Mathematical modeling of linearly piezoelectric slender rods

Thibaut Weller, Christian Licht

MATHEMATICAL MODELING OF LINEARLY PIEZOELECTRIC SLENDER RODS

Thibaut Weller, Christian Licht
Laboratoire de Mécanique et Génie Civil, UMR 5508 CNRS - UMII, Université Montpellier II, c.c. 48, Place Eugène Bataillon, 34095 Montpellier cedex 5, France

Summary
The piezoelectric thin plate modeling already derived by the authors is extended to rod-like structures. Two models corresponding to sensor or actuator behavior are obtained. The conditions of existence of non local terms in the limit models are discussed.

Introduction

The mathematical modeling of elastic thin plates or slender rods through asymptotic analysis has become classical : the thickness or the diameter is assigned to a role of parameter whose aim is to tend to zero ([1]-[3]). We have extended this method to linear piezoelectric and electromagneto-elastic plates ([4], [5]). But, because beam modeling requires to condense on a line the properties of slender 3D objects having one dimension prevailing on the others, it is more challenging than plate modeling. Depending on the boundary conditions, two limit models, corresponding to *sensors* or *actuators*, appear. They involve a greater number of state variables than the couple (displacement/electrical potential) of the genuine 3D physical problem. We therefore exhibit reduced formulations where the number of variables drops to one or two, one reduced problem being purely mechanical ! We discuss the conditions for which the elimination of additional variables leads to non standard equations involving non local terms.

Setting the problem

The reference configuration of a linearly piezoelectric slender rod is the closure in $\mathbb{R}^3$ of the set $\Omega^\varepsilon := \varepsilon\omega \times (0, L)$ where ω is a bounded domain of $\mathbb{R}^2$ with Lipschitz boundary $\partial\omega$, L is the length of the rod and ε a small positive number. The Euclidean physical space whose orthonormal basis is assumed to be the principal frame of inertia of the rod is identified with $\mathbb{R}^3$. Let S^N the set of $N \times N$ symmetric matrices and $\mathcal{H} := S^3 \times \mathbb{R}^3$. Greek coordinate indexes will run in $\{1, 2\}$ and Latin ones in $\{1, 2, 3\}$; for all $\xi = (\xi_1, \xi_2, \xi_3)$ of $\mathbb{R}^3$, $\hat{\xi}$, ξ^R stand for (ξ_1, ξ_2) and $(-\xi_2, \xi_1)$. Let $\Gamma^\varepsilon_{\text{lat}} := \varepsilon\partial\omega \times (0, L)$, $\Gamma^\varepsilon_0 := \varepsilon\omega \times \{0\}$, $\Gamma^\varepsilon_L := \varepsilon\omega \times \{L\}$, $\Gamma_{0,L} := \Gamma_0 \cup \Gamma_L$, and two partitions of $\partial\Omega^\varepsilon$: $(\Gamma^\varepsilon_{mD}, \Gamma^\varepsilon_{mN})$, $(\Gamma^\varepsilon_{eD}, \Gamma^\varepsilon_{eN})$ with Γ^ε_{mD}, Γ^ε_{eD} of strictly positive surface measures. The rod is clamped along Γ^ε_{mD} and at an electrical potential φ^ε_0 on Γ^ε_{eD}. It is subjected to body forces f^ε in Ω^ε, surface forces g^ε on Γ^ε_{mN}, electrical loading w^ε on Γ^ε_{eN}. We denote the outward unit normal to $\partial\Omega^\varepsilon$ by n^ε. The piezoelectric state $s^\varepsilon := (u^\varepsilon, \varphi^\varepsilon)$ at equilibrium satisfies :

$$\mathcal{P}(\Omega^\varepsilon) \ \begin{cases} \operatorname{div} \sigma^\varepsilon + f^\varepsilon = 0 \text{ in } \Omega^\varepsilon, \operatorname{div} D^\varepsilon = 0 \text{ in } \Omega^\varepsilon, \ \sigma^\varepsilon n^\varepsilon = g^\varepsilon \text{ on } \Gamma^\varepsilon_{mN}, \ u^\varepsilon = 0 \text{ on } \Gamma^\varepsilon_{mD}, \\ D^\varepsilon \cdot n^\varepsilon = w^\varepsilon \text{ on } \Gamma^\varepsilon_{eN}, \ \varphi^\varepsilon = \varphi^\varepsilon_0 \text{ on } \Gamma^\varepsilon_{eD}, (\sigma^\varepsilon, D^\varepsilon) = M^\varepsilon(x)(e(u^\varepsilon), \nabla\varphi^\varepsilon) \text{ in } \Omega^\varepsilon, \end{cases}$$

