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# SYMMETRIC ITINERARY SETS

MICHAEL F. BARNESLEY AND NICOLAE MIHALACHE

ABSTRACT. We consider a one parameter family of dynamical systems  $W : [0, 1] \rightarrow [0, 1]$  constructed from a pair of monotone increasing diffeomorphisms  $W_i$ , such that  $W_i^{-1} : [0, 1] \rightarrow [0, 1]$ , ( $i = 0, 1$ ). We characterize the set of symbolic itineraries of  $W$  using an attractor  $\bar{\Omega}$  of an iterated closed relation, in the terminology of McGehee, and prove that there is a member of the family for which  $\bar{\Omega}$  is symmetrical.

## 1. INTRODUCTION

Let  $W_0 : [0, a] \rightarrow [0, 1]$  and  $W_1 : [1-b, 1] \rightarrow [0, 1]$  be continuous and differentiable, and such that  $a + b > 1$ ,  $W_0(0) = W_1(1-b) = 0$ ,  $W_0(a) = W_1(1) = 1$ . Let the derivatives  $W'_i(x)$  ( $i = 0, 1$ ) be uniformly bounded below by  $d > 1$ .

For  $\rho \in [1-b, a]$  we define  $W : [0, 1] \rightarrow [0, 1]$  by

$$[0, 1] \ni x \mapsto \begin{cases} W_0(x) & \text{if } x \in [0, \rho] \\ W_1(x) & \text{otherwise.} \end{cases}$$

See Figure 1. Similarly, we define  $W_+ : [0, 1] \rightarrow [0, 1]$  by replacing  $[0, \rho]$  by  $[0, \rho)$ .

Let  $I = \{0, 1\}$ . Let  $I^\infty = \{0, 1\} \times \{0, 1\} \times \dots$  have the product topology induced from the discrete topology on  $I$ . For  $\sigma \in I^\infty$  write  $\sigma = \sigma_0\sigma_1\sigma_2\dots$ , where  $\sigma_k \in I$  for all  $k \in \mathbb{N}$ . The product topology on  $I^\infty$  is the same as the topology induced by the metric  $d(\omega, \sigma) = 2^{-k}$  where  $k$  is the least index such that  $\omega_k \neq \sigma_k$ . It is well known that  $(I^\infty, d)$  is a compact metric space. We define a total order relation  $\preceq$  on  $I^\infty$ , and on  $I^n$  for any  $n \in \mathbb{N}$ , by  $\sigma \prec \omega$  if  $\sigma \neq \omega$  and  $\sigma_k < \omega_k$  where  $k$  is the least index such that  $\sigma_k \neq \omega_k$ . For  $\sigma \in I^\infty$  and  $n \in \mathbb{N}$  we write  $\sigma|_n = \sigma_0\sigma_1\sigma_2\dots\sigma_n$ .  $I^\infty$  is the appropriate space in which to embed and study the itineraries of the family of discontinuous dynamical systems  $W : [0, 1] \rightarrow [0, 1]$ .

For  $W_{(+)} \in \{W, W_+\}$  let  $W_{(+)}^k$  denote  $W_{(+)}$  composed with itself  $k$  times, for  $k \in \mathbb{N}$ , and let  $W_{(+)}^{-k} = (W_{(+)}^k)^{-1}$ . We define a map  $\tau : [0, 1] \rightarrow I^\infty$ , using all of the orbits of  $W$ , by

$$\tau(x) = \sigma_0\sigma_1\sigma_2\dots$$

where  $\sigma_k$  equals 0, or 1, according as  $W^k(x) \in [0, \rho]$ , or  $(\rho, 1]$ , respectively. We call  $\tau(x)$  the *itinerary* of  $x$  under  $W$ , or an *address* of  $x$ , and we call  $\Omega = \tau([0, 1])$  an *address space* for  $[0, 1]$ . Similarly, we define  $\tau^+ : [0, 1] \rightarrow I^\infty$  so that  $\tau^+(x)_k$  equals 0, or 1, according as  $W_+^k(x) \in [0, \rho)$ , or  $[\rho, 1]$ , respectively; and we define  $\Omega_+ = \tau^+([0, 1])$ . Note that  $W$ ,  $W_+$ ,  $\Omega$ ,  $\Omega_+$ ,  $\tau$ , and  $\tau^+$  all depend on  $\rho$ .

The main goals of this paper are to characterise  $\bar{\Omega}$  and to show that there exists a value of  $\rho$  such that  $\bar{\Omega}$  is symmetric.

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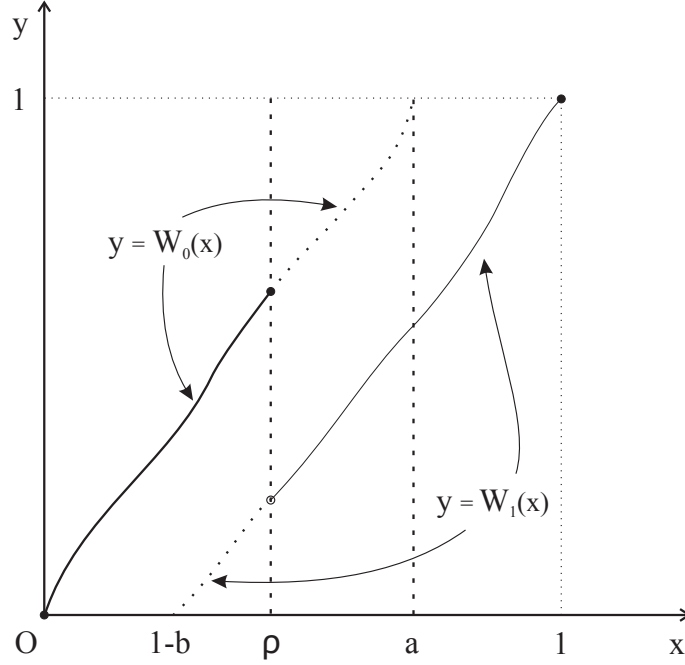


FIGURE 1. The piecewise continuous dynamical system  $W:[0,1] \rightarrow [0,1]$  is defined in terms of two monotone strictly increasing differentiable functions  $W_0(x)$  and  $W_1(x)$ , and a real parameter  $\rho$ .

