# SYMMETRIC ITINERARY SETS 

Michael Barnsley, Nicolae Mihalache

## To cite this version:

Michael Barnsley, Nicolae Mihalache. SYMMETRIC ITINERARY SETS. 2011. hal-00796905

## HAL Id: hal-00796905

## https://hal.science/hal-00796905

Preprint submitted on 5 Mar 2013

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# SYMMETRIC ITINERARY SETS 

MICHAEL F. BARNSLEY AND NICOLAE MIHALACHE


#### Abstract

We consider a one parameter family of dynamical systems $W$ : $[0,1] \rightarrow[0,1]$ constructed from a pair of monotone increasing diffeomorphisms $W_{i}$, such that $W_{i}^{-1}:[0,1] \rightarrow[0,1],(i=0,1)$. We characterize the set of symbolic itineraries of $W$ using an attractor $\bar{\Omega}$ of an iterated closed relation, in the terminology of McGehee, and prove that there is a member of the family for which $\bar{\Omega}$ is symmetrical.


## 1. Introduction

Let $W_{0}:[0, a] \rightarrow[0,1]$ and $W_{1}:[1-b, 1] \rightarrow[0,1]$ be continuous and differentiable, and such that $a+b>1, W_{0}(0)=W_{1}(1-b)=0, W_{0}(a)=W_{1}(1)=1$. Let the derivatives $W_{i}^{\prime}(x)(i=0,1)$ be uniformly bounded below by $d>1$.

For $\rho \in[1-b, a]$ we define $W:[0,1] \rightarrow[0,1]$ by

$$
[0,1] \ni x \mapsto\left\{\begin{array}{l}
W_{0}(x) \text { if } x \in[0, \rho] \\
W_{1}(x) \text { otherwise }
\end{array}\right.
$$

See Figure 1. Similarly, we define $W_{+}:[0,1] \rightarrow[0,1]$ by replacing $[0, \rho]$ by $[0, \rho)$.
Let $I=\{0,1\}$. Let $I^{\infty}=\{0,1\} \times\{0,1\} \times \cdots$ have the product topology induced from the discrete topology on $I$. For $\sigma \in I^{\infty}$ write $\sigma=\sigma_{0} \sigma_{1} \sigma_{2} \ldots$, where $\sigma_{k} \in I$ for all $k \in \mathbb{N}$. The product topology on $I^{\infty}$ is the same as the topology induced by the metric $d(\omega, \sigma)=2^{-k}$ where $k$ is the least index such that $\omega_{k} \neq \sigma_{k}$. It is well known that $\left(I^{\infty}, d\right)$ is a compact metric space. We define a total order relation $\preceq$ on $I^{\infty}$, and on $I^{n}$ for any $n \in \mathbb{N}$, by $\sigma \prec \omega$ if $\sigma \neq \omega$ and $\sigma_{k}<\omega_{k}$ where $k$ is the least index such that $\sigma_{k} \neq \omega_{k}$. For $\sigma \in I^{\infty}$ and $n \in \mathbb{N}$ we write $\left.\sigma\right|_{n}=\sigma_{0} \sigma_{1} \sigma_{2} \ldots \sigma_{n} . I^{\infty}$ is the appropriate space in which to embed and study the itineraries of the family of discontinuous dynamical systems $W:[0,1] \rightarrow[0,1]$.

For $W_{(+)} \in\left\{W, W_{+}\right\}$let $W_{(+)}^{k}$ denote $W_{(+)}$composed with itself $k$ times, for $k \in \mathbb{N}$, and let $W_{(+)}^{-k}=\left(W_{(+)}^{k}\right)^{-1}$. We define a map $\tau:[0,1] \rightarrow I^{\infty}$, using all of the orbits of $W$, by

$$
\tau(x)=\sigma_{0} \sigma_{1} \sigma_{2} \ldots
$$

where $\sigma_{k}$ equals 0 , or 1 , according as $W^{k}(x) \in[0, \rho]$, or $(\rho, 1]$, respectively. We call $\tau(x)$ the itinerary of $x$ under $W$, or an address of $x$, and we call $\Omega=\tau([0,1])$ an address space for $[0,1]$. Similarly, we define $\tau^{+}:[0,1] \rightarrow I^{\infty}$ so that $\tau^{+}(x)_{k}$ equals 0 , or 1 , according as $W_{+}^{k}(x) \in[0, \rho)$, or $[\rho, 1]$, respectively; and we define $\Omega_{+}=\tau^{+}([0,1])$. Note that $W, W_{+}, \Omega, \Omega_{+}, \tau$, and $\tau^{+}$all depend on $\rho$.

The main goals of this paper are to characterise $\bar{\Omega}$ and to show that there exists a value of $\rho$ such that $\bar{\Omega}$ is symmetric.

[^0]

Figure 1. The piecewise continuous dynamical system $\mathrm{W}:[0,1] \rightarrow[0,1]$ is defined in terms of two monotone strictly increasing differentiable functions $\mathrm{W}_{0}(\mathrm{x})$ and $\mathrm{W}_{1}(\mathrm{x})$, and a real parameter $\rho$.

Theorem 1. Let an iterated closed relation $r \subset I^{\infty} \times I^{\infty}$ be defined by

$$
r:=\left\{(\sigma, 0 \sigma) \in I^{\infty} \times I^{\infty}: \sigma \preceq \alpha\right\} \cup\left\{(\sigma, 1 \sigma) \in I^{\infty} \times I^{\infty}: \sigma \succeq \beta\right\}
$$

where $\alpha=\tau\left(W_{0}(\rho)\right)$ and $\beta=\tau^{+}\left(W_{1}(\rho)\right)$. The only attractors of $r$ are $\{\overline{0}\}$, $\{\overline{1}\},\{\overline{0}, \overline{1}\}$, and $\bar{\Omega}$. The corresponding dual repellers are $\left\{\sigma \in I^{\infty}: \beta \preceq \sigma\right\},\{\sigma \in$ $\left.I^{\infty}: \sigma \preceq \alpha\right\},\left\{\sigma \in I^{\infty}: \beta \preceq \sigma\right\} \cup\left\{\sigma \in I^{\infty}: \sigma \preceq \alpha\right\}$, and the empty set, respectively. The chain recurrent set for $r$ is $\{\overline{0}, \overline{1}\} \cup\{\sigma \in \bar{\Omega}: \beta \preceq \sigma \preceq \alpha\}$.

We write $\bar{E}$ to denote the closure of a set $E$. But we write $\overline{0}=000 \ldots, \overline{1}=$ $111 \ldots \in I^{\infty}$. For $\sigma=\sigma_{0} \sigma_{1} \sigma_{2} \ldots \in I^{\infty}$ we write $0 \sigma$ to mean $0 \sigma_{0} \sigma_{1} \sigma_{2} \ldots \in I^{\infty}$ and $1 \sigma=0 \sigma_{0} \sigma_{1} \sigma_{2} \ldots \in I^{\infty}$.

