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Maximal Regularity for Evolution Equations Governed by Non-Autonomous Forms

Wolfgang Arendt, Dominik Dier, Hafida Laasri, El Maati Ouhabaz*

March 5, 2013

Abstract

We consider a non-autonomous evolutionary problem

$$\dot{u}(t) + \mathcal{A}(t)u(t) = f(t), \quad u(0) = u_0$$

where the operator $\mathcal{A}(t) : V \rightarrow V'$ is associated with a form $\mathfrak{a}(t, \cdot, \cdot) : V \times V \rightarrow \mathbb{R}$ and $u_0 \in V$. Our main concern is to prove well-posedness with maximal regularity which means the following. Given a Hilbert space H such that V is continuously and densely embedded into H and given $f \in L^2(0, T; H)$ we are interested in solutions $u \in H^1(0, T; H) \cap L^2(0, T; V)$. We do prove well-posedness in this sense whenever the form is piecewise Lipschitz-continuous and symmetric. Moreover, we show that each solution is in $C([0, T]; V)$. We apply the results to non-autonomous Robin-boundary conditions and also use maximal regularity to solve a quasilinear problem.

Key words: Sesquilinear forms, non-autonomous evolution equations, maximal regularity, non-linear heat equations.

MSC: 35K90, 35K50, 35K45, 47D06.

1 Introduction

The aim of this article is to study non-autonomous evolution equations governed by forms. We consider Hilbert spaces H and V such that V is continuously embedded into H and a form

$$\mathfrak{a} : [0, T] \times V \times V \rightarrow \mathbb{C}$$

such that $\mathfrak{a}(t, \cdot, \cdot)$ is sesquilinear for all $t \in [0, T]$, $\mathfrak{a}(\cdot, u, v) : [0, T] \rightarrow \mathbb{C}$ is measurable for all $u, v \in V$,

$$|\mathfrak{a}(t, u, v)| \leq M \|u\|_V \|v\|_V \quad (t \in [0, T]) \quad (V\text{-boundedness})$$

and such that

$$\operatorname{Re} \mathfrak{a}(t, u, u) \geq \alpha \|u\|_V^2 \quad (u \in V, t \in [0, T]) \quad (\text{coerciveness})$$

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where $M \geq 0$ and $\alpha > 0$. For fixed $t \in [0, T]$ there is a unique operator $\mathcal{A}(t) \in \mathcal{L}(V, V')$ such that $\mathfrak{a}(t, u, v) = \langle \mathcal{A}(t)u, v \rangle$ for all $u, v \in V$. Given $f \in L^2(0, T; V')$, $u_0 \in H$, the Cauchy problem

$$\dot{u}(t) + \mathcal{A}(t)u(t) = f(t) \quad (t \in (0, T)), \quad u(0) = u_0 \quad (1.1)$$

is well-posed in V' ; i.e. there exists a unique solution $u \in MR(V, V') := L^2(0, T; V) \cap H^1(0, T; V')$. Note that $MR(V, V') \hookrightarrow C([0, T], H)$ so that the initial condition makes sense. This is a well-known result due to J. L. Lions, see [Sho97, p. 112], [DL88, XVIII Chapter 3, p. 620]. However, the result is not really satisfying since in concrete situations one is interested in solutions which take values in H and not merely in V' (note that $H \hookrightarrow V'$ by the canonical identification). In fact, if we consider boundary value problems, only the part $A(t)$ of $\mathcal{A}(t)$ in H does really realize the boundary conditions in question. So the general problem is whether maximal regularity in H is valid in the following sense:

Problem 1.1. If $f \in L^2(0, T; H)$ and $u_0 \in V$, is the solution of (1.1) in $MR(V, H) := H^1(0, T; H) \cap L^2(0, T; V)$?

One has to distinguish the two cases $u_0 = 0$ and $u_0 \neq 0$. For $u_0 = 0$ Problem 1.1 is explicitly asked by Lions [Lio61, p. 68] and seems to be open up to today. A positive answer is given by Lions if \mathfrak{a} is symmetric (i.e. $\mathfrak{a}(t, u, v) = \overline{\mathfrak{a}(t, v, u)}$) and $\mathfrak{a}(\cdot, u, v) \in C^1[0, T]$ for all $u, v \in V$. By a completely different approach a positive answer is also given in [OS10] for general forms such that $\mathfrak{a}(\cdot, u, v) \in C^\alpha[0, T]$ for all $u, v \in V$ and some $\alpha > \frac{1}{2}$. Again, the result in [OS10] concerns the case $u_0 = 0$.

Concerning $u_0 \neq 0$ it seems natural to assume $u_0 \in V$ as we did in Problem 1.1. However, already in the autonomous case, i.e. $A(t) \equiv A$, the solution is in $MR(V, H)$ if and only if $u_0 \in D(A^{1/2})$, and it may happen that $V \not\subset D(A^{1/2})$. So one has to impose a stronger condition on the initial value u_0 or the form (e.g. symmetry). Lions [Lio61, p. 94] gave a positive answer for $u_0 \in D(A(0))$ provided that $\mathfrak{a}(\cdot, u, v) \in C^2[0, T]$ for all $u, v \in V$ and $f \in H^1(0, T; H)$. Moreover, he asked the following particular case of Problem 1.1 (see [Lio61, p. 95]):

Problem 1.2. For $u_0 \in D(A(0))$, is the solution of (1.1) in $MR(V, H) = H^1(0, T; H) \cap L^2(0, T; V)$ provided $\mathfrak{a}(\cdot, u, v) \in C^1[0, T]$ for all $u, v \in V$?

A little bit hidden in his book one finds a solution to Problem 1.2 essentially in the case where the form is symmetric. In fact, a combination of [Lio61, Theorem 1.1, p. 129] and [Lio61, Theorem 5.1, p. 138] shows that Problem 1.2 has a positive answer for $u_0 \in V$, $f \in L^2(0, T; H)$ if $\mathfrak{a}(\cdot, u, v) \in C^1[0, T]$ and $\mathfrak{a}(t, u, v) = \overline{\mathfrak{a}(t, v, u)}$ for all $u, v \in V$, $t \in [0, T]$.

Now we explain our contribution to the problem of maximal regularity formulated in Problem 1.1 (and Problem 1.2). We suppose that the sesquilinear form \mathfrak{a} can be written as $\mathfrak{a}(t, u, v) = \mathfrak{a}_1(t, u, v) + \mathfrak{a}_2(t, u, v)$ where \mathfrak{a}_1 is symmetric, V -bounded and coercive as above and piecewise Lipschitz-continuous on $[0, T]$, whereas $\mathfrak{a}_2 : [0, T] \times V \times H \rightarrow \mathbb{C}$ satisfies $|\mathfrak{a}_2(t, u, v)| \leq M_2 \|u\|_V \|v\|_H$ and $|\mathfrak{a}_2(\cdot, u, v)|$ is measurable for all $u \in V$, $v \in H$. Furthermore we consider a more general Cauchy problem than (1.1) introducing a multiplicative perturbation $B : [0, T] \rightarrow \mathcal{L}(H)$ which is strongly measurable such that $0 < \beta_0 \leq (B(t)g | g)_H \leq \beta_1$ for $g \in H$, $\|g\|_H = 1$, $t \in [0, T]$ and study the problem

$$B(t)\dot{u}(t) + A(t)u(t) = f(t) \quad (t \in (0, T)), \quad u(0) = u_0 \quad (1.2)$$

(where $A(t)$ is the part of $\mathcal{A}(t)$ in H). The multiplicative perturbation is needed for several applications to non-linear problems (see below). Our main result on maximal regularity is the following (Corollary 5.2): Given $f \in L^2(0, T; H)$, $u_0 \in V$ there is a unique solution $u \in H^1(0, T; H) \cap L^2(0, T; V)$ of (1.2). This extends the result of Lions mentioned above. One of our other results, established in Section 4, shows that the solution is automatically in $C([0, T], V)$. In fact, the classical result of Lions says that

$$MR(V, V') \hookrightarrow C([0, T]; H), \quad (1.3)$$

and also that for $u \in MR(V, V')$ the function $\|u(\cdot)\|_H^2$ is in $W^{1,1}(0, T)$ with

$$(\|u\|_H^2)' = 2 \operatorname{Re} \langle \dot{u}, u \rangle, \quad (1.4)$$

see [Sho97, p. 106]. In the non-autonomous situation considered here we prove that

$$MR_{\mathbf{a}}(H) := \{u \in H^1(0, T; H) \cap L^2(0, T; V) : \mathcal{A}(\cdot)u(\cdot) \in L^2(0, T; H)\}$$

is continuously imbedded into $C([0, T], V)$ and that for $u \in MR_{\mathbf{a}}(H)$ the function $\mathbf{a}(\cdot, u(\cdot), u(\cdot))$ is in $W^{1,1}(0, T)$ with

$$(\mathbf{a}(\cdot, u(\cdot), u(\cdot)))' = \dot{\mathbf{a}}(\cdot, u(\cdot), u(\cdot)) + 2 \operatorname{Re}(A(\cdot)u(\cdot) | \dot{u}(\cdot))_H.$$

Note that if $u \in MR(V, H)$ is a solution of (1.1), then automatically $u \in MR_{\mathbf{a}}(H)$. It is this continuity with values in V which allows us to weaken the regularity assumption on the form $\mathbf{a}(t, \cdot, \cdot)$ from Lipschitz-continuity in Theorem 5.1 to piecewise Lipschitz continuity on $[0, T]$ in Corollary 5.2.

We illustrate our abstract results by three applications. One of them concerns the heat equation with non-autonomous Robin-boundary-conditions

$$\partial_{\nu} u(t) + \beta(t)u(t)|_{\partial\Omega} = 0 \quad (1.5)$$

on a bounded Lipschitz domain Ω . Here ∂_{ν} denotes the normal derivative. Under appropriate assumptions on β we prove *maximal regularity*, i.e., that the solution is in $MR(H^1(\Omega), L^2(\Omega))$. This is of great importance if non-linear problems are considered. As an example we prove existence of a solution of the problem

$$\begin{cases} \dot{u}(t) = m(t, u(t))\Delta u(t) + f(t) \\ u(0) = u_0 \in H^1(\Omega) \\ \partial_{\nu} u(t) + \beta(t, \cdot)u(t) = 0 \text{ on } \partial\Omega \end{cases}$$

i.e., a quasilinear problem with non-autonomous Robin boundary conditions on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$. It is here that we need well-posedness and maximal regularity of problem (1.2) with multiplicative perturbation (of the form $Bg = \frac{1}{m(u(\cdot))}g$). Previous results (see [AC10]) did not allow non-autonomous boundary conditions.

