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Stability of Quasi-Static Crack Evolution through Dimensional Reduction

Jean-François Babadjian

Abstract This paper deals with quasi-static crack growth in thin films. We show that, when the thickness of the film tends to zero, any three-dimensional quasi-static crack evolution converges to a two-dimensional one, in a sense related to the Γ -convergence of the associated total energy. We extend the prior analysis of [2] by adding conservative body and surface forces which allow us to remove the boundedness assumption on the deformation field.

1 Introduction

In this paper, we study the evolution of cracks in thin structures in a quasi-static setting. Our approach of fracture mechanics is based on a variational model proposed in [10] (see also the monograph [4]) where the (quasi-static) evolution results from the competition – at each time – between a bulk and a surface energy, under a growth constraint on the crack. Many existence results have been obtained (see e.g. [8, 9, 11] and references therein).

Sometimes a small parameter is involved in the model, and it is an interesting question to study the asymptotic behavior of the model when the parameter tends to zero (see e.g. [14] for the homogenization and [2] for the dimension reduction of quasi-static crack evolution). When dealing static problems, the notion of Γ -convergence (see [7]) has proven to be a powerful tool to capture the asymptotic behavior of minimizers, or even minimizing sequences. It turns out that even in the quasi-static case, one can define a notion of convergence related to the Γ -convergence of the associated total energy (see [16] for an abstract theory in the more general framework of rate independent processes).

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We present here an extension of the result in [2] on the convergence of a quasi-static crack evolution in thin films, as the thickness tends to zero. In [2], an empirical L^∞ bound was done on the deformation field (as in [9]) in order to gain compactness in the space SBV^p of special functions of bounded variation. It is sometimes possible to justify this assumption as in the antiplanar case (see [11] when $N = 2$) where it follows from a consequence of the maximum principle. Unfortunately, in the full three-dimensional elasticity, the maximum principle does not hold anymore. We propose here to remove this hypothesis adding suitable conservative bulk and surface forces as in [8]. The price to pay is that the deformation field is not anymore compact in SBV^p but in a larger subspace $GSBV_q^p$ of generalized special functions of bounded variation. The arguments we use in the present paper are very close to those of [2], and for this reason we will only mention the main differences without giving the precise proofs of the results.

The paper is organized as follows: in Section 2, we will describe the model in the physical configuration and state the existence result of [8]. Then, in Section 3, we will reformulate the problem on a rescaled configuration in order to work on a fixed domain. In Section 4, we will perform an asymptotic analysis of the total energy of the system in a static setting, thanks to a Γ -convergence method. Finally, we will address the asymptotic of the quasi-static problem in Section 5, proving that it converges to a quasi-static evolution associated to the Γ -limit model.

2 Description of the Model

We consider a homogeneous thin film occupying in its reference configuration the cylinder $\Omega^\varepsilon := \omega \times (-\varepsilon, \varepsilon)$, where $\varepsilon > 0$ and ω is a bounded open subset of \mathbf{R}^2 with Lipschitz boundary. The Dirichlet part of the boundary where the deformation is prescribed is the lateral boundary $\partial_D \Omega^\varepsilon := \partial\omega \times (-\varepsilon, \varepsilon)$, while the Neumann part $\partial_N \Omega^\varepsilon = \omega \times \{-\varepsilon, \varepsilon\}$ is made of the lower and upper sections.

On the lateral boundary $\partial_D \Omega^\varepsilon$, we impose a time dependent boundary deformation $\phi^\varepsilon(t)$ on a finite time interval $[0, T]$, where

$$t \mapsto \phi^\varepsilon(t) \in W^{1,1}([0, T]; W^{1,p}(\Omega^\varepsilon; \mathbf{R}^3) \cap L^q(\Omega^\varepsilon; \mathbf{R}^3)),$$

for some $p > 1$ and $q \geq 1$.

On the remaining part of the boundary $\partial_N \Omega^\varepsilon$, we impose a time dependent surface conservative force which will be described in Section 2.3.2.

2.1 Admissible Cracks

We fix an open subset Ω_B^ε of Ω^ε of the form $\Omega_B^\varepsilon := \omega \times (-\varepsilon + \varepsilon\eta, \varepsilon - \varepsilon\eta)$ for some $\eta \in (0, 1)$, so that the set $\overline{\Omega_B^\varepsilon}$ represents the brittle part of the body. The set of all

admissible cracks is given by

$$\mathcal{R}(\overline{\Omega_B^\varepsilon}) := \{K : K \text{ is rectifiable, } K \tilde{\subset} \overline{\Omega_B^\varepsilon} \text{ and } \mathcal{H}^2(K) < +\infty\}.$$

Note that any admissible crack must lie far enough from the upper and lower sections. The safety region $\Omega^\varepsilon \setminus \overline{\Omega_B^\varepsilon}$ can be interpreted as a layer of unbreakable material (see [8, remark 3.8]).

We denote by \mathcal{H}^{N-1} the $(N - 1)$ -dimensional Hausdorff measure in \mathbf{R}^N (we shall only consider the cases $N = 2$ or 3), and by $\tilde{\subset}$ (resp. \cong) inclusion (resp. equality) up to a set of zero \mathcal{H}^{N-1} -measure.

We assume that the energy spent to produce a crack K is of Griffith type, i.e.,

$$\mathcal{K}(\varepsilon)(K) := \mathcal{H}^2(K). \quad (1)$$

2.2 Admissible Deformations

We refer to [1] for the usual definitions and results on geometric measure theory, BV , SBV and $GSBV$ spaces. Precise definitions of the jump set S_u and of the approximate gradient ∇u of a function $u \in GSBV(U; \mathbf{R}^d)$, where U is an open subset of \mathbf{R}^N , can be found in that reference. Following Dal Maso et al. [8], we further define for $p > 1$

$$GSBV^p(U; \mathbf{R}^d) := \{u \in GSBV(U; \mathbf{R}^d) : \nabla u \in L^p(U; \mathbf{R}^{d \times N}) \text{ and } \mathcal{H}^{N-1}(S_u) < +\infty\},$$

and if $q \geq 1$, $GSBV_q^p(U; \mathbf{R}^d) := GSBV^p(U; \mathbf{R}^d) \cap L^q(U; \mathbf{R}^d)$. Moreover, we say that a sequence $u_n \rightharpoonup u$ in $GSBV_q^p(U; \mathbf{R}^d)$ if $u_n \rightarrow u$ a.e. in U , $u_n \rightharpoonup u$ in $L^q(U; \mathbf{R}^d)$, $\nabla u_n \rightharpoonup \nabla u$ in $L^p(U; \mathbf{R}^{d \times N})$ and $\mathcal{H}^{N-1}(S_{u_n})$ is uniformly bounded.