**Theorem 1.** *Let an iterated closed relation  $r \subset I^\infty \times I^\infty$  be defined by*

$$r := \{(\sigma, 0\sigma) \in I^\infty \times I^\infty : \sigma \preceq \alpha\} \cup \{(\sigma, 1\sigma) \in I^\infty \times I^\infty : \sigma \succeq \beta\}$$

*where  $\alpha = \tau(W_0(\rho))$  and  $\beta = \tau^+(W_1(\rho))$ . The only attractors of  $r$  are  $\{\bar{0}\}$ ,  $\{\bar{1}\}$ ,  $\{\bar{0}, \bar{1}\}$ , and  $\bar{\Omega}$ . The corresponding dual repellers are  $\{\sigma \in I^\infty : \beta \preceq \sigma\}$ ,  $\{\sigma \in I^\infty : \sigma \preceq \alpha\}$ ,  $\{\sigma \in I^\infty : \beta \preceq \sigma\} \cup \{\sigma \in I^\infty : \sigma \preceq \alpha\}$ , and the empty set, respectively. The chain recurrent set for  $r$  is  $\{\bar{0}, \bar{1}\} \cup \{\sigma \in \bar{\Omega} : \beta \preceq \sigma \preceq \alpha\}$ .*

We write  $\bar{E}$  to denote the closure of a set  $E$ . But we write  $\bar{0} = 000\dots$ ,  $\bar{1} = 111\dots \in I^\infty$ . For  $\sigma = \sigma_0\sigma_1\sigma_2\dots \in I^\infty$  we write  $0\sigma$  to mean  $0\sigma_0\sigma_1\sigma_2\dots \in I^\infty$  and  $1\sigma = 1\sigma_0\sigma_1\sigma_2\dots \in I^\infty$ .

Define a symmetry function  $*$  :  $I^\infty \rightarrow I^\infty$  by  $\sigma^* = \omega$  where  $\omega_k = 1 - \sigma_k$  for all  $k$ .

**Theorem 2.** *There exists a unique  $\rho \in [1-b, a]$  such that  $\bar{\Omega}^* = \bar{\Omega}$ .*

Theorem 1 tells us that  $\bar{\Omega}$  is fixed by itineraries of two inverse images of the critical point  $\rho$ , and provides the basis for a stable algorithm to determine  $\bar{\Omega}$ . It relates the address spaces of dynamical systems of the form of  $W$  to the beautiful theory of iterated closed relations on compact Hausdorff spaces [3], and hence to the work of Charles Conley.

Theorem 2 is interesting in its own right and also because it has applications in digital imaging, as explained and demonstrated, in the special case of affine maps, in [1]. It enables the construction of parameterized families of nondifferentiable

homeomorphisms on  $[0, 1]$ , using pairs of overlapping iterated function systems, see Proposition 4. Theorem 2 generalizes results in [1] to nonlinear  $W_i$ 's. The proof uses symbolic dynamics in place of the geometrical construction outlined in [1]. The approach and results open up the mathematics underlying [1] and [2].

To tie the present work into [1], note that  $\tau$  is a section, as defined in [1], for the hyperbolic iterated function system

$$\mathcal{F} := ([0, 1]; W_0^{-1}, W_1^{-1}).$$

Our observations interrelate to, but are more specialized than, the work of Parry [5]. Our point of view is topological rather than measure-theoretic, and our main results appear to be new.

## 2. BASIC PROPERTIES OF $\tau$

The following list of properties is relatively easy to check. Below the list we elaborate on points 1, 2, and 3.

- (1)  $W^n$  is piecewise differentiable and its derivative is uniformly bounded below by  $d^n$ ; each, except the leftmost branch of  $W^n$ , is defined on an interval of the form  $(r, s]$ .  $W_+^n$  is piecewise differentiable and its derivative is uniformly bounded below by  $d^n$ ; each, except for the rightmost branch of  $W_+^n$ , is defined on an interval of the form  $[r, s)$ .
- (2) If  $(r, s)$  is the interior of the definition domain of a branch of  $W^n$  (and of  $W_+^n$ ) then  $\tau(x)|_n$  is constant on  $(r, s]$ ,  $\tau^+(x)|_n$  is constant on  $[r, s)$ , and  $\tau(x)|_n = \tau^+(x)|_n$  for all  $x \in (r, s)$ .
- (3) The boundary of the definition domain of a branch of  $W^n$  is contained in  $\{0, 1\} \cup \bigcup_{k=0}^{n-1} W^{-k}(\rho)$ ; by (1), the length of such a domain is at most  $d^{-n}$ .
- (4) The set  $\bigcup_{k \in \mathbb{N}} W^{-k}(\rho)$  is dense in  $[0, 1]$ . This follows from (3).
- (5)  $\tau(x) = \tau^+(x)$  unless  $x \in \bigcup_{k \in \mathbb{N}} W^{-k}(\rho)$  in which case  $\tau(x) \prec \tau^+(x)$ .
- (6) Both  $\tau(x)$  and  $\tau^+(x)$  are strictly increasing functions of  $x \in [0, 1]$  and  $\tau(x) \preceq \tau^+(x)$ . This follows from (4) and (5).
- (7) For all  $x \in [0, 1]$ ,  $\tau(x)$  is continuous from the left,  $\tau^+(x)$  is continuous from the right. Moreover, for all  $x \in (0, 1)$ ,

$$\tau(x) = \lim_{\varepsilon \rightarrow 0^+} \tau^+(x - \varepsilon) \text{ and } \tau^+(x) = \lim_{\varepsilon \rightarrow 0^+} \tau(x + \varepsilon).$$

These assertions follow from (2), (3) and (4).

- (8) Each  $x \in W^{-n}(\rho)$ , such that  $\tau(x)|_n$  is constant, moves continuously with respect to  $\rho$  with positive velocity bounded above by  $d^{-n}$ . This follows from (1).
- (9) For  $x \in (0, 1) \setminus \bigcup_{k=0}^n W^{-k}(\rho)$ ,  $\tau(x)|_n = \tau^+(x)|_n$  is locally constant with respect to  $\rho$ ; moreover, this holds if  $x$  depends continuously on  $\rho$ . This follows from (2), (3) and (6).
- (10) The symmetry function  $*$  :  $I^\infty \rightarrow I^\infty$  is strictly decreasing and continuous.
- (11) For any  $\sigma|_n \in I^n$ ,  $n \in \mathbb{N}$ , the set

$$\mathcal{I}(\sigma|_n) := \{x \in [0, 1] : \tau(x)|_n = \sigma|_n \text{ or } \tau^+(x)|_n = \sigma|_n\},$$

is either empty or a non-degenerate compact interval of length at most  $d^{-n}$ .  
This follows from (2), (3) and (6).