Define a symmetry function ${ }^{*}: I^{\infty} \rightarrow I^{\infty}$ by $\sigma^{*}=\omega$ where $\omega_{k}=1-\sigma_{k}$ for all $k$.

Theorem 2. There exists a unique $\rho \in[1-b, a]$ such that $\bar{\Omega}^{*}=\bar{\Omega}$.
Theorem 1 tells us that $\bar{\Omega}$ is fixed by itineraries of two inverse images of the critical point $\rho$, and provides the basis for a stable algorithm to determine $\bar{\Omega}$. It relates the address spaces of dynamical systems of the form of $W$ to the beautiful theory of iterated closed relations on compact Hausdorff spaces [3], and hence to the work of Charles Conley.

Theorem 2 is interesting in its own right and also because it has applications in digital imaging, as explained and demonstrated, in the special case of affine maps, in [1]. It enables the construction of parameterized families of nondifferentiable
homeomorphisms on $[0,1]$, using pairs of overlapping iterated function systems, see Proposition 4. Theorem 2 generalizes results in [1] to nonlinear $W_{i}$ 's. The proof uses symbolic dynamics in place of the geometrical construction outlined in [1]. The approach and results open up the mathematics underlying [1] and [2].

To tie the present work into [1], note that $\tau$ is a section, as defined in [1], for the hyperbolic iterated function system

$$
\mathcal{F}:=\left([0,1] ; W_{0}^{-1}, W_{1}^{-1}\right)
$$

Our observations interrelate to, but are more specialized than, the work of Parry [5]. Our point of view is topological rather than measure-theoretic, and our main results appear to be new.

## 2. Basic properties of $\tau$

The following list of properties is relatively easy to check. Below the list we elaborate on points 1,2 , and 3 .
(1) $W^{n}$ is piecewise differentiable and its derivative is uniformly bounded below by $d^{n}$; each, except the leftmost branch of $W^{n}$, is defined on an interval of the form $(r, s] . W_{+}^{n}$ is piecewise differentiable and its derivative is uniformly bounded below by $d^{n}$; each, except for the rightmost branch of $W_{+}^{n}$, is defined on an interval of the form $[r, s)$.
(2) If $(r, s)$ is the interior of the definition domain of a branch of $W^{n}$ (and of $W_{+}^{n}$ ) then $\left.\tau(x)\right|_{n}$ is constant on $(r, s],\left.\tau^{+}(x)\right|_{n}$ is constant on $[r, s)$, and $\left.\tau(x)\right|_{n}=\left.\tau^{+}(x)\right|_{n}$ for all $x \in(r, s)$.
(3) The boundary of the definition domain of a branch of $W^{n}$ is contained in $\{0,1\} \cup \bigcup_{k=0}^{n-1} W^{-k}(\rho)$; by (1), the length of such a domain is at most $d^{-n}$.
(4) The set $\bigcup_{k \in \mathbb{N}} W^{-k}(\rho)$ is dense in $[0,1]$. This follows from (3).
(5) $\tau(x)=\tau^{+}(x)$ unless $x \in \bigcup_{k \in \mathbb{N}} W^{-k}(\rho)$ in which case $\tau(x) \prec \tau^{+}(x)$.
(6) Both $\tau(x)$ and $\tau^{+}(x)$ are strictly increasing functions of $x \in[0,1]$ and $\tau(x) \preceq \tau^{+}(x)$. This follows from (4) and (5).
(7) For all $x \in[0,1], \tau(x)$ is continuous from the left, $\tau^{+}(x)$ is continuous from the right. Moreover, for all $x \in(0,1)$,

$$
\tau(x)=\lim _{\varepsilon \rightarrow 0^{+}} \tau^{+}(x-\varepsilon) \text { and } \tau^{+}(x)=\lim _{\varepsilon \rightarrow 0^{+}} \tau(x+\varepsilon)
$$

These assertions follow from (2), (3) and (4).
(8) Each $x \in W^{-n}(\rho)$, such that $\left.\tau(x)\right|_{n}$ is constant, moves continuously with respect to $\rho$ with positive velocity bounded above by $d^{-n}$. This follows from (1).
(9) For $x \in(0,1) \backslash \bigcup_{k=0}^{n} W^{-k}(\rho),\left.\tau(x)\right|_{n}=\left.\tau^{+}(x)\right|_{n}$ is locally constant with respect to $\rho$; moreover, this holds if $x$ depends continuously on $\rho$. This follows from (2), (3) and (6).
(10) The symmetry function ${ }^{*}: I^{\infty} \rightarrow I^{\infty}$ is strictly decreasing and continuous.
(11) For any $\left.\sigma\right|_{n} \in I^{n}, n \in \mathbb{N}$, the set

$$
\mathcal{I}\left(\left.\sigma\right|_{n}\right):=\left\{x \in[0,1]:\left.\tau(x)\right|_{n}=\left.\sigma\right|_{n} \text { or }\left.\tau^{+}(x)\right|_{n}=\left.\sigma\right|_{n}\right\}
$$

is either empty or a non-degenerate compact interval of length at most $d^{-n}$. This follows from (2), (3) and (6).
(12) The projection $\hat{\pi}: I^{\infty} \rightarrow[0,1]$ is well-defined by

$$
\hat{\pi}(\sigma)=\sup \left\{x \in[0,1]: \tau^{+}(x) \preceq \sigma\right\}=\inf \{x \in[0,1]: \tau(x) \succeq \sigma\}
$$

This follows from (6).
(13) The projection $\hat{\pi}: I^{\infty} \rightarrow[0,1]$ is increasing, by (6); continuous, by (11); and, by (7),

$$
\begin{aligned}
& \hat{\pi}(\tau(x))=\hat{\pi}\left(\tau^{+}(x)\right)=x \text { for all } x \in[0,1] \\
& \tau(\hat{\pi}(\sigma)) \preceq \sigma \preceq \tau^{+}(\hat{\pi}(\sigma)) \text { for all } \sigma \in I^{\infty}
\end{aligned}
$$