For a given admissible crack $K \in \mathcal{R}(\overline{\Omega_B^\varepsilon})$ and a boundary deformation $\phi \in W^{1,p}(\Omega^\varepsilon; \mathbf{R}^3) \cap L^q(\Omega^\varepsilon; \mathbf{R}^3)$, we define the set of admissible deformations with finite energy relative to (K, ϕ) by

$$AD^\varepsilon(\phi, K) := \{u \in GSBV_q^p(\Omega^\varepsilon; \mathbf{R}^3) : S_u \tilde{\subset} K, u = \phi \text{ } \mathcal{H}^2\text{-a.e. on } \partial_D \Omega^\varepsilon \setminus K\}.$$

The associate bulk energy is defined by

$$\mathcal{W}(\varepsilon)(\nabla u) := \int_{\Omega^\varepsilon} W(\nabla u(x)) dx, \quad (2)$$

where $W : \mathbf{R}^{3 \times 3} \rightarrow [0, +\infty)$, the stored energy density, is a quasiconvex function of class \mathcal{C}^1 satisfying standard p -growth and p -coercivity conditions ($p > 1$): there exist $0 < \beta' < \beta < +\infty$ such that

$$\beta' |\xi|^p \leq W(\xi) \leq \beta(1 + |\xi|^p) \quad \text{for every } \xi \in \mathbf{R}^{3 \times 3}. \quad (3)$$

In particular, the functional $\mathcal{W}(\varepsilon) : L^p(\Omega^\varepsilon; \mathbf{R}^{3 \times 3}) \rightarrow [0, +\infty)$ defined by

$$\mathcal{W}(\varepsilon)(\Phi) := \int_{\Omega^\varepsilon} W(\Phi(x)) dx$$

is differentiable on $L^p(\Omega^\varepsilon; \mathbf{R}^{3 \times 3})$, and its differential $D\mathcal{W}(\varepsilon) : L^p(\Omega^\varepsilon; \mathbf{R}^{3 \times 3}) \rightarrow L^{p'}(\Omega^\varepsilon; \mathbf{R}^{3 \times 3})$, with $p' = p/(p-1)$, is given by

$$\langle D\mathcal{W}(\varepsilon)(\Phi), \Psi \rangle = \int_{\Omega^\varepsilon} DW(\Phi(x)) : \Psi(x) dx \quad \text{for every } \Phi, \Psi \in L^p(\Omega^\varepsilon; \mathbf{R}^{3 \times 3}).$$

On the left-hand side of the previous equality, we have denoted by $\langle \cdot, \cdot \rangle$ the duality pairing between $L^p(\Omega^\varepsilon; \mathbf{R}^{3 \times 3})$ and $L^{p'}(\Omega^\varepsilon; \mathbf{R}^{3 \times 3})$.

2.3 The Forces

We assume that the body is subjected to the action of conservative body and surface forces with potentials F and $G^\varepsilon = \varepsilon G$ respectively. Note that the order of magnitude of the applied forces are exactly those inducing a limiting membrane model (see [12, 13]).

2.3.1 The Body Forces

Let $q \geq 1$, the density of the applied body forces per unit volume at time $t \in [0, T]$ is given by $D_z F(t, u(x))$, where $F : [0, T] \times \mathbf{R}^3 \rightarrow \mathbf{R}$ and the map $z \mapsto F(t, z)$ belongs to $\mathcal{C}^1(\mathbf{R}^3)$ for every $t \in [0, T]$. We suppose that for every $t \in [0, T]$, the functional

$$\mathcal{F}(\varepsilon)(t)(u) := \int_{\Omega^\varepsilon} F(t, u(x)) dx \quad (4)$$

is of class \mathcal{C}^1 on the space $L^q(\Omega^\varepsilon; \mathbf{R}^3)$, and its differential $D\mathcal{F}(\varepsilon)(t) : L^q(\Omega^\varepsilon; \mathbf{R}^3) \rightarrow L^{q'}(\Omega^\varepsilon; \mathbf{R}^3)$, with $q' := q/(q-1)$, is given by

$$\langle D\mathcal{F}(\varepsilon)(t)(u), v \rangle = \int_{\Omega^\varepsilon} D_z F(t, u(x)) \cdot v(x) dx \quad \text{for every } u, v \in L^q(\Omega^\varepsilon; \mathbf{R}^3).$$

We have denoted by $\langle \cdot, \cdot \rangle$ the duality pairing between $L^q(\Omega^\varepsilon; \mathbf{R}^3)$ and $L^{q'}(\Omega^\varepsilon; \mathbf{R}^3)$.

Concerning the regularity in time, we assume that there exist an exponent $\dot{q} < q$ and, for a.e. $t \in [0, T]$, a functional $\dot{\mathcal{F}}(\varepsilon)(t) : L^{\dot{q}}(\Omega^\varepsilon; \mathbf{R}^3) \rightarrow \mathbf{R}$ of class \mathcal{C}^1 , with differential $D\dot{\mathcal{F}}(\varepsilon)(t) : L^{\dot{q}}(\Omega^\varepsilon; \mathbf{R}^3) \rightarrow L^{\dot{q}'}(\Omega^\varepsilon; \mathbf{R}^3)$, where $\dot{q}' = \dot{q}/(\dot{q}-1)$, such that for every $u, v \in L^q(\Omega^\varepsilon; \mathbf{R}^3)$, the functions $t \mapsto \dot{\mathcal{F}}(\varepsilon)(t)(u)$ and $t \mapsto \langle D\dot{\mathcal{F}}(\varepsilon)(t)(u), v \rangle$ are integrable on $[0, T]$, and

$$\mathcal{F}(\varepsilon)(t)(u) = \mathcal{F}(\varepsilon)(0)(u) + \int_0^t \dot{\mathcal{F}}(\varepsilon)(s)(u) ds, \quad (5)$$

$$\langle D\mathcal{F}(\varepsilon)(t)(u), v \rangle = \langle D\mathcal{F}(\varepsilon)(0)(u), v \rangle + \int_0^t \langle D\dot{\mathcal{F}}(\varepsilon)(s)(u), v \rangle ds \quad (6)$$

for every $t \in [0, T]$. We further assume that $\mathcal{F}(\varepsilon)(t)$ is upper semicontinuous in $L^q(\Omega^\varepsilon; \mathbf{R}^3)$ with respect to the pointwise almost everywhere convergence.