- (12) The projection  $\hat{\pi} : I^\infty \rightarrow [0, 1]$  is well-defined by

$$\hat{\pi}(\sigma) = \sup\{x \in [0, 1] : \tau^+(x) \preceq \sigma\} = \inf\{x \in [0, 1] : \tau(x) \succeq \sigma\}.$$

This follows from (6).

- (13) The projection  $\hat{\pi} : I^\infty \rightarrow [0, 1]$  is increasing, by (6); continuous, by (11); and, by (7),

$$\begin{aligned}\hat{\pi}(\tau(x)) &= \hat{\pi}(\tau^+(x)) = x \text{ for all } x \in [0, 1], \\ \tau(\hat{\pi}(\sigma)) &\preceq \sigma \preceq \tau^+(\hat{\pi}(\sigma)) \text{ for all } \sigma \in I^\infty.\end{aligned}$$

- (14) Let  $S : I^\infty \rightarrow I^\infty$  denote the left-shift map  $\sigma_0\sigma_1\sigma_2\ldots \mapsto \sigma_1\sigma_2\sigma_3\ldots$ . For all  $\sigma \in I^\infty$  such that  $\sigma \preceq \tau(\rho)$  or  $\sigma \geq \tau^+(\rho)$ ,

$$\hat{\pi}(S(\sigma)) = W(\hat{\pi}(\sigma)).$$

Also  $\hat{\pi}(\tau^+(\rho)) = \rho$  and  $\hat{\pi}(S(\tau^+(\rho))) = W_1(\rho)$ . These statements follow from (7).

Here we elaborate on points (1), (2) and (3). Consider the piecewise continuous function  $W^k(x)$ , for  $k \in \{1, 2, \dots\}$ . Its discontinuities are at  $\rho$  and, for  $k > 1$ , other points in  $(0, 1)$ , each of which can be written in the form  $W_{\sigma_0}^{-1} \circ W_{\sigma_2}^{-1} \circ \dots \circ W_{\sigma_{l-1}}^{-1}(\rho)$  for some  $\sigma_0\sigma_1\ldots\sigma_{l-1} \in \{0, 1\}^l$  for some  $l \in \{1, 2, \dots, k-1\}$ . We denote these discontinuities, together with the points 0 and 1, by

$$D_{k,0} := 0 < D_{k,1} < D_{k,2} < \dots < D_{k,D(k)-1} < 1 =: D_{k,D(k)},$$

where  $D(1) = 3, D(2) = 5 < D(3) < D(4) \dots$ . For each  $k \geq 1$ , one of the  $D_{k,j}$ 's is equal to  $\rho$ . For  $k \geq 1$  we have  $W^k(x) = W_0^k(x)$  for  $x \in [D_{k,0}, D_{k,1}]$  and  $W_+^k(x) = W_0^k(x)$  for  $x \in [D_{k,0}, D_{k,1})$ . Similarly  $W^k(x) = W_1^k(x)$  for all  $x \in (D_{k,D(k)-1}, D_{k,D(k)}]$  and  $W_+^k(x) = W_1^k(x)$  for  $x \in [D_{k,D(k)-1}, D_{k,D(k)})$ .

For all  $x \in (D_{k,l}, D_{k,l+1})$  ( $l = 0, 1, \dots, D(k)-1$ ),  $W^k(x) = W_+^k(x) = W_{\theta_k} \circ W_{\theta_{k-1}} \circ \dots \circ W_{\theta_1}(x)$  for some fixed  $\theta_1\theta_2\ldots\theta_k \in \{0, 1\}^k$ . We refer to  $\theta_1\theta_2\ldots\theta_k$  as the *address of the interval*  $(D_{k,l}, D_{k,l+1})$ , we say  $(D_{k,l}, D_{k,l+1})$  that "has address  $\theta_1\theta_2\ldots\theta_k$ ", and we write, by slight abuse of notation,  $\tau((D_{k,l}, D_{k,l+1})) = \theta_1\theta_2\ldots\theta_k$ .

Let  $k > 1$ . Consider two adjacent intervals,  $(D_{k,m-1}, D_{k,m}]$  and  $(D_{k,m}, D_{k,m+1}]$  for  $m \in \{1, 2, \dots, D(k)-1\}$  and  $k > 1$ . Let the one on the right have address  $\theta_0\theta_1\ldots\theta_{k-1}$  and the one on the left have address  $\eta_0\eta_1\ldots\eta_{k-1}$ . Then  $\eta_0\eta_1\ldots\eta_{k-1} \prec \theta_0\theta_1\ldots\theta_{k-1}$  and we have

$$\begin{aligned}\tau(x)|_{k-1} &= \eta_0\eta_1\ldots\eta_{k-1} \text{ for all } x \in (D_{k,m-1}, D_{k,m}], \\ \tau^+(x)|_{k-1} &= \eta_0\eta_1\ldots\eta_{k-1} \text{ for all } x \in [D_{k,m-1}, D_{k,m}), \\ \tau(x)|_{k-1} &= \theta_0\theta_1\ldots\theta_{k-1} \text{ for all } x \in (D_{k,m}, D_{k,m+1}], \\ \tau^+(x)|_{k-1} &= \theta_0\theta_1\ldots\theta_{k-1} \text{ for all } x \in [D_{k,m}, D_{k,m+1}).\end{aligned}$$

In particular,  $\tau(x)|_{k-1}$  and  $\tau^+(x)|_{k-1}$  are constant and equal on each of the open intervals  $(D_{k,m-1}, D_{k,m})$  and have distinct values at the discontinuity points  $\{D_{k,m}\}_{m=1}^{D(k)-1}$ .

3. THE STRUCTURES OF  $\Omega$ ,  $\Omega_+$  AND  $\overline{\Omega}$ .

In this section we characterize  $\Omega$  and  $\Omega_+$  as certain inverse limits, and we characterize  $\overline{\Omega}$  as an attractor of an iterated closed relation on  $I^\infty$ . These inverse limits are natural and they clarify the structures of  $\Omega$  and  $\Omega_+$ . They are implied by the shift invariance of  $\Omega$  and  $\Omega_+$ . Recall that  $S : I^\infty \rightarrow I^\infty$  denotes the left-shift map  $\sigma_0\sigma_1\sigma_2\ldots \mapsto \sigma_1\sigma_2\sigma_3\ldots$ .