(14) Let $S: I^{\infty} \rightarrow I^{\infty}$ denote the left-shift map $\sigma_{0} \sigma_{1} \sigma_{2} \ldots \mapsto \sigma_{1} \sigma_{2} \sigma_{3} \ldots$. For all $\sigma \in I^{\infty}$ such that $\sigma \preceq \tau(\rho)$ or $\sigma \geq \tau^{+}(\rho)$,

$$
\hat{\pi}(S(\sigma))=W(\hat{\pi}(\sigma))
$$

Also $\hat{\pi}\left(\tau^{+}(\rho)\right)=\rho$ and $\hat{\pi}\left(S\left(\tau^{+}(\rho)\right)\right)=W_{1}(\rho)$. These statements follow from (7).
Here we elaborate on points (1), (2) and (3). Consider the piecewise continuous function $W^{k}(x)$, for $k \in\{1,2, \ldots\}$. Its discontinuities are at $\rho$ and, for $k>1$, other points in $(0,1)$, each of which can be written in the form $W_{\sigma_{0}}^{-1} \circ W_{\sigma_{2}}^{-1} \circ$ $\ldots W_{\sigma_{l-1}}^{-1}(\rho)$ for some $\sigma_{0} \sigma_{1} \ldots \sigma_{l-1} \in\{0,1\}^{l}$ for some $l \in\{1,2, \ldots k-1\}$. We denote these discontinuities, together with the points 0 and 1 , by

$$
D_{k, 0}:=0<D_{k, 1}<D_{k, 2}<\ldots<D_{k, D(k)-1}<1=: D_{k, D(k)}
$$

where $D(1)=3, D(2)=5<D(3)<D(4) \ldots$. For each $k \geq 1$, one of the $D_{k, j}$ 's is equal to $\rho$. For $k \geq 1$ we have $W^{k}(x)=W_{0}^{k}(x)$ for $x \in\left[D_{k, 0}, D_{k, 1}\right]$ and $W_{+}^{k}(x)=W_{0}^{k}(x)$ for $x \in\left[D_{k, 0}, D_{k, 1}\right)$. Similarly $W^{k}(x)=W_{1}^{k}(x)$ for all $x \in\left(D_{k, D(k)-1}, D_{k, D(k)}\right]$ and $W_{+}^{k}(x)=W_{1}^{k}(x)$ for $x \in\left[D_{D(k)-1}, D_{D(k)}\right]$.

For all $x \in\left(D_{k, l}, D_{k, l+1}\right)(l=0,1, \ldots, D(k)-1), W^{k}(x)=W_{+}^{k}(x)=W_{\theta_{k}} \circ W_{\theta_{k-1}} \circ$ $\ldots W_{\theta_{1}}(x)$ for some fixed $\theta_{1} \theta_{2} \ldots \theta_{k} \in\{0,1\}^{k}$. We refer to $\theta_{1} \theta_{2} \ldots \theta_{k}$ as the address of the interval $\left(D_{k, l}, D_{k, l+1}\right)$, we say $\left(D_{k, l}, D_{k, l+1}\right)$ that "has address $\theta_{1} \theta_{2} \ldots \theta_{k}$ ", and we write, by slight abuse of notation, $\tau\left(\left(D_{k, l}, D_{k, l+1}\right)\right)=\theta_{1} \theta_{2} \ldots \theta_{k}$.

Let $k>1$. Consider two adjacent intervals, $\left(D_{k, m-1}, D_{k, m}\right]$ and $\left(D_{k, m}, D_{k, m+1}\right]$ for $m \in\{1,2, \ldots, D(k)-1\}$ and $k>1$. Let the one on the right have address $\theta_{0} \theta_{1} \ldots \theta_{k-1}$ and the one on the left have address $\eta_{0} \eta_{1} \ldots \eta_{k-1}$. Then $\eta_{0} \eta_{1} \ldots \eta_{k-1} \prec$ $\theta_{0} \theta_{1} \ldots \theta_{k-1}$ and we have

$$
\begin{aligned}
\left.\tau(x)\right|_{k-1} & =\eta_{0} \eta_{1} \ldots \eta_{k-1} \text { for all } x \in\left(D_{k, m-1}, D_{k, m}\right], \\
\left.\tau^{+}(x)\right|_{k-1} & =\eta_{0} \eta_{1} \ldots \eta_{k-1} \text { for all } x \in\left[D_{k, m-1}, D_{k, m}\right), \\
\left.\tau(x)\right|_{k-1} & =\theta_{0} \theta_{1} \ldots \theta_{k-1} \text { for all } x \in\left(D_{k, m}, D_{k, m+1}\right] \\
\left.\tau^{+}(x)\right|_{k-1} & =\theta_{0} \theta_{1} \ldots \theta_{k-1} \text { for all } x \in\left[D_{k, m}, D_{k, m+1}\right) .
\end{aligned}
$$

In particular, $\left.\tau(x)\right|_{k-1}$ and $\left.\tau^{+}(x)\right|_{k-1}$ are constant and equal on each of the open intervals $\left(D_{k, m-1}, D_{k, m}\right)$ and have distinct values at the discontinuity points $\left\{D_{k, m}\right\}_{m=1}^{D(k)-1}$.
3. The structures of $\Omega, \Omega_{+}$and $\bar{\Omega}$.

In this section we characterize $\Omega$ and $\Omega_{+}$as certain inverse limits, and we characterize $\bar{\Omega}$ as an attractor of an iterated closed relation on $I^{\infty}$. These inverse limits are natural and they clarify the structures of $\Omega$ and $\Omega_{+}$. They are implied by the shift invariance of $\Omega$ and $\Omega_{+}$. Recall that $S: I^{\infty} \rightarrow I^{\infty}$ denotes the left-shift map $\sigma_{0} \sigma_{1} \sigma_{2} \ldots \mapsto \sigma_{1} \sigma_{2} \sigma_{3} \ldots$

Proposition 1. (i) $\tau(W(x))=S(\tau(x))$ and $\tau^{+}\left(W_{+}(x)\right)=S\left(\tau^{+}(x)\right)$ for all $x \in$ $[0,1]$.
(ii) $S(\Omega)=\Omega$ and $S\left(\Omega_{+}\right)=\Omega_{+}$.

Proof. (i) This follows at once from the definitions of $\tau$ and $\tau^{+}$. (ii) This follows from (i) together with $W([0,1])=W_{+}([0,1])=[0,1]$.

We say that $\Lambda \subset I^{\infty}$ is closed from the left if, whenever $\left\{x_{n}\right\}_{n=0}^{\infty}$ is an nondecreasing sequence of points in $\Lambda, \lim x_{n} \in \Lambda$. We say that $\Lambda \subset I^{\infty}$ is closed from the right if, whenever $\left\{x_{n}\right\}_{n=0}^{\infty}$ is non-increasing sequence in $\Lambda$, $\lim x_{n} \in \Lambda$. For $S \subset X$, where $X=I^{\infty}$ or $[0,1]$, we write $L(S)=\{\sigma \in X$ : there is a non-decreasing sequence $\left\{z_{n}\right\}_{n=0}^{\infty} \subset S$ with $\left.\sigma=\lim z_{n}\right\}$ to denote the closure of $S$ from the left. Analogously, we define $R(S)$ for the closure of $S$ from the right.

Proposition 2. (i) $\Omega$ is closed from the left and $\Omega_{+}$is closed from the right;
(ii) $\bar{\Omega}=\overline{\Omega_{+}}=\Omega \cup \Omega_{+}=\overline{\Omega \cap \Omega_{+}}$

Proof. Proof of (i): By (6) $\tau:[0,1] \rightarrow I^{\infty}$ is monotone strictly increasing. By (7) $\tau$ is continuous from the left. Let $\left\{z_{n}\right\}_{n=0}^{\infty}$ be a non-decreasing sequence of points in $\Omega$. Let $y_{n}=\tau^{-1}\left(z_{n}\right)$. Let $y=\lim y_{n} \in[0,1]$. Since $\tau$ is continuous from the left, $\Omega \ni \tau(y)=\tau\left(\lim y_{n}\right)=\lim \tau\left(y_{n}\right)=\lim z_{n}$. It follows that $\Omega$ is closed from the left. Similarly, $\Omega_{+}$is closed from the right.