The paper is organized as follows. In Section 2 we put together preliminary results on operators associated with forms. Also Section 3 has preliminary character. We prove Lions' Riesz-Representation Theorem which will be used later. In Section 4 we prove that $MR_{\mathbf{a}}(H)$ is continuously imbedded into $C([0, T]; V)$, and in Section 5 we finally prove maximal regularity in H . In Section 6 a series of examples concerning parabolic equations are given, where the main point concerns non-autonomous boundary conditions. A product rule for vector-valued one-dimensional Sobolev spaces is proved in the appendix.

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2 Forms and associated operators

Throughout this paper the underlying field is $\mathbb{K} = \mathbb{C}$ or \mathbb{R} . This means that all results are valid no matter whether the underlying field is \mathbb{R} or \mathbb{C} . Let V, H be two Hilbert spaces over \mathbb{K} . Their scalar products and the corresponding norms will be denoted by $(\cdot | \cdot)_H$, $(\cdot | \cdot)_V$, $\|\cdot\|_H$ and $\|\cdot\|_V$, respectively. We assume that

$$V \xhookrightarrow{d} H;$$

i.e., V is a dense subspace of H such that for some constant $c_H > 0$,

$$\|u\|_H \leq c_H \|u\|_V \quad (u \in V). \quad (2.1)$$

Let

$$\mathfrak{a} : V \times V \rightarrow \mathbb{K}$$

be sesquilinear and *continuous*; i.e.

$$|\mathfrak{a}(u, v)| \leq M \|u\|_V \|v\|_V \quad (u, v \in V) \quad (2.2)$$

for some constant M . We assume that \mathfrak{a} is *quasi-coercive*; i.e. there exist constants $\alpha > 0$, $\omega \in \mathbb{R}$ such that

$$\operatorname{Re} \mathfrak{a}(u, u) + \omega \|u\|_H^2 \geq \alpha \|u\|_V^2 \quad (u \in V). \quad (2.3)$$

If $\omega = 0$, we say that the form \mathfrak{a} is *coercive*. The operator $\mathcal{A} \in \mathcal{L}(V, V')$ associated with \mathfrak{a} is defined by

$$\langle \mathcal{A}u, v \rangle = \mathfrak{a}(u, v) \quad (u, v \in V).$$

Here V' denotes the antidual of V when $\mathbb{K} = \mathbb{C}$ and the dual when $\mathbb{K} = \mathbb{R}$. The duality between V' and V is denoted by $\langle \cdot, \cdot \rangle$. As usual, we identify H with a dense subspace of V' (associating to $f \in H$ the antilinear form $v \mapsto (f | v)_H$). Then V' is a Hilbert space for a suitable scalar product.

Seen as an unbounded operator on V' with domain $D(\mathcal{A}) = V$ the operator $-\mathcal{A}$ generates a holomorphic semigroup on V' . In the case where $\mathbb{K} = \mathbb{R}$ we mean by this that the \mathbb{C} -linear extension of $-\mathcal{A}$ on the complexification of V' generates a holomorphic C_0 -semigroup. The semigroup is bounded on a sector

if $\omega = 0$, in which case \mathcal{A} is an isomorphism. Denote by A the part of \mathcal{A} on H ; i.e.,

$$\begin{aligned} D(A) &:= \{u \in V : \mathcal{A}u \in H\} \\ Au &= \mathcal{A}u. \end{aligned}$$

Then $-A$ generates a holomorphic C_0 -semigroup on H (the restriction of the semigroup generated by $-\mathcal{A}$ to H). For all this, see e.g. the monographs [Ouh05, Chap. 1], [Tan79, Chap. 2].

Next we want to consider the symmetric case. The form \mathfrak{a} is called *symmetric* if

$$\mathfrak{a}(u, v) = \overline{\mathfrak{a}(v, u)}$$

for all $u, v \in V$. In that case the operator A is self-adjoint. By the Spectral Theorem A is unitarily equivalent to a multiplication operator. In terms of the form this leads to the following spectral representation.

Theorem 2.1 (Spectral Representation of Symmetric Forms). *Let \mathfrak{a} be symmetric and coercive. Then up to unitary equivalence $H, V, \mathcal{A}, A, \mathfrak{a}$ are given as follows. There exists a measure space (Ω, Σ, μ) and a measurable function $m : \Omega \rightarrow [\delta, \infty)$ where $\delta = \frac{\alpha}{c_H^2}$ such that*

$$H = L^2(\Omega, \mu), \quad V = L^2(\Omega, m\mu), \quad V' = L^2(\Omega, \frac{\mu}{m}).$$

Moreover

$$\langle f, u \rangle = \int_{\Omega} f \bar{u} d\mu \quad (f \in V', u \in V), \quad \mathfrak{a}(u, v) = \int_{\Omega} u \bar{v} m d\mu \quad (u, v \in V)$$

and

$$\begin{aligned} D(A) &= L^2(\Omega, m^2\mu), \quad Au = mu \quad (u \in D(A)), \\ \mathcal{A}u &= mu \quad (u \in V). \end{aligned}$$

Via this representation theorem we may identify H with $L^2(\Omega, \mu)$ with the scalar product $(u | v)_H = \int_{\Omega} u \bar{v} d\mu$. Then $V = L^2(\Omega, m\mu)$ with a norm which is equivalent to the usual norm of $L^2(\Omega, m\mu)$. More precisely,

$$\alpha \|u\|_V^2 \leq \|u\|_{L^2(\Omega, m\mu)}^2 = a(u, u) \leq M \|u\|_V^2. \quad (2.4)$$

We define the operator $\mathcal{A}^{1/2}$ via this spectral representation by

$$\mathcal{A}^{1/2}u = m^{1/2}u.$$

Then $\mathcal{A}^{1/2} \in \mathcal{L}(H, V')$ is an isomorphism such that $\mathcal{A}^{1/2}V = H$. The definition of $\mathcal{A}^{1/2}$ does not depend on the spectral representation. The following estimates follow from (2.4).

Proposition 2.2. *Assume that the form is symmetric and coercive. Then*

- a) $\|(\lambda + \mathcal{A})^{-1}\|_{\mathcal{L}(V)} \leq c_1(1 + \lambda)^{-1}$ for all $\lambda \geq 0$,
- b) $\|(\lambda + \mathcal{A})^{-1}\|_{\mathcal{L}(V', V)} \leq 1/\alpha$ for all $\lambda \geq 0$,
- c) $\|\mathcal{A}^{-1/2}\|_{\mathcal{L}(H, V)} \leq 1/\sqrt{\alpha}$ and finally,
- d) $\|\mathcal{A}^{1/2}\|_{\mathcal{L}(H, V')} \leq \sqrt{M}$,

where $c_1 = \sqrt{M/\alpha} \max\{1, c_H^2/\alpha\}$.

3 Lions' Representation Theorem

In this (still preliminary) section we give Lions' Representation Theorem, which will be used later and include its short, elegant proof for convenience.

Theorem 3.1 (Lions' Representation Theorem [Lio59, p. 156], [Lio61, p. 61]). *Let \mathcal{V} be a Hilbert space, \mathcal{W} a pre-Hilbert space such that $\mathcal{W} \hookrightarrow \mathcal{V}$. Let $E : \mathcal{V} \times \mathcal{W} \rightarrow \mathbb{K}$ be sesquilinear such that*

- a) *for all $w \in \mathcal{W}$, $E(\cdot, w)$ is a continuous linear functional on \mathcal{V} ;*
- b) *$|E(w, w)| \geq \alpha \|w\|_{\mathcal{W}}^2$ for all $w \in \mathcal{W}$*

for some $\alpha > 0$. Let $L \in \mathcal{W}'$. Then there exists $u \in \mathcal{V}$ such that

$$Lw = E(u, w)$$

for all $w \in \mathcal{W}$.

Proof. By the Riesz Representation Theorem there exists a linear map $T : \mathcal{W} \rightarrow \mathcal{V}$ such that $E(v, w) = (v | Tw)_{\mathcal{V}}$ for all $v \in \mathcal{V}$, $w \in \mathcal{W}$. It follows from coerciveness that

$$\alpha \|w\|_{\mathcal{W}}^2 \leq |E(w, w)| = |(w | Tw)_{\mathcal{V}}| \leq \|w\|_{\mathcal{V}} \|Tw\|_{\mathcal{V}}.$$

Since $\mathcal{W} \hookrightarrow \mathcal{V}$, there exists $\alpha' > 0$ such that

$$\alpha' \|w\|_{\mathcal{W}} \leq \|Tw\|_{\mathcal{V}} \quad (w \in \mathcal{W}).$$

We may assume that $T\mathcal{W}$ is dense in \mathcal{V} (otherwise we replace \mathcal{V} by $\overline{T\mathcal{W}}$). Then there exists a unique bounded operator S from \mathcal{V} into the completion $\tilde{\mathcal{W}}$ of \mathcal{W} such that

$$STw = w \quad (w \in \mathcal{W}).$$

By the Riesz Representation Theorem there exists $\tilde{w} \in \tilde{\mathcal{W}}$ such that

$$Lw = (\tilde{w} | w)_{\tilde{\mathcal{W}}} \quad (w \in \mathcal{W}).$$

For $u \in \mathcal{V}$ one has the desired property

$$Lw = E(u, w) \quad (w \in \mathcal{W})$$

if and only if

$$(\tilde{w} | STw)_{\tilde{\mathcal{W}}} = (\tilde{w} | w)_{\tilde{\mathcal{W}}} = (u | Tw)_{\mathcal{V}} \quad (w \in \mathcal{W})$$

i.e.,

$$(\tilde{w} | Sv)_{\tilde{\mathcal{W}}} = (u | v)_{\mathcal{V}}$$

for all $v \in T\mathcal{W}$; or equivalently for all $v \in \mathcal{V}$. Thus $u := S^*\tilde{w}$ has the desired property. \square

Remark 3.2 (uniqueness). The vector u is unique if and only if for $v \in \mathcal{V}$, $E(v, w) = 0$ for all $w \in \mathcal{W}$ implies $v = 0$. This is the same as saying that $T\mathcal{W}$ is dense in \mathcal{V} , where T is the mapping of the proof.