**Proposition 1.** (i)  $\tau(W(x)) = S(\tau(x))$  and  $\tau^+(W_+(x)) = S(\tau^+(x))$  for all  $x \in [0, 1]$ .

(ii)  $S(\Omega) = \Omega$  and  $S(\Omega_+) = \Omega_+$ .

*Proof.* (i) This follows at once from the definitions of  $\tau$  and  $\tau^+$ . (ii) This follows from (i) together with  $W([0, 1]) = W_+([0, 1]) = [0, 1]$ .  $\square$

We say that  $\Lambda \subset I^\infty$  is *closed from the left* if, whenever  $\{x_n\}_{n=0}^\infty$  is a non-decreasing sequence of points in  $\Lambda$ ,  $\lim x_n \in \Lambda$ . We say that  $\Lambda \subset I^\infty$  is *closed from the right* if, whenever  $\{x_n\}_{n=0}^\infty$  is a non-increasing sequence in  $\Lambda$ ,  $\lim x_n \in \Lambda$ . For  $S \subset X$ , where  $X = I^\infty$  or  $[0, 1]$ , we write  $L(S) = \{\sigma \in X : \text{there is a non-decreasing sequence } \{z_n\}_{n=0}^\infty \subset S \text{ with } \sigma = \lim z_n\}$  to denote the closure of  $S$  from the left. Analogously, we define  $R(S)$  for the closure of  $S$  from the right.

**Proposition 2.** (i)  $\Omega$  is closed from the left and  $\Omega_+$  is closed from the right;

(ii)  $\overline{\Omega} = \overline{\Omega_+} = \Omega \cup \Omega_+ = \overline{\Omega \cap \Omega_+}$

*Proof.* Proof of (i): By (6)  $\tau : [0, 1] \rightarrow I^\infty$  is monotone strictly increasing. By (7)  $\tau$  is continuous from the left. Let  $\{z_n\}_{n=0}^\infty$  be a non-decreasing sequence of points in  $\Omega$ . Let  $y_n = \tau^{-1}(z_n)$ . Let  $y = \lim y_n \in [0, 1]$ . Since  $\tau$  is continuous from the left,  $\Omega \ni \tau(y) = \tau(\lim y_n) = \lim \tau(y_n) = \lim z_n$ . It follows that  $\Omega$  is closed from the left. Similarly,  $\Omega_+$  is closed from the right.

Proof of (ii): Let  $Q = \{x \in [0, 1] : \tau(x) = \tau^+(x)\}$ . Then by (4)  $\overline{Q} = [0, 1]$ . Also, by (5),

$$\Omega \cap \Omega_+ = \tau([0, 1]) \cap \tau^+([0, 1]) = \tau(Q) = \tau^+(Q).$$

Hence

$$\overline{\Omega \cap \Omega_+} = \overline{\tau(Q)} = \overline{\tau^+(Q)} = \overline{\Omega} = \overline{\Omega_+}.$$

Finally,  $\Omega \cup \Omega_+ = L(\tau(Q)) \cup R(\tau^+(Q)) = L(\tau(Q)) \cup R(\tau(Q)) = \overline{\tau(Q)} = \overline{\Omega}$ .  $\square$

We define  $s_i : I^\infty \rightarrow I^\infty$  by  $s_i(\sigma) = i\sigma$  ( $i = 0, 1$ ). Note that both  $s_0$ , and  $s_1$ , are contractions with contractivity  $1/2$ . We write  $2^{I^\infty}$  to denote the set of all subsets of  $I^\infty$ . For  $\sigma, \omega \in I^\infty$  we define

$$\begin{aligned} [\sigma, \omega] &:= \{\zeta \in I^\infty : \sigma \preceq \zeta \preceq \omega\}, \\ (\sigma, \omega) &:= \{\zeta \in I^\infty : \sigma \prec \zeta \prec \omega\}, \\ (\sigma, \omega] &:= \{\zeta \in I^\infty : \sigma \prec \zeta \preceq \omega\}, \\ [\sigma, \omega) &:= \{\zeta \in I^\infty : \sigma \preceq \zeta \prec \omega\}. \end{aligned}$$

**Proposition 3.** Let  $\alpha = S(\tau(\rho))$  and  $\beta = S(\tau^+(\rho))$ .

(i)  $\Omega = \bigcap_{k \in \mathbb{N}} \Psi^k([\overline{0}, \overline{1}])$  where  $\Psi : 2^{I^\infty} \rightarrow 2^{I^\infty}$  is defined by

$$2^{I^\infty} \ni \Lambda \mapsto s_0(\Lambda \cap [\overline{0}, \alpha]) \cup s_1(\Lambda \cap (\beta, \overline{1})).$$

(ii)  $\Omega_+ = \bigcap_{k \in \mathbb{N}} \Psi_+^k([\bar{0}, \bar{1}])$  where  $\Psi_+ : 2^{I^\infty} \rightarrow 2^{I^\infty}$  is defined by

$$2^{I^\infty} \ni \Lambda \mapsto s_0(\Lambda \cap [\bar{0}, \alpha)) \cup s_1(\Lambda \cap [\beta, \bar{1}]).$$

(iii)  $\bar{\Omega} = \overline{\Omega_+} = \bigcap_{k \in \mathbb{N}} \bar{\Psi}^k([\bar{0}, \bar{1}])$  where  $\bar{\Psi} : 2^{I^\infty} \rightarrow 2^{I^\infty}$  is defined by

$$2^{I^\infty} \ni \Lambda \mapsto s_0(\Lambda \cap [\bar{0}, \alpha]) \cup s_1(\Lambda \cap [\beta, \bar{1}]).$$

*Proof.* Proof of (i): Let  $S|_\Omega : \Omega \rightarrow \Omega$  denote the domain and range restricted shift map. It is readily found that the branches of  $S|_\Omega^{-1} : \Omega \rightarrow \Omega$  are  $s_0|_\Omega : [\bar{0}, \alpha] \cap \Omega \rightarrow \Omega$  where

$$s_0|_\Omega(\sigma) = s_0(\sigma) = 0\sigma \text{ for all } \sigma \in [\bar{0}, \alpha] \cap \Omega,$$

and  $s_1|_\Omega : (\beta, \bar{1}] \cap \Omega \rightarrow \Omega$  where

$$s_1|_\Omega(\sigma) = s_1(\sigma) = 1\sigma \text{ for all } \sigma \in (\beta, \bar{1}] \cap \Omega.$$