Proof of (ii): Let $Q=\left\{x \in[0,1]: \tau(x)=\tau^{+}(x)\right\}$. Then by (4) $\bar{Q}=[0,1]$. Also, by (5),

$$
\Omega \cap \Omega_{+}=\tau([0,1]) \cap \tau^{+}([0,1])=\tau(Q)=\tau^{+}(Q)
$$

Hence

$$
\overline{\Omega \cap \Omega_{+}}=\overline{\tau(Q)}=\overline{\tau^{+}(Q)}=\bar{\Omega}=\overline{\Omega_{+}} .
$$

Finally, $\Omega \cup \Omega_{+}=L(\tau(Q)) \cup R\left(\tau^{+}(Q)\right)=L(\tau(Q)) \cup R(\tau(Q))=\overline{\tau(Q)}=\bar{\Omega}$.
We define $s_{i}: I^{\infty} \rightarrow I^{\infty}$ by $s_{i}(\sigma)=i \sigma(i=0,1)$. Note that both $s_{0}$, and $s_{1}$, are contractions with contractivity $1 / 2$. We write $2^{I^{\infty}}$ to denote the set of all subsets of $I^{\infty}$. For $\sigma, \omega \in I^{\infty}$ we define

$$
\begin{aligned}
{[\sigma, \omega] } & :=\left\{\zeta \in I^{\infty}: \sigma \preceq \zeta \preceq \omega\right\}, \\
(\sigma, \omega) & :=\left\{\zeta \in I^{\infty}: \sigma \prec \zeta \prec \omega\right\}, \\
(\sigma, \omega] & :=\left\{\zeta \in I^{\infty}: \sigma \prec \zeta \preceq \omega\right\}, \\
{[\sigma, \omega) } & :=\left\{\zeta \in I^{\infty}: \sigma \preceq \zeta \prec \omega\right\} .
\end{aligned}
$$

Proposition 3. Let $\alpha=S(\tau(\rho))$ and $\beta=S\left(\tau^{+}(\rho)\right)$.
(i) $\Omega=\bigcap_{k \in \mathbb{N}} \Psi^{k}([\overline{0}, \overline{1}])$ where $\Psi: 2^{I^{\infty}} \rightarrow 2^{I^{\infty}}$ is defined by

$$
2^{I^{\infty}} \ni \Lambda \mapsto s_{0}(\Lambda \cap[\overline{0}, \alpha]) \cup s_{1}(\Lambda \cap(\beta, \overline{1}])
$$

(ii) $\Omega_{+}=\bigcap_{k \in \mathbb{N}} \Psi_{+}^{k}([\overline{0}, \overline{1}])$ where $\Psi_{+}: 2^{I^{\infty}} \rightarrow 2^{I^{\infty}}$ is defined by

$$
2^{I^{\infty}} \ni \Lambda \mapsto s_{0}(\Lambda \cap[\overline{0}, \alpha)) \cup s_{1}(\Lambda \cap[\beta, \overline{1}])
$$

(iii) $\bar{\Omega}=\overline{\Omega_{+}}=\bigcap_{k \in \mathbb{N}} \bar{\Psi}^{k}([\overline{0}, \overline{1}])$ where $\bar{\Psi}: 2^{I^{\infty}} \rightarrow 2^{I^{\infty}}$ is defined by
$2^{I^{\infty}} \ni \Lambda \mapsto s_{0}(\Lambda \cap[\overline{0}, \alpha]) \cup s_{1}(\Lambda \cap[\beta, \overline{1}])$.

Proof. Proof of (i): Let $\left.S\right|_{\Omega}: \Omega \rightarrow \Omega$ denote the domain and range restricted shift map. It is readily found that the branches of $\left.S\right|_{\Omega} ^{-1}: \Omega \rightarrow \Omega$ are $\left.s_{0}\right|_{\Omega}:[\overline{0}, \alpha] \cap \Omega \rightarrow \Omega$ where

$$
\left.s_{0}\right|_{\Omega}(\sigma)=s_{0}(\sigma)=0 \sigma \text { for all } \sigma \in[\overline{0}, \alpha] \cap \Omega
$$

and $\left.s_{1}\right|_{\Omega}:(\beta, \overline{1}] \cap \Omega \rightarrow \Omega$ where

$$
\left.s_{1}\right|_{\Omega}(\sigma)=s_{1}(\sigma)=1 \sigma \text { for all } \sigma \in(\beta, \overline{1}] \cap \Omega
$$

(Note that $\alpha_{0}=1, \beta_{0}=0$ and $\beta \prec \alpha$.) It follows that

$$
\left.S\right|_{\Omega} ^{-1}(\Lambda)=s_{0}(\Lambda \cap[\overline{0}, \alpha]) \cup s_{1}(\Lambda \cap(\beta, \overline{1}])=\Psi(\Lambda)
$$

for all $\Lambda \subset \Omega$. Since $\Omega \subset[\overline{0}, \overline{1}]$ it follows that

$$
\Omega=\left.S\right|_{\Omega} ^{-1}(\Omega)=\Psi(\Omega) \subset \Psi([\overline{0}, \overline{1}])
$$

Also, since $\Psi([\overline{0}, \overline{1}]) \subset[\overline{0}, \overline{1}]$ it follows that $\left\{\Psi^{k}([\overline{0}, \overline{1}])\right\}$ is a decreasing (nested) sequence of sets, each of which contains $\Omega$; hence

$$
\Omega \subset \bigcap_{k \in \mathbb{N}} \Psi^{k}([\overline{0}, \overline{1}])
$$

It remains to prove that $\Omega \supset \bigcap_{k \in \mathbb{N}} \Psi^{k}([\overline{0}, \overline{1}])$. We note that $s_{0}([\overline{0}, \alpha])=[\overline{0}, \tau(\rho)]$ and $s_{1}((\beta, \overline{1}])=\left(\tau^{+}(\rho), \overline{1}\right]$, from which it follows that

$$
\begin{equation*}
\bigcap_{k \in \mathbb{N}} \Psi^{k}([\overline{0}, \overline{1}])=\bigcap_{k \in \mathbb{N}}\left\{\sigma \in I^{\infty}: S^{k}(\sigma) \in[\overline{0}, \tau(\rho)] \cup\left(\tau^{+}(\rho), \overline{1}\right]\right\} \tag{3.1}
\end{equation*}
$$

