Stability of quasi-static crack evolution through dimensional reduction

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Abstract This paper deals with quasi-static crack growth in thin films. We show that, when the thickness of the film tends to zero, any three-dimensional quasi-static crack evolution converges to a two-dimensional one, in a sense related to the Γ-convergence of the associated total energy. We extend the prior analysis of [2] by adding conservative body and surface forces which allow us to remove the boundedness assumption on the deformation field.

1 Introduction

In this paper, we study the evolution of cracks in thin structures in a quasi-static setting. Our approach of fracture mechanics is based on a variational model proposed in [10] (see also the monograph [4]) where the (quasi-static) evolution results from the competition – at each time – between a bulk and a surface energy, under a growth constraint on the crack. Many existence results have been obtained (see e.g. [8, 9, 11] and references therein).

Sometimes a small parameter is involved in the model, and it is an interesting question to study the asymptotic behavior of the model when the parameter tends to zero (see e.g. [14] for the homogenization and [2] for the dimension reduction of quasi-static crack evolution). When dealing static problems, the notion of Γ-convergence (see [7]) has proven to be a powerful tool to capture the asymptotic behavior of minimizers, or even minimizing sequences. It turns out that even in the quasi-static case, one can define a notion of convergence related to the Γ-convergence of the associated total energy (see [16] for an abstract theory in the more general framework of rate independent processes).

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We present here an extension of the result in [2] on the convergence of a quasi-static crack evolution in thin films, as the thickness tends to zero. In [2], an empirical L^∞ bound was done on the deformation field (as in [9]) in order to gain compactness in the space SBV^p of special functions of bounded variation. It is sometimes possible to justify this assumption as in the antiplanar case (see [11] when $N = 2$) where it follows from a consequence of the maximum principle. Unfortunately, in the full three-dimensional elasticity, the maximum principle does not hold anymore. We propose here to remove this hypothesis adding suitable conservative bulk and surface forces as in [8]. The price to pay is that the deformation field in not anymore compact in SBV^p but in a larger subspace $GSBV^p_q$ of generalized special functions of bounded variation. The arguments we use in the present paper are very close to those of [2], and for this reason we will only mention the main differences without giving the precise proofs of the results.

The paper is organized as follows: in Section 2, we will describe the model in the physical configuration and state the existence result of [8]. Then, in Section 3, we will reformulate the problem on a rescaled configuration in order to work on a fixed domain. In Section 4, we will perform an asymptotic analysis of the total energy of the system in a static setting, thanks to a Γ-convergence method. Finally, we will address the asymptotic of the quasi-static problem in Section 5, proving that it converges to a quasi-static evolution associated to the Γ-limit model.

2 Description of the Model

We consider a homogeneous thin film occupying in its reference configuration the cylinder $\Omega^\varepsilon := \omega \times (-\varepsilon, \varepsilon)$, where $\varepsilon > 0$ and ω is a bounded open subset of $\mathbf{R}^2$ with Lipschitz boundary. The Dirichlet part of the boundary where the deformation is prescribed is the lateral boundary $\partial_D\Omega^\varepsilon := \partial\omega \times (-\varepsilon, \varepsilon)$, while the Neumann part $\partial_N\Omega^\varepsilon = \omega \times \{-\varepsilon, \varepsilon\}$ is made of the lower and upper sections.

On the lateral boundary $\partial_D\Omega^\varepsilon$, we impose a time dependent boundary deformation $\phi^\varepsilon(t)$ on a finite time interval $[0, T]$, where

$$t \mapsto \phi^\varepsilon(t) \in W^{1,1}([0, T]; W^{1,p}(\Omega^\varepsilon; \mathbf{R}^3) \cap L^q(\Omega^\varepsilon; \mathbf{R}^3)),$$

for some $p > 1$ and $q \geq 1$.

On the remaining part of the boundary $\partial_N\Omega^\varepsilon$, we impose a time dependent surface conservative force which will be described in Section 2.3.2.

2.1 Admissible Cracks

We fix an open subset Ω^ε_B of Ω^ε of the form $\Omega^\varepsilon_B := \omega \times (-\varepsilon + \varepsilon\eta, \varepsilon - \varepsilon\eta)$ for some $\eta \in (0, 1)$, to that the set $\overline{\Omega^\varepsilon_B}$ represents the brittle part of the body. The set of all admissible cracks is given by

$$\mathcal{R}(\overline{\Omega^\varepsilon_B}) := \{K : K \text{ is rectifiable}, K \,\tilde{\subset}\, \overline{\Omega^\varepsilon_B} \text{ and } \mathcal{H}^2(K) < +\infty\}.$$

Note that any admissible crack must lie far enough from the upper and lower sections. The safety region $\Omega^\varepsilon \setminus \overline{\Omega^\varepsilon_B}$ can be interpreted as a layer of unbreakable material (see [8, remark 3.8]).

We denote by $\mathcal{H}^{N-1}$ the $(N-1)$-dimensional Hausdorff measure in $\mathbf{R}^N$ (we shall only consider the cases $N = 2$ or 3), and by $\tilde{\subset}$ (resp. $\tilde{=}$) inclusion (resp. equality) up to a set of zero $\mathcal{H}^{N-1}$-measure.

We assume that the energy spent to produce a crack K is of Griffith type, i.e.,

$$\mathcal{K}(\varepsilon)(K) := \mathcal{H}^2(K). \tag{1}$$

2.2 Admissible Deformations

We refer to [1] for the usual definitions and results on geometric measure theory, BV, SBV and $GSBV$ spaces. Precise definitions of the jump set S_u and of the approximate gradient ∇u of a function $u \in GSBV(U; \mathbf{R}^d)$, where U is an open subset of $\mathbf{R}^N$, can be found in that reference. Following Dal Maso et al. [8], we further define for $p > 1$

$$GSBV^p(U; \mathbf{R}^d) : = \big\{u \in GSBV(U; \mathbf{R}^d) : \nabla u \in L^p(U; \mathbf{R}^{d\times N}) \\ \text{and } \mathcal{H}^{N-1}(S_u) < +\infty\big\},$$

and if $q \geq 1$, $GSBV^p_q(U; \mathbf{R}^d) := GSBV^p(U; \mathbf{R}^d) \cap L^q(U; \mathbf{R}^d)$. Moreover, we say that a sequence $u_n \rightharpoonup u$ in $GSBV^p_q(U; \mathbf{R}^d)$ if $u_n \to u$ a.e. in U, $u_n \rightharpoonup u$ in $L^q(U; \mathbf{R}^d)$, $\nabla u_n \rightharpoonup \nabla u$ in $L^p(U; \mathbf{R}^{d\times N})$ and $\mathcal{H}^{N-1}(S_{u_n})$ is uniformly bounded.