4 Embedding into continuous functions

In this section we show that in a non-autonomous framework a mixed Sobolev space embeds into a space of continuous functions (extending (1.3) to a non-autonomous setting). Let V, H be separable Hilbert spaces over $\mathbb{K} = \mathbb{R}$ or \mathbb{C} such that $V \xrightarrow{d} H$. Let $T > 0$ and

$$\mathfrak{a} : [0, T] \times V \times V \rightarrow \mathbb{K}$$

be a function such that $\mathfrak{a}(t, \cdot, \cdot) : V \times V \rightarrow \mathbb{K}$ is sesquilinear for all $t \in [0, T]$. We assume that \mathfrak{a} is V -bounded, and coercive, see Introduction. In addition we assume in addition that \mathfrak{a} is *symmetric*; i.e.,

$$\mathfrak{a}(t, u, v) = \overline{\mathfrak{a}(t, v, u)} \quad (t \in [0, T], u, v \in V),$$

and that \mathfrak{a} is *Lipschitz continuous*; i.e., there exists a positive constant \dot{M} such that

$$|\mathfrak{a}(t, u, v) - \mathfrak{a}(s, u, v)| \leq \dot{M}|t - s| \|u\|_V \|v\|_V \quad (t, s \in [0, T], u, v \in V).$$

Remark 4.1. It follows from the Uniform Boundedness Principle that \mathfrak{a} is Lipschitz continuous whenever $\mathfrak{a}(\cdot, u, v) : [0, T] \rightarrow \mathbb{K}$ is Lipschitz continuous for all $u, v \in V$.

We denote by $\mathcal{A}(t) \in \mathcal{L}(V, V')$ the operator associated with $\mathfrak{a}(t, \cdot, \cdot)$. We consider the following maximal regularity space

$$MR_{\mathfrak{a}}(H) := \{u \in H^1(0, T; H) \cap L^2(0, T; V) : \mathcal{A}(\cdot)u(\cdot) \in L^2(0, T; H)\}.$$

It is a Hilbert space for the norm $\|\cdot\|_{MR_{\mathfrak{a}}(H)}$ given by

$$\|u\|_{MR_{\mathfrak{a}}(H)}^2 := \|u\|_{L^2(0, T; V)}^2 + \|\dot{u}\|_{L^2(0, T; H)}^2 + \|\mathcal{A}(\cdot)u(\cdot)\|_{L^2(0, T; H)}^2.$$

Under the above assumptions on the form \mathfrak{a} our main result of this section says the following.

Theorem 4.2. *The space $MR_{\mathfrak{a}}(H)$ is continuously embedded into $C([0, T]; V)$. Moreover, if $u \in MR_{\mathfrak{a}}(H)$, then $\mathfrak{a}(\cdot, u(\cdot), u(\cdot)) \in W^{1,1}(0, T)$ and*

$$(\mathfrak{a}(\cdot, u(\cdot), u(\cdot)))' = \dot{\mathfrak{a}}(\cdot, u(\cdot), u(\cdot)) + 2 \operatorname{Re}(A(\cdot)u(\cdot) | \dot{u}(\cdot))_H. \quad (4.1)$$

Observe that for $u \in MR_{\mathfrak{a}}(H)$ one has $u(t) \in D(A(t))$ a.e. and $A(\cdot)u(\cdot) = \mathcal{A}(\cdot)u(\cdot) \in L^2(0, T; H)$. Thus $(A(\cdot)u(\cdot) | \dot{u}(\cdot))_H \in L^1(0, T)$. This explains the second term on the right hand side of (4.1). The definition of $\dot{\mathfrak{a}}$ becomes clear from the following lemma. In fact, by our assumption $\mathcal{A} : [0, T] \rightarrow \mathcal{L}(V, V')$ is Lipschitz continuous. By Lemma 4.3 a) below there exists $\dot{\mathcal{A}} : [0, T] \rightarrow \mathcal{L}(V, V')$, strongly measurable and bounded, such that

$$\dot{\mathcal{A}}(t)u = \frac{d}{dt}\mathcal{A}(t)u \quad \text{a.e.}$$

for all $u \in V$. We define $\dot{\mathfrak{a}}$ by

$$\dot{\mathfrak{a}}(t, u, v) = \langle \dot{\mathcal{A}}(t)u, v \rangle$$

for all $t \in [0, T]$, $u, v \in V$. Thus, for $u \in L^2(0, T; V)$, $\dot{\mathfrak{a}}(t, u(\cdot), u(\cdot)) = \langle \dot{\mathcal{A}}(\cdot)u(\cdot), u(\cdot) \rangle \in L^1(0, T)$. Thus the right hand side of (4.1) is in $L^1(0, T)$.

For the proof of Theorem 4.2 we need several auxiliary results.

Lemma 4.3. *Let $S : [0, T] \rightarrow \mathcal{L}(V, V')$ be Lipschitz continuous. Then the following holds.*

- a) *There exists a bounded, strongly measurable function $\dot{S} : [0, T] \rightarrow \mathcal{L}(V, V')$ such that*

$$\frac{d}{dt}S(t)u = \dot{S}(t)u \quad (u \in V)$$

for a.e. $t \in [0, T]$ and

$$\|\dot{S}(t)\|_{\mathcal{L}(V, V')} \leq L \quad (t \in [0, T])$$

where L is the Lipschitz constant of S .

- b) *If $u \in H^1(0, T; V)$, then $Su := S(\cdot)u(\cdot) \in H^1(0, T; V')$ and*

$$(Su)' = \dot{S}(\cdot)u(\cdot) + S(\cdot)\dot{u}(\cdot). \quad (4.2)$$

For the proof of Lemma 4.3 we recall the following. If a function $u : [0, T] \rightarrow V$ is absolutely continuous, then $\dot{u}(t) := \frac{d}{dt}u(t)$ exists almost everywhere and $u(t) = u(0) + \int_0^t \dot{u}(s) ds$ [ABHN11, Proposition 1.2.3 and Corollary 1.2.7]. In fact, the space of all absolutely continuous functions on $[0, T]$ with values in V is the same as the Sobolev space $W^{1,1}(0, T; V)$ and \dot{u} coincides with the weak derivative (this is true for a Banach space V if and only if it has the Radon-Nikodým property). The function u is in $H^1(0, T; V)$ if and only if $u \in W^{1,1}(0, T; V)$ and $\dot{u} \in L^2(0, T; V)$.

Proof of Lemma 4.3. a) Since for $u \in V$, $S(\cdot)u$ is Lipschitz continuous, the derivative $\frac{d}{dt}S(t)u$ exists a.e. (see [ABHN11, Sec. 1.2]). Let V_0 be a countable dense subset of V . There exists a Borel null set $N \subset [0, T]$ such that $\frac{d}{dt}S(t)u$ exists in V' for all $t \notin N$ and all $u \in V_0$. Since S is Lipschitz-continuous it follows easily that $\frac{d}{dt}S(t)u$ exists also for all $u \in \overline{V_0} = V$ and $t \notin N$. Let

$$\dot{S}(t)u = \begin{cases} \frac{d}{dt}S(t)u & \text{if } t \notin N \text{ and} \\ 0 & \text{if } t \in N. \end{cases}$$

Let L be the Lipschitz constant of S . Then $\dot{S}(t) \in \mathcal{L}(V, V')$, with $\|\dot{S}(t)\|_{\mathcal{L}(V, V')} \leq L$ for all $t \in [0, T]$ and $\dot{S}(\cdot)u$ is measurable for all $u \in V$.

- b) Let $u \in H^1(0, T; V)$. Then $u \in C([0, T]; V)$ and

$$\|u\|_\infty := \sup_{t \in [0, T]} \|u(t)\|_V < \infty.$$

Denote the supremum norm of S by

$$\|S\|_\infty := \sup_{t \in [0, T]} \|S(t)\|_{\mathcal{L}(V, V')}.$$

We first show that Su is absolutely continuous. Let $\epsilon > 0$. Since u is absolutely continuous there exists a $\delta > 0$ such that

$$\sum_i \|u(b_i) - u(a_i)\| \leq \|S\|_\infty^{-1} \frac{\epsilon}{2}$$

for each finite collection of non-overlapping intervals (a_i, b_i) in $(0, T)$ satisfying $\sum_i (b_i - a_i) < \delta$. We may take $\delta > 0$ so small that $L\|u\|_\infty \delta < \frac{\epsilon}{2}$. Then

$$\begin{aligned}
& \sum_i \|S(b_i)u(b_i) - S(a_i)u(a_i)\| \\
& \leq \sum_i \|(S(b_i) - S(a_i))u(b_i)\| + \sum_i \|S(a_i)(u(b_i) - u(a_i))\| \\
& \leq L \sum_i (b_i - a_i) \|u\|_\infty + \|S\|_\infty \|S\|_\infty^{-1} \frac{\epsilon}{2} \\
& < L \delta \|u\|_\infty + \frac{\epsilon}{2} \leq \epsilon.
\end{aligned}$$

Thus Su is absolutely continuous. Moreover

$$\begin{aligned}
(Su)'(t) &= \lim_{h \rightarrow 0} \frac{1}{h} (S(t+h)u(t+h) - S(t)u(t)) \\
&= \lim_{h \rightarrow 0} \left[\frac{1}{h} (S(t+h) - S(t))(u(t+h) - u(t)) \right. \\
&\quad \left. + \frac{1}{h} (S(t+h) - S(t))u(t) \right. \\
&\quad \left. + \frac{1}{h} S(t)(u(t+h) - u(t)) \right] \\
&= \dot{S}(t)u(t) + S(t)\dot{u}(t) \quad \text{a.e.}
\end{aligned}$$

Thus $(Su)' \in L^2(0, T; V')$ and so $Su \in H^1(0, T; V')$. \square

Recall from Section 2 that $\mathcal{A}(t)^{-1/2} \in \mathcal{L}(V', H)$ is invertible and $\mathcal{A}(t)^{-1/2}H = V$. One has

$$\mathcal{A}(t)^{-1/2}u = \frac{1}{\pi} \int_0^\infty \lambda^{-1/2} (\lambda + \mathcal{A}(t))^{-1} u \, d\lambda \quad (u \in V'),$$

see [ABHN11, (3.52)] or [Paz83, Sec. 2.6 p. 69]. The inverse operator is denoted by $\mathcal{A}(t)^{1/2}$ with domain $D(\mathcal{A}(t)^{1/2}) = V$.

Lemma 4.4. *The mappings*

- a) $\mathcal{A}^{-1/2} : [0, T] \rightarrow \mathcal{L}(V)$ and
- b) $\mathcal{A}^{1/2} : [0, T] \rightarrow \mathcal{L}(V, V')$

are Lipschitz continuous.