Let $\omega \in \bigcap_{k \in \mathbb{N}} \Psi^{k}([\overline{0}, \overline{1}])$. Suppose $\omega \notin \Omega$. Let

$$
\omega_{-}=\sup \{\sigma \in \Omega: \sigma \preceq \omega\} \text { and } \omega_{+}=\inf \{\sigma \in \Omega: \omega \preceq \sigma\}
$$

so that

$$
\omega_{-} \preceq \omega \preceq \omega_{+} .
$$

But $\omega_{-} \in \Omega$ (since $\Omega$ is closed from the left), so

$$
\omega_{-} \prec \omega \preceq \omega_{+} .
$$

Note that, since $\inf \{\sigma \in \Omega: \omega \preceq \sigma\}=\inf \left\{\sigma \in \Omega_{+}: \omega \preceq \sigma\right\}$, and $\Omega_{+}$is closed from the right, we have $\omega_{+} \in \Omega_{+}$. Let $K=\min \left\{k \in \mathbb{N}:\left(\omega_{-}\right)_{k} \neq\left(\omega_{+}\right)_{k}\right\}$. Then $S^{K}\left(\omega_{-}\right) \prec S^{K}(\omega) \preceq S^{K}\left(\omega_{+}\right)$and we must have $S^{K}\left(\omega_{-}\right)=\tau(\rho)$ and $S^{K}\left(\omega_{+}\right)=$ $\tau^{+}(\rho)$. So

$$
\tau(\rho) \prec S^{K}(\omega) \preceq \tau^{+}(\rho)
$$

therefore $\omega \notin\left\{\sigma \in I^{\infty}: S^{K}(\sigma) \in[\overline{0}, \tau(\rho)] \cup\left(\tau^{+}(\rho), \overline{1}\right]\right\}$ which, because of (3.1), contradicts our assumption that $\omega \in \bigcap_{k \in \mathbb{N}} \Psi^{k}([\overline{0}, \overline{1}])$. Hence $\omega \in \Omega$ and we have

$$
\Omega \supset \bigcap_{k \in \mathbb{N}} \Psi^{k}([\overline{0}, \overline{1}])
$$

This completes the proof of (i).
Proof of (ii): similar to the proof of (i), with the role of $[\overline{0}, \tau(\rho)]$ played by $[\overline{0}, \tau(\rho))$ and the role of $\left(\tau^{+}(\rho), \overline{1}\right]$ played by $\left[\tau^{+}(\rho), \overline{1}\right]$.

Proof of (iii): similar to the proofs of (i) and (ii).
It is helpful to note that the addresses $\alpha$ and $\beta$ in Proposition 3 obey

$$
\begin{aligned}
\alpha & =\tau\left(W_{0}(\rho)\right), \beta=\tau\left(W_{1}(\rho)\right) \\
\tau(\rho) & =0 \alpha=01 \alpha_{1} \alpha_{2} \ldots \text { and } \tau^{+}(\rho)=0 \beta=10 \beta_{1} \beta_{2} \ldots
\end{aligned}
$$

Let $M>0$ be such that $D_{k, M+1}=\rho$. It follows from the discussion at the end of Section 2 that $\tau\left(\left(D_{k, M}, \rho\right)\right)=\tau^{+}\left(\left(D_{k, M}, \rho\right)\right)=01 \alpha_{1} \alpha_{2} . . \alpha_{k-2}$ and $\tau\left(\left(\rho, D_{k, M+2}\right)\right)=$ $\tau^{+}\left(\left(\rho, D_{k, M+2}\right)\right)=10 \beta_{1} \beta_{2} \ldots \beta_{k-2}$.
Corollary 1. Let $k \geq 1, \alpha=\tau\left(W_{0}(\rho)\right), \beta=\tau\left(W_{1}(\rho)\right)$, and let $M>0$ be such that $D_{k, M+1}=\rho$. The set of addresses $\left\{\tau\left(\left(D_{k, l}, D_{k, l+1}\right)\right)\right\}_{l=0}^{D(k)-1}$ is uniquely determined by $\left.\alpha\right|_{k-1}$ and $\left.\beta\right|_{k-1}$. For some $n_{1}, n_{2}$ such that $0 \leq n_{1}<M<$ $n_{2} \leq D(k)-1, \tau\left(\left(D_{k, n_{1}}, D_{k, n_{1}+1}\right)\right)=\beta_{0} \beta_{1} \ldots \beta_{k-2} \beta_{k-1}$ and $\tau\left(\left(D_{k, n_{2}}, D_{k, n_{2}+1}\right)\right)=$ $\alpha_{0} \alpha_{1} \ldots \alpha_{k-2} \alpha_{k-1}$. The set of addresses $\left\{\tau\left(\left(D_{k, l}, D_{k, l+1}\right)\right): l \in\{0,1, \ldots, D(k)-\right.$ $\left.1\}, l \neq n_{1}, l \neq n\right\}$ are uniquely determined by $\left.\alpha\right|_{k-2}$ and $\left.\beta\right|_{k-2}$; for example, $\tau\left(\left(D_{k, M}, \rho\right)\right)=0 \alpha_{0} \alpha_{1} \ldots \alpha_{k-2}$, and $\tau\left(\left(\rho, D_{k, M+2}\right)\right)=1 \beta_{0} \beta_{1} \ldots \beta_{k-2}$.
Proof. It follows from Proposition 3 that the set of addresses at level $k$, namely $\left\{\tau\left(\left(D_{k, l}, D_{k, l+1}\right)\right)\right\}_{l=0}^{D(k)-1}$, is invariant under the following operation: put a " 0 " in front of each address that is less than or equal to $\alpha$, then truncate back to length $k$; take the union of the resulting set of addresses with the set of addresses obtained by: put a " 1 " in front of each address that is greater than or equal to $\beta$, and drop the last digit.

## 4. SYMMETRY OF $\bar{\Omega}$ AND A CONSEQUENT HOMEOMORPHISM OF $[0,1]$

Lemma 1. $\bar{\Omega}=\left\{\sigma \in I^{\infty}:\right.$ for all $k \in \mathbb{N}, \sigma_{k}=0 \Rightarrow S^{k}(\sigma) \preceq \tau(\rho)$ and $\sigma_{0}=1 \Rightarrow$ $\left.\tau^{+}(\rho) \preceq S^{k}(\sigma)\right\}$.
Proof. This is an immediate consequence of Proposition 3.
Corollary 2. $\bar{\Omega}$ is symmetric if and only if $\alpha=\beta^{*}$ (or equivalently $\tau(\rho)=$ $\left.\left(\tau^{+}(\rho)\right)^{*}\right)$.
Lemma 2. The maps $\tau(\rho)$ and $\tau^{+}(\rho)$ are strictly increasing as functions of $\rho \in$ $[a, b]$ to $I^{\infty}$.
Proof. Note that $\tau(\rho)$ depends both implicitly and explicitly on $\rho$. Let $1-b \leq \rho<$ $\rho^{\prime} \leq a$ be such that $\tau(\rho) \succeq \tau\left(\rho^{\prime}\right)$. Observe that $\left.\tau(\rho)\right|_{0}=\left.\tau\left(\rho^{\prime}\right)\right|_{0}$.