(Note that  $\alpha_0 = 1$ ,  $\beta_0 = 0$  and  $\beta \prec \alpha$ .) It follows that

$$S|_\Omega^{-1}(\Lambda) = s_0(\Lambda \cap [\bar{0}, \alpha]) \cup s_1(\Lambda \cap (\beta, \bar{1}]) = \Psi(\Lambda)$$

for all  $\Lambda \subset \Omega$ . Since  $\Omega \subset [\bar{0}, \bar{1}]$  it follows that

$$\Omega = S|_\Omega^{-1}(\Omega) = \Psi(\Omega) \subset \Psi([\bar{0}, \bar{1}]).$$

Also, since  $\Psi([\bar{0}, \bar{1}]) \subset [\bar{0}, \bar{1}]$  it follows that  $\{\Psi^k([\bar{0}, \bar{1}])\}$  is a decreasing (nested) sequence of sets, each of which contains  $\Omega$ ; hence

$$\Omega \subset \bigcap_{k \in \mathbb{N}} \Psi^k([\bar{0}, \bar{1}]).$$

It remains to prove that  $\Omega \supset \bigcap_{k \in \mathbb{N}} \Psi^k([\bar{0}, \bar{1}])$ . We note that  $s_0([\bar{0}, \alpha]) = [\bar{0}, \tau(\rho)]$

and  $s_1((\beta, \bar{1}]) = (\tau^+(\rho), \bar{1}]$ , from which it follows that

$$(3.1) \quad \bigcap_{k \in \mathbb{N}} \Psi^k([\bar{0}, \bar{1}]) = \bigcap_{k \in \mathbb{N}} \{\sigma \in I^\infty : S^k(\sigma) \in [\bar{0}, \tau(\rho)] \cup (\tau^+(\rho), \bar{1}]\}.$$

Let  $\omega \in \bigcap_{k \in \mathbb{N}} \Psi^k([\bar{0}, \bar{1}])$ . Suppose  $\omega \notin \Omega$ . Let

$$\omega_- = \sup\{\sigma \in \Omega : \sigma \preceq \omega\} \text{ and } \omega_+ = \inf\{\sigma \in \Omega : \omega \preceq \sigma\},$$

so that

$$\omega_- \preceq \omega \preceq \omega_+.$$

But  $\omega_- \in \Omega$  (since  $\Omega$  is closed from the left), so

$$\omega_- \prec \omega \preceq \omega_+.$$

Note that, since  $\inf\{\sigma \in \Omega : \omega \preceq \sigma\} = \inf\{\sigma \in \Omega_+ : \omega \preceq \sigma\}$ , and  $\Omega_+$  is closed from the right, we have  $\omega_+ \in \Omega_+$ . Let  $K = \min\{k \in \mathbb{N} : (\omega_-)_k \neq (\omega_+)_k\}$ . Then  $S^K(\omega_-) \prec S^K(\omega) \preceq S^K(\omega_+)$  and we must have  $S^K(\omega_-) = \tau(\rho)$  and  $S^K(\omega_+) = \tau^+(\rho)$ . So

$$\tau(\rho) \prec S^K(\omega) \preceq \tau^+(\rho),$$

therefore  $\omega \notin \{\sigma \in I^\infty : S^K(\sigma) \in [\bar{0}, \tau(\rho)] \cup (\tau^+(\rho), \bar{1}]\}$  which, because of (3.1), contradicts our assumption that  $\omega \in \bigcap_{k \in \mathbb{N}} \Psi^k([\bar{0}, \bar{1}])$ . Hence  $\omega \in \Omega$  and we have

$$\Omega \supset \bigcap_{k \in \mathbb{N}} \Psi^k([\bar{0}, \bar{1}]).$$

This completes the proof of (i).

Proof of (ii): similar to the proof of (i), with the role of  $[\bar{0}, \tau(\rho)]$  played by  $[\bar{0}, \tau(\rho))$  and the role of  $(\tau^+(\rho), \bar{1}]$  played by  $[\tau^+(\rho), \bar{1}]$ .

Proof of (iii): similar to the proofs of (i) and (ii).  $\square$

It is helpful to note that the addresses  $\alpha$  and  $\beta$  in Proposition 3 obey

$$\begin{aligned}\alpha &= \tau(W_0(\rho)), \beta = \tau(W_1(\rho)), \\ \tau(\rho) &= 0\alpha = 01\alpha_1\alpha_2\dots \text{ and } \tau^+(\rho) = 0\beta = 10\beta_1\beta_2\dots\end{aligned}$$

Let  $M > 0$  be such that  $D_{k,M+1} = \rho$ . It follows from the discussion at the end of Section 2 that  $\tau((D_{k,M}, \rho)) = \tau^+((D_{k,M}, \rho)) = 01\alpha_1\alpha_2\dots\alpha_{k-2}$  and  $\tau((\rho, D_{k,M+2})) = \tau^+((\rho, D_{k,M+2})) = 10\beta_1\beta_2\dots\beta_{k-2}$ .

**Corollary 1.** *Let  $k \geq 1$ ,  $\alpha = \tau(W_0(\rho))$ ,  $\beta = \tau(W_1(\rho))$ , and let  $M > 0$  be such that  $D_{k,M+1} = \rho$ . The set of addresses  $\{\tau((D_{k,l}, D_{k,l+1}))\}_{l=0}^{D(k)-1}$  is uniquely determined by  $\alpha|_{k-1}$  and  $\beta|_{k-1}$ . For some  $n_1, n_2$  such that  $0 \leq n_1 < M < n_2 \leq D(k) - 1$ ,  $\tau((D_{k,n_1}, D_{k,n_1+1})) = \beta_0\beta_1\dots\beta_{k-2}\beta_{k-1}$  and  $\tau((D_{k,n_2}, D_{k,n_2+1})) = \alpha_0\alpha_1\dots\alpha_{k-2}\alpha_{k-1}$ . The set of addresses  $\{\tau((D_{k,l}, D_{k,l+1})) : l \in \{0, 1, \dots, D(k) - 1\}, l \neq n_1, l \neq n_2\}$  are uniquely determined by  $\alpha|_{k-2}$  and  $\beta|_{k-2}$ ; for example,  $\tau((D_{k,M}, \rho)) = 0\alpha_0\alpha_1\dots\alpha_{k-2}$ , and  $\tau((\rho, D_{k,M+2})) = 1\beta_0\beta_1\dots\beta_{k-2}$ .*

*Proof.* It follows from Proposition 3 that the set of addresses at level  $k$ , namely  $\{\tau((D_{k,l}, D_{k,l+1}))\}_{l=0}^{D(k)-1}$ , is invariant under the following operation: put a "0" in front of each address that is less than or equal to  $\alpha$ , then truncate back to length  $k$ ; take the union of the resulting set of addresses with the set of addresses obtained by: put a "1" in front of each address that is greater than or equal to  $\beta$ , and drop the last digit.  $\square$

#### 4. SYMMETRY OF $\bar{\Omega}$ AND A CONSEQUENT HOMEOMORPHISM OF $[0, 1]$

**Lemma 1.**  $\bar{\Omega} = \{\sigma \in I^\infty : \text{for all } k \in \mathbb{N}, \sigma_k = 0 \Rightarrow S^k(\sigma) \preceq \tau(\rho) \text{ and } \sigma_0 = 1 \Rightarrow \tau^+(\rho) \preceq S^k(\sigma)\}$ .