Proof. a) Let $u \in V$. Then by Proposition 2.2 a) and b),

$$\begin{aligned}
& \|\mathcal{A}^{-1/2}(t)u - \mathcal{A}^{-1/2}(s)u\|_V \\
&= \left\| \frac{1}{\pi} \int_0^\infty \lambda^{-1/2} [(\lambda + \mathcal{A}(t))^{-1} - (\lambda + \mathcal{A}(s))^{-1}] u \, d\lambda \right\|_V \\
&= \left\| \frac{1}{\pi} \int_0^\infty \lambda^{-1/2} (\lambda + \mathcal{A}(t))^{-1} (\mathcal{A}(s) - \mathcal{A}(t)) (\lambda + \mathcal{A}(s))^{-1} u \, d\lambda \right\|_V \\
&\leq \frac{1}{\alpha} \int_0^\infty \lambda^{-1/2} \|(\mathcal{A}(s) - \mathcal{A}(t)) (\lambda + \mathcal{A}(s))^{-1} u\|_{V'} \, d\lambda \\
&\leq \frac{M}{\alpha} |s - t| \int_0^\infty \lambda^{-1/2} \|(\lambda + \mathcal{A}(s))^{-1} u\|_V \, d\lambda \\
&\leq c_1 \frac{M}{\alpha} |s - t| \int_0^\infty \lambda^{-1/2} (\lambda + 1)^{-1} \, d\lambda \|u\|_V.
\end{aligned}$$

b) Let $u \in V$. Then by Proposition 2.2 c)

$$\begin{aligned}
& \|\mathcal{A}^{1/2}(t)u - \mathcal{A}^{1/2}(s)u\|_{V'} \\
&= \|\mathcal{A}(t)\mathcal{A}^{-1/2}(t)u - \mathcal{A}(s)\mathcal{A}^{-1/2}(s)u\|_{V'} \\
&\leq \|(\mathcal{A}(t) - \mathcal{A}(s))\mathcal{A}^{-1/2}(t)u\|_{V'} + \|\mathcal{A}(s)(\mathcal{A}^{-1/2}(t)u - \mathcal{A}^{-1/2}(s)u)\|_{V'} \\
&\leq |t - s|\dot{M}\|\mathcal{A}^{-1/2}(t)u\|_V + M\|\mathcal{A}^{-1/2}(t)u - \mathcal{A}^{-1/2}(s)u\|_V \\
&\leq |t - s|\text{const}\|u\|_V \quad \text{by a).} \quad \square
\end{aligned}$$

Lemma 4.5. *The mappings*

a) $\mathcal{A}^{-1/2} : [0, T] \rightarrow \mathcal{L}(H, V)$ and

b) $\mathcal{A}^{1/2} : [0, T] \rightarrow \mathcal{L}(H, V')$

are strongly continuous.

Proof. a) We know from Proposition 2.2 c) that

$$\|\mathcal{A}^{-1/2}(t)u\|_V \leq \frac{1}{\sqrt{\alpha}}\|u\|_H \quad (u \in H, t \in [0, T]).$$

Since by Lemma 4.4 a) $\mathcal{A}^{-1/2}(\cdot)u : [0, T] \rightarrow V$ is continuous for $u \in V$, the claim follows by a 3ϵ -argument.

b) By Proposition 2.2 d) one has

$$\|\mathcal{A}^{1/2}(t)u\|_{V'} \leq \sqrt{M}\|u\|_H \quad (u \in H, t \in [0, T]).$$

Since by Lemma 4.4 b) $\mathcal{A}^{1/2}(\cdot)u : [0, T] \rightarrow V'$ is continuous for $u \in V$, it is also continuous for $u \in H$ by a 3ϵ -argument. \square

Next we consider the Hilbert space

$$MR(V, H) := L^2(0, T; V) \cap H^1(0, T; H)$$

with norm

$$\|u\|_{MR(V, H)}^2 := \|u\|_{L^2(0, T; V)}^2 + \|u\|_{H^1(0, T; H)}^2.$$

Lemma 4.6. $H^1(0, T; V)$ is dense in $MR(V, H)$.

Proof. We use the spectral representation Theorem 2.1. Let

$$\Omega_n := \{x \in \Omega : m(x) \leq n\}$$

and

$$P_n f = \mathbf{1}_{\Omega_n} f \quad (f \in H = L^2(\Omega, \mu)).$$

Then $P_n^2 = P_n = P_n^* \in \mathcal{L}(H)$, $P_n H \subset V$, $\lim_{n \rightarrow \infty} P_n f = f$ in H for all $f \in H$ and $\lim_{n \rightarrow \infty} P_n f = f$ in V for all $f \in V$. Now let $u \in MR(V, H)$. Then $u_n = P_n \circ u \in H^1(0, T; V)$ and $\dot{u}_n = P_n \circ \dot{u}$. Thus $u_n \rightarrow u$ in $L^2(0, T; V)$, $\dot{u}_n \rightarrow \dot{u}$ in $L^2(0, T; H)$. \square

Similarly, we define the Hilbert space

$$MR(H, V') = L^2(0, T; H) \cap H^1(0, T; V')$$

with norm

$$\|u\|_{MR(H, V')}^2 = \|u\|_{L^2(0, T; H)}^2 + \|u\|_{H^1(0, T; V')}^2.$$

Proposition 4.7. *Let $u \in MR(V, H)$. Then $\mathcal{A}^{1/2}(\cdot)u(\cdot) \in MR(H, V')$ and*

$$(\mathcal{A}^{1/2}(\cdot)u(\cdot))' = \dot{\mathcal{A}}^{1/2}(\cdot)u(\cdot) + \mathcal{A}^{1/2}(\cdot)\dot{u}(\cdot). \quad (4.3)$$

Recall that by Lemma 4.4 b) $S := \mathcal{A}^{1/2} : [0, T] \rightarrow \mathcal{L}(V, V')$ is Lipschitz continuous. Thus by Lemma 4.3 b) there exists $\dot{S} : [0, T] \rightarrow \mathcal{L}(V, V')$, which is strongly measurable and bounded. For typographical reasons we let $\dot{\mathcal{A}}^{1/2}(\cdot) := \dot{S}(\cdot)$. Thus for $u \in L^2(0, T; V)$, $\dot{\mathcal{A}}^{1/2}(\cdot)u(\cdot) \in L^2(0, T; V')$, which explains that the first term on the right hand side of (4.3) is well-defined. Concerning the second, recall from Lemma 4.5 that $\mathcal{A}^{1/2} : [0, T] \rightarrow \mathcal{L}(H, V')$ is strongly measurable and bounded by Proposition 2.2 c). Thus, for $u \in H^1(0, T; H)$ one has $\mathcal{A}^{1/2}(\cdot)\dot{u}(\cdot) \in L^2(0, T; V')$. Thus the right hand side of (4.3) is indeed in $L^2(0, T; V')$.

Proof of Proposition 4.7. a) Let $u \in H^1(0, T; V)$. Since $S := \mathcal{A}^{1/2} : [0, T] \rightarrow \mathcal{L}(V, V')$ is Lipschitz continuous (Lemma 4.4 b)), it follows from Lemma 4.3 b) that $S(\cdot)u(\cdot) \in H^1(0, T; V')$ and

$$\frac{d}{dt}S(t)u(t) = \dot{S}(t)u(t) + S(t)\dot{u}(t).$$

Since $\|\dot{S}(t)\|_{\mathcal{L}(V, V')}$ is bounded on $[0, T]$, it follows that

$$\|\dot{S}(\cdot)u(\cdot)\|_{L^2(0, T; V')} \leq \text{const} \|u\|_{L^2(0, T; V)}.$$

Since $\|S(t)\|_{\mathcal{L}(H, V')} \leq \text{const}$ (Proposition 2.2 d)), it follows that

$$\|S(\cdot)\dot{u}(\cdot)\|_{L^2(0, T; V')} \leq \text{const} \|\dot{u}\|_{L^2(0, T; H)}.$$

Finally, since $\|S(t)\|_{\mathcal{L}(V, H)} \leq \text{const}$, it follows that

$$\|S(\cdot)u(\cdot)\|_{L^2(0, T; H)} \leq \text{const} \|u\|_{L^2(0, T; V)}.$$

We have shown that

$$\|S(\cdot)u(\cdot)\|_{MR(H, V')} \leq \text{const} \|u\|_{MR(V, H)} \quad (u \in H^1(0, T; V)). \quad (4.4)$$

b) Let $u \in MR(V, H)$. By Lemma 4.6 there exist $u_n \in H^1(0, T; V)$ such that $u_n \rightarrow u$ in $MR(V, H)$. It follows from (4.4) that $(S(\cdot)u_n(\cdot))_{n \in \mathbb{N}}$ is a Cauchy sequence in $MR(H, V')$. Let $w = \lim_{n \rightarrow \infty} S(\cdot)u_n(\cdot)$ in $MR(H, V')$. Since $u_n \rightarrow u$ in $L^2(0, T; V)$, passing to a subsequence we can assume that $u_n(t) \rightarrow u(t)$ a.e. in V . Thus $S(t)u_n(t) \rightarrow S(t)u(t)$ a.e. in H . Since $w = \lim_{n \rightarrow \infty} S(\cdot)u_n(\cdot)$ in $L^2(0, T; H)$, it follows that $w = S(\cdot)u(\cdot)$. Thus $S(\cdot)u(\cdot) \in MR(H, V')$ and

$$\dot{w} = \lim_{n \rightarrow \infty} (S(\cdot)u_n(\cdot))' = \lim_{n \rightarrow \infty} (\dot{S}(\cdot)u_n(\cdot) + S(\cdot)\dot{u}_n(\cdot)) = \dot{S}(\cdot)u(\cdot) + S(\cdot)\dot{u}(\cdot)$$

in $L^2(0, T; V')$. This proves the proposition. \square

Now we are in the position to prove Theorem 4.2.

Proof of Theorem 4.2. Let $u \in MR_{\mathbf{a}}(H)$; i.e. $u \in MR(V, H)$ and $\mathcal{A}(\cdot)u(\cdot) \in L^2(0, T; H)$. Then by Lemma 4.5,

$$\mathcal{A}^{1/2}(\cdot)u(\cdot) = \mathcal{A}^{-1/2}(\cdot)\mathcal{A}(\cdot)u(\cdot) \in L^2(0, T; V).$$

Moreover by Proposition 4.7, $\mathcal{A}^{1/2}(\cdot)u(\cdot) \in MR(H, V')$. Thus one has even $\mathcal{A}^{1/2}(\cdot)u(\cdot) \in MR(V, V')$. Consequently the classical continuity result (1.3) implies that $\mathcal{A}^{1/2}(\cdot)u(\cdot) \in C([0, T]; H)$. Now Lemma 4.5 a) implies that

$$u = \mathcal{A}^{-1/2}(\cdot)\mathcal{A}^{1/2}(\cdot)u(\cdot) \in C([0, T]; V)$$

which is the first assertion of Theorem 4.2. In order to prove the second we deduce from (1.4) that $\mathbf{a}(\cdot, u(\cdot), u(\cdot)) = \|\mathcal{A}^{1/2}(\cdot)u(\cdot)\|_H^2 \in W^{1,1}(0, T)$ and

$$(\mathbf{a}(\cdot, u(\cdot), u(\cdot)))' = 2 \operatorname{Re}\langle \mathcal{A}^{1/2}(\cdot)u(\cdot), \mathcal{A}^{1/2}(\cdot)u(\cdot) \rangle.$$