where u^ε, φ^ε, σ^ε, $e(u^\varepsilon)$ and D^ε are respectively the displacement, the electric potential field, the stress tensor, the tensor of small strains and the electrical displacement. The operator M^ε stands for the classical piezoelectric constitutive equations, *i.e.* $\sigma^\varepsilon = M^\varepsilon_{mm} e(u^\varepsilon) - M^\varepsilon_{me} \nabla\varphi^\varepsilon$, $D^\varepsilon = M^{\varepsilon T}_{me} e(u^\varepsilon) + M^\varepsilon_{ee} \nabla\varphi^\varepsilon$, where M^ε_{mm}, M^ε_{me} and M^ε_{ee} are respectively the elastic, piezoelectric and dielectric tensors while the superscript T denotes the transposition. Of course, M^ε is not symmetric but under realistic assumption of boundedness of M^ε and of uniform ellipticity of M^ε_{mm}, M^ε_{ee}, the physical problem $\mathcal{P}(\Omega^\varepsilon)$ has a unique weak solution.
Our piezoelectric rod models will be obtained by studying the limit behavior of s^ε when $\varepsilon \to 0$.

Our asymptotic models

As in [4], we will show that two different limit behaviors, indexed by $p = 1$ or 2, appear according to the type of electric boundary conditions and to the magnitude of the electrical external loading. In the sequel, for all $G \subset \mathbb{R}^N$, $H^1_g(G)$ denotes the subset of the Sobolev space $H^1(G)$ whose elements vanish on $g \subset \partial G$, except $H^1_m(\omega)$ which is the set of the elements of $H^1(\omega)$ with zero average on ω. The process is as follows : first we come down to a fixed open set $\Omega := \omega \times (0, L)$ through the bijection $x = (x_1, x_2, x_3) \in \overline{\Omega} \mapsto x^\varepsilon = \pi^\varepsilon(x) = (\varepsilon x_1, \varepsilon x_2, x_3) \in \overline{\Omega}^\varepsilon$. We drop the index ε for the images by $(\pi^\varepsilon)^{-1}$ of the geometric sets defined *supra*. We also assume that the electro-elastic coefficients and the loading satisfy :

$$\begin{cases} M^\varepsilon(\pi^\varepsilon x) =: M(x), \ M \in L^\infty(\Omega, \operatorname{Lin}(\mathcal{H})), \ \exists \kappa > 0 \ : \ M(x)\, h \cdot h \geq \kappa \,|\, h\,|^2_{\mathcal{H}}, \ \forall h \in \mathcal{H}, \ a.e. \ x \in \Omega, \\ \hat{f}^\varepsilon(\pi^\varepsilon x) = \varepsilon^2 \hat{f}(x), \ f^\varepsilon_3(\pi^\varepsilon x) = \varepsilon f_3(x), \ \forall x \in \Omega, \hat{g}^\varepsilon(\pi^\varepsilon x) = \varepsilon^2 \hat{g}(x), \ g^\varepsilon_3(\pi^\varepsilon x) = \varepsilon g_3(x), \ \forall x \in \Gamma_{mN} \cap \Gamma_{0,L}, \\ \hat{g}^\varepsilon(\pi^\varepsilon x) = \varepsilon^3 \hat{g}(x), \ g^\varepsilon_3(\pi^\varepsilon x) = \varepsilon^2 g_3(x), \ \forall x \in \Gamma_{mN} \cap \Gamma_{\text{lat}}, \ \varphi^\varepsilon_0(\pi^\varepsilon x) = \varepsilon^p \varphi_0(x), \ \forall x \in \Gamma_{eD}, \\ w^\varepsilon(\pi^\varepsilon x) = \varepsilon^{2-p} d(x), \ \forall x \in \Gamma_{eN} \cap \Gamma_{0,L}, \ w^\varepsilon(\pi^\varepsilon x) = \varepsilon^{3-p} d(x), \ \forall x \in \Gamma_{eN} \cap \Gamma_{\text{lat}}, \end{cases}$$

where (f, g, w) is an element (independent of ε) of $L^2(\Omega)^3 \times L^2(\Gamma_{mN})^3 \times L^2(\Gamma_{eN})$. We also suppose that φ_0 has an $H^1(\Omega)$ extension into Ω still denoted by φ_0 and :

$$\begin{cases} \text{if } p=1 : \text{the extension of } \varphi_0 \text{ into } \Omega \text{ does not depend on } \widehat{x} \text{ and } \Gamma_{eD} \subset \Gamma_{0,L}. \\ \text{if } p=2 : \exists\, \gamma_{eD} \subset \gamma \text{ with positive length such that } (\gamma \setminus \gamma_{eD}) \times (0,L) \subset \Gamma_{eN} \text{ and either } \Gamma_{eN} \cap \Gamma_{0,L} = \emptyset \\ \qquad\qquad \text{or } w=0 \text{ on } \Gamma_{eN} \cap \Gamma_{0,L}. \end{cases}$$