For a given admissible crack $K \in \mathcal{R}(\overline{\Omega^\varepsilon_B})$ and a boundary deformation $\phi \in W^{1,p}(\Omega^\varepsilon; \mathbf{R}^3) \cap L^q(\Omega^\varepsilon; \mathbf{R}^3)$, we define the set of admissible deformations with finite energy relative to (K, ϕ) by

$$AD^\varepsilon(\phi, K) := \{u \in GSBV^p_q(\Omega^\varepsilon; \mathbf{R}^3) : \; S_u \,\tilde{\subset}\, K, \; u = \phi \; \mathcal{H}^2\text{-a.e. on } \partial_D\Omega^\varepsilon \setminus K\}.$$

The associate bulk energy is defined by

$$\mathcal{W}(\varepsilon)(\nabla u) := \int_{\Omega^\varepsilon} W(\nabla u(x))\, dx, \tag{2}$$

where $W : \mathbf{R}^{3\times 3} \to [0, +\infty)$, the stored energy density, is a quasiconvex function of class $\mathcal{C}^1$ satisfying standard p-growth and p-coercivity conditions ($p > 1$): there exist $0 < \beta' < \beta < +\infty$ such that

$$\beta'|\xi|^p \leq W(\xi) \leq \beta(1 + |\xi|^p) \quad \text{for every } \xi \in \mathbf{R}^{3\times 3}. \tag{3}$$

In particular, the functional $\mathcal{W}(\varepsilon) : L^p(\Omega^\varepsilon; \mathbf{R}^{3\times 3}) \to [0, +\infty)$ defined by

$$\mathcal{W}(\varepsilon)(\Phi) := \int_{\Omega^\varepsilon} W(\Phi(x))\, dx$$

is differentiable on $L^p(\Omega^\varepsilon; \mathbf{R}^{3\times 3})$, and its differential $D\mathcal{W}(\varepsilon) : L^p(\Omega^\varepsilon; \mathbf{R}^{3\times 3}) \to L^{p'}(\Omega^\varepsilon; \mathbf{R}^{3\times 3})$, with $p' = p/(p-1)$, is given by

$$\langle D\mathcal{W}(\varepsilon)(\Phi), \Psi\rangle = \int_{\Omega^\varepsilon} DW(\Phi(x)) : \Psi(x)\, dx \quad \text{for every } \Phi,\ \Psi \in L^p(\Omega^\varepsilon; \mathbf{R}^{3\times 3}).$$

On the left-hand side of the previous equality, we have denoted by $\langle\cdot,\cdot\rangle$ the duality pairing between $L^p(\Omega^\varepsilon; \mathbf{R}^{3\times 3})$ and $L^{p'}(\Omega^\varepsilon; \mathbf{R}^{3\times 3})$.

2.3 The Forces

We assume that the body is subjected to the action of conservative body and surface forces with potentials F and $G^\varepsilon = \varepsilon G$ respectively. Note that the order of magnitude of the applied forces are exactly those inducing a limiting membrane model (see [12, 13]).

2.3.1 The Body Forces

Let $q \geq 1$, the density of the applied body forces per unit volume at time $t \in [0, T]$ is given by $D_z F(t, u(x))$, where $F : [0, T] \times \mathbf{R}^3 \to \mathbf{R}$ and the map $z \mapsto F(t, z)$ belongs to $\mathcal{C}^1(\mathbf{R}^3)$ for every $t \in [0, T]$. We suppose that for every $t \in [0, T]$, the functional

$$\mathcal{F}(\varepsilon)(t)(u) := \int_{\Omega^\varepsilon} F(t, u(x))\, dx \tag{4}$$

is of class $\mathcal{C}^1$ on the space $L^q(\Omega^\varepsilon; \mathbf{R}^3)$, and its differential $D\mathcal{F}(\varepsilon)(t) : L^q(\Omega^\varepsilon; \mathbf{R}^3) \to L^{q'}(\Omega^\varepsilon; \mathbf{R}^3)$, with $q' := q/(q-1)$, is given by

$$\langle D\mathcal{F}(\varepsilon)(t)(u), v\rangle = \int_{\Omega^\varepsilon} D_z F(t, u(x)) \cdot v(x)\, dx \quad \text{for every } u,\ v \in L^q(\Omega^\varepsilon; \mathbf{R}^3).$$

We have denoted by $\langle\cdot,\cdot\rangle$ the duality pairing between $L^q(\Omega^\varepsilon; \mathbf{R}^3)$ and $L^{q'}(\Omega^\varepsilon; \mathbf{R}^3)$.

Concerning the regularity in time, we assume that there exist an exponent $\dot{q} < q$ and, for a.e. $t \in [0, T]$, a functional $\dot{\mathcal{F}}(\varepsilon)(t) : L^{\dot{q}}(\Omega^\varepsilon; \mathbf{R}^3) \to \mathbf{R}$ of class $\mathcal{C}^1$, with differential $D\dot{\mathcal{F}}(\varepsilon)(t) : L^{\dot{q}}(\Omega^\varepsilon; \mathbf{R}^3) \to L^{\dot{q}'}(\Omega^\varepsilon; \mathbf{R}^3)$, where $\dot{q}' = \dot{q}/(\dot{q}-1)$, such that for every $u,\ v \in L^q(\Omega^\varepsilon; \mathbf{R}^3)$, the functions $t \mapsto \dot{\mathcal{F}}(\varepsilon)(t)(u)$ and $t \mapsto \langle D\dot{\mathcal{F}}(\varepsilon)(t)(u), v\rangle$ are integrable on $[0, T]$, and

$$\mathcal{F}(\varepsilon)(t)(u) = \mathcal{F}(\varepsilon)(0)(u) + \int_0^t \dot{\mathcal{F}}(\varepsilon)(s)(u)\,ds, \tag{5}$$

$$\langle D\mathcal{F}(\varepsilon)(t)(u), v\rangle = \langle D\mathcal{F}(\varepsilon)(0)(u), v\rangle + \int_0^t \langle D\dot{\mathcal{F}}(\varepsilon)(s)(u), v\rangle\,ds \tag{6}$$

for every $t \in [0, T]$. We further assume that $\mathcal{F}(\varepsilon)(t)$ is upper semicontinuous in $L^q(\Omega^\varepsilon; \mathbf{R}^3)$ with respect to the pointwise almost everywhere convergence.