Finally, we suppose that $\mathcal{F}(\varepsilon)(t)$, $D\mathcal{F}(\varepsilon)(t)$, $\dot{\mathcal{F}}(\varepsilon)(t)$ and $D\dot{\mathcal{F}}(\varepsilon)(t)$ satisfy suitable q -growth conditions: there exist constants $a_0 > 0$, $a_1 > 0$, $a_2 > 0$, $b_0 \geq 0$, $b_1 \geq 0$, $b_2 \geq 0$, and nonnegative integrable functions on $[0, T]$, a_3 , a_4 , b_3 and b_4 (uniform in ε) such that

$$\begin{cases} a_0 \|u\|_{L^q(\Omega^\varepsilon; \mathbf{R}^3)}^q - b_0 \leq -\mathcal{F}(\varepsilon)(t)(u) \leq a_1 \|u\|_{L^q(\Omega^\varepsilon; \mathbf{R}^3)}^q + b_1, \\ |\langle D\mathcal{F}(\varepsilon)(t)(u), v \rangle| \leq (a_2 \|u\|_{L^q(\Omega^\varepsilon; \mathbf{R}^3)}^{q-1} + b_2) \|v\|_{L^q(\Omega^\varepsilon; \mathbf{R}^3)}, \\ |\dot{\mathcal{F}}(\varepsilon)(t)(u)| \leq a_3(t) \|u\|_{L^{\dot{q}}(\Omega^\varepsilon; \mathbf{R}^3)}^{\dot{q}} + b_3(t), \\ |\langle D\dot{\mathcal{F}}(\varepsilon)(t)(u), v \rangle| \leq (a_4(t) \|u\|_{L^{\dot{q}}(\Omega^\varepsilon; \mathbf{R}^3)}^{\dot{q}-1} + b_4(t)) \|v\|_{L^{\dot{q}}(\Omega^\varepsilon; \mathbf{R}^3)}. \end{cases} \quad (7)$$

2.3.2 The Surface Forces

The density of the surface forces on $\partial_N \Omega^\varepsilon$ at time $t \in [0, T]$, under the deformation u is given by $\varepsilon D_z G(t, u(x))$, where $G : [0, T] \times \mathbf{R}^3 \rightarrow \mathbf{R}$ is such that $z \mapsto G(t, z)$ is of class $\mathcal{C}^1(\mathbf{R}^3)$ for every $t \in [0, T]$.

We fix an exponent r , related to the trace theorem in Sobolev spaces, such that $r \in [p, p/(3-p)]$ if $p < 3$, while $r \geq p$ if $p \geq 3$. We assume that for every $t \in [0, T]$, the functional

$$\mathcal{G}(\varepsilon)(t)(u) := \varepsilon \int_{\partial_N \Omega^\varepsilon} G(t, u(x)) d\mathcal{H}^2(x) \quad (8)$$

is of class \mathcal{C}^1 on $L^r(\partial_N \Omega^\varepsilon; \mathbf{R}^3)$, with differential $D\mathcal{G}(\varepsilon)(t) : L^r(\partial_N \Omega^\varepsilon; \mathbf{R}^3) \rightarrow L^{r'}(\partial_N \Omega^\varepsilon; \mathbf{R}^3)$, where $r' = r/(r-1)$, given by

$$\langle \mathcal{G}(\varepsilon)(t)(u), v \rangle = \varepsilon \int_{\partial_N \Omega^\varepsilon} D_z G(t, u(x)) \cdot v(x) d\mathcal{H}^2(x)$$

for all $u, v \in L^r(\partial_N \Omega^\varepsilon; \mathbf{R}^3)$,

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $L^r(\partial_N \Omega^\varepsilon; \mathbf{R}^3)$ and $L^{r'}(\partial_N \Omega^\varepsilon; \mathbf{R}^3)$.

As for the regularity in time, we suppose that for a.e. $t \in [0, T]$, there exists a functional $\dot{\mathcal{G}}(\varepsilon)(t) : L^r(\partial_N \Omega^\varepsilon; \mathbf{R}^3) \rightarrow \mathbf{R}$ of class \mathcal{C}^1 , with differential $D\dot{\mathcal{G}}(\varepsilon)(t) : L^r(\partial_N \Omega^\varepsilon; \mathbf{R}^3) \rightarrow L^{r'}(\partial_N \Omega^\varepsilon; \mathbf{R}^3)$, such that for every $u, v \in L^r(\partial_N \Omega^\varepsilon; \mathbf{R}^3)$, the mappings $t \mapsto \dot{\mathcal{G}}(\varepsilon)(t)(u)$ and $t \mapsto \langle D\dot{\mathcal{G}}(\varepsilon)(t)(u), v \rangle$ are integrable on $[0, T]$, and

$$\mathcal{G}(\varepsilon)(t)(u) = \mathcal{G}(\varepsilon)(0)(u) + \int_0^t \dot{\mathcal{G}}(\varepsilon)(s)(u) ds, \quad (9)$$

$$\langle D\mathcal{G}(\varepsilon)(t)(u), v \rangle = \langle D\mathcal{G}(\varepsilon)(0)(u), v \rangle + \int_0^t \langle D\dot{\mathcal{G}}(\varepsilon)(s)(u), v \rangle ds \quad (10)$$

for every $t \in [0, T]$.