Next, with the true physical state $s^\varepsilon = (u^\varepsilon, \varphi^\varepsilon)$ defined on Ω^ε, we associate a *scaled* piezoelectric state $s_p(\varepsilon) := (u_p(\varepsilon), \varphi_p(\varepsilon))$ defined by $\hat{u}^\varepsilon(x^\varepsilon) = (\hat{u}_p(\varepsilon))(x)$, $u_3^\varepsilon(x^\varepsilon) = \varepsilon(u_p(\varepsilon))_3(x)$, $\varphi^\varepsilon(x^\varepsilon) = \varepsilon^p\, \varphi_p(\varepsilon)(x)$, $\forall\, x = \pi^\varepsilon(x) \in \Omega$, so that $s_p(\varepsilon)$ is the unique solution of the following mathematical problem, equivalent to the genuine physical one :

$$\mathcal{P}(\varepsilon,\Omega)_p : s_p(\varepsilon) \in (0,\varphi_0) + \mathbf{V}; m_p(\varepsilon)(s_p(\varepsilon), r) = L(r),\ \forall\, r \in \mathbf{V} := \{r = (v,\psi) \in H^1_{\Gamma_{mD}}(\Omega)^3 \times H^1_{\Gamma_{eD}}(\Omega)\},$$

with

$$\begin{cases} m_p(\varepsilon)(s,r) := \int_\Omega M(x)\, k_p(\varepsilon,s)\cdot k_p(\varepsilon,r)\, dx,\ k_p(\varepsilon,r) := k_p(\varepsilon,(v,\psi)) = (e(\varepsilon,v), \nabla_p(\varepsilon,\psi)), \\ e(\varepsilon,v)_{\alpha\beta} := \varepsilon^{-2} e(v)_{\alpha\beta},\ e(\varepsilon,v)_{\alpha 3} := \varepsilon^{-1} e(v)_{\alpha 3},\ e(\varepsilon,v)_{33} := e(v)_{33}, \\ 2e_{ij} := \partial_i v_j + \partial_j v_i,\ \nabla_p(\varepsilon,\phi)_\alpha := \varepsilon^{p-2}\partial_\alpha \varphi,\ \nabla_p(\varepsilon,\varphi)_3 := \varepsilon^{p-1}\partial_3\varphi, \\ L(r) := L(v,\psi) = \int_\Omega f\cdot v\, dx + \int_{\Gamma_{mN}} g\cdot v\, dx + \int_{\Gamma_{eN}} w\psi\, dx. \end{cases}$$

Finding the limit problems is a little bit more difficult than in the case of plates because the limit problems involve a greater number of state variables : $\widetilde{s}_1 = (v,w,\psi)$ and $\widetilde{s}_2 = (v,w)$ are added to the initial state variable $s = (u,\phi)$ and we let $\mathsf{s}_p = (s, \widetilde{s}_p)$; they belong to the following spaces, some of them being classical in rod theory ([2]) :

$$\begin{cases} D_m := \{x_3 \in \{0,L\}; \omega \times \{x_3\} \supset \Gamma_{mD} \cap \Gamma_{0,L}\}, D_e := \{x_3 \in \{0,L\}; \omega \times \{x_3\} \supset \Gamma_{eD} \cap \Gamma_{0,L}\}, \\ V_{BN}(\Omega) := \{u \in H^1_{\Gamma_{mD}}(\Omega)^3; e_{\alpha\beta}(u) = e_{\alpha 3}(u) = 0\} \\ R(\Omega) = \{v\ : \exists c \in H^1_{D_m}(0,L),\ \widehat{v}(x) = c(x_3)x^R, v_3 \in L^2(0,L;H^1_m(\omega))\}, \\ RD_2^\perp(\Omega) := \{w : \widehat{w} \in L^2(0,L;H^1_m(\omega)), w_3(x) = 0 \text{ and } \int_\omega x^R\cdot \widehat{w}(x)\, d\widehat{x} = 0, a.e.\ x_3 \in (0,L)\}, \\ \Phi_1 = \{\phi; \exists\, \varphi \in H^1_{D_e}(0,L) : \phi(x) = \varphi(x_3)\},\ \Phi_2 = L^2(0,L;H^1_{\gamma_e}(\omega)),\ \Psi_1 = L^2(0,L;H^1_m(\omega)), \\ V_1 := V_{BN}(\Omega)\times\Phi_1\times R(\Omega)\times RD_2^\perp(\Omega)\times\Psi_1,\ V_2 := V_{BN}(\Omega)\times\Phi_2\times R(\Omega)\times RD_2^\perp(\Omega). \end{cases}$$