Finally, we suppose that $\mathcal{F}(\varepsilon)(t)$, $D\mathcal{F}(\varepsilon)(t)$, $\dot{\mathcal{F}}(\varepsilon)(t)$ and $D\dot{\mathcal{F}}(\varepsilon)(t)$ satisfy suitable q-growth conditions: there exist constants $a_0 > 0$, $a_1 > 0$, $a_2 > 0$, $b_0 \geq 0$, $b_1 \geq 0$, $b_2 \geq 0$, and nonnegative integrable functions on $[0, T]$, a_3, a_4, b_3 and b_4 (uniform in ε) such that

$$\begin{cases} a_0\|u\|^q_{L^q(\Omega^\varepsilon;\mathbf{R}^3)} - b_0 \leq -\mathcal{F}(\varepsilon)(t)(u) \leq a_1\|u\|^q_{L^q(\Omega^\varepsilon;\mathbf{R}^3)} + b_1, \\ |\langle D\mathcal{F}(\varepsilon)(t)(u), v\rangle| \leq (a_2\|u\|^{q-1}_{L^q(\Omega^\varepsilon;\mathbf{R}^3)} + b_2)\|v\|_{L^q(\Omega^\varepsilon;\mathbf{R}^3)}, \\ |\dot{\mathcal{F}}(\varepsilon)(t)(u)| \leq a_3(t)\|u\|^{\dot{q}}_{L^{\dot{q}}(\Omega^\varepsilon;\mathbf{R}^3)} + b_3(t), \\ |\langle D\dot{\mathcal{F}}(\varepsilon)(t)(u), v\rangle| \leq (a_4(t)\|u\|^{\dot{q}-1}_{L^{\dot{q}}(\Omega^\varepsilon;\mathbf{R}^3)} + b_4(t))\|v\|_{L^{\dot{q}}(\Omega^\varepsilon;\mathbf{R}^3)}. \end{cases} \tag{7}$$

2.3.2 The Surface Forces

The density of the surface forces on $\partial_N\Omega^\varepsilon$ at time $t \in [0, T]$, under the deformation u is given by $\varepsilon D_z G(t, u(x))$, where $G : [0, T] \times \mathbf{R}^3 \to \mathbf{R}$ is such that $z \mapsto G(t, z)$ is of class $\mathcal{C}^1(\mathbf{R}^3)$ for every $t \in [0, T]$.

We fix an exponent r, related to the trace theorem in Sobolev spaces, such that $r \in [p, p/(3 - p)]$ if $p < 3$, while $r \geq p$ if $p \geq 3$. We assume that for every $t \in [0, T]$, the functional

$$\mathcal{G}(\varepsilon)(t)(u) := \varepsilon \int_{\partial_N\Omega^\varepsilon} G(t, u(x))\,d\mathcal{H}^2(x) \tag{8}$$

is of class $\mathcal{C}^1$ on $L^r(\partial_N\Omega^\varepsilon; \mathbf{R}^3)$, with differential $D\mathcal{G}(\varepsilon)(t) : L^r(\partial_N\Omega^\varepsilon; \mathbf{R}^3) \to L^{r'}(\partial_N\Omega^\varepsilon; \mathbf{R}^3)$, where $r' = r/(r - 1)$, given by

$$\langle \mathcal{G}(\varepsilon)(t)(u), v\rangle = \varepsilon \int_{\partial_N\Omega^\varepsilon} D_z G(t, u(x)) \cdot v(x)\,d\mathcal{H}^2(x)$$
$$\text{for all } u,\ v \in L^r(\partial_N\Omega^\varepsilon; \mathbf{R}^3),$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $L^r(\partial_N\Omega^\varepsilon; \mathbf{R}^3)$ and $L^{r'}(\partial_N\Omega^\varepsilon; \mathbf{R}^3)$.

As for the regularity in time, we suppose that for a.e. $t \in [0, T]$, there exists a functional $\dot{\mathcal{G}}(\varepsilon)(t) : L^r(\partial_N\Omega^\varepsilon; \mathbf{R}^3) \to \mathbf{R}$ of class $\mathcal{C}^1$, with differential $D\dot{\mathcal{G}}(\varepsilon)(t) : L^r(\partial_N\Omega^\varepsilon; \mathbf{R}^3) \to L^{r'}(\partial_N\Omega^\varepsilon; \mathbf{R}^3)$, such that for every $u, v \in L^r(\partial_N\Omega^\varepsilon; \mathbf{R}^3)$, the mappings $t \mapsto \dot{\mathcal{G}}(\varepsilon)(t)(u)$ and $t \mapsto \langle D\dot{\mathcal{G}}(\varepsilon)(t)(u), v\rangle$ are integrable on $[0, T]$, and

$$\mathcal{G}(\varepsilon)(t)(u) = \mathcal{G}(\varepsilon)(0)(u) + \int_0^t \dot{\mathcal{G}}(\varepsilon)(s)(u)\,ds, \tag{9}$$

$$\langle D\mathcal{G}(\varepsilon)(t)(u), v\rangle = \langle D\mathcal{G}(\varepsilon)(0)(u), v\rangle + \int_0^t \langle D\dot{\mathcal{G}}(\varepsilon)(s)(u), v\rangle\,ds \tag{10}$$

for every $t \in [0, T]$.

