^\varepsilon \\ b^{\varepsilon T} & c^\varepsilon & \beta^\varepsilon \\ \alpha^{\varepsilon T} & \beta^{\varepsilon T} & \gamma^\varepsilon \end{pmatrix} \quad (2)$$

which describes the electromechanical coupling with electric field gradient effect, or second order piezoelectric coupling. More precisely, a^ε , b^ε and c^ε are respectively the elastic, piezoelectric and dielectric tensors while α^ε , β^ε and γ^ε describe the second order couplings (recall that θ^T denotes the transpose of any tensor θ). We have $(a^\varepsilon, b^\varepsilon, c^\varepsilon, \alpha^\varepsilon, \beta^\varepsilon, \gamma^\varepsilon) \in L(S^3) \times L(R^3, S^3) \times L(R^3) \times L(S^3) \times L(S^3, R^3) \times L(S^3)$, so that $M^\varepsilon \in L(H)$.

We are looking for the electromechanical state $(u^\varepsilon, \varphi^\varepsilon)$ living in the second order piezoelectric plate at equilibrium, where u^ε denotes the displacement field. For this purpose, we make the following regularity hypothesis on the exterior loading:

$$\begin{aligned} \text{i) } & (f^\varepsilon, q^\varepsilon, d^\varepsilon, F^\varepsilon, q_s^\varepsilon, d_s^\varepsilon) \in L^2(\Omega^\varepsilon)^3 \times L^2(\Omega^\varepsilon) \\ & \times L^2(\Omega^\varepsilon)^3 \times L^2(\Gamma_{mN}^\varepsilon)^3 \times L^2(\Gamma_{eN,1}^\varepsilon) \times L^2(\Gamma_{eN,2}^\varepsilon), \\ \text{ii) } & \varphi_0 \in H^2(\Omega^\varepsilon) \end{aligned} \quad (3)$$

and define

$$H_{\Gamma_{eD}}^2(\Omega^\varepsilon) = \left\{ \psi \in H^2(\Omega^\varepsilon) : \begin{aligned} & \psi = 0 \text{ on } \Gamma_{eD,1}^\varepsilon, \\ & \partial_n \psi = 0 \text{ on } \Gamma_{eD,2}^\varepsilon. \end{aligned} \right\}. \quad (4)$$

Now, on the space of electromechanical states $V^\varepsilon = H_{\Gamma_{mD}}^1(\Omega^\varepsilon)^3 \times H_{\Gamma_{eD}}^2(\Omega^\varepsilon)^3$ we define a bilinear form m^ε :

$$\begin{aligned} m^\varepsilon(r, t) &= m^\varepsilon((v, \psi), (w, \phi)) \\ &= \int_{\Omega^\varepsilon} M^\varepsilon(e(v), \nabla\psi, \nabla^2\psi) \cdot (e(w), \nabla\phi, \nabla^2\phi) dx^\varepsilon \end{aligned} \quad (5)$$

and a linear form L^ε :

$$\begin{aligned} L^\varepsilon(r) &= L^\varepsilon((v, \psi)) \\ &= \int_{\Omega^\varepsilon} (f^\varepsilon \cdot v + q^\varepsilon \psi + d^\varepsilon \cdot \nabla\psi) dx^\varepsilon \\ &+ \int_{\Gamma_{mN}^\varepsilon} F^\varepsilon \cdot v ds^\varepsilon + \int_{\Gamma_{eN,1}^\varepsilon} q_s^\varepsilon \psi ds^\varepsilon + \int_{\Gamma_{eN,2}^\varepsilon} d_s^\varepsilon \partial_n \psi ds^\varepsilon. \end{aligned} \quad (6)$$

The genuine electromechanical problem then takes the form

$$P(\Omega^\varepsilon) : \left\{ \begin{aligned} & \text{Find } s^\varepsilon = (u^\varepsilon, \varphi^\varepsilon) \in V^\varepsilon \text{ such that} \\ & m^\varepsilon(s^\varepsilon, r) = L^\varepsilon(r), \quad \forall r \in V^\varepsilon. \end{aligned} \right. \quad (7)$$

Thus, with the additional and realistic assumptions of boundedness of $a^\varepsilon, b^\varepsilon, c^\varepsilon, \alpha^\varepsilon, \beta^\varepsilon, \gamma^\varepsilon$ and of uniform ellipticity of $a^\varepsilon, c^\varepsilon$ and γ^ε :

$$\begin{aligned} M^\varepsilon \in L^\infty(\Omega^\varepsilon, L(H)), \exists \kappa^\varepsilon > 0: \\ M^\varepsilon(x^\varepsilon)h \cdot h \geq \kappa^\varepsilon \|h\|_H^2, \forall h \in H, \text{ a.e. } x^\varepsilon \in \Omega^\varepsilon \end{aligned} \quad (8)$$

the Stampacchia's theorem (cf. [11]) implies the

THEOREM 1: *Under assumptions (3) and (8), the genuine electromechanical problem $P(\Omega^\varepsilon)$ has a unique solution.*

To derive simplified accurate models, the very question is to study the behavior of s^ε when ε , regarded as a parameter, tends to zero.