Assume first that there is a largest $n>0$ such that $\left.\tau(\rho)\right|_{n}=\left.\tau\left(\rho^{\prime}\right)\right|_{n}:=\theta_{0} \theta_{1} \ldots \theta_{n}$. Then $\tau(\rho)=\theta_{0} \theta_{1} \ldots \theta_{n} 1 \ldots$ and $\tau^{+}(\rho)=\theta_{0} \theta_{1} \ldots \theta_{n} 0 \ldots$, which implies

$$
\begin{equation*}
W_{\rho}^{n}(\rho) \geq \rho \text { and } W_{\rho^{\prime}}^{n}\left(\rho^{\prime}\right) \leq \rho^{\prime} \tag{4.1}
\end{equation*}
$$

(We write $W=W_{\rho}$ when we want to note the dependence on $\rho$.) We may assume that $\left.\tau(\rho)\right|_{n}$ is constant on $\left[\rho, \rho^{\prime}\right]$ for otherwise we can restrict to a smaller interval with a strictly smaller value of $n$. As a consequence, at every iteration, we apply the same branch $W_{0}$ or $W_{1}$ to $W_{\xi}$ to compute compute $g(\xi):=W_{\xi}^{n}(\xi)$ for all $\xi \in\left[\rho, \rho^{\prime}\right]$. Therefore $g$ is continuous with derivative at least $d^{n}>1$, which contradicts (4.1).

The only remaining possibility is that $\tau(\rho)=\tau\left(\rho^{\prime}\right)$. We may assume that $\tau(\rho)$ is constant on $\left[\rho, \rho^{\prime}\right]$, otherwise we can reduce the problem to the previous case. This would mean that for arbitrarily large $n$, the image of the interval $\left[\rho, \rho^{\prime}\right]$ under $g$ is at an interval of size least $d^{n}\left(\rho^{\prime}-\rho\right)$, a contradiction.

Essentially the same argument, with the role of $\tau$ played by $\tau^{+}$and the role of played by $W_{+}$, proves that $\tau^{+}(\rho)$ is strictly increasing as a function of $\rho \in[1-b, a]$ to $I^{\infty}$.

Corollary 3. The map $\rho \mapsto \tau(\rho)$ is left continuous and the map $\rho \mapsto \tau^{+}(\rho)$ is right continuous.

Proof. Fix a parameter $\rho_{0}$ and let $\varepsilon>0$. Then by (7) there is $x<\rho_{0}$ which is not a preimage of $\rho_{0}$ for any order and such that

$$
d\left(\tau_{\rho_{0}}^{+}(x), \tau_{\rho_{0}}\left(\rho_{0}\right)\right)<\frac{\varepsilon}{2}
$$

By (9), for any $n \in \mathbb{N}$ there exists $\delta>0$ such that the prefix $\left.\tau_{\rho}^{+}(x)\right|_{n}$ is constant when $\rho \in\left(\rho_{0}-\delta, \rho_{0}+\delta\right)$. Let $n$ be such that $2^{-n}<\varepsilon$, and let $\rho>x$ and $\rho \in\left(\rho_{0}-\delta, \rho_{0}\right)$. We have that $\tau_{\rho}^{+}(x) \prec \tau_{\rho}^{+}(\rho)$ and

$$
d\left(\tau_{\rho}^{+}(x), \tau_{\rho_{0}}^{+}(x)\right)<\frac{\varepsilon}{2}
$$

Combining the two inequalities we obtain

$$
d\left(\tau_{\rho}^{+}(x), \tau_{\rho_{0}}\left(\rho_{0}\right)\right)<\varepsilon
$$

and by Lemma 2 we also have

$$
\tau_{\rho}^{+}(x) \prec \tau_{\rho}(\rho) \prec \tau_{\rho_{0}}\left(\rho_{0}\right)
$$

The distance $d$ has the property that if $\sigma \prec \zeta \prec \sigma^{\prime}$ then $d(\sigma, \zeta) \leq d\left(\sigma, \sigma^{\prime}\right)$ and $d\left(\zeta, \sigma^{\prime}\right) \leq d\left(\sigma, \sigma^{\prime}\right)$. This shows that $\rho \mapsto \tau(\rho)$ is left continuous. The right continuity of $\rho \mapsto \tau^{+}(\rho)$ admits an analagous proof.

As a consequence of Corollary 2, Lemma 2 and (10), we obtain the unicity of $\rho$ for which $\bar{\Omega}$ is symmetric.As a consequence of Corollary 2 , Lemma 2 and (10), we obtain the unicity of $\rho$ for which $\bar{\Omega}$ is symmetric.
Corollary 4. There is at most one $\rho \in[1-b, a]$ such that $\bar{\Omega}=\bar{\Omega}^{*}$.
Proof of Theorem 2. By Lemma 2 and (10), we may define

$$
\rho_{0}:=\sup \left\{\rho \in[1-b, a]: \tau(\rho) \preceq \tau^{+}(\rho)^{*}\right\}=\inf \left\{\rho \in[1-b, a]: \tau(\rho)^{*} \preceq \tau^{+}(\rho)\right\} .
$$

Assume $\tau\left(\rho_{0}\right) \prec \tau^{+}\left(\rho_{0}\right)^{*}$. It is straighfoward to check $1-b<\rho_{0}<a$.
There is a largest $n \geq 2$ such that $\left.\tau\left(\rho_{0}\right)\right|_{n}=\left.\tau^{+}\left(\rho_{0}\right)^{*}\right|_{n}=: \eta=01 \ldots$.
Observe that $\tau\left(\rho_{0}\right)=0 \tau\left(W_{0}\left(\rho_{0}\right)\right)$ and $\tau^{+}\left(\rho_{0}\right)=1 \tau^{+}\left(W_{1}\left(\rho_{0}\right)\right)$. If neither $W_{0}\left(\rho_{0}\right)$
nor $W_{1}\left(\rho_{0}\right)$ belongs to $\{0,1\} \cup \bigcup_{k=0}^{n-1} W^{-k}\left(\rho_{0}\right)$, then by ( 9 ) both $\left.\tau(\rho)\right|_{n+1}$ and $\left.\tau^{+}(\rho)\right|_{n+1}$ are constant on a neighborhood of $\rho_{0}$ which contradicts the definition of $\rho_{0}$.

Let us consider the projection $\hat{\pi}\left(\tau^{+}\left(W_{1}\left(\rho_{0}\right)\right)^{*}\right)$. If $\hat{\pi}\left(\tau^{+}\left(W_{1}\left(\rho_{0}\right)\right)^{*}\right)>W_{0}\left(\rho_{0}\right)$ then by the continuity of $W_{0}$, of $\hat{\pi}$ by (13), of $\rho \mapsto \tau_{\rho}^{+}(\rho)$ (Corollary 3) there is a $\rho>\rho_{0}$ such that $\hat{\pi}_{\rho}\left(\tau_{\rho}^{+}\left(W_{1}\left(\rho_{0}\right)\right)^{*}\right)>W_{0}(\rho)$. By (6) and (13) this implies $\tau_{\rho}(\rho) \prec \tau_{\rho}^{+}(\rho)^{*}$, which again contradicts the definition of $\rho_{0}$.