Finally, we suppose that $\mathcal{G}(\varepsilon)(t)$, $D\mathcal{G}(\varepsilon)(t)$, $\dot{\mathcal{G}}(\varepsilon)(t)$ and $D\dot{\mathcal{G}}(\varepsilon)(t)$ satisfy suitable r-growth conditions: there exist nonnegative constants α_0, α_1, α_2, β_0, β_1, β_2, and nonnegative integrable functions on $[0, T]$, α_3, α_4, β_3 and β_4 such that

$$\begin{cases} -\alpha_0\varepsilon\|u\|^r_{L^r(\partial_N\Omega^\varepsilon;\mathbf{R}^3)} - \beta_0\varepsilon \le -\mathcal{G}(\varepsilon)(t)(u) \le \alpha_1\varepsilon\|u\|^q_{L^r(\partial_N\Omega^\varepsilon;\mathbf{R}^3)} + \beta_1\varepsilon, \\ |\langle D\mathcal{G}(\varepsilon)(t)(u), v\rangle| \le (\alpha_2\varepsilon\|u\|^{r-1}_{L^r(\partial_N\Omega^\varepsilon;\mathbf{R}^3)} + \beta_2\varepsilon)\|v\|_{L^r(\partial_N\Omega^\varepsilon;\mathbf{R}^3)}, \\ |\dot{\mathcal{G}}(\varepsilon)(t)(u)| \le \alpha_3(t)\varepsilon\|u\|^r_{L^r(\partial_N\Omega^\varepsilon;\mathbf{R}^3)} + \beta_3(t)\varepsilon, \\ |\langle D\dot{\mathcal{G}}(\varepsilon)(t)(u), v\rangle| \le (\alpha_4(t)\varepsilon\|u\|^{r-1}_{L^r(\partial_N\Omega^\varepsilon;\mathbf{R}^3)} + \beta_4(t)\varepsilon)\|v\|_{L^r(\partial_N\Omega^\varepsilon;\mathbf{R}^3)}. \end{cases} \tag{11}$$

The reason why all the previous coercivity and growth constants/functions are of order ε is due to the fact that the surface force density $G_\varepsilon = \varepsilon G$ of scales like ε.

2.4 Quasi-Static Evolution

For a given admissible crack $K \in \mathcal{R}(\overline{\Omega^\varepsilon_B})$ and a given boundary deformation $\phi \in W^{1,p}(\Omega^\varepsilon;\mathbf{R}^3) \cap L^q(\Omega^\varepsilon;\mathbf{R}^3)$, the total energy of the configuration (K, u), with $u \in AD^\varepsilon(\phi, K)$, at time $t \in [0, T]$ is given by

$$\mathcal{E}(\varepsilon)(t)(u, K) := \mathcal{W}(\varepsilon)(\nabla u) - \mathcal{F}(\varepsilon)(t)(u) - \mathcal{G}(\varepsilon)(t)(u) + \mathcal{K}(\varepsilon)(K).$$

We define a quasi-static evolution with boundary condition $t \mapsto \phi^\varepsilon(t)$ as a map $t \mapsto (v^\varepsilon(t), K^\varepsilon(t))$ from $[0, T]$ to $GSBV^p_q(\Omega^\varepsilon;\mathbf{R}^3) \times \mathcal{R}(\overline{\Omega^\varepsilon_B})$ with the following properties:

(i) *Global stability*: for all $t \in [0, T]$, we have $v^\varepsilon(t) \in AD^\varepsilon(\phi^\varepsilon(t), K^\varepsilon(t))$ and

$$\mathcal{E}(\varepsilon)(t)(v^\varepsilon(t), K^\varepsilon(t)) = \min\{\mathcal{E}(\varepsilon)(v', K') : K' \in \mathcal{R}(\overline{\Omega^\varepsilon_B}),\ K^\varepsilon(t) \tilde{\subset} K' \text{ and } v' \in AD^\varepsilon(\phi^\varepsilon(t), K')\}.$$

(ii) *Irreversibility*: $K^\varepsilon(s) \tilde{\subset} K^\varepsilon(t)$ whenever $s \le t$.

(iii) *Energy balance*: the mapping $t \mapsto E(\varepsilon)(t) := \mathcal{E}(\varepsilon)(t)(v^\varepsilon(t), K^\varepsilon(t))$ is absolutely continuous on $[0, T]$ and

$$\dot{E}(\varepsilon)(t) = \langle D\mathcal{W}(\varepsilon)(\nabla v^\varepsilon(t)), \nabla\dot{\phi}^\varepsilon(t)\rangle - \langle D\mathcal{F}(\varepsilon)(t)(v^\varepsilon(t)), \dot{\phi}^\varepsilon(t)\rangle - \dot{\mathcal{F}}(\varepsilon)(t)(v^\varepsilon(t))$$

$$-\langle D\mathcal{G}(\varepsilon)(t)(v^\varepsilon(t)), \dot{\phi}^\varepsilon(t)\rangle - \dot{\mathcal{G}}(\varepsilon)(t)(v^\varepsilon(t)).$$

The following existence result has been proven in [8].

Theorem 1. *Let* $K_0^\varepsilon \in \mathcal{R}(\overline{\Omega_B^\varepsilon})$ *and* $v_0^\varepsilon \in AD^\varepsilon(\phi^\varepsilon(0), K_0^\varepsilon)$ *such that*

$$\mathcal{E}(\varepsilon)(0)(v_0^\varepsilon, K_0^\varepsilon) \leq \mathcal{E}(\varepsilon)(0)(v', K')$$

for every $K' \in \mathcal{R}(\overline{\Omega_B^\varepsilon})$ *with* $K_0^\varepsilon \mathrel{\tilde{\subset}} K'$*, and every* $v' \in AD^\varepsilon(\phi^\varepsilon(0), K')$*. Then there exists a quasi-static evolution* $t \mapsto (v^\varepsilon(t), K^\varepsilon(t))$ *with boundary deformation* $\phi^\varepsilon(t)$ *such that* $(v^\varepsilon(0), K^\varepsilon(0)) = (v_0^\varepsilon, K_0^\varepsilon)$.