*Proof.* This is an immediate consequence of Proposition 3.  $\square$

**Corollary 2.**  $\bar{\Omega}$  is symmetric if and only if  $\alpha = \beta^*$  (or equivalently  $\tau(\rho) = (\tau^+(\rho))^*$ ).

**Lemma 2.** The maps  $\tau(\rho)$  and  $\tau^+(\rho)$  are strictly increasing as functions of  $\rho \in [a, b]$  to  $I^\infty$ .

*Proof.* Note that  $\tau(\rho)$  depends both implicitly and explicitly on  $\rho$ . Let  $1 - b \leq \rho < \rho' \leq a$  be such that  $\tau(\rho) \succeq \tau(\rho')$ . Observe that  $\tau(\rho)|_0 = \tau(\rho')|_0$ .

Assume first that there is a largest  $n > 0$  such that  $\tau(\rho)|_n = \tau(\rho')|_n := \theta_0\theta_1\dots\theta_n$ . Then  $\tau(\rho) = \theta_0\theta_1\dots\theta_n1\dots$  and  $\tau^+(\rho) = \theta_0\theta_1\dots\theta_n0\dots$ , which implies

$$(4.1) \quad W_\rho^n(\rho) \geq \rho \text{ and } W_{\rho'}^n(\rho') \leq \rho'.$$

(We write  $W = W_\rho$  when we want to note the dependence on  $\rho$ .) We may assume that  $\tau(\rho)|_n$  is constant on  $[\rho, \rho']$  for otherwise we can restrict to a smaller interval with a strictly smaller value of  $n$ . As a consequence, at every iteration, we apply the same branch  $W_0$  or  $W_1$  to  $W_\xi$  to compute  $g(\xi) := W_\xi^n(\xi)$  for all  $\xi \in [\rho, \rho']$ . Therefore  $g$  is continuous with derivative at least  $d^n > 1$ , which contradicts (4.1).



The only remaining possibility is that  $\tau(\rho) = \tau(\rho')$ . We may assume that  $\tau(\rho)$  is constant on  $[\rho, \rho']$ , otherwise we can reduce the problem to the previous case. This would mean that for arbitrarily large  $n$ , the image of the interval  $[\rho, \rho']$  under  $g$  is at an interval of size least  $d^n(\rho' - \rho)$ , a contradiction.

Essentially the same argument, with the role of  $\tau$  played by  $\tau^+$  and the role of played by  $W_+$ , proves that  $\tau^+(\rho)$  is strictly increasing as a function of  $\rho \in [1 - b, a]$  to  $I^\infty$ .  $\square$

**Corollary 3.** *The map  $\rho \mapsto \tau(\rho)$  is left continuous and the map  $\rho \mapsto \tau^+(\rho)$  is right continuous.*

*Proof.* Fix a parameter  $\rho_0$  and let  $\varepsilon > 0$ . Then by (7) there is  $x < \rho_0$  which is not a preimage of  $\rho_0$  for any order and such that

$$d(\tau_{\rho_0}^+(x), \tau_{\rho_0}(\rho_0)) < \frac{\varepsilon}{2}.$$

By (9), for any  $n \in \mathbb{N}$  there exists  $\delta > 0$  such that the prefix  $\tau_\rho^+(x)|_n$  is constant when  $\rho \in (\rho_0 - \delta, \rho_0 + \delta)$ . Let  $n$  be such that  $2^{-n} < \varepsilon$ , and let  $\rho > x$  and  $\rho \in (\rho_0 - \delta, \rho_0)$ . We have that  $\tau_\rho^+(x) \prec \tau_\rho^+(\rho)$  and

$$d(\tau_\rho^+(x), \tau_{\rho_0}^+(x)) < \frac{\varepsilon}{2}.$$

Combining the two inequalities we obtain

$$d(\tau_\rho^+(x), \tau_{\rho_0}(\rho_0)) < \varepsilon,$$

and by Lemma 2 we also have

$$\tau_\rho^+(x) \prec \tau_\rho(\rho) \prec \tau_{\rho_0}(\rho_0).$$

The distance  $d$  has the property that if  $\sigma \prec \zeta \prec \sigma'$  then  $d(\sigma, \zeta) \leq d(\sigma, \sigma')$  and  $d(\zeta, \sigma') \leq d(\sigma, \sigma')$ . This shows that  $\rho \mapsto \tau(\rho)$  is left continuous. The right continuity of  $\rho \mapsto \tau^+(\rho)$  admits an analogous proof.  $\square$

As a consequence of Corollary 2, Lemma 2 and (10), we obtain the unicity of  $\rho$  for which  $\bar{\Omega}$  is symmetric. As a consequence of Corollary 2, Lemma 2 and (10), we obtain the unicity of  $\rho$  for which  $\bar{\Omega}$  is symmetric.

**Corollary 4.** *There is at most one  $\rho \in [1 - b, a]$  such that  $\bar{\Omega} = \bar{\Omega}^*$ .*

*Proof of Theorem 2.* By Lemma 2 and (10), we may define

$$\rho_0 := \sup\{\rho \in [1 - b, a] : \tau(\rho) \preceq \tau^+(\rho)^*\} = \inf\{\rho \in [1 - b, a] : \tau(\rho)^* \preceq \tau^+(\rho)\}.$$

Assume  $\tau(\rho_0) \prec \tau^+(\rho_0)^*$ . It is straightforward to check  $1 - b < \rho_0 < a$ .

There is a largest  $n \geq 2$  such that  $\tau(\rho_0)|_n = \tau^+(\rho_0)^*|_n =: \eta = 01\dots$ .