As $\hat{\pi}$ is increasing, (13) and $\tau\left(W_{0}\left(\rho_{0}\right)\right) \prec \tau^{+}\left(W_{1}\left(\rho_{0}\right)\right)^{*}$, we have $\hat{\pi}\left(\tau^{+}\left(W_{1}\left(\rho_{0}\right)\right)^{*}\right)=$ $W_{0}\left(\rho_{0}\right)$. Let $0<m<n$ be minimal such that $W^{m} \circ W_{0}\left(\rho_{0}\right)=\rho_{0}$ or $W^{m} \circ W_{1}\left(\rho_{0}\right)=$ $\rho_{0}$. We may apply (14) $m$ times and obtain

$$
\begin{equation*}
W^{m} \circ W_{0}\left(\rho_{0}\right)=\hat{\pi}\left(S^{m}\left(\tau^{+}\left(W_{1}\left(\rho_{0}\right)\right)^{*}\right)\right)=\hat{\pi}\left(\tau^{+}\left(W^{m} \circ W_{1}\left(\rho_{0}\right)\right)^{*}\right) \tag{4.2}
\end{equation*}
$$

As $\tau^{+}\left(\rho_{0}\right)=1 \ldots$, if $W^{m} \circ W_{1}\left(\rho_{0}\right)=\rho_{0}$ then we have

$$
\tau\left(\rho_{0}\right) \prec \tau^{+}\left(\rho_{0}\right)^{*}=\tau^{+}\left(W^{m} \circ W_{1}\left(\rho_{0}\right)\right)^{*} \prec \tau^{+}\left(\rho_{0}\right)
$$

which by (6) and equation (4.2) implies $W^{m} \circ W_{0}\left(\rho_{0}\right)=\rho_{0}$. Therefore $\tau\left(\rho_{0}\right)=$ $\tau^{+}\left(\rho_{0}\right)^{*}$ as both are periodic of period $m+1$ and have the same prefix of length $n>m$, a contradiction.

If $W^{m} \circ W_{1}\left(\rho_{0}\right) \neq \rho_{0}$ then $W^{m} \circ W_{0}\left(\rho_{0}\right)=\rho_{0}$ thus by (13), (10) and equality (4.2) we obtain

$$
\tau\left(\rho_{0}\right) \prec \tau^{+}\left(\rho_{0}\right)^{*} \preceq \tau^{+}\left(W^{m} \circ W_{1}\left(\rho_{0}\right)\right):=\sigma^{\prime}
$$

By (6), this means that $\rho_{0} \leq W^{m} \circ W_{1}\left(\rho_{0}\right)$ so in fact

$$
\rho_{0}<W^{m} \circ W_{1}\left(\rho_{0}\right)
$$

As $W^{m+1}\left(\rho_{0}\right)=\rho_{0}, \tau\left(\rho_{0}\right)=\kappa \kappa \kappa \ldots:=\kappa^{\infty}$ where $\kappa=\left.\tau\left(\rho_{0}\right)\right|_{m+1}=\left.\tau^{+}\left(\rho_{0}\right)^{*}\right|_{m+1}$, as $m+1 \leq n$. We can write $\tau^{+}\left(\rho_{0}\right)=\kappa^{*} \sigma^{\prime}$ therefore $\kappa^{*} \sigma^{\prime} \prec \sigma^{\prime}$ by (6) and the previous inequality. By induction we get $\kappa^{* \infty} \prec \sigma^{\prime}$ so

$$
\tau^{+}\left(\rho_{0}\right)^{*}=\kappa\left(\sigma^{*}\right) \prec \kappa^{\infty}=\tau\left(\rho_{0}\right)
$$

a contradiction.
The case $\tau\left(\rho_{0}\right) \succ \tau^{+}\left(\rho_{0}\right)^{*}$ is analagous by the symmetric definition of $\rho_{0}$, therefore $\bar{\Omega}_{\rho_{0}}$ is symmetric.

Proposition 4. If $\bar{\Omega}=\bar{\Omega}^{*}$ then the map $h:[0,1] \rightarrow[0,1]$ defined by $h(x)=$ $\hat{\pi}\left(\tau(x)^{*}\right)$ is a homeomorphism and $h \circ \hat{\pi}=\hat{\pi} \circ^{*}$ on $I^{\infty}$.

Proof. First by Corollary 2, we have $\tau(\rho)=\tau^{+}(\rho)^{*}$ and points $x$ for which $\tau(x) \neq$ $\tau^{+}(x)$ are exactly preimages of $\rho$. In this case, there is $n \geq 0$ such that $\tau(x)$ and $\tau^{+}(x)$ have the same initial prefix $\kappa:=\left.\tau(x)\right|_{n}=\left.\tau^{+}(x)\right|_{n}$, and $\tau(x)=\kappa \tau(\rho), \tau^{+}(x)=$ $\kappa \tau^{+}(\rho)$. Therefore, by (13), for all $x \in[0,1]$, we have

$$
\tau(h(x))=\tau^{+}(x)^{*} \text { and } \tau^{+}(h(x))=\tau(x)^{*},
$$

thus $h \circ h(x)=x$. By (6), (10) and (13), $h$ is also decreasing. Therefore $h:[0,1] \rightarrow$ $[0,1]$ is a homeomorphism.

Let $\sigma \in I^{\infty}$ and $x=\hat{\pi}(\sigma)$. By (13) we have $\tau(x) \preceq \sigma \preceq \tau^{+}(x)$. As $\bar{\Omega}=\bar{\Omega}^{*}$, by Proposition 2, we obtain that there exists $y \in[0,1]$ such that $\tau(x)^{*}=\tau^{+}(y)$. By Lemma 1 and Corollary 2 we also have that $\tau^{+}(x)^{*}=\tau(y)$. We may compute $h \circ \hat{\pi}(\sigma)=h(x)=\hat{\pi}\left(\tau(x)^{*}\right)=\hat{\pi}(\tau(y))=y$, which is also equal to $\hat{\pi}\left(\sigma^{*}\right)$ as $\tau(y) \preceq$ $\sigma^{*} \preceq \tau^{+}(y)$.

## 5. Iterated Closed Relations and Conley Decomposition for Itineraries of $W$

Theorem 1 follows from Proposition 3, but some extra language is needed. In explaining this language we describe the Conley-McGehee-Wiandt decomposition theorem, [3, Theorem 13.1].