Hence by Proposition 4.7

$$\begin{aligned} (\mathbf{a}(\cdot, u(\cdot), u(\cdot)))' &= 2 \operatorname{Re}\langle \dot{\mathcal{A}}^{1/2}(\cdot)u(\cdot) + \mathcal{A}^{1/2}(\cdot)\dot{u}(\cdot), \mathcal{A}^{1/2}(\cdot)u(\cdot) \rangle \\ &= 2 \operatorname{Re}\langle \dot{\mathcal{A}}^{1/2}(\cdot)u(\cdot), \mathcal{A}^{1/2}(\cdot)u(\cdot) \rangle + 2 \operatorname{Re}\langle \mathcal{A}(\cdot)u(\cdot) | \dot{u}(\cdot) \rangle_H \\ &= \dot{\mathbf{a}}(\cdot, u(\cdot), u(\cdot)) + 2 \operatorname{Re}\langle \mathcal{A}(\cdot)u(\cdot) | \dot{u}(\cdot) \rangle_H. \end{aligned} \quad \square$$

5 Well-posedness in H

Let V, H be separable Hilbert spaces such that $V \xrightarrow{d} H$ and let

$$\mathbf{a} : [0, T] \times V \times V \rightarrow \mathbb{K}$$

be a form on which we impose the following conditions. It can be written as the sum of two non-autonomous forms

$$\mathbf{a}(t, u, v) = \mathbf{a}_1(t, u, v) + \mathbf{a}_2(t, u, v) \quad (t \in [0, T], u, v \in V)$$

where

$$\mathbf{a}_1 : [0, T] \times V \times V \rightarrow \mathbb{K}$$

satisfies the assumptions considered in Section 4; i.e.,

- a) $|\mathbf{a}_1(t, u, v)| \leq M_1 \|u\|_V \|v\|_V$ for all $u, v \in V, t \in [0, T]$;
- b) $\mathbf{a}_1(t, u, u) \geq \alpha \|u\|_V^2$ for all $u \in V, t \in [0, T]$ with $\alpha > 0$;
- c) $\mathbf{a}_1(t, u, v) = \overline{\mathbf{a}_1(t, v, u)}$ for all $u, v \in V, t \in [0, T]$;
- d) \mathbf{a}_1 is Lipschitz-continuous; i.e.,

$$|\mathbf{a}_1(t, u, v) - \mathbf{a}_1(s, u, v)| \leq \dot{M}_1 |t - s| \|u\|_V \|v\|_V$$

for all $u, v \in V, s, t \in [0, T]$,

and

$$\mathbf{a}_2 : [0, T] \times V \times H \rightarrow \mathbb{K}$$

satisfies

e) $|\mathfrak{a}_2(t, u, v)| \leq M_2 \|u\|_V \|v\|_H$ for all $u \in V, v \in H, t \in [0, T]$,

f) $\mathfrak{a}_2(\cdot, u, v) : V \times V \rightarrow \mathbb{K}$ is measurable for all $u, v \in V$.

We denote by $\mathcal{A}(t)$ the operator given by $\langle \mathcal{A}(t)u, v \rangle = a(t, u, v)$ and by $A(t)$ we denote the part of $\mathcal{A}(t)$ in H . Let $B : [0, T] \rightarrow \mathcal{L}(H)$ be a strongly measurable function satisfying

$$\beta_0 \|g\|_H^2 \leq \operatorname{Re}(B(t)g | g)_H \leq \beta_1 \|g\|_H^2$$

for all $g \in H, t \in [0, T]$ where $0 < \beta_0 \leq \beta_1$ are constants. This implies that $B(t)$ is invertible and $\|B(t)^{-1}\| \leq \frac{1}{\beta_0}$ for a.e. $t \in [0, T]$. Now we state our results on existence and uniqueness.

Theorem 5.1. *Let $u_0 \in V, f \in L^2(0, T; H)$. Then there exists a unique*

$$u \in H^1(0, T; H) \cap L^2(0, T; V)$$

satisfying

$$\begin{aligned} B(t)\dot{u}(t) + \mathcal{A}(t)u(t) &= f(t) \quad \text{a.e.} \\ u(0) &= u_0. \end{aligned}$$

Moreover, $u \in C([0, T]; V)$ and

$$\|u\|_{MR(V, H)} \leq C \left[\|u_0\|_V + \|f\|_{L^2(0, T; H)} \right], \quad (5.1)$$

where the constant C depends merely on $\beta_0, M_1, M_2, \alpha, T$ and \dot{M}_1 .

Note that $u(t) \in D(A(t))$ a.e., since $\mathcal{A}(t)u(t) = f(t) - B(t)\dot{u}(t) \in H$ a.e. So we may replace $\mathcal{A}(t)$ by $A(t)$ in the equation. The fact that the solution u is in $C([0, T]; V)$ allows us to relax the continuity condition on \mathfrak{a}_1 allowing a finite number of jumps. We say that a non-autonomous form $\mathfrak{a}_1 : [0, T] \times V \times V \rightarrow \mathbb{K}$ is *piecewise Lipschitz-continuous* if there exist $0 = t_0 < t_1 < \dots < t_n = b$ such that on each interval (t_{i-1}, t_i) the form \mathfrak{a}_1 is the restriction of a Lipschitz-continuous form on $[t_{i-1}, t_i] \times V \times V$. Then Theorem 5.1 remains true.

Corollary 5.2. *Assume instead of d) that \mathfrak{a}_1 is merely piecewise Lipschitz-continuous. Let $u_0 \in V, f \in L^2(0, T; H)$. Then there exists a unique*

$$u \in H^1(0, T; H) \cap L^2(0, T; V)$$

satisfying

$$\begin{aligned} B(t)\dot{u}(t) + \mathcal{A}(t)u(t) &= f(t) \quad \text{a.e.} \\ u(0) &= u_0. \end{aligned}$$

Moreover, $u \in C([0, T]; V)$.

Proof. By Theorem 5.1 there is a solution $u_1 \in H^1(0, t_1; H) \cap L^2(0, t_1; V)$ on $(0, t_1)$ satisfying $u_1(0) = u_0$, and $u_1 \in C([0, t_1]; V)$. Since $u_1(t_1) \in V$ we find a solution $u_2 \in H^1(t_1, t_2; H) \cap L^2(t_1, t_2; V) \cap C([t_1, t_2]; V)$ with $u_2(t_1) = u_1(t_1)$. Solving successively we obtain solutions $u_i \in H^1(t_{i-1}, t_i; H) \cap L^2(t_{i-1}, t_i; V) \cap C([t_{i-1}, t_i]; V)$ with $u_i(t_{i-1}) = u_{i-1}(t_{i-1})$ $i = 1, \dots, n$. Letting $u(t) = u_i(t)$ for $t \in [t_{i-1}, t_i)$ we obtain a solution. Uniqueness follows from uniqueness in Theorem 5.1. \square

Proof of Theorem 5.1. a) Existence: Let

$$\mathcal{V} := \{u \in H^1(0, T; H) \cap L^2(0, T; V) : u(0) \in V\}.$$

Then \mathcal{V} is a Hilbert space for the norm

$$\|u\|_{\mathcal{V}}^2 := \int_0^T \|\dot{u}(t)\|_H^2 dt + \int_0^T \|u(t)\|_V^2 dt + \|u(0)\|_V^2.$$

Let $\mathcal{W} := H^1(0, T; V)$ with the same norm. Consider the sesquilinear form $E : \mathcal{V} \times \mathcal{W} \rightarrow \mathbb{K}$ given by

$$\begin{aligned} E(u, w) &:= \int_0^T (B(t)\dot{u}(t) | \dot{w}(t))_H e^{-\gamma t} dt \\ &\quad + \int_0^T \mathbf{a}(t, u(t), \dot{w}(t)) e^{-\gamma t} dt \\ &\quad + \mathbf{a}_1(0, u(0), w(0)), \end{aligned}$$

where γ will be determined later. Clearly $E(\cdot, w) \in \mathcal{V}'$ for all $w \in \mathcal{W}$. We show coerciveness of E . For that we denote by $\mathcal{A}_1(t) \in \mathcal{L}(V, V')$ the operator associated with \mathbf{a}_1 . By assumption d) we have

$$\|\mathcal{A}_1(t) - \mathcal{A}_1(s)\|_{\mathcal{L}(V, V')} \leq \dot{M}_1 |t - s|. \quad (5.2)$$

By Lemma 4.3 there exists a strongly measurable function $\dot{\mathcal{A}}_1 : [0, T] \rightarrow \mathcal{L}(V, V')$ such that $\dot{\mathcal{A}}_1(t)u = \frac{d}{dt}\mathcal{A}_1(t)u$ for all $u \in V$ and $t \notin N$ where $N \subset [0, T]$ is a null set, and

$$\|\dot{\mathcal{A}}_1(t)\|_{\mathcal{L}(V, V')} \leq \dot{M}_1 \quad (t \in [0, T]).$$

Let $w \in \mathcal{W}$. Recall the definition of $\dot{\mathbf{a}}_1$ and $\dot{\mathcal{A}}_1$ before Lemma 4.3. Note that by Lemma 4.3 the function $\mathcal{A}_1(\cdot)w(\cdot)$ is in $H^1(0, T; V')$ and $(\mathcal{A}_1(\cdot)w(\cdot))' = \dot{\mathcal{A}}_1(\cdot)w(\cdot) + \mathcal{A}_1(\cdot)\dot{w}(\cdot)$. Thus it follows from the product rule Lemma 7.2 that $\mathbf{a}_1(\cdot, w(\cdot), w(\cdot))' = \langle \mathcal{A}_1(\cdot)w(\cdot), w(\cdot) \rangle \in W^{1,1}(0, T)$ and

$$\begin{aligned} \mathbf{a}_1(\cdot, w(\cdot), w(\cdot))' &= \langle \dot{\mathcal{A}}_1(\cdot)w(\cdot), w(\cdot) \rangle + \langle \mathcal{A}_1(\cdot)\dot{w}(\cdot), w(\cdot) \rangle + \langle \mathcal{A}_1(\cdot)w(\cdot), \dot{w}(\cdot) \rangle \\ &= \dot{\mathbf{a}}_1(\cdot, w(\cdot), w(\cdot)) + 2 \operatorname{Re} \mathbf{a}_1(\cdot, w(\cdot), \dot{w}(\cdot)). \end{aligned}$$