3 The Rescaled Configuration

Our goal is to perform an asymptotic analysis of the quasi-static evolution as the thickness of the film $\varepsilon \to 0$. As usual in dimension reduction problems (see e.g. [6, 15]) we rescale the problem into an equivalent one with the advantage of being stated over a fixed domain.

Before doing this we shall make some assumptions on the initial crack. We assume that it is compatible with the geometry of the problem, i.e., that $K_0^\varepsilon = \gamma_0 \times (-\varepsilon + \varepsilon\eta, \varepsilon - \varepsilon\eta)$, for some countably $\mathcal{H}^1$-rectifiable set $\gamma_0 \subset \omega$.

We now define $\Omega := \Omega^1$, $\Omega_B := \Omega_B^1$, $\partial_D\Omega := \partial_D\Omega^1$ and $\partial_N\Omega := \partial_N\Omega^1$. For $x \in \Omega$, we denote by $x_\alpha := (x_1, x_2) \in \omega$ the in-plane variable. We set

$$\begin{aligned}
\psi^\varepsilon(t, x_\alpha, x_3) &:= \phi^\varepsilon(t, x_\alpha, \varepsilon x_3),\\
u_0^\varepsilon(x_\alpha, x_3) &:= v_0^\varepsilon(x_\alpha, \varepsilon x_3),\\
u^\varepsilon(t)(x_\alpha, x_3) &:= v^\varepsilon(t)(x_\alpha, \varepsilon x_3),\\
\Gamma^\varepsilon(t) &:= \{(x_\alpha, x_3) \in \overline{\Omega_B} : (x_\alpha, \varepsilon x_3) \in K^\varepsilon(t)\}.
\end{aligned}$$

Changing variables in (1), (2), (4) and (8) leads to

$$\begin{aligned}
\mathcal{W}(\varepsilon)(\nabla v^\varepsilon(t)) &= \varepsilon \int_\Omega W\left(\nabla_\alpha u^\varepsilon(t)\Big|\frac{1}{\varepsilon}\nabla_3 u^\varepsilon(t)\right) dx =: \varepsilon \mathcal{W}^\varepsilon(\nabla u^\varepsilon(t)),\\
\mathcal{K}(\varepsilon)(K^\varepsilon(t)) &= \varepsilon \int_{\Gamma^\varepsilon(t)} \left|\left(\left(\nu_{\Gamma^\varepsilon(t)}\right)_\alpha \Big| \frac{1}{\varepsilon}\left(\nu_{\Gamma^\varepsilon(t)}\right)_3\right)\right| d\mathcal{H}^2 =: \varepsilon \mathcal{K}^\varepsilon(\Gamma^\varepsilon(t)),\\
\mathcal{F}(\varepsilon)(t)(v^\varepsilon(t)) &= \varepsilon \int_\Omega F(t, u^\varepsilon(t))\, dx =: \varepsilon \mathcal{F}(t)(u^\varepsilon(t)),\\
\mathcal{G}(\varepsilon)(t)(v^\varepsilon(t)) &= \varepsilon \int_{\partial_N\Omega} G(t, u^\varepsilon(t))\, d\mathcal{H}^2(x) =: \varepsilon \mathcal{G}(t)(u^\varepsilon(t)),
\end{aligned}$$

where ∇_α (resp. ∇_3) denotes the approximate gradient with respect to x_α (resp. x_3), and $\nu_{\Gamma^\varepsilon(t)} = ((\nu_{\Gamma^\varepsilon(t)})_\alpha, (\nu_{\Gamma^\varepsilon(t)})_3)$ is the normal to $\Gamma^\varepsilon(t)$.

As in Section 2.3, we define the functionals $\dot{\mathcal{F}}(t)$ and $\dot{\mathcal{G}}(t)$ so that there hold the analogue of (5), (6), (9) and (10) with $\mathcal{F}$, $\dot{\mathcal{F}}$, $\mathcal{G}$ and $\dot{\mathcal{G}}$ instead of $\mathcal{F}(\varepsilon)$, $\dot{\mathcal{F}}(\varepsilon)$, $\mathcal{G}(\varepsilon)$ and $\dot{\mathcal{G}}(\varepsilon)$. Note that $\mathcal{G}$, $\dot{\mathcal{G}}$, $D\mathcal{G}$, $D\dot{\mathcal{G}}$, $\mathcal{F}$, $\dot{\mathcal{F}}$, $D\mathcal{F}$ and $D\dot{\mathcal{F}}$ satisfy analogue growth and coercivity conditions as (7)–(11) with the same exponents, and with coercivity and growth constants/functions independent of ε.

We write $\mathcal{E}^\varepsilon(t)$ for the rescaled total energy at time t. Then, we deduce that the mapping $t \mapsto (u^\varepsilon(t), \Gamma^\varepsilon(t))$ from $[0,T]$ to $GSBV^p_q(\Omega;\mathbf{R}^3) \times \mathcal{R}(\overline{\Omega_B})$ satisfies the following properties:

(i) *Global stability*: for all $t \in [0,T]$, we have $u^\varepsilon(t) \in AD^1(\psi^\varepsilon(t), \Gamma^\varepsilon(t))$ and

$$\mathcal{E}^\varepsilon(t)(u^\varepsilon(t), \Gamma^\varepsilon(t)) = \min\{\mathcal{E}^\varepsilon(u', \Gamma') : \Gamma' \in \mathcal{R}(\overline{\Omega_B}),\ \Gamma^\varepsilon(t) \tilde{\subset} \Gamma' \text{ and } u' \in AD^1(\psi^\varepsilon(t), \Gamma')\}.$$

(ii) *Irreversibility*: $\Gamma^\varepsilon(s) \tilde{\subset} \Gamma^\varepsilon(t)$ whenever $s \le t$.