If $\mathsf{s}_{1_0} := (0,\varphi_0,0,0,0)$, $\mathsf{s}_{2_0} := (0,\varphi_0,0,0)$ the limit problems read as

$$\overline{\mathsf{P}}(\Omega)_p : \overline{\mathsf{s}}_p \in \mathsf{s}_{p_0} + V_p; \int_\Omega M(x)\, \mathsf{k}_p(\overline{\mathsf{s}}_p)\cdot \mathsf{k}_p(\mathsf{s}'_p)\, dx = L(\mathsf{s}'_p),\ \forall\, \mathsf{s}'_p \in V_p,$$

$\mathsf{k}_1(\mathsf{s}_1), \mathsf{k}_2(\mathsf{s}_2) \in L^2(\Omega;\mathcal{H})$ being represented by $(e_{\alpha\beta}(w), e_{\alpha 3}(v), e_{33}(u), \widehat{\nabla}\psi, \frac{d\phi}{dx_3})$, $(e_{\alpha\beta}(w), e_{\alpha 3}(v), e_{33}(u), \widehat{\nabla}\phi, 0)$. We have the following convergence result :

When $\varepsilon \to 0$, the familly $(s_p(\varepsilon))_{\varepsilon>0}$ of the unique solutions of $\mathcal{P}(\varepsilon,\Omega)_p$ is such that $(s_p(\varepsilon), k(\varepsilon, s_p(\varepsilon))$ converges strongly in $V_{BN}(\Omega)\times\Phi_p\times L^2(\Omega;\mathcal{H})$ to $(\overline{s}_p, \mathsf{k}_p(\overline{\mathsf{s}}_p))$, where $\overline{\mathsf{s}}_p = (\overline{s}_p, \overline{\widetilde{s}}_p)$ is the unique solution of $\overline{\mathsf{P}}(\Omega)_p$.

Due to the V_p-ellipticity of the bilinear forms involved in $\overline{\mathcal{P}}(\Omega)_p$ the state variables $(\overline{v}_1, \overline{w}_1, \overline{\psi}_1)$ and $(\overline{\phi}_2, \overline{v}_2, \overline{w}_2)$ can be eliminated so that $\overline{s}_1$ and $\overline{u}_2$ solve *monodimensional* variational problems $\overline{\mathcal{P}}(\Omega)_p$ involving *non local* terms. They appear only under the conjunction of the three conditions : the crystalline class of the material is either 1 or (if $p=1$) m, its heterogeneity in the x_3 direction and the clamping condition on the two bases of the rod. Furthermore, when the crystalline class is neither 1 nor m and with a transversaly homogeneous material if the cardinal of D_m equals 1 or a homogeneous material if the cardinal of D_m equals 2, the local reduced constitutive equations involve a matrix $\widetilde{M_p} \in S^{3-p}$ whose entries depend only on those of M. When $p=1$, there is a piezoelectric decoupling for the classes 222, 32, $\overline{4}$, 422, $\overline{4}2m$, $\overline{6}$, 622, $\overline{6}m2$ and 23. Moreover, $\widetilde{M}_{1_{11}}$ is a purely mechanical entry, whereas $\widetilde{M}_{1_{22}}$ is purely dielectric for classes 32, 422, $\overline{6}$, 622 and $\overline{6}m2$. For all crystalline classes, $\widetilde{M}_{1_{11}} = \widetilde{M}_2$.

As usual ([4], [5]), we get piezoelectric slender rods models with cross section $\varepsilon\omega$ by a descaling of $\overline{\mathsf{P}}(\Omega)_p$ and $\overline{\mathcal{P}}(\Omega)_p$.

References

[1] P.G. Ciarlet, Mathematical Elasticity, Vol. II, North-Holland, 1997.

[2] F. Murat, A. Sili, Comportement asymptotique des solutions du système de l'élasticité linéarisée anisotrope hétérogène dans des cylindres minces, C. R. Acad. Sci. Paris, Sér. I 328 (1999) 179-184.

[3] F. Murat, A. Sili, Effets non locaux dans le passage 3d-1d en élasticité linéarisée anisotrope hétérogène, C. R. Acad. Sci. Paris, Sér. I 330 (2000) 745-750.

[4] T. Weller, C. Licht, Analyse asymptotique de plaques minces linéairement piézoélectriques, C. R. Acad. Sci. Paris, Sér. I 335 (2002) 309-314.

[5] T. Weller, C. Licht, Modeling of linearly electromagneto-elastic thin plates, C. R. Mecanique, 335 (2007) 201-206.