Multiplying by $e^{-\gamma \cdot}$ (and using the scalar product rule) we finally obtain that $\mathbf{a}_1(\cdot, w(\cdot), w(\cdot))e^{-\gamma \cdot} \in W^{1,1}(0, T)$ and

$$\begin{aligned} [\mathbf{a}_1(\cdot, w(\cdot), w(\cdot))e^{-\gamma \cdot}]' &= \dot{\mathbf{a}}_1(\cdot, w(\cdot), w(\cdot))e^{-\gamma \cdot} \\ &\quad + 2 \operatorname{Re} \mathbf{a}_1(\cdot, w(\cdot), \dot{w}(\cdot))e^{-\gamma \cdot} - \gamma \mathbf{a}_1(\cdot, w(\cdot), w(\cdot))e^{-\gamma \cdot}. \end{aligned} \quad (5.3)$$

Now we prove the coerciveness estimate as follows. For $w \in \mathcal{W}$ one has

$$\begin{aligned}
|E(w, w)| &\geq \operatorname{Re} E(w, w) \\
&\geq \beta_0 \int_0^T \|\dot{w}(t)\|_H^2 e^{-\gamma t} dt + \operatorname{Re} \int_0^T \mathbf{a}_1(t, w(t), \dot{w}(t)) e^{-\gamma t} dt \\
&\quad - M_2 \int_0^T \|w(t)\|_V \|\dot{w}(t)\|_H e^{-\gamma t} dt + \mathbf{a}_1(0, w(0), w(0)) \\
&= \beta_0 \int_0^T \|\dot{w}(t)\|_H^2 e^{-\gamma t} dt + \frac{1}{2} \int_0^T [\mathbf{a}_1(t, w(t), w(t)) e^{-\gamma t}] dt \\
&\quad - \frac{1}{2} \int_0^T \dot{\mathbf{a}}_1(t, w(t), w(t)) e^{-\gamma t} dt + \frac{\gamma}{2} \int_0^T \mathbf{a}_1(t, w(t), w(t)) e^{-\gamma t} dt \\
&\quad - M_2 \int_0^T \|w(t)\|_V \|\dot{w}(t)\|_H e^{-\gamma t} dt + \mathbf{a}_1(0, w(0), w(0)) \\
&\geq \beta_0 \int_0^T \|\dot{w}(t)\|_H^2 e^{-\gamma t} dt + \frac{1}{2} \mathbf{a}_1(T, w(T), w(T)) e^{-\gamma T} - \frac{1}{2} \mathbf{a}_1(0, w(0), w(0)) \\
&\quad - \frac{1}{2} \dot{M}_1 \int_0^T \|w(t)\|_V^2 e^{-\gamma t} dt + \frac{\gamma}{2} \alpha \int_0^T \|w(t)\|_V^2 e^{-\gamma t} dt \\
&\quad - \epsilon \int_0^T \|\dot{w}(t)\|_H^2 e^{-\gamma t} dt - \frac{M_2^2}{4\epsilon} \int_0^T \|w(t)\|_V^2 e^{-\gamma t} dt + \mathbf{a}_1(0, w(0), w(0)) \\
&\geq (\beta_0 - \epsilon) \int_0^T \|\dot{w}(t)\|_H^2 e^{-\gamma t} dt + \frac{1}{2} \mathbf{a}_1(0, w(0), w(0)) \\
&\quad + \frac{1}{2} \left(\gamma \alpha - \dot{M}_1 - \frac{M_2^2}{2\epsilon} \right) \int_0^T \|w(t)\|_V^2 e^{-\gamma t} dt \\
&\geq \delta \|w\|_{\mathcal{V}}^2
\end{aligned}$$

if ϵ is chosen in $(0, \beta_0)$ and $\gamma > 0$ is chosen so large that

$$\gamma \alpha - \dot{M}_1 - \frac{M_2^2}{2\epsilon} > 0,$$

for

$$\delta = \min \left\{ \frac{\alpha}{2}, (\beta_0 - \epsilon) e^{-\gamma T}, \frac{1}{2} \left(\gamma \alpha - \dot{M}_1 - \frac{M_2^2}{2\epsilon} \right) e^{-\gamma T} \right\}.$$

This proves coerciveness.

Now define $L \in \mathcal{W}'$ by

$$Lw = \mathbf{a}_1(0, u_0, w(0)) + \int_0^T (f(t) | \dot{w}(t))_H e^{-\gamma t} dt,$$

where $u_0 \in V$ is the given initial value and $f \in L^2(0, T; H)$ the given inhomogeneity. By Lions' Representation Theorem there exists $u \in \mathcal{V}$ such that

$$E(u, w) = Lw \tag{5.4}$$

for all $w \in \mathcal{W}$. Let $\psi \in \mathcal{D}(0, T)$, $v \in V$. Choose $w(t) = \int_0^t \psi(s) ds \cdot v$. Then

$w \in \mathcal{W}$. Hence by (5.4),

$$\begin{aligned} \int_0^T (B(t)\dot{u}(t) | v)_H \psi(t) e^{-\gamma t} dt + \int_0^T \mathbf{a}(t, u(t), v) \psi(t) e^{-\gamma t} dt \\ = \int_0^T (f(t) | v)_H \psi(t) e^{-\gamma t} dt. \end{aligned}$$

Since $\psi \in \mathcal{D}(0, T)$ is arbitrary, it follows that

$$(B(t)\dot{u}(t) | v)_H + \mathbf{a}(t, u(t), v) = (f(t) | v)_H \quad (5.5)$$

a.e. for all $v \in V$. Let V_0 be a countable, dense subset of V . We find a null set $N \subset [0, T]$ such that (5.5) holds for all $t \in [0, T] \setminus N$ and all $v \in V_0$, and hence for all $v \in \overline{V_0} = V$. Thus, for $t \in [0, T] \setminus N$ one has $u(t) \in D(A(t))$ and

$$B(t)\dot{u}(t) + A(t)u(t) = f(t). \quad (5.6)$$

We introduce (5.6) into (5.4) and find that for all $w \in \mathcal{W}$,

$$\begin{aligned} \int_0^T (f(t) | \dot{w}(t))_H e^{-\gamma t} dt + \mathbf{a}_1(0, u(0), w(0)) = E(u, w) \\ = Lw = \int_0^T (f(t) | \dot{w}(t))_H e^{-\gamma t} dt + \mathbf{a}_1(0, u_0, w(0)). \end{aligned}$$

Consequently

$$\mathbf{a}_1(0, u(0), w(0)) = \mathbf{a}_1(0, u_0, w(0))$$

for all $w \in \mathcal{W}$. Letting $w(t) \equiv u(0) - u_0$ we conclude that

$$\alpha \|u(0) - u_0\|_V^2 \leq \mathbf{a}_1(0, u(0) - u_0, u(0) - u_0) = 0.$$

Thus $u(0) = u_0$. We have shown that u is a solution.

b) Next we prove that $u \in C([0, T]; V)$. Let $u \in H^1(0, T; H) \cap L^2(0, T; V)$ be a solution. Note that

$$\|\mathcal{A}_2(t)\|_{\mathcal{L}(V, H)} \leq M_2 \quad (t \in [0, T]).$$

Since

$$\mathcal{A}_1 u(t) = f(t) - B(t)\dot{u}(t) - \mathcal{A}_2(t)u(t),$$

it follows that $\mathcal{A}_1 u \in L^2(0, T; H)$. Thus $u \in MR_{\mathbf{a}_1}(H) \subset C([0, T]; V)$ by Theorem 4.2.

c) We prove the estimate (5.1). Since $u \in MR_{\mathbf{a}_1}$ by b) it follows from Theorem 4.2 that $\mathbf{a}_1(\cdot, u(\cdot), u(\cdot))e^{\gamma \cdot} \in W^{1,1}(0, T)$ and

$$\begin{aligned} [\mathbf{a}_1(\cdot, u(\cdot), u(\cdot))e^{-\gamma \cdot}]' = \dot{\mathbf{a}}_1(\cdot, u(\cdot), u(\cdot))e^{-\gamma \cdot} \\ + 2 \operatorname{Re}(A_1(\cdot)u(\cdot) | \dot{u}(\cdot))_H e^{-\gamma \cdot} - \gamma \mathbf{a}_1(\cdot, u(\cdot), u(\cdot))e^{-\gamma \cdot}. \end{aligned} \quad (5.7)$$

Now even though u might not be in \mathcal{W} the above coerciveness estimate goes

through if we use (5.7) instead of (5.3). In fact

$$\begin{aligned}
& \|f\|_{L^2(0,T;H)} \|\dot{u}\|_{L^2(0,T;H)} + \frac{1}{2} M_1 \|u_0\|_V^2 \\
& \geq \operatorname{Re} \int_0^T (f(t) | \dot{u}(t))_H e^{-\gamma t} dt + \frac{1}{2} \mathbf{a}_1(0, u(0), u(0)) \\
& = \operatorname{Re} \int_0^T (B(t)\dot{u}(t) | \dot{u}(t))_H e^{-\gamma t} dt + \operatorname{Re} \int_0^T (A(t)u(t) | \dot{u}(t))_H e^{-\gamma t} dt \\
& \quad + \frac{1}{2} \mathbf{a}_1(0, u(0), u(0)) \\
& \geq \beta_0 \int_0^T \|\dot{u}(t)\|_H^2 e^{-\gamma t} dt + \operatorname{Re} \int_0^T (A_1(t)u(t) | \dot{u}(t))_H e^{-\gamma t} dt \\
& \quad - M_2 \int_0^T \|u(t)\|_V \|\dot{u}(t)\|_H e^{-\gamma t} dt + \frac{1}{2} \mathbf{a}_1(0, u(0), u(0)) \\
& = \beta_0 \int_0^T \|\dot{u}(t)\|_H^2 e^{-\gamma t} dt + \frac{1}{2} \int_0^T [\mathbf{a}_1(t, u(t), u(t)) e^{-\gamma t}] dt \\
& \quad - \frac{1}{2} \int_0^T \dot{\mathbf{a}}_1(t, u(t), u(t)) e^{-\gamma t} dt + \frac{\gamma}{2} \int_0^T \mathbf{a}_1(t, u(t), u(t)) e^{-\gamma t} dt \\
& \quad - M_2 \int_0^T \|u(t)\|_V \|\dot{u}(t)\|_H e^{-\gamma t} dt + \frac{1}{2} \mathbf{a}_1(0, u(0), u(0)) \\
& \geq \beta_0 \int_0^T \|\dot{u}(t)\|_H^2 e^{-\gamma t} dt + \frac{1}{2} \mathbf{a}_1(T, u(T), u(T)) e^{-\gamma T} - \frac{1}{2} \mathbf{a}_1(0, u(0), u(0)) \\
& \quad - \frac{1}{2} \dot{M}_1 \int_0^T \|u(t)\|_V^2 e^{-\gamma t} dt + \frac{\gamma}{2} \alpha \int_0^T \|u(t)\|_V^2 e^{-\gamma t} dt \\
& \quad - \epsilon \int_0^T \|\dot{u}(t)\|_H^2 e^{-\gamma t} dt - \frac{M_2^2}{4\epsilon} \int_0^T \|u(t)\|_V^2 e^{-\gamma t} dt + \frac{1}{2} \mathbf{a}_1(0, u(0), u(0)) \\
& \geq (\beta_0 - \epsilon) \int_0^T \|\dot{u}(t)\|_H^2 e^{-\gamma t} dt + \frac{1}{2} \left(\gamma\alpha - \dot{M}_1 - \frac{M_2^2}{2\epsilon} \right) \int_0^T \|u(t)\|_V^2 e^{-\gamma t} dt \\
& \geq \delta \|u\|_{MR(V,H)}^2
\end{aligned}$$

if ϵ is chosen in $(0, \beta_0)$ and $\gamma > 0$ is chosen so large that

$$\gamma\alpha - \dot{M}_1 - \frac{M_2^2}{2\epsilon} > 0,$$

for

$$\delta = \min \left\{ (\beta_0 - \epsilon) e^{-\gamma T}, \frac{1}{2} \left(\gamma\alpha - \dot{M}_1 - \frac{M_2^2}{2\epsilon} \right) e^{-\gamma T} \right\}.$$