(iii) *Energy balance*: the mapping $t \mapsto E^\varepsilon(t) := \mathcal{E}^\varepsilon(t)(u^\varepsilon(t), \Gamma^\varepsilon(t))$ is absolutely continuous on $[0,T]$ and

$$\begin{aligned}\dot{E}^\varepsilon(t) = {} & \langle D\mathcal{W}^\varepsilon(\nabla u^\varepsilon(t)), \nabla\dot{\psi}^\varepsilon(t)\rangle \\ & -\langle D\mathcal{F}(t)(u^\varepsilon(t)), \dot{\psi}^\varepsilon(t)\rangle - \dot{\mathcal{F}}(t)(u^\varepsilon(t)) \\ & -\langle D\mathcal{G}(t)(u^\varepsilon(t)), \dot{\psi}^\varepsilon(t)\rangle - \dot{\mathcal{G}}(t)(u^\varepsilon(t)).\end{aligned}$$

4 Analysis of Static Problem by Γ-Convergence

Before going to the study of the quasi-static problem, we will discuss the asymptotic behavior of the total energy as $\varepsilon \to 0$ thanks to a Γ-convergence method. Since the work of external forces $\mathcal{F} + \mathcal{G}$ corresponds to a continuous perturbation of the sum of the bulk and surface energies, we will not take it into account. We will actually study a weak formulation of the problem replacing the crack by the jump set of the deformation field. Indeed, Let us define $\mathcal{I}_\varepsilon : L^1(\Omega;\mathbf{R}^3) \to [0,+\infty]$ by

$$\mathcal{I}_\varepsilon(u) := \int_\Omega W\left(\nabla_\alpha u \Big| \frac{1}{\varepsilon}\nabla_3 u\right) dx + \int_{S_u} \left|\left((\nu_u)_\alpha \,\Big|\, \frac{1}{\varepsilon}\,(\nu_u)_3\right)\right| d\mathcal{H}^2$$

if $u \in GSBV^p(\Omega;\mathbf{R}^3)$, and $+\infty$ if $u \in L^1(\Omega;\mathbf{R}^3) \setminus GSBV^p(\Omega;\mathbf{R}^3)$. Then, the following Γ-convergence result holds:

Theorem 2. *Let ω be a bounded open subset of $\mathbf{R}^2$ and $W : \mathbf{R}^{3\times 3} \to \mathbf{R}$ be a continuous function satisfying (3). Then the functional $\mathcal{I}_\varepsilon$ Γ-converges for the strong $L^1(\Omega;\mathbf{R}^3)$-topology to $\mathcal{I} : L^1(\Omega;\mathbf{R}^3) \to [0,+\infty]$ defined by*

$$\mathcal{I}(u) := \begin{cases} 2\int_\omega QW_0(\nabla_\alpha u)\,dx_\alpha + 2\mathcal{H}^1(S_u) & \text{if } u \in GSBV^p(\omega;\mathbf{R}^3),\\ +\infty & \text{otherwise,}\end{cases}$$

where $W_0(\overline{\xi}) := \inf\{W(\overline{\xi}|z) : z \in \mathbf{R}^3\}$ *for every* $\overline{\xi} \in \mathbf{R}^{3\times 2}$, *and* QW_0 *is the quasiconvexification of* W_0.

This result has been proven in [2] (see also [3]) in a SBV^p framework, and one can notice that much easier arguments lead to the analogue in $GSBV^p$ as stated in Theorem 2. Indeed, in $GSBV^p$, there is no lack of compactness and it is not necessary to appeal to a truncation argument as in [2, lemma 3.3]. It follows immediately from the $GSBV$-compactness theorem [1, theorem 4.36] that any minimizing sequence $(u_\varepsilon) \subset GSBV^p(\Omega;\mathbf{R}^3)$ with uniformly bounded energy is relatively compact in $GSBV^p(\Omega;\mathbf{R}^3)$, and that any accumulation point is independent of x_3 (we identify those functions to $GSBV^p(\omega;\mathbf{R}^3)$). The proof of the lower bound is exactly the same than [2, lemma 3.9] using the lower semicontinuity result in $GSBV^p$ (see e.g. [8, theorem 2.8]). The construction of a recovery $(\bar{u}_\varepsilon)$ can be performed as in [15]: it suffices to take $\bar{u}_\varepsilon(x_\alpha, x_3) := u(x_\alpha) + \varepsilon x_3 b_\varepsilon(x_\alpha)$ for some suitable function $b_\varepsilon \in \mathcal{C}_c^\infty(\omega;\mathbf{R}^3)$, and then we appeal to a classical relaxation result in $GSBV^p$. We also refer to [5] for an alternative proof using a singular perturbation argument.

5 Analysis of the Quasi-Static Problem

In view of Theorem 2, one can guess that the 3D quasi-static evolution – whose existence is ensured by Theorem 1 – will converge in a certain sense to a 2D quasi-static evolution associated to the Γ-limit model.

We assume that $\psi^\varepsilon \to \psi$ in $W^{1,1}([0,T]; W^{1,p}(\Omega;\mathbf{R}^3) \cap L^q(\Omega;\mathbf{R}^3))$ and that the sequence $((1/\varepsilon)\nabla_3\psi^\varepsilon)$ is strongly converging in $W^{1,1}([0,T]; L^p(\Omega;\mathbf{R}^3))$. In particular, the limit function $\psi \in W^{1,1}([0,T]; W^{1,p}(\omega;\mathbf{R}^3) \cap L^q(\omega;\mathbf{R}^3))$ is independent of x_3.