Finally, we suppose that $\mathcal{G}(\varepsilon)(t)$, $D\mathcal{G}(\varepsilon)(t)$, $\dot{\mathcal{G}}(\varepsilon)(t)$ and $D\dot{\mathcal{G}}(\varepsilon)(t)$ satisfy suitable r -growth conditions: there exist nonnegative constants $\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2$, and nonnegative integrable functions on $[0, T]$, $\alpha_3, \alpha_4, \beta_3$ and β_4 such that

$$\begin{cases} -\alpha_0 \varepsilon \|u\|_{L^r(\partial_N \Omega^\varepsilon; \mathbf{R}^3)}^r - \beta_0 \varepsilon \leq -\mathcal{G}(\varepsilon)(t)(u) \leq \alpha_1 \varepsilon \|u\|_{L^r(\partial_N \Omega^\varepsilon; \mathbf{R}^3)}^q + \beta_1 \varepsilon, \\ |\langle D\mathcal{G}(\varepsilon)(t)(u), v \rangle| \leq (\alpha_2 \varepsilon \|u\|_{L^r(\partial_N \Omega^\varepsilon; \mathbf{R}^3)}^{r-1} + \beta_2 \varepsilon) \|v\|_{L^r(\partial_N \Omega^\varepsilon; \mathbf{R}^3)}, \\ |\dot{\mathcal{G}}(\varepsilon)(t)(u)| \leq \alpha_3(t) \varepsilon \|u\|_{L^r(\partial_N \Omega^\varepsilon; \mathbf{R}^3)}^r + \beta_3(t) \varepsilon, \\ |\langle D\dot{\mathcal{G}}(\varepsilon)(t)(u), v \rangle| \leq (\alpha_4(t) \varepsilon \|u\|_{L^r(\partial_N \Omega^\varepsilon; \mathbf{R}^3)}^{r-1} + \beta_4(t) \varepsilon) \|v\|_{L^r(\partial_N \Omega^\varepsilon; \mathbf{R}^3)}. \end{cases} \quad (11)$$

The reason why all the previous coercivity and growth constants/functions are of order ε is due to the fact that the surface force density $G_\varepsilon = \varepsilon G$ of scales like ε .

2.4 Quasi-Static Evolution

For a given admissible crack $K \in \mathcal{R}(\overline{\Omega_B^\varepsilon})$ and a given boundary deformation $\phi \in W^{1,p}(\Omega^\varepsilon; \mathbf{R}^3) \cap L^q(\Omega^\varepsilon; \mathbf{R}^3)$, the total energy of the configuration (K, u) , with $u \in AD^\varepsilon(\phi, K)$, at time $t \in [0, T]$ is given by

$$\mathcal{E}(\varepsilon)(t)(u, K) := \mathcal{W}(\varepsilon)(\nabla u) - \mathcal{F}(\varepsilon)(t)(u) - \mathcal{G}(\varepsilon)(t)(u) + \mathcal{K}(\varepsilon)(K).$$

We define a quasi-static evolution with boundary condition $t \mapsto \phi^\varepsilon(t)$ as a map $t \mapsto (v^\varepsilon(t), K^\varepsilon(t))$ from $[0, T]$ to $GSBV_q^p(\Omega^\varepsilon; \mathbf{R}^3) \times \mathcal{R}(\overline{\Omega_B^\varepsilon})$ with the following properties:

- (i) *Global stability*: for all $t \in [0, T]$, we have $v^\varepsilon(t) \in AD^\varepsilon(\phi^\varepsilon(t), K^\varepsilon(t))$ and

$$\begin{aligned} \mathcal{E}(\varepsilon)(t)(v^\varepsilon(t), K^\varepsilon(t)) &= \min\{\mathcal{E}(\varepsilon)(v', K') : K' \in \mathcal{R}(\overline{\Omega_B^\varepsilon}), K^\varepsilon(t) \widetilde{\subset} K' \\ &\quad \text{and } v' \in AD^\varepsilon(\phi^\varepsilon(t), K')\}. \end{aligned}$$

- (ii) *Irreversibility*: $K^\varepsilon(s) \widetilde{\subset} K^\varepsilon(t)$ whenever $s \leq t$.

- (iii) *Energy balance*: the mapping $t \mapsto E(\varepsilon)(t) := \mathcal{E}(\varepsilon)(t)(v^\varepsilon(t), K^\varepsilon(t))$ is absolutely continuous on $[0, T]$ and

$$\begin{aligned} \dot{E}(\varepsilon)(t) &= \langle D\mathcal{W}(\varepsilon)(\nabla v^\varepsilon(t)), \nabla \dot{\phi}^\varepsilon(t) \rangle \\ &\quad - \langle D\mathcal{F}(\varepsilon)(t)(v^\varepsilon(t)), \dot{\phi}^\varepsilon(t) \rangle - \dot{\mathcal{F}}(\varepsilon)(t)(v^\varepsilon(t)) \end{aligned}$$

$$-\langle D\mathcal{G}(\varepsilon)(t)(v^\varepsilon(t)), \dot{\phi}^\varepsilon(t) \rangle - \dot{\mathcal{G}}(\varepsilon)(t)(v^\varepsilon(t)).$$

The following existence result has been proven in [8].

Theorem 1. *Let $K_0^\varepsilon \in \mathcal{R}(\overline{\Omega_B^\varepsilon})$ and $v_0^\varepsilon \in AD^\varepsilon(\phi^\varepsilon(0), K_0^\varepsilon)$ such that*

$$\mathcal{E}(\varepsilon)(0)(v_0^\varepsilon, K_0^\varepsilon) \leq \mathcal{E}(\varepsilon)(0)(v', K')$$

for every $K' \in \mathcal{R}(\overline{\Omega_B^\varepsilon})$ with $K_0^\varepsilon \widetilde{\subset} K'$, and every $v' \in AD^\varepsilon(\phi^\varepsilon(0), K')$. Then there exists a quasi-static evolution $t \mapsto (v^\varepsilon(t), K^\varepsilon(t))$ with boundary deformation $\phi^\varepsilon(t)$ such that $(v^\varepsilon(0), K^\varepsilon(0)) = (v_0^\varepsilon, K_0^\varepsilon)$.