Observe that  $\tau(\rho_0) = 0\tau(W_0(\rho_0))$  and  $\tau^+(\rho_0) = 1\tau^+(W_1(\rho_0))$ . If neither  $W_0(\rho_0)$  nor  $W_1(\rho_0)$  belongs to  $\{0, 1\} \cup \bigcup_{k=0}^{n-1} W^{-k}(\rho_0)$ , then by (9) both  $\tau(\rho)|_{n+1}$  and  $\tau^+(\rho)|_{n+1}$  are constant on a neighborhood of  $\rho_0$  which contradicts the definition of  $\rho_0$ .

Let us consider the projection  $\hat{\pi}(\tau^+(W_1(\rho_0))^*)$ . If  $\hat{\pi}(\tau^+(W_1(\rho_0))^*) > W_0(\rho_0)$  then by the continuity of  $W_0$ , of  $\hat{\pi}$  by (13), of  $\rho \mapsto \tau_\rho^+(\rho)$  (Corollary 3) there is a  $\rho > \rho_0$  such that  $\hat{\pi}_\rho(\tau_\rho^+(W_1(\rho_0))^*) > W_0(\rho)$ . By (6) and (13) this implies  $\tau_\rho(\rho) \prec \tau_\rho^+(\rho)^*$ , which again contradicts the definition of  $\rho_0$ .

As  $\hat{\pi}$  is increasing, (13) and  $\tau(W_0(\rho_0)) \prec \tau^+(W_1(\rho_0))^*$ , we have  $\hat{\pi}(\tau^+(W_1(\rho_0))^*) = W_0(\rho_0)$ . Let  $0 < m < n$  be minimal such that  $W^m \circ W_0(\rho_0) = \rho_0$  or  $W^m \circ W_1(\rho_0) = \rho_0$ . We may apply (14)  $m$  times and obtain

$$(4.2) \quad W^m \circ W_0(\rho_0) = \hat{\pi}(S^m(\tau^+(W_1(\rho_0))^*)) = \hat{\pi}(\tau^+(W^m \circ W_1(\rho_0))^*).$$

As  $\tau^+(\rho_0) = 1\dots$ , if  $W^m \circ W_1(\rho_0) = \rho_0$  then we have

$$\tau(\rho_0) \prec \tau^+(\rho_0)^* = \tau^+(W^m \circ W_1(\rho_0))^* \prec \tau^+(\rho_0),$$

which by (6) and equation (4.2) implies  $W^m \circ W_0(\rho_0) = \rho_0$ . Therefore  $\tau(\rho_0) = \tau^+(\rho_0)^*$  as both are periodic of period  $m+1$  and have the same prefix of length  $n > m$ , a contradiction.

If  $W^m \circ W_1(\rho_0) \neq \rho_0$  then  $W^m \circ W_0(\rho_0) = \rho_0$  thus by (13), (10) and equality (4.2) we obtain

$$\tau(\rho_0) \prec \tau^+(\rho_0)^* \preceq \tau^+(W^m \circ W_1(\rho_0)) := \sigma'.$$

By (6), this means that  $\rho_0 \leq W^m \circ W_1(\rho_0)$  so in fact

$$\rho_0 < W^m \circ W_1(\rho_0).$$

As  $W^{m+1}(\rho_0) = \rho_0$ ,  $\tau(\rho_0) = \kappa\kappa\kappa\dots := \kappa^\infty$  where  $\kappa = \tau(\rho_0)|_{m+1} = \tau^+(\rho_0)^*|_{m+1}$ , as  $m+1 \leq n$ . We can write  $\tau^+(\rho_0) = \kappa^*\sigma'$  therefore  $\kappa^*\sigma' \prec \sigma'$  by (6) and the previous inequality. By induction we get  $\kappa^{*\infty} \prec \sigma'$  so

$$\tau^+(\rho_0)^* = \kappa(\sigma'^*) \prec \kappa^\infty = \tau(\rho_0),$$

a contradiction.

The case  $\tau(\rho_0) \succ \tau^+(\rho_0)^*$  is analagous by the symmetric definition of  $\rho_0$ , therefore  $\bar{\Omega}_{\rho_0}$  is symmetric.  $\square$

**Proposition 4.** *If  $\bar{\Omega} = \bar{\Omega}^*$  then the map  $h : [0, 1] \rightarrow [0, 1]$  defined by  $h(x) = \hat{\pi}(\tau(x)^*)$  is a homeomorphism and  $h \circ \hat{\pi} = \hat{\pi} \circ *$  on  $I^\infty$ .*

*Proof.* First by Corollary 2, we have  $\tau(\rho) = \tau^+(\rho)^*$  and points  $x$  for which  $\tau(x) \neq \tau^+(x)$  are exactly preimages of  $\rho$ . In this case, there is  $n \geq 0$  such that  $\tau(x)$  and  $\tau^+(x)$  have the same initial prefix  $\kappa := \tau(x)|_n = \tau^+(x)|_n$ , and  $\tau(x) = \kappa\tau(\rho)$ ,  $\tau^+(x) = \kappa\tau^+(\rho)$ . Therefore, by (13), for all  $x \in [0, 1]$ , we have

$$\tau(h(x)) = \tau^+(x)^* \text{ and } \tau^+(h(x)) = \tau(x)^*,$$

thus  $h \circ h(x) = x$ . By (6), (10) and (13),  $h$  is also decreasing. Therefore  $h : [0, 1] \rightarrow [0, 1]$  is a homeomorphism.

Let  $\sigma \in I^\infty$  and  $x = \hat{\pi}(\sigma)$ . By (13) we have  $\tau(x) \preceq \sigma \preceq \tau^+(x)$ . As  $\bar{\Omega} = \bar{\Omega}^*$ , by Proposition 2, we obtain that there exists  $y \in [0, 1]$  such that  $\tau(x)^* = \tau^+(y)$ . By Lemma 1 and Corollary 2 we also have that  $\tau^+(x)^* = \tau(y)$ . We may compute  $h \circ \hat{\pi}(\sigma) = h(x) = \hat{\pi}(\tau(x)^*) = \hat{\pi}(\tau(y)) = y$ , which is also equal to  $\hat{\pi}(\sigma^*)$  as  $\tau(y) \preceq \sigma^* \preceq \tau^+(y)$ .  $\square$

## 5. ITERATED CLOSED RELATIONS AND CONLEY DECOMPOSITION FOR ITINERARIES OF $W$

Theorem 1 follows from Proposition 3, but some extra language is needed. In explaining this language we describe the Conley-McGehee-Wiandt decomposition theorem, [3, Theorem 13.1].