Now Young's inequality implies (5.1).

d) Uniqueness: The difference u of two solutions is in $MR_\alpha(H)$ and satisfies

$$\begin{aligned}
B(t)\dot{u}(t) + A(t)u(t) &= 0 \quad \text{a.e.} \\
u(0) &= 0.
\end{aligned}$$

Thus (5.1) shows that $u \equiv 0$. □

Our proof of Theorem 5.1 is partly inspired by Lions' proof of [Lio61, Théorème 6.1, p. 65] which is valid for $u_0 = 0$ if \mathbf{a} is symmetric and C^1 for $B \equiv Id$. One difference is that we have to take care of the fact that $u_0 \neq 0$. This is done by applying Lions' Representation Theorem to a different Hilbert space and modified form which incorporate initial values.

Remark 5.3. Theorem 5.1 remains true if \mathbf{a}_1 is merely quasi-coercive instead of coercive. In fact, then we may replace \mathbf{a}_2 by $\tilde{\mathbf{a}}_2(t, u, v) = \mathbf{a}_2(t, u, v) - \omega(u|v)_H$ and \mathbf{a}_1 by $\tilde{\mathbf{a}}_1(t, u, v) = \mathbf{a}_1(t, u, v) + \omega(u|v)_H$ and have $\mathbf{a} = \tilde{\mathbf{a}}_1 + \tilde{\mathbf{a}}_2$ in the desired form.

Remark 5.4 (Square root property). For $f \equiv 0$ problem (1.1) does not always have a positive answer even in the autonomous case. If $\mathbf{a}(t, u, v) = \mathbf{a}(u, v)$ does not depend on time, then $\mathcal{A} = \mathcal{A}(t) \in \mathcal{L}(V, V')$ does not depend on time. Denote by A the part of \mathcal{A} in H . Then $-A$ generates a holomorphic C_0 -semigroup $(T(t))_{t \geq 0}$ on H . Let $f \equiv 0$. Then for $u_0 \in H$, $u(t) = T(t)u_0$ defines the solution $u \in H^1(0, T; V') \cap L^2(0, T; V)$ of $\dot{u}(t) + Au(t) = 0$ a.e. with $u(0) = u_0$.

One has $u \in H^1(0, T; H)$ if and only if $u_0 \in D(A^{1/2})$ (see [Are04, 4.4.10, 5.3.1, 6.2.2]). However by an example due to A. McIntosh it may happen that $V \not\subset D(A^{1/2})$. We say that the form has *the square root property* if $V = D(A^{1/2})$. The square root property holds for second order differential operators with measurable coefficients and Dirichlet boundary conditions on a Lipschitz domain. This is a version of the famous Kato's square root problem, solved in [AT03].

If the sesquilinear form \mathbf{a} is of the special form $\mathbf{a} = \mathbf{a}_1 + \mathbf{a}_2$ with \mathbf{a}_1 symmetric and \mathbf{a}_2 continuous on $V \times H$, then it has the square root property by a result in [McI72]. Our results give an alternative proof of this statement. In fact, Theorem 5.1 shows that $V \subset D(A^{1/2})$ and Theorem 4.2 that $D(A^{1/2}) \subset V$. However, our methods do not work for the more general case where $|\mathbf{a}_2(u, u)| \leq C\|u\|_V\|u\|_H$ (i.e., $u = v$) considered by McIntosh.

6 Applications

This section is devoted to applications of our results on existence and maximal regularity of Section 5 to concrete evolution equations. We show how they can be applied to both linear and non-linear evolution equations. We give examples illustrating the theory without seeking for generality. In all examples the underlying field is \mathbb{R} .

6.1 The Laplacian with non-autonomous Robin boundary conditions

Let Ω be a bounded domain of \mathbb{R}^d with Lipschitz boundary Γ . Denote by σ be the $(d-1)$ -dimensional Hausdorff measure on Γ . Let

$$\beta : [0, T] \times \Gamma \rightarrow \mathbb{R}$$

be a bounded measurable function which is Lipschitz continuous w.r.t. the first variable, i.e.,

$$|\beta(t, x) - \beta(s, x)| \leq M|t - s| \tag{6.1}$$

for some constant M and all $t, s \in [0, T]$, $x \in \Gamma$. We consider the symmetric form

$$\mathfrak{a} : [0, T] \times H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{R}$$

defined by

$$\mathfrak{a}(t, u, v) = \int_{\Omega} \nabla u \nabla v \, dx + \int_{\Gamma} \beta(t, \cdot) uv \, d\sigma. \quad (6.2)$$

In the second integral we omitted the trace symbol; we should write $u|_{\Gamma}v|_{\Gamma}$ if we want to be more precise. The form \mathfrak{a} is $H^1(\Omega)$ -bounded and quasi-coercive. The first statement follows readily from the continuity of the trace operator and the boundedness of β . The second one is a consequence of the inequality

$$\int_{\Gamma} |u|^2 \, d\sigma \leq \epsilon \|u\|_{H^1}^2 + c_{\epsilon} \|u\|_{L^2(\Omega)}^2, \quad (6.3)$$

which is valid for all $\epsilon > 0$ (c_{ϵ} is a constant depending on ϵ). Note that (6.3) is a consequence of compactness of the trace as an operator from $H^1(\Omega)$ into $L^2(\Gamma, d\sigma)$, see [Nec67, Chap. 2 § 6, Theorem 6.2].

The operator $A(t)$ associated with $\mathfrak{a}(t, \cdot, \cdot)$ on $H := L^2(\Omega)$ is (minus) the Laplacian with time dependent Robin boundary conditions

$$\partial_{\nu} u(t) + \beta(t, \cdot) u = 0 \text{ on } \Gamma.$$

Here we use the following weak definition of the normal derivative. Let $v \in H^1(\Omega)$ such that $\Delta v \in L^2(\Omega)$. Let $h \in L^2(\Gamma, d\sigma)$. Then $\partial_{\nu} v = h$ by definition if $\int_{\Omega} \nabla v \nabla w + \int_{\Omega} \Delta v w = \int_{\Gamma} h w \, d\sigma$ for all $w \in H^1(\Omega)$. Based on this definition, the domain of $A(t)$ is the set

$$D(A(t)) = \{v \in H^1(\Omega) : \Delta v \in L^2(\Omega), \partial_{\nu} v + \beta(t)v|_{\Gamma} = 0\},$$

and for $v \in D(A(t))$ the operator is given by $A(t)v = -\Delta v$.

By Theorem 5.1, the heat equation

$$\begin{cases} \dot{u}(t) - \Delta u(t) = f(t) \\ u(0) = u_0 \in H^1(\Omega) \\ \partial_{\nu} u(t) + \beta(t, \cdot) u = 0 \text{ on } \Gamma \end{cases}$$

has a unique solution $u \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ whenever $f \in L^2(0, T, L^2(\Omega))$. This example is also valid for more general elliptic operators than the Laplacian. We could even include elliptic operators with time dependent coefficients.

6.2 Schrödinger operators with time-dependent potentials

Let $0 \leq m_0 \in L^1_{loc}(\mathbb{R}^d)$ and $m : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a measurable function for which there exist positive constants α_1, α_2 and M such that for a.e. x

$$\alpha_1 m_0(x) \leq m(t, x) \leq \alpha_2 m_0(x)$$

and

$$|m(t, x) - m(s, x)| \leq M|t - s|m_0(x) \quad x\text{-a.e.}$$

for all $t, s \in [0, T]$. We define the form

$$\mathfrak{a}(t, u, v) = \int_{\mathbb{R}^d} \nabla u \nabla v \, dx + \int_{\mathbb{R}^d} m(t, x) uv \, dx$$

with domain

$$V = \left\{ u \in H^1(\mathbb{R}^d) : \int_{\mathbb{R}^d} m_0(x) |u|^2 \, dx < \infty \right\}.$$

It is clear that V is a Hilbert space for the norm $\|u\|_V$ given by

$$\|u\|_V^2 = \int_{\mathbb{R}^d} |\nabla u|^2 \, dx + \int_{\mathbb{R}^d} m_0(x) |u|^2 \, dx.$$

In addition, \mathfrak{a} is V -bounded and coercive. Its associated operator on $L^2(\mathbb{R}^d)$ is formally given by

$$A(t) = -\Delta + m(t, \cdot).$$

Given $f \in L^2(0, T, L^2(\mathbb{R}^d))$ and $u_0 \in V$, we apply Theorem 5.1 and obtain a unique solution $u \in H^1(0, T; L^2(\mathbb{R}^d)) \cap L^2(0, T; V)$ of the evolution equation

$$\begin{cases} \dot{u}(t) - \Delta u(t) + m(t, \cdot)u(t) = f(t) & \text{a.e.} \\ u(0) = u_0. \end{cases}$$

6.3 A quasi-linear heat equation

In this subsection we consider the non-linear evolution equation

$$(NLCP) \begin{cases} \dot{u}(t) = m(t, u(t))\Delta u(t) + f(t) \\ u(0) = u_0 \in H^1(\Omega) \\ \partial_\nu u(t) + \beta(t, \cdot)u(t) = 0 \text{ on } \Gamma \end{cases}$$

The function m is supposed to be continuous from $[0, T] \times \mathbb{R}$ with values in $[\delta, \frac{1}{\delta}]$ for some constant $\delta > 0$. The domain $\Omega \subset \mathbb{R}^d$ is bounded with Lipschitz boundary and the function β satisfies (6.1). By a *solution* u of (NLCP) we mean a function $u \in H^1(0, T, L^2(\Omega)) \cap L^2(0, T, H^1(\Omega))$ such that $\Delta u(t) \in L^2(\Omega)$ t -a.e. and the equality $\dot{u}(t) = m(t, u(t))\Delta u(t) + f(t)$ holds for a.e. $t \in [0, T]$. We have the following result.