3 The Rescaled Configuration

Our goal is to perform an asymptotic analysis of the quasi-static evolution as the thickness of the film $\varepsilon \rightarrow 0$. As usual in dimension reduction problems (see e.g. [6, 15]) we rescale the problem into an equivalent one with the advantage of being stated over a fixed domain.

Before doing this we shall make some assumptions on the initial crack. We assume that it is compatible with the geometry of the problem, i.e., that $K_0^\varepsilon = \gamma_0 \times (-\varepsilon + \varepsilon\eta, \varepsilon - \varepsilon\eta)$, for some countably \mathcal{H}^1 -rectifiable set $\gamma_0 \subset \omega$.

We now define $\Omega := \Omega^1$, $\Omega_B := \Omega_B^1$, $\partial_D \Omega := \partial_D \Omega^1$ and $\partial_N \Omega := \partial_N \Omega^1$. For $x \in \Omega$, we denote by $x_\alpha := (x_1, x_2) \in \omega$ the in-plane variable. We set

$$\begin{aligned} \psi^\varepsilon(t, x_\alpha, x_3) &:= \phi^\varepsilon(t, x_\alpha, \varepsilon x_3), \\ u_0^\varepsilon(x_\alpha, x_3) &:= v_0^\varepsilon(x_\alpha, \varepsilon x_3), \\ u^\varepsilon(t)(x_\alpha, x_3) &:= v^\varepsilon(t)(x_\alpha, \varepsilon x_3), \\ \Gamma^\varepsilon(t) &:= \{(x_\alpha, x_3) \in \overline{\Omega_B} : (x_\alpha, \varepsilon x_3) \in K^\varepsilon(t)\}. \end{aligned}$$

Changing variables in (1), (2), (4) and (8) leads to

$$\begin{aligned} \mathcal{W}(\varepsilon)(\nabla v^\varepsilon(t)) &= \varepsilon \int_{\Omega} W \left(\nabla_\alpha u^\varepsilon(t) \left| \frac{1}{\varepsilon} \nabla_3 u^\varepsilon(t) \right. \right) dx =: \varepsilon \mathcal{W}^\varepsilon(\nabla u^\varepsilon(t)), \\ \mathcal{K}(\varepsilon)(K^\varepsilon(t)) &= \varepsilon \int_{\Gamma^\varepsilon(t)} \left| \left((v_{\Gamma^\varepsilon(t)})_\alpha \left| \frac{1}{\varepsilon} (v_{\Gamma^\varepsilon(t)})_3 \right. \right) \right| d\mathcal{H}^2 =: \varepsilon \mathcal{K}^\varepsilon(\Gamma^\varepsilon(t)), \\ \mathcal{F}(\varepsilon)(t)(v^\varepsilon(t)) &= \varepsilon \int_{\Omega} F(t, u^\varepsilon(t)) dx =: \varepsilon \mathcal{F}(t)(u^\varepsilon(t)), \\ \mathcal{G}(\varepsilon)(t)(v^\varepsilon(t)) &= \varepsilon \int_{\partial_N \Omega} G(t, u^\varepsilon(t)) d\mathcal{H}^2(x) =: \varepsilon \mathcal{G}(t)(u^\varepsilon(t)), \end{aligned}$$

where ∇_α (resp. ∇_3) denotes the approximate gradient with respect to x_α (resp. x_3), and $\nu_{\Gamma^\varepsilon(t)} = ((\nu_{\Gamma^\varepsilon(t)})_\alpha, (\nu_{\Gamma^\varepsilon(t)})_3)$ is the normal to $\Gamma^\varepsilon(t)$.

As in Section 2.3, we define the functionals $\tilde{\mathcal{F}}(t)$ and $\dot{\mathcal{G}}(t)$ so that there hold the analogue of (5), (6), (9) and (10) with \mathcal{F} , $\tilde{\mathcal{F}}$, \mathcal{G} and $\dot{\mathcal{G}}$ instead of $\mathcal{F}(\varepsilon)$, $\tilde{\mathcal{F}}(\varepsilon)$, $\mathcal{G}(\varepsilon)$ and $\dot{\mathcal{G}}(\varepsilon)$. Note that \mathcal{G} , $\dot{\mathcal{G}}$, $D\mathcal{G}$, $D\dot{\mathcal{G}}$, \mathcal{F} , $\tilde{\mathcal{F}}$, $D\mathcal{F}$ and $D\tilde{\mathcal{F}}$ satisfy analogue growth and coercivity conditions as (7)–(11) with the same exponents, and with coercivity and growth constants/functions independent of ε .

We write $\mathcal{E}^\varepsilon(t)$ for the rescaled total energy at time t . Then, we deduce that the mapping $t \mapsto (u^\varepsilon(t), \Gamma^\varepsilon(t))$ from $[0, T]$ to $GSBV_q^p(\Omega; \mathbf{R}^3) \times \mathcal{R}(\overline{\Omega_B})$ satisfies the following properties:

- (i) *Global stability*: for all $t \in [0, T]$, we have $u^\varepsilon(t) \in AD^1(\psi^\varepsilon(t), \Gamma^\varepsilon(t))$ and

$$\begin{aligned} \mathcal{E}^\varepsilon(t)(u^\varepsilon(t), \Gamma^\varepsilon(t)) &= \min\{\mathcal{E}^\varepsilon(u', \Gamma') : \Gamma' \in \mathcal{R}(\overline{\Omega_B}), \Gamma^\varepsilon(t) \tilde{\subset} \Gamma' \\ &\text{and } u' \in AD^1(\psi^\varepsilon(t), \Gamma')\}. \end{aligned}$$

- (ii) *Irreversibility*: $\Gamma^\varepsilon(s) \tilde{\subset} \Gamma^\varepsilon(t)$ whenever $s \leq t$.