For  $X$  a compact Hausdorff space, let  $2^X$  be the subsets of  $X$ . A *relation*  $r$  on  $X$  is simply a subset of  $X \times X$ . A relation  $r$  on  $X$  is called a *closed relation* if  $r$  is a closed of  $X \times X$ . For example the set  $r \subset I^\infty \times I^\infty$  defined in Theorem 1, namely

$$r = \{(0\sigma, \sigma) \in I^\infty \times I^\infty : \sigma \preceq \alpha\} \cup \{(1\sigma, \sigma) \in I^\infty \times I^\infty : \beta \preceq \sigma\},$$

is a closed relation. Following [3], a relation  $r \in 2^X$  provides a mapping  $r : 2^X \rightarrow 2^X$  defined by

$$r(C) = \{y \in X : (x, y) \in r \text{ for some } x \in C\}.$$

Notice that the image of a nonempty set may be empty. Iterated relations are defined by  $r^0 = X \times X$  and, for all  $k \in \mathbb{N}$ ,

$$r^{k+1} = r \circ r^k = \{(x, z) : (x, y) \in r, (y, z) \in r^k \text{ for some } y \in X\}.$$

The *omega limit set* of  $C \subset X$  under a closed relation  $r \subset X \times X$  is

$$\omega(C) = \cap \mathfrak{K}(C)$$

where

$$\mathfrak{K}(C) = \{D \text{ is a closed subset of } X : r(D) \cup r^n(C) \subset D \text{ for some } n \in \mathbb{N}\}.$$

By definition, an *attractor* of a closed relation  $r$  is a closed set  $A$  such that the following two conditions hold:

- (i)  $r(A) = A$ ;
- (ii) there is a closed neighborhood  $\overline{\mathcal{N}}(A)$  of  $A$  such that  $\omega(C) \subset A$  for all  $C \subset \overline{\mathcal{N}}(A)$ .

The basin  $\mathcal{B}(A)$  of an attractor  $A$  for a closed relation  $r$  on a compact Hausdorff space  $X$  is the union of all open sets  $O \subset X$  such that  $\omega(C) \subset A$  for all  $C \subset O$ .

Given an attractor  $A$  for a closed relation  $r$  on a compact Hausdorff space  $X$ , there exists a corresponding *attractor block*, namely a closed set  $E \subset X$  such that  $E$  contains both  $A$  and  $r(E)$  in its interior, and  $A = \omega(E)$ . Also, there exists a unique *dual repeller*  $A^* = X \setminus \mathcal{B}(A)$ . This repeller is an attractor for the transpose relation  $r^* = \{(y, x) : (x, y) \in r\}$ . The set of connecting orbits associated with the attractor/repeller pair  $A, A^*$  is  $\mathcal{C}(A) = X \setminus (A \cup A^*)$ .

If  $r$  is a closed relation on a compact Hausdorff space  $X$ , then  $x \in X$  is called *chain-recurrent* for  $r$  if for every closed neighborhood  $f$  of  $r$ ,  $x$  is periodic for  $f$  (i.e. there exists a finite sequence of points  $\{x_n\}_{n=0}^{p-1} \subset X$  such that  $x_0 = x$ ,  $(x_{p-1}, x_0) \in f$  and  $(x_{n-1}, x_n) \in f$  for  $n = 1, 2, \dots, p-1$ ). The chain recurrent set  $\mathcal{R}$  for  $r$  is the union of all the points that are chain recurrent for  $r$ . A transitive component of  $\mathcal{R}$  is a member of the equivalence class on  $\mathcal{R}$  defined by  $x \sim y$  when for every closed neighborhood  $f$  of  $r$  there is an orbit from  $x$  to  $y$  under  $f$  (i.e. there exists a finite sequence of points  $\{x_n\}_{n=0}^{p-1} \subset \mathcal{R}$  such that  $x_0 = x$ ,  $x_{p-1} = y$ , and  $(x_n, x_{n+1}) \in f$  for all  $n \in \{0, 1, \dots, p-1\}$ ).

**Theorem 3** (Conley-McGehee-Wiandt). *If  $r$  is a closed relation on a compact Hausdorff space  $X$ , then*

$$\mathcal{R} = \bigcup_{A \in \mathcal{U}} \mathcal{C}(A)$$

where  $\mathcal{R}$  is the chain-recurrent set and  $\mathcal{U}$  is the set of attractors.

**Proof of Theorem 1.** This follows at once from Proposition 3 together with Theorem 3, but see [3].  $\square$

We note the following.  $\bar{\Omega}$  can be embedded in  $[0, 1] \subset \mathbb{R}$  using the (continuous and surjective) coding map  $\pi : I^\infty \rightarrow [0, 1]$  associated with the iterated function system  $([0, 1]; x \mapsto x/2, x \mapsto (1+x)/2)$ . This coding map  $\pi$  is defined, for all  $\sigma$ , by

$$\pi(\sigma) = \sum_{k \in \mathbb{N}} \frac{\sigma_k}{2^{k+1}}.$$

$\pi$  provides a homeomorphism between  $\bar{\Omega}$  and  $\pi(\bar{\Omega})$ . The point  $\sigma \in \bar{\Omega}$  is uniquely and unambiguously represented by the binary real number  $0.\sigma$ . In the representation provided by  $\pi$ , the map  $\bar{\Psi} : 2^{I^\infty} \rightarrow 2^{I^\infty}$  becomes the action of the iterated closed relation  $\tilde{r} \subset [0, 1] \times [0, 1] \subset \mathbb{R}^2$  defined by

$$\tilde{r} := \{(x, x/2) : x \in [0, \pi(\alpha)]\} \cup \{(x, (x+1)/2) : x \in [\pi(\beta), 1]\}$$

on subsets of  $[0, 1]$ . It follows from Proposition 3 (iii) that  $\pi(\bar{\Omega})$  is the maximal attractor, as defined in [3], of  $\tilde{r}$ . The corresponding dual repeller is the empty set. It is also easy to see that  $\{0\}$  and  $\{1\}$  are the only other attractors, with corresponding dual repellers  $[\pi(\alpha), 1]$  and  $[0, \pi(\beta)]$  respectively. It follows from Theorem 3 that the chain recurrent set of  $\tilde{r}$  is  $\{0, 1\} \cup (\pi(\bar{\Omega}) \cap (\pi(\beta), \pi(\alpha)))$ .

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