We first derive some compactness of $(u^\varepsilon(t), \Gamma^\varepsilon(t))$. Indeed taking $(\psi^\varepsilon(t), \Gamma^\varepsilon(t))$ as competitor in the minimality, and using the growth an coercivity properties satisfied by the functionals $\mathcal{W}_\varepsilon$, $\mathcal{F}$ and $\mathcal{G}$ implies that the sequence of approximate scaled gradients $(\nabla_\alpha u^\varepsilon(t)|(1/\varepsilon)\nabla_3 u^\varepsilon(t))$ is bounded in $L^p(\Omega;\mathbf{R}^{3\times 3})$, and the sequence $(u^\varepsilon(t))$ is bounded in $L^{\dot{q}}(\Omega;\mathbf{R}^3) \cap L^q(\Omega;\mathbf{R}^3)$. Moreover, since $u^\varepsilon(t) \in W^{1,p}(\Omega \setminus \overline{\Omega_B};\mathbf{R}^3)$, the trace theorem and the choice of the exponent r ensures that $(u^\varepsilon(t))$ is compact in $L^r(\partial_N\Omega;\mathbf{R}^3)$. Then we use the energy balance together with the growth and coercivity conditions satisfied by $D\mathcal{F}(t)$, $D\mathcal{G}(t)$, $\dot{\mathcal{F}}(t)$ and $\dot{\mathcal{G}}(t)$ to ensure that

$$\sup_{\varepsilon>0}\int_{\Gamma^\varepsilon(t)}\left|\left((\nu_{\Gamma^\varepsilon(t)})_\alpha\Big|\frac{1}{\varepsilon}(\nu_{\Gamma^\varepsilon(t)})_3\right)\right| d\mathcal{H}^2 < +\infty.$$

At this step, we are in position to use a mean convergence for rectifiable sets introduced in [2], very close to the σ^p-convergence in [8, 9]:

Definition 1. Let $\varepsilon_n \searrow 0^+$, $\Gamma_n \subset \Omega$ be a sequence of countably $\mathcal{H}^2$-rectifiable sets, and $\gamma \subset \omega$ be a countably $\mathcal{H}^1$-rectifiable set. We say that Γ_n converges to γ in Ω if

$$\int_{\Gamma_n} \left| \left((\nu_{\Gamma_n})_\alpha \Big| \frac{1}{\varepsilon_n} (\nu_{\Gamma_n})_3 \right) \right| d\mathcal{H}^2 \leq C,$$

and the following properties hold:

(a) if $u_k \rightharpoonup u$ in $SBV^p(\Omega)$, $S_{u_k} \mathrel{\tilde{\subset}} \Gamma_{n_k}$ and $\int_\Omega \left|\left(\nabla_\alpha u_k | (1/\varepsilon_{n_k}) \nabla_3 u_k\right)\right|^p dx \leq C$, for some subsequence $(\varepsilon_{n_k}) \subset (\varepsilon_n)$, then $u \in SBV^p(\omega)$ and $S_u \mathrel{\tilde{\subset}} \gamma$;

(b) there exist a function $u \in SBV^p(\omega)$ and a sequence $(u_n) \subset SBV^p(\Omega)$ such that $u_n \rightharpoonup u$ in $SBV^p(\Omega)$, $S_{u_n} \mathrel{\tilde{\subset}} \Gamma_n$, $\int_\Omega \left|\left(\nabla_\alpha u_n | (1/\varepsilon_n) \nabla_3 u_n\right)\right|^p dx \leq C$, and $S_u \mathrel{\tilde{=}} \gamma$.

Then, using [2, proposition 4.3], one can extract a subsequence (ε_n) (independently of t) and find a countably $\mathcal{H}^1$-rectifiable set $\gamma(t)$ increasing with respect to t such that $\Gamma^{\varepsilon_n}(t)$ converges to $\gamma(t)$ in sense of Definition 1. Note that it is possible to prove that $\gamma(0) = \gamma_0$. Moreover the estimates we have on the sequence $(u^{\varepsilon_n}(t))$ allow us to apply the $GSBV$-Compactness Theorem (see [1, theorem 4.36]) which ensures, for each $t \in [0, T]$, the existence of a t-dependent subsequence $(\varepsilon_{n_t}) \subset (\varepsilon_n)$ such that $u^{\varepsilon_{n_t}}(t) \rightharpoonup u(t)$ in $GSBV^p_q(\Omega; \mathbf{R}^3)$. Moreover the limit deformation $u \in GSBV^p_q(\omega; \mathbf{R}^3)$, i.e., it is independent of x_3.

We claim that the pair $(u(t), \gamma(t))$ is a quasi-static evolution associated to the Γ-limit model. We already proved the irreversibility condition. To show the minimality property we use the following jump transfer theorem whose proof can be obtained exactly as in [2, theorem 4.4], using [8, theorem 5.3] instead of [11, theorem 2.1]. Let $\omega' \subset \mathbf{R}^2$ a bounded open set containing $\overline{\omega}$, and define $\Omega' := \omega' \times (-1, 1)$.

Theorem 3. *Let $\Gamma_n \in \mathcal{R}(\overline{\Omega_B})$ be a sequence of countably $\mathcal{H}^2$-rectifiable sets converging to γ in the sense of Definition 1. Then, for every $v \in GSBV^p_q(\omega'; \mathbf{R}^3)$, there exists a sequence $(v_n) \subset GSBV^p_q(\Omega'; \mathbf{R}^3)$ such that $v_n = v$ a.e. on $\Omega' \setminus \overline{\Omega}$,*

- $v_n \to v$ *in* $L^q(\Omega'; \mathbf{R}^3)$,
- $\left(\nabla_\alpha v_n \Big| \frac{1}{\varepsilon_n} \nabla_3 v_n\right) \to (\nabla_\alpha v | 0)$ *in* $L^p(\Omega'; \mathbf{R}^{3\times 3})$,
- $\displaystyle\limsup_{n\to+\infty} \int_{S_{v_n} \setminus \Gamma_n} \left| \left((\nu_{v_n})_\alpha \Big| \frac{1}{\varepsilon_n} (\nu_{v_n})_3 \right) \right| d\mathcal{H}^2 \leq 2\mathcal{H}^1(S_v \setminus \gamma).$