- (iii) *Energy balance*: the mapping $t \mapsto E^\varepsilon(t) := \mathcal{E}^\varepsilon(t)(u^\varepsilon(t), \Gamma^\varepsilon(t))$ is absolutely continuous on $[0, T]$ and

$$\begin{aligned} \dot{E}^\varepsilon(t) &= \langle D\mathcal{W}^\varepsilon(\nabla u^\varepsilon(t)), \nabla \dot{\psi}^\varepsilon(t) \rangle \\ &\quad - \langle D\mathcal{F}(t)(u^\varepsilon(t)), \dot{\psi}^\varepsilon(t) \rangle - \tilde{\mathcal{F}}(t)(u^\varepsilon(t)) \\ &\quad - \langle D\mathcal{G}(t)(u^\varepsilon(t)), \dot{\psi}^\varepsilon(t) \rangle - \dot{\mathcal{G}}(t)(u^\varepsilon(t)). \end{aligned}$$

4 Analysis of Static Problem by Γ -Convergence

Before going to the study of the quasi-static problem, we will discuss the asymptotic behavior of the total energy as $\varepsilon \rightarrow 0$ thanks to a Γ -convergence method. Since the work of external forces $\mathcal{F} + \mathcal{G}$ corresponds to a continuous perturbation of the sum of the bulk and surface energies, we will not take it into account. We will actually study a weak formulation of the problem replacing the crack by the jump set of the deformation field. Indeed, Let us define $\mathcal{I}_\varepsilon : L^1(\Omega; \mathbf{R}^3) \rightarrow [0, +\infty]$ by

$$\mathcal{I}_\varepsilon(u) := \int_\Omega W\left(\nabla_\alpha u \left| \frac{1}{\varepsilon} \nabla_3 u \right.\right) dx + \int_{S_u} \left| \left((v_u)_\alpha \left| \frac{1}{\varepsilon} (v_u)_3 \right. \right) \right| d\mathcal{H}^2$$

if $u \in GSBV^p(\Omega; \mathbf{R}^3)$, and $+\infty$ if $u \in L^1(\Omega; \mathbf{R}^3) \setminus GSBV^p(\Omega; \mathbf{R}^3)$. Then, the following Γ -convergence result holds:

Theorem 2. *Let ω be a bounded open subset of \mathbf{R}^2 and $W : \mathbf{R}^{3 \times 3} \rightarrow \mathbf{R}$ be a continuous function satisfying (3). Then the functional \mathcal{I}_ε Γ -converges for the strong $L^1(\Omega; \mathbf{R}^3)$ -topology to $\mathcal{I} : L^1(\Omega; \mathbf{R}^3) \rightarrow [0, +\infty]$ defined by*

$$\mathcal{I}(u) := \begin{cases} 2 \int_{\omega} QW_0(\nabla_{\alpha} u) dx_{\alpha} + 2\mathcal{H}^1(S_u) & \text{if } u \in GSBV^p(\omega; \mathbf{R}^3), \\ +\infty & \text{otherwise,} \end{cases}$$

where $W_0(\bar{\xi}) := \inf\{W(\bar{\xi}|z) : z \in \mathbf{R}^3\}$ for every $\bar{\xi} \in \mathbf{R}^{3 \times 2}$, and QW_0 is the quasiconvexification of W_0 .

This result has been proven in [2] (see also [3]) in a SBV^p framework, and one can notice that much easier arguments lead to the analogue in $GSBV^p$ as stated in Theorem 2. Indeed, in $GSBV^p$, there is no lack of compactness and it is not necessary to appeal to a truncation argument as in [2, lemma 3.3]. It follows immediately from the $GSBV$ -compactness theorem [1, theorem 4.36] that any minimizing sequence $(u_{\varepsilon}) \subset GSBV^p(\Omega; \mathbf{R}^3)$ with uniformly bounded energy is relatively compact in $GSBV^p(\Omega; \mathbf{R}^3)$, and that any accumulation point is independent of x_3 (we identify those functions to $GSBV^p(\omega; \mathbf{R}^3)$). The proof of the lower bound is exactly the same than [2, lemma 3.9] using the lower semicontinuity result in $GSBV^p$ (see e.g. [8, theorem 2.8]). The construction of a recovery (\bar{u}_{ε}) can be performed as in [15]: it suffices to take $\bar{u}_{\varepsilon}(x_{\alpha}, x_3) := u(x_{\alpha}) + \varepsilon x_3 b_{\varepsilon}(x_{\alpha})$ for some suitable function $b_{\varepsilon} \in C_c^{\infty}(\omega; \mathbf{R}^3)$, and then we appeal to a classical relaxation result in $GSBV^p$. We also refer to [5] for an alternative proof using a singular perturbation argument.

5 Analysis of the Quasi-Static Problem

In view of Theorem 2, one can guess that the 3D quasi-static evolution – whose existence is ensured by Theorem 1 – will converge in a certain sense to a 2D quasi-static evolution associated to the Γ -limit model.

We assume that $\psi^{\varepsilon} \rightarrow \psi$ in $W^{1,1}([0, T]; W^{1,p}(\Omega; \mathbf{R}^3) \cap L^q(\Omega; \mathbf{R}^3))$ and that the sequence $((1/\varepsilon)\nabla_3 \psi^{\varepsilon})$ is strongly converging in $W^{1,1}([0, T]; L^p(\Omega; \mathbf{R}^3))$. In particular, the limit function $\psi \in W^{1,1}([0, T]; W^{1,p}(\omega; \mathbf{R}^3) \cap L^q(\omega; \mathbf{R}^3))$ is independent of x_3 .

We first derive some compactness of $(u^{\varepsilon}(t), \Gamma^{\varepsilon}(t))$. Indeed taking $(\psi^{\varepsilon}(t), \Gamma^{\varepsilon}(t))$ as competitor in the minimality, and using the growth and coercivity properties satisfied by the functionals $\mathcal{W}_{\varepsilon}$, \mathcal{F} and \mathcal{G} implies that the sequence of approximate scaled gradients $(\nabla_{\alpha} u^{\varepsilon}(t) | (1/\varepsilon)\nabla_3 u^{\varepsilon}(t))$ is bounded in $L^p(\Omega; \mathbf{R}^{3 \times 3})$, and the sequence $(u^{\varepsilon}(t))$ is bounded in $L^q(\Omega; \mathbf{R}^3) \cap L^q(\Omega; \mathbf{R}^3)$. Moreover, since $u^{\varepsilon}(t) \in W^{1,p}(\Omega \setminus \overline{\Omega_B}; \mathbf{R}^3)$, the trace theorem and the choice of the exponent r ensures that $(u^{\varepsilon}(t))$ is compact in $L^r(\partial_N \Omega; \mathbf{R}^3)$. Then we use the energy balance together with the growth and coercivity conditions satisfied by $D\mathcal{F}(t)$, $D\mathcal{G}(t)$, $\dot{\mathcal{F}}(t)$ and $\dot{\mathcal{G}}(t)$ to ensure that

$$\sup_{\varepsilon > 0} \int_{\Gamma^{\varepsilon}(t)} \left| \left((v_{\Gamma^{\varepsilon}(t)})_{\alpha} \left| \frac{1}{\varepsilon} (v_{\Gamma^{\varepsilon}(t)})_3 \right| \right) \right| d\mathcal{H}^2 < +\infty.$$