3 The three different models

It is not possible to present here the details of the asymptotic procedure that leads to our modeling. We refer the reader to [1] for full description and rigorous proofs. Let us just say that three different limit behaviors of s^ε appear according to both the type of electric boundary conditions and the magnitude of electric external loading. These three different behaviors can be indexed by $p=1, 2$ or 3 .

The sketch of the method is classical (see [12]). The first step is to find a way to avoid working on variable Sobolev spaces. Indeed, until now, all the functional spaces that have been introduced are defined on Ω^ε . As ε is regarded as a small *parameter* whose aim is to tend to zero, these functional spaces are *variable* and this fact implies very heavy technical difficulties. We therefore first come down to a *fixed* open set $\Omega = \omega \times (-1, 1)$ through the mapping π^ε :

$$x = (x_1, x_2, x_3) \in \bar{\Omega} \mapsto \pi^\varepsilon x = (x_1, x_2, \varepsilon x_3) \in \bar{\Omega}^\varepsilon \quad (9)$$

This is the so-called *zoom technique*.

To get physically meaningful results, we also have to make various kinds of assumptions. They deal with the electromechanical coefficients and loading but also with the boundedness of the work of the exterior loading (see [1] but also [9]).

Finally, with the true physical electromechanical state $s^\varepsilon = (u^\varepsilon, \varphi^\varepsilon)$ defined on Ω^ε , we associate a *scaled* electromechanical state $s_p(\varepsilon) = (u_p(\varepsilon), \varphi_p(\varepsilon))$ defined on Ω by:

$$\begin{aligned} u_\alpha^\varepsilon(x^\varepsilon) &= \varepsilon(u_p(\varepsilon))_\alpha(x), \quad u_3^\varepsilon(x^\varepsilon) = (u_p(\varepsilon))_3(x), \\ \varphi^\varepsilon(x^\varepsilon) &= \varepsilon^p \varphi_p(\varepsilon)(x), \quad \forall x^\varepsilon \in \bar{\Omega}^\varepsilon. \end{aligned} \quad (10)$$

This scaled electromechanical state $s_p(\varepsilon)$ is then the unique solution of a so-called 'scaled problem' indexed by p , say $P(\varepsilon, \Omega)_p$. The asymptotic analysis of this problem shows that $s_p(\varepsilon)$ strongly converges in a suitable topology to an electromechanical state s_p . It is possible to show that s_p is the unique solution of a *limit problem* denoted $P(\Omega)_p$. These three problems, once written on Ω^ε , represent our three models!

For $p=2$ or $p=3$, the problem $P(\Omega)_p$ is governed by a limit constitutive law \bar{M}_p , in the same way that $P(\Omega^\varepsilon)$ is governed by M^ε (see (7), (5) and (2)).

4 Some properties of \bar{M}_p

It is interesting to give some properties of the operators \bar{M}_p which supply the constitutive relations of the electromechanical plate with electric field gradient. We recall that these operators are defined for $p=2$ and $p=3$. In the case $p=2$, the limit model involves a coupling between the displacement field, the electric field and the electric field gradient while in the case $p=3$, the coupling involves the displacement field and the electric field gradient only. Because of the explicit expression of \bar{M}_p as a Schur complement (see [1] and [9]), it is possible to show that

$$\bar{M}_2 = \begin{pmatrix} \bar{a}_2 & \bar{b}_2 & \bar{\alpha}_2 \\ \bar{b}_2^T & \bar{c}_2 & \bar{\beta}_2 \\ \bar{\alpha}_2^T & \bar{\beta}_2^T & \bar{\gamma}_2 \end{pmatrix}, \quad \bar{M}_3 = \begin{pmatrix} \bar{a}_3 & \bar{\alpha}_3 \\ \bar{\alpha}_3^T & \bar{\gamma}_3 \end{pmatrix} \quad (11)$$

Where $(\bar{a}_2, \bar{b}_2, \bar{c}_2, \bar{\alpha}_2, \bar{\beta}_2, \bar{\gamma}_2) \in L(S^2) \times L(R, S^2) \times L(R) \times L(R^2, S^2) \times L(R^2, R) \times L(R^2)$ and $(\bar{a}_3, \bar{\alpha}_3, \bar{\gamma}_3) \in L(S^2) \times L(R, S^2) \times L(R)$. Considering the influence of crystallographic classes, it can be shown that in the case of a polarization normal to the plate we have the following properties:

- \bar{a}_3 involves mechanical terms only,
- $\bar{\alpha}_3$ never vanishes,
- \bar{b}_2 vanishes for the crystalline classes $m, 32, 422, \bar{6}, 622$ and $\bar{6}m2$,
- $\bar{\alpha}_2$ always vanishes except for the classes $3, 32$ and $3m$,
- $\bar{\beta}_2$ always vanishes except for the class m ,
- when $p=2$, there is a complete decoupling ($\bar{b}_2 = \bar{\alpha}_2 = \bar{\beta}_2 = 0$) for the classes $422, \bar{6}, 622$ and $\bar{6}m2$,

nevertheless the operators \bar{a}_2 , \bar{c}_2 and $\bar{\gamma}_2$ involve a mixture of elastic, dielectric and second gradient

coupling coefficients. In these cases, \bar{M}_2 is symmetric which involves a quadratic convex energy. *For plates made of these piezoelectric monocrystals, the first and second order coupling effects disappear at the structural level!* This phenomenon is described as a ‘structural switch-off’.

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