Arguing exactly as in the proof of [2, lemma 5.5] and using the upper semicontinuity property of $\mathcal{F}(t)$ together with the continuity of $\mathcal{G}(t)$ (which comes from the trace theorem in $W^{1,p}$ and the choice of the exponent r), one can show that for every $t \in [0, T]$, $u(t)$ minimizes

$$v \mapsto 2 \int_\omega QW_0(\nabla_\alpha v)\, dx_\alpha + 2\mathcal{H}^1(S_v \setminus \gamma(t)) - 2\mathcal{F}(t)(v) - 2\mathcal{G}(t)(v),$$

among $\{v \in GSBV^p_q(\omega; \mathbf{R}^3) : v = \psi(t)\ \mathcal{H}^1\text{-a.e. on } \partial\omega\}$. Moreover one has convergence of the bulk energy (for the sequence ε_n)

$$\int_\Omega W\left(\nabla_\alpha u^{\varepsilon_n}(t)\Big|\frac{1}{\varepsilon_n}\nabla_3 u^{\varepsilon_n}(t)\right)dx \to 2\int_\omega QW_0(\nabla_\alpha u(t))\,dx_\alpha,$$

as well as weak convergence of the stress (for the subsequence ε_{n_t})

$$DW\left(\nabla_\alpha u^{\varepsilon_{n_t}}(t)\Big|\frac{1}{\varepsilon_{n_t}}\nabla_3 u^{\varepsilon_{n_t}}(t)\right) \rightharpoonup \big(D(QW_0)(\nabla_\alpha u(t)|0)\big) \quad \text{in } L^{p'}(\Omega;\mathbf{R}^{3\times 3}) \tag{12}$$

at every time. Remark that by [2, proposition 4.7], the function QW_0 is of class $\mathcal{C}^1$. For every $v \in GSBV_q^p(\omega;\mathbf{R}^3)$ such that $v = \psi(t)$ $\mathcal{H}^1$-a.e. on $\partial\omega$, and every countably $\mathcal{H}^1$-rectifiable set $\gamma \subset \omega$, we define

$$\mathcal{E}(t)(v,\gamma) = 2\int_\omega QW_0(\nabla_\alpha v)\,dx_\alpha + 2\mathcal{H}^1(\gamma) - 2\mathcal{F}(t)(v) - 2\mathcal{G}(t)(v).$$

The minimality property proven above exactly says that

$$\begin{aligned}\mathcal{E}(t)(u(t),\gamma(t)) = \min\{&\mathcal{E}(t)(v,\gamma) : \gamma \text{ countably } \mathcal{H}^1\text{-rectifiable set in } \omega,\\ &v \in GSBV_q^p(\omega;\mathbf{R}^3) \text{ such that } v = \psi(t)\ \mathcal{H}^1\text{-a.e. on } \partial\omega\}\end{aligned}$$

(see e.g. [2, remark 5.4]). Moreover, since we have reduced dimension, the functional $\mathcal{G}(t)$ becomes a bulk force as well as $\mathcal{F}(t)$.

It remains to prove the energy balance. Arguing word for word as in [8], approximating Bochner integrals by suitable Riemann sums, one can show that

$$\begin{aligned}\mathcal{E}(t)(u(t),\gamma(t)) \geq \mathcal{E}(0)(u(0),\gamma(0)) + 2\int_0^t \Big(&\langle D\mathcal{W}_0(\nabla_\alpha u(s)), \nabla_\alpha \dot\psi(s)\rangle\rangle\\ &-\langle D\mathcal{F}(t)(u(s)), \dot\psi(s)\rangle - \dot{\mathcal{F}}(s)(u(s))\\ &-\langle D\mathcal{G}(s)(u(s)), \dot\psi(s)\rangle - \dot{\mathcal{G}}(s)(u(s))\Big)\,ds,\end{aligned}$$

where $\mathcal{W}_0 : L^p(\omega;\mathbf{R}^{3\times 2}) \to [0,+\infty)$ is defined by $\mathcal{W}_0(\Phi) := \int_\omega QW_0(\Phi(x))\,dx_\alpha$. By [2, Proposition 4.7], we deduce that $\mathcal{W}_0$ is of class $\mathcal{C}^1$ with differential $D\mathcal{W}_0 : L^p(\omega;\mathbf{R}^{3\times 2}) \to L^{p'}(\omega;\mathbf{R}^{3\times 2})$ given by

$$\langle D\mathcal{W}_0(\Phi), \Psi\rangle = \int_\omega D(QW_0)(\Phi(x)) : \Psi(x)\,dx_\alpha.$$

We prove the other inequality exactly as in [2, lemma 5.8], using the upper semicontinuity property of the functional $\mathcal{F}(t)$ and the weak convergence of the stresses (12) already mentioned above. We also deduce the convergence of the surface energy (for the sequence ε_n):

$$\int_{\Gamma^{\varepsilon_n}(t)} \left|\left((\nu_{\Gamma^{\varepsilon_n}(t)})_\alpha \Big| \frac{1}{\varepsilon_n}(\nu_{\Gamma^{\varepsilon_n}(t)})_3\right)\right| d\mathcal{H}^2 \to 2\mathcal{H}^1(\gamma(t)).$$

In conclusion, we have proven the following result which states that any 3D quasi-static crack evolution converges to a 2D quasi-static evolution associated to the Γ-limit model:

Theorem 4. *There exist a two-dimensional quasi-static evolution $t \mapsto (u(t), \gamma(t))$ relative to the boundary data $\psi(t)$ for the Γ-limit model, and a sequence $\varepsilon_n \searrow 0^+$ such that for any $t \in [0, T]$,*

- *$\Gamma^{\varepsilon_n}(t)$ converges to $\gamma(t)$ in sense of Definition 1;*
- *$u^{\varepsilon_{n_t}}(t) \rightharpoonup u(t)$ in $GSBV^p_q(\Omega; \mathbf{R}^3)$ for some t-dependent subsequence $(\varepsilon_{n_t}) \subset (\varepsilon_n)$;*
- *the total energy $\mathcal{E}^{\varepsilon_n}(t)$ converges to $\mathcal{E}(t)$, and more precisely*

$$\int_\Omega W\left(\nabla_\alpha u^{\varepsilon_n}(t)\Big|\frac{1}{\varepsilon_n}\nabla_3 u^{\varepsilon_n}(t)\right) dx \to 2\int_\omega QW_0(\nabla_\alpha u(t))\, dx_\alpha,$$

$$\int_{\Gamma^{\varepsilon_n}(t)} \left|\left((\nu_{\Gamma^{\varepsilon_n}(t)})_\alpha \Big| \frac{1}{\varepsilon_n}(\nu_{\Gamma^{\varepsilon_n}(t)})_3\right)\right| d\mathcal{H}^2 \to 2\mathcal{H}^1(\gamma(t)).$$

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