Theorem 6.1. *Let $f \in L^2(0, T, L^2(\Omega))$ and $u_0 \in H^1(\Omega)$. Then there exists a solution $u \in H^1(0, T, L^2(\Omega)) \cap L^2(0, T, H^1(\Omega))$ of (NLCP).*

We shall use Schauder's fixed point theorem to prove this result. This idea is classical in PDE but it is here that we need in an essential way the maximal regularity result for the corresponding non-autonomous linear evolution equation established in Section 5. Some of our arguments are similar to those in [AC10]. We emphasize that we could replace in (NLCP) the Laplacian by an elliptic operator with time-dependent coefficients (with an appropriate Lipschitz continuity with respect to t). Again, we do not search for further generality in order to make the ideas in the proof more transparent.

Proof of Theorem 6.1. Let us denote by H the Hilbert space $L^2(\Omega)$, let $V = H^1(\Omega)$ and denote by $A(t)$ the operator on H associated with the form $\mathfrak{a}(t, \cdot, \cdot)$ defined by (6.2). As made precise in Subsection 6.1, $A(t)$ is the negative Laplacian with boundary conditions $\partial_\nu u(t) + \beta(t, \cdot)u(t) = 0$ on Γ . Given $v \in L^2(0, T, H)$ we set for $g \in H$

$$B_v(t)g = \frac{1}{m(t, v(t))}g.$$

Note that

$$\delta \|g\|_H^2 \leq (B_v(t)g | g)_H \leq \frac{1}{\delta} \|g\|_H^2. \quad (6.4)$$

By Theorem 5.1 there exists a unique $u \in MR(V, H) = H^1(0, T, H) \cap L^2(0, T, V)$ such that

$$\begin{cases} B_v(t)\dot{u}(t) = -\mathcal{A}(t)u(t) + \frac{1}{m(t, v(t))}f(t) \\ u(0) = u_0 \in V \end{cases}$$

Now we consider the mapping

$$S : L^2(0, T, H) \rightarrow L^2(0, T, H), \quad Sv = u.$$

By the estimate (5.1) of Theorem 5.1, we have

$$\|u\|_{MR(V, H)} \leq C \left[\|f\|_{L^2(0, T, H)} + \|u_0\|_V \right], \quad (6.5)$$

with a constant C which is independent of v . In particular, the image of S is bounded in $MR(V, H)$. Since $V = H^1(\Omega)$ is compactly embedded into $H = L^2(\Omega)$ (recall that Ω is bounded and has Lipschitz boundary), we obtain from the Aubin-Lions lemma that $MR(V, H)$ is compactly embedded into $L^2(0, T, H)$, see [Sho97, p. 106]. As a consequence, it is enough to prove continuity of S and then apply Schauder's fixed point theorem to find $u \in MR(V, H)$ such that $Su = u$. Such u is a solution of (NLCP).

Now we prove continuity of S . For this, we consider a sequence (v_n) which converges to v in $L^2(0, T, H)$ and let $u_n = S(v_n)$. It is enough to prove that (u_n) has a subsequence which converges to Sv . For each $n \in \mathbb{N}$, u_n is the solution of

$$(CP)_n \begin{cases} B_{v_n}(t)\dot{u}_n(t) = -\mathcal{A}(t)u_n(t) + \frac{1}{m(t, v(t))}f(t) \\ u_n(0) = u_0 \in V \end{cases}$$

By (6.5), the sequence (u_n) is bounded in $MR(V, H)$ and hence by extracting a subsequence we may assume that (u_n) converges weakly to some u in $MR(V, H)$. Then $(u_n)_{n \in \mathbb{N}}$ converges in norm to u in $L^2(0, T, H)$ by the Aubin-Lions lemma. By extracting a subsequence again we can also assume that $v_n(t)(x) \rightarrow v(t)(x)$ a.e. with respect to t and to x . Now let $g \in V$ and $\psi \in \mathcal{D}(0, T)$, and consider

$$\begin{aligned} \int_0^T (B_{v_n}(t)\dot{u}_n(t) | g)_H \psi(t) \, dt &= \int_0^T (\dot{u}_n(t) | B_{v_n}(t)g)_H \psi(t) \, dt \\ &= (\dot{u}_n(\cdot) | B_{v_n}(\cdot)g\psi(\cdot))_{L^2(0, T, H)}. \end{aligned}$$

The last term converges to $(\dot{u}(\cdot) | B_v(\cdot)g\psi(\cdot))_{L^2(0,T,H)}$. In fact \dot{u}_n converges weakly to \dot{u} in $L^2(0,T,H)$ and $B_{v_n}(\cdot)g$ converges in $L^2(0,T,H)$ to $B_v(\cdot)g$, by the Dominated Convergence Theorem. We have proved that

$$\int_0^T (B_{v_n}(t)\dot{u}_n(t) | g)_H \psi(t) dt \rightarrow \int_0^T (B_v(t)\dot{u}(t) | g)_H \psi(t) dt \quad (n \rightarrow \infty). \quad (6.6)$$

On the other hand,

$$\begin{aligned} \int_0^T \langle \mathcal{A}(t)u_n(t), g \rangle \psi(t) dt &= \int_0^T \mathbf{a}(t, u_n(t), g) \psi(t) dt \\ &= \int_0^T \mathbf{a}(t, g, u_n(t)) \psi(t) dt \\ &= \int_0^T \langle \mathcal{A}(t)g, u_n(t) \rangle \psi(t) dt \\ &= \langle \mathcal{A}(\cdot)g, u_n(\cdot) \rangle_{L^2(0,T,V'), L^2(0,T,V)}. \end{aligned}$$

Since (u_n) converges weakly to u in $L^2(0,T,V)$ it follows that the last term converges to $\langle \mathcal{A}(\cdot)g, u(\cdot) \rangle_{L^2(0,T,V'), L^2(0,T,V)}$. Hence

$$\int_0^T \langle \mathcal{A}(t)u_n(t), g \rangle \psi(t) dt \rightarrow \int_0^T \langle \mathcal{A}(t)u(t), g \rangle \psi(t) dt \quad (n \rightarrow \infty). \quad (6.7)$$

Therefore, we obtain from $(CP)_n$, (6.7) and (6.6) that

$$\int_0^T (B_v(t)\dot{u}(t) | g)_H \psi(t) dt = \int_0^T (-\mathcal{A}(t)u(t) + \frac{1}{m(t, v(t))}f(t) | g)_H \psi(t) dt.$$

Since this true for all $\psi \in \mathcal{D}(0,T)$ and all $g \in V$ it follows that

$$B_v(t)\dot{u}(t) = -\mathcal{A}(t)u(t) + \frac{1}{m(t, v(t))}f(t)$$

for a.e. $t \in [0, T]$. Finally, the fact that $MR(V, H) \hookrightarrow C([0, T]; H)$ together with the weak convergence in $MR(V, H)$ of (u_n) to u imply

$$u_0 = u_n(0) \rightarrow u(0).$$

We conclude that $u = Sv$ which is the desired identity. \square

7 Appendix: Vector-valued 1-dimensional Sobolev spaces

In this section we consider Sobolev spaces defined on an interval $(0, T)$, where $T > 0$, with values in a Hilbert space H . Given $u \in L^2(0, T; H)$ a function $\dot{u} \in L^2(0, T; H)$ is called *the weak derivative* of u if

$$-\int_0^T u(s)\dot{\varphi}(s) ds = \int_0^T \dot{u}(s)\varphi(s) ds$$

for all $\varphi \in C_c^\infty(0, T)$. Thus we merely test with scalar-valued test functions φ on $(0, T)$. It is clear that the weak derivative \dot{u} of u is unique whenever it exists. We let

$$H^1(0, T; H) := \{u \in L^2(0, T; H) : u \text{ has a weak derivative } \dot{u} \in L^2(0, T; H)\}.$$

It is easy to see that $H^1(0, T; H)$ is a Hilbert space for the scalar product

$$(u | v)_{H^1(0, T; H)} := \int_0^T \left[(u(t) | v(t))_H + (\dot{u}(t) | \dot{v}(t))_H \right] dt.$$

As in the scalar case [Bre11, Section 8.2] one shows the following.

Proposition 7.1. a) *Let $u \in H^1(0, T; H)$. Then there exists a unique $w \in C([0, T]; H)$ such that $u(t) = w(t)$ a.e. and*

$$w(t) = w(0) + \int_0^t \dot{u}(s) ds.$$

b) *Conversely, if $w \in C([0, T]; H)$, $v \in L^2(0, T; H)$ such that $w(t) = w(0) + \int_0^t v(s) ds$, then $w \in H^1(0, T; H)$ and $\dot{w} = v$.*

In the following we always identify $u \in H^1(0, T; H)$ with its unique continuous representative w according to a).

Now we use Proposition 7.1 to prove an integration by parts formula. Let V, H be Hilbert spaces such that $V \xrightarrow{d} H$. We consider H as a dense subspace of V' , cf. Section 2.

Lemma 7.2. *Let $u \in H^1(0, T; V)$ and $v \in H^1(0, T; V')$. Then $\langle v(\cdot), u(\cdot) \rangle \in W^{1,1}(0, T)$ and*

$$\langle v(\cdot), u(\cdot) \rangle' = \langle \dot{v}(\cdot), u(\cdot) \rangle + \langle v(\cdot), \dot{u}(\cdot) \rangle$$

Proof. By Fubini's Theorem we have

$$\begin{aligned} \int_0^t \langle \dot{v}(s), u(s) \rangle ds &= \int_0^t \left\langle \dot{v}(s), u(0) + \int_0^s \dot{u}(r) dr \right\rangle ds \\ &= \langle v(t), u(0) \rangle - \langle v(0), u(0) \rangle + \int_0^t \int_0^s \langle \dot{v}(s), \dot{u}(r) \rangle dr ds \\ &= \langle v(t), u(0) \rangle - \langle v(0), u(0) \rangle + \int_0^t \int_r^t \langle \dot{v}(s), \dot{u}(r) \rangle ds dr \\ &= \langle v(t), u(0) \rangle - \langle v(0), u(0) \rangle + \int_0^t \langle v(t), \dot{u}(r) \rangle - \langle v(r), \dot{u}(r) \rangle dr \\ &= \langle v(t), u(t) \rangle - \langle v(0), u(0) \rangle - \int_0^t \langle v(r), \dot{u}(r) \rangle dr. \end{aligned}$$

Thus

$$\langle v(t), u(t) \rangle = \langle v(0), u(0) \rangle + \int_0^t \langle \dot{v}(s), u(s) \rangle ds + \int_0^t \langle v(s), \dot{u}(s) \rangle ds$$

which proves the claim. \square

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