At this step, we are in position to use a mean convergence for rectifiable sets introduced in [2], very close to the σ^p -convergence in [8, 9]:

Definition 1. Let $\varepsilon_n \searrow 0^+$, $\Gamma_n \subset \Omega$ be a sequence of countably \mathcal{H}^2 -rectifiable sets, and $\gamma \subset \omega$ be a countably \mathcal{H}^1 -rectifiable set. We say that Γ_n converges to γ in Ω if

$$\int_{\Gamma_n} \left| \left((v_{\Gamma_n})_\alpha \left| \frac{1}{\varepsilon_n} (v_{\Gamma_n})_3 \right| \right) \right| d\mathcal{H}^2 \leq C,$$

and the following properties hold:

- (a) if $u_k \rightharpoonup u$ in $SBV^p(\Omega)$, $S_{u_k} \tilde{\subset} \Gamma_{n_k}$ and $\int_{\Omega} |(\nabla_\alpha u_k|(1/\varepsilon_{n_k})\nabla_3 u_k)|^p dx \leq C$, for some subsequence $(\varepsilon_{n_k}) \subset (\varepsilon_n)$, then $u \in SBV^p(\omega)$ and $S_u \tilde{\subset} \gamma$;
- (b) there exist a function $u \in SBV^p(\omega)$ and a sequence $(u_n) \subset SBV^p(\Omega)$ such that $u_n \rightharpoonup u$ in $SBV^p(\Omega)$, $S_{u_n} \tilde{\subset} \Gamma_n$, $\int_{\Omega} |(\nabla_\alpha u_n|(1/\varepsilon_n)\nabla_3 u_n)|^p dx \leq C$, and $S_u \tilde{=} \gamma$.

Then, using [2, proposition 4.3], one can extract a subsequence (ε_n) (independently of t) and find a countably \mathcal{H}^1 -rectifiable set $\gamma(t)$ increasing with respect to t such that $\Gamma^{\varepsilon_n}(t)$ converges to $\gamma(t)$ in sense of Definition 1. Note that it is possible to prove that $\gamma(0) = \gamma_0$. Moreover the estimates we have on the sequence $(u^{\varepsilon_n}(t))$ allow us to apply the $GSBV$ -Compactness Theorem (see [1, theorem 4.36]) which ensures, for each $t \in [0, T]$, the existence of a t -dependent subsequence $(\varepsilon_{n_t}) \subset (\varepsilon_n)$ such that $u^{\varepsilon_{n_t}}(t) \rightharpoonup u(t)$ in $GSBV_q^p(\Omega; \mathbf{R}^3)$. Moreover the limit deformation $u \in GSBV_q^p(\omega; \mathbf{R}^3)$, i.e., it is independent of x_3 .

We claim that the pair $(u(t), \gamma(t))$ is a quasi-static evolution associated to the Γ -limit model. We already proved the irreversibility condition. To show the minimality property we use the following jump transfer theorem whose proof can be obtained exactly as in [2, theorem 4.4], using [8, theorem 5.3] instead of [11, theorem 2.1]. Let $\omega' \subset \mathbf{R}^2$ a bounded open set containing $\overline{\omega}$, and define $\Omega' := \omega' \times (-1, 1)$.

Theorem 3. Let $\Gamma_n \in \mathcal{R}(\overline{\Omega_B})$ be a sequence of countably \mathcal{H}^2 -rectifiable sets converging to γ in the sense of Definition 1. Then, for every $v \in GSBV_q^p(\omega'; \mathbf{R}^3)$, there exists a sequence $(v_n) \subset GSBV_q^p(\Omega'; \mathbf{R}^3)$ such that $v_n = v$ a.e. on $\Omega' \setminus \overline{\Omega}$,

- $v_n \rightarrow v$ in $L^q(\Omega'; \mathbf{R}^3)$,
- $\left(\nabla_\alpha v_n \left| \frac{1}{\varepsilon_n} \nabla_3 v_n \right| \right) \rightarrow (\nabla_\alpha v|0)$ in $L^p(\Omega'; \mathbf{R}^{3 \times 3})$,
- $\limsup_{n \rightarrow +\infty} \int_{S_{v_n} \setminus \Gamma_n} \left| \left((v_{v_n})_\alpha \left| \frac{1}{\varepsilon_n} (v_{v_n})_3 \right| \right) \right| d\mathcal{H}^2 \leq 2\mathcal{H}^1(S_v \setminus \gamma).$