For $X$ a compact Hausdorff space, let $2^{X}$ be the subsets of $X$. A relation $r$ on $X$ is simply a subset of $X \times X$. A relation $r$ on $X$ is called a closed relation if $r$ is a closed of $X \times X$. For example the set $r \subset I^{\infty} \times I^{\infty}$ defined in Theorem 1, namely

$$
r=\left\{(0 \sigma, \sigma) \in I^{\infty} \times I^{\infty}: \sigma \preceq \alpha\right\} \cup\left\{(1 \sigma, \sigma) \in I^{\infty} \times I^{\infty}: \beta \preceq \sigma\right\}
$$

is a closed relation. Following [3], a relation $r \in 2^{X}$ provides a mapping $r: 2^{X} \rightarrow 2^{X}$ defined by

$$
r(C)=\{y \in X:(x, y) \in r \text { for some } x \in C\}
$$

Notice that the image of a nonempty set may be empty. Iterated relations are defined by $r^{0}=X \times X$ and, for all $k \in \mathbb{N}$,

$$
r^{k+1}=r \circ r^{k}=\left\{(x, z):(x, y) \in r,(y, z) \in r^{k} \text { for some } y \in X\right\}
$$

The omega limit set of $C \subset X$ under a closed relation $r \subset X \times X$ is

$$
\omega(C)=\cap \mathfrak{K}(C)
$$

where

$$
\mathfrak{K}(C)=\left\{D \text { is a closed subset of } X: r(D) \cup r^{n}(C) \subset D \text { for some } n \in \mathbb{N}\right\} .
$$

By definition, an attractor of a closed relation $r$ is a closed set $A$ such that the following two conditions hold:
(i) $r(A)=A$;
(ii) there is a closed neighborhood $\overline{\mathcal{N}}(A)$ of $A$ such that $\omega(C) \subset A$ for all $C \subset$ $\overline{\mathcal{N}}(A)$.

The basin $\mathcal{B}(A)$ of an attractor $A$ for a closed relation $r$ on a compact Hausdorff space $X$ is the union of all open sets $O \subset X$ such that $\omega(C) \subset A$ for all $C \subset O$.

Given an attractor $A$ for a closed relation $r$ on a compact Hausdorff space $X$, there exists a corresponding attractor block, namely a closed set $E \subset X$ such that $E$ contains both $A$ and $r(E)$ in its interior, and $A=\omega(E)$. Also, there exists a unique dual repeller $A^{*}=X \backslash \mathcal{B}(A)$. This repeller is an attractor for the transpose relation $r^{*}=\{(y, x):(x, y) \in r\}$. The set of connecting orbits associated with the attractor/repeller pair $A, A^{*}$ is $\mathcal{C}(A)=X \backslash\left(A \cup A^{*}\right)$.

If $r$ is a closed relation on a compact Hausdorff space $X$, then $x \in X$ is called chain-recurrent for $r$ if for every closed neighborhood $f$ of $r, x$ is periodic for $f$ (i.e. there exists a finite sequence of points $\left\{x_{n}\right\}_{n=0}^{p-1} \subset X$ such that $x_{0}=x$, $\left(x_{p-1}, x_{0}\right) \in f$ and $\left(x_{n-1}, x_{n}\right) \in f$ for $\left.n=1,2, \ldots, p-1\right)$. The chain recurrent set $\mathcal{R}$ for $r$ is the union of all the points that are chain recurrent for $r$. A transitive component of $\mathcal{R}$ is a member of the equivalence class on $\mathcal{R}$ defined by $x \sim y$ when for every closed neighborhood $f$ of $r$ there is an orbit from $x$ to $y$ under $f$ (i.e. there exists a finite sequence of points $\left\{x_{n}\right\}_{n=0}^{p-1} \subset \mathcal{R}$ such that $x_{0}=x, x_{p-1}=y$, and $\left(x_{n}, x_{n+1}\right) \in f$ for all $n \in\{0,1, \ldots p-1\}$.)

Theorem 3 (Conley-McGehee-Wiandt). If $r$ is a closed relation on a compact Hausdorff space $X$, then

$$
\mathcal{R}=\bigcup_{A \in \mathcal{U}} \mathcal{C}(A)
$$

where $\mathcal{R}$ is the chain-recurrent set and $\mathcal{U}$ is the set of attractors.
Proof of Theorem 1. This follows at once from Proposition 3 together with Theorem 3, but see [3].

We note the following. $\bar{\Omega}$ can be embedded in $[0,1] \subset \mathbb{R}$ using the (continuous and surjective) coding map $\pi: I^{\infty} \rightarrow[0,1]$ associated with the iterated function system $([0,1] ; x \mapsto x / 2, x \mapsto(1+x) / 2)$. This coding map $\pi$ is defined, for all $\sigma$, by

$$
\pi(\sigma)=\sum_{k \in \mathbb{N}} \frac{\sigma_{k}}{2^{k+1}}
$$

$\pi$ provides a homeomorphism between $\bar{\Omega}$ and $\pi(\bar{\Omega})$. The point $\sigma \in \bar{\Omega}$ is uniquely and unambiguously represented by the binary real number $0 . \sigma$. In the representation provided by $\pi$, the map $\bar{\Psi}: 2^{I^{\infty}} \rightarrow 2^{I^{\infty}}$ becomes the action of the iterated closed relation $\widetilde{r} \subset[0,1] \times[0,1] \subset \mathbb{R}^{2}$ defined by

$$
\tilde{r}:=\{(x, x / 2): x \in[0, \pi(\alpha)]\} \cup\{(x,(x+1) / 2): x \in[\pi(\beta), 1]\}
$$

on subsets of $[0,1]$. It follows from Proposition 3 (iii) that $\pi(\bar{\Omega})$ is the maximal attractor, as defined in [3], of $\widetilde{r}$. The corresponding dual repeller is the empty set. It is also easy to see that $\{0\}$ and $\{1\}$ are the only other attractors, with corresponding dual repellers $[\pi(\alpha), 1]$ and $[0, \pi(\beta)]$ respectively. It follows from Theorem 3 that the chain recurrent set of $\widetilde{r}$ is $\{0,1\} \cup(\pi(\bar{\Omega}) \cap(\pi(\beta), \pi(\alpha)))$.
Acknowledgement 1. We thank Konstantin Igudesman for helpful comments.

## References

[1] M. F. Barnsley, B. Harding and K. Igudesman, How to filter and transform images using iterated function systems, SIAM Journal of Imaging Science, 4 (2011) 1001-1024.
[2] K. B. Igudesman, A certain family of self-similar sets, Russian Mathematics (Iz. Vuz) 55 (2011) 26-38.
[3] Richard P. McGehee and T. Wiandt, Conley decomposition for closed relations, Difference Equations and Applications, 12 (2006) 1-47.
[4] John E. Hutchinson, Fractals and self-similarity, Indiana Univ. Math. J. 30 (1981) 713-747.
[5] William Parry, Symbolic dynamics and transformations of the unit interval, Trans. Am. Math. Soc, 122 (1966) 368-378.

Department of Mathematics, Australian National University, Canberra, ACt, AusTRALIA

E-mail address: michael.barnsley@maths.anu.edu.au
Université Paris-Est Créteil, LAMA, 94010 Créteil, France
E-mail address: nicolae.mihalache@u-pec.fr


[^0]:    Date: 25th November 2011.
    2010 Mathematics Subject Classification. Primary 37B10, 37B40; Secondary 28A80.