Arguing exactly as in the proof of [2, lemma 5.5] and using the upper semicontinuity property of $\mathcal{F}(t)$ together with the continuity of $\mathcal{G}(t)$ (which comes from the trace theorem in $W^{1,p}$ and the choice of the exponent r), one can show that for every $t \in [0, T]$, $u(t)$ minimizes

$$v \mapsto 2 \int_{\omega} QW_0(\nabla_\alpha v) dx_\alpha + 2\mathcal{H}^1(S_v \setminus \gamma(t)) - 2\mathcal{F}(t)(v) - 2\mathcal{G}(t)(v),$$

among $\{v \in GSBV_q^p(\omega; \mathbf{R}^3) : v = \psi(t) \text{ } \mathcal{H}^1\text{-a.e. on } \partial\omega\}$. Moreover one has convergence of the bulk energy (for the sequence ε_n)

$$\int_{\Omega} W \left(\nabla_{\alpha} u^{\varepsilon_n}(t) \middle| \frac{1}{\varepsilon_n} \nabla_3 u^{\varepsilon_n}(t) \right) dx \rightarrow 2 \int_{\omega} QW_0(\nabla_{\alpha} u(t)) dx_{\alpha},$$

as well as weak convergence of the stress (for the subsequence ε_{n_i})

$$DW \left(\nabla_{\alpha} u^{\varepsilon_{n_i}}(t) \middle| \frac{1}{\varepsilon_{n_i}} \nabla_3 u^{\varepsilon_{n_i}}(t) \right) \rightharpoonup (D(QW_0)(\nabla_{\alpha} u(t)|0)) \quad \text{in } L^{p'}(\Omega; \mathbf{R}^{3 \times 3}) \quad (12)$$

at every time. Remark that by [2, proposition 4.7], the function QW_0 is of class \mathcal{C}^1 . For every $v \in GSBV_q^p(\omega; \mathbf{R}^3)$ such that $v = \psi(t)$ \mathcal{H}^1 -a.e. on $\partial\omega$, and every countably \mathcal{H}^1 -rectifiable set $\gamma \subset \omega$, we define

$$\mathcal{E}(t)(v, \gamma) = 2 \int_{\omega} QW_0(\nabla_{\alpha} v) dx_{\alpha} + 2\mathcal{H}^1(\gamma) - 2\mathcal{F}(t)(v) - 2\mathcal{G}(t)(v).$$

The minimality property proven above exactly says that

$$\begin{aligned} \mathcal{E}(t)(u(t), \gamma(t)) &= \min \{ \mathcal{E}(t)(v, \gamma) : \gamma \text{ countably } \mathcal{H}^1\text{-rectifiable set in } \omega, \\ &\quad v \in GSBV_q^p(\omega; \mathbf{R}^3) \text{ such that } v = \psi(t) \text{ } \mathcal{H}^1\text{-a.e. on } \partial\omega \} \end{aligned}$$

(see e.g. [2, remark 5.4]). Moreover, since we have reduced dimension, the functional $\mathcal{G}(t)$ becomes a bulk force as well as $\mathcal{F}(t)$.

It remains to prove the energy balance. Arguing word for word as in [8], approximating Bochner integrals by suitable Riemann sums, one can show that

$$\begin{aligned} \mathcal{E}(t)(u(t), \gamma(t)) &\geq \mathcal{E}(0)(u(0), \gamma(0)) + 2 \int_0^t \left(\langle D\mathcal{W}_0(\nabla_{\alpha} u(s)), \nabla_{\alpha} \dot{\psi}(s) \rangle \right. \\ &\quad \left. - \langle D\mathcal{F}(t)(u(s)), \dot{\psi}(s) \rangle - \dot{\mathcal{F}}(s)(u(s)) \right. \\ &\quad \left. - \langle D\mathcal{G}(s)(u(s)), \dot{\psi}(s) \rangle - \dot{\mathcal{G}}(s)(u(s)) \right) ds, \end{aligned}$$

where $\mathcal{W}_0 : L^p(\omega; \mathbf{R}^{3 \times 2}) \rightarrow [0, +\infty)$ is defined by $\mathcal{W}_0(\Phi) := \int_{\omega} QW_0(\Phi(x)) dx_{\alpha}$. By [2, Proposition 4.7], we deduce that \mathcal{W}_0 is of class \mathcal{C}^1 with differential $D\mathcal{W}_0 : L^p(\omega; \mathbf{R}^{3 \times 2}) \rightarrow L^{p'}(\omega; \mathbf{R}^{3 \times 2})$ given by

$$\langle D\mathcal{W}_0(\Phi), \Psi \rangle = \int_{\omega} D(QW_0)(\Phi(x)) : \Psi(x) dx_{\alpha}.$$

We prove the other inequality exactly as in [2, lemma 5.8], using the upper semi-continuity property of the functional $\mathcal{F}(t)$ and the weak convergence of the stresses (12) already mentioned above. We also deduce the convergence of the surface energy (for the sequence ε_n):

$$\int_{\Gamma^{\varepsilon_n}(t)} \left| \left((v_{\Gamma^{\varepsilon_n}(t)})_{\alpha} \middle| \frac{1}{\varepsilon_n} (v_{\Gamma^{\varepsilon_n}(t)})_3 \right) \right| d\mathcal{H}^2 \rightarrow 2\mathcal{H}^1(\gamma(t)).$$

In conclusion, we have proven the following result which states that any 3D quasi-static crack evolution converges to a 2D quasi-static evolution associated to the Γ -limit model:

Theorem 4. *There exist a two-dimensional quasi-static evolution $t \mapsto (u(t), \gamma(t))$ relative to the boundary data $\psi(t)$ for the Γ -limit model, and a sequence $\varepsilon_n \searrow 0^+$ such that for any $t \in [0, T]$,*

- $\Gamma^{\varepsilon_n}(t)$ converges to $\gamma(t)$ in sense of Definition 1;
- $u^{\varepsilon_n}(t) \rightharpoonup u(t)$ in $GSBV_q^p(\Omega; \mathbf{R}^3)$ for some t -dependent subsequence $(\varepsilon_{n_t}) \subset (\varepsilon_n)$;
- the total energy $\mathcal{E}^{\varepsilon_n}(t)$ converges to $\mathcal{E}(t)$, and more precisely

$$\int_{\Omega} W \left(\nabla_{\alpha} u^{\varepsilon_n}(t) \left| \frac{1}{\varepsilon_n} \nabla_3 u^{\varepsilon_n}(t) \right. \right) dx \rightarrow 2 \int_{\omega} QW_0(\nabla_{\alpha} u(t)) dx_{\alpha},$$

$$\int_{\Gamma^{\varepsilon_n}(t)} \left| \left((v_{\Gamma^{\varepsilon_n}(t)})_{\alpha} \left| \frac{1}{\varepsilon_n} (v_{\Gamma^{\varepsilon_n}(t)})_3 \right. \right) \right| d\mathcal{H}^2 \rightarrow 2\mathcal{H}^1(\gamma(t)).$$

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