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SYMMETRIC ITINERARY SETS

MICHAEL F. BARNSLEY AND NICOLAE MIHALACHE

ABSTRACT. We consider a one parameter family of dynamical systems $W : [0, 1] \rightarrow [0, 1]$ constructed from a pair of monotone increasing diffeomorphisms W_i , such that $W_i^{-1} : [0, 1] \rightarrow [0, 1]$, ($i = 0, 1$). We characterize the set of symbolic itineraries of W using an attractor $\overline{\Omega}$ of an iterated closed relation, in the terminology of McGehee, and prove that there is a member of the family for which $\overline{\Omega}$ is symmetrical.

1. INTRODUCTION

Let $W_0 : [0, a] \rightarrow [0, 1]$ and $W_1 : [1-b, 1] \rightarrow [0, 1]$ be continuous and differentiable, and such that $a + b > 1$, $W_0(0) = W_1(1 - b) = 0$, $W_0(a) = W_1(1) = 1$. Let the derivatives $W_i'(x)$ ($i = 0, 1$) be uniformly bounded below by $d > 1$.

For $\rho \in [1 - b, a]$ we define $W : [0, 1] \rightarrow [0, 1]$ by

$$[0, 1] \ni x \mapsto \begin{cases} W_0(x) & \text{if } x \in [0, \rho] \\ W_1(x) & \text{otherwise.} \end{cases}$$

See Figure 1. Similarly, we define $W_+ : [0, 1] \rightarrow [0, 1]$ by replacing $[0, \rho]$ by $[0, \rho)$.

Let $I = \{0, 1\}$. Let $I^\infty = \{0, 1\} \times \{0, 1\} \times \dots$ have the product topology induced from the discrete topology on I . For $\sigma \in I^\infty$ write $\sigma = \sigma_0\sigma_1\sigma_2\dots$, where $\sigma_k \in I$ for all $k \in \mathbb{N}$. The product topology on I^∞ is the same as the topology induced by the metric $d(\omega, \sigma) = 2^{-k}$ where k is the least index such that $\omega_k \neq \sigma_k$. It is well known that (I^∞, d) is a compact metric space. We define a total order relation \preceq on I^∞ , and on I^n for any $n \in \mathbb{N}$, by $\sigma \prec \omega$ if $\sigma \neq \omega$ and $\sigma_k < \omega_k$ where k is the least index such that $\sigma_k \neq \omega_k$. For $\sigma \in I^\infty$ and $n \in \mathbb{N}$ we write $\sigma|_n = \sigma_0\sigma_1\sigma_2\dots\sigma_n$. I^∞ is the appropriate space in which to embed and study the itineraries of the family of discontinuous dynamical systems $W : [0, 1] \rightarrow [0, 1]$.

For $W_{(+)} \in \{W, W_+\}$ let $W_{(+)}^k$ denote $W_{(+)}$ composed with itself k times, for $k \in \mathbb{N}$, and let $W_{(+)}^{-k} = (W_{(+)}^k)^{-1}$. We define a map $\tau : [0, 1] \rightarrow I^\infty$, using all of the orbits of W , by

$$\tau(x) = \sigma_0\sigma_1\sigma_2\dots$$

where σ_k equals 0, or 1, according as $W^k(x) \in [0, \rho]$, or $(\rho, 1]$, respectively. We call $\tau(x)$ the *itinerary* of x under W , or an *address* of x , and we call $\Omega = \tau([0, 1])$ an *address space* for $[0, 1]$. Similarly, we define $\tau^+ : [0, 1] \rightarrow I^\infty$ so that $\tau^+(x)_k$ equals 0, or 1, according as $W_+^k(x) \in [0, \rho)$, or $[\rho, 1]$, respectively; and we define $\Omega_+ = \tau^+([0, 1])$. Note that W , W_+ , Ω , Ω_+ , τ , and τ^+ all depend on ρ .

The main goals of this paper are to characterise $\overline{\Omega}$ and to show that there exists a value of ρ such that $\overline{\Omega}$ is symmetric.

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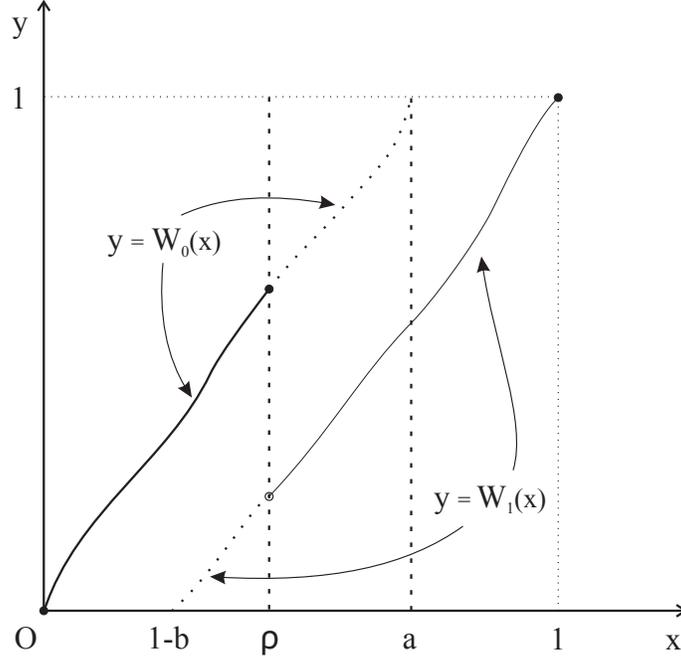


FIGURE 1. The piecewise continuous dynamical system $W:[0,1]\rightarrow[0,1]$ is defined in terms of two monotone strictly increasing differentiable functions $W_0(x)$ and $W_1(x)$, and a real parameter ρ .

Theorem 1. *Let an iterated closed relation $r \subset I^\infty \times I^\infty$ be defined by*

$$r := \{(\sigma, 0\sigma) \in I^\infty \times I^\infty : \sigma \preceq \alpha\} \cup \{(\sigma, 1\sigma) \in I^\infty \times I^\infty : \sigma \succeq \beta\}$$

where $\alpha = \tau(W_0(\rho))$ and $\beta = \tau^+(W_1(\rho))$. *The only attractors of r are $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{0}, \bar{1}\}$, and $\bar{\Omega}$. The corresponding dual repellers are $\{\sigma \in I^\infty : \beta \preceq \sigma\}$, $\{\sigma \in I^\infty : \sigma \preceq \alpha\}$, $\{\sigma \in I^\infty : \beta \preceq \sigma\} \cup \{\sigma \in I^\infty : \sigma \preceq \alpha\}$, and the empty set, respectively. The chain recurrent set for r is $\{\bar{0}, \bar{1}\} \cup \{\sigma \in \bar{\Omega} : \beta \preceq \sigma \preceq \alpha\}$.*

We write \bar{E} to denote the closure of a set E . But we write $\bar{0} = 000\dots$, $\bar{1} = 111\dots \in I^\infty$. For $\sigma = \sigma_0\sigma_1\sigma_2\dots \in I^\infty$ we write 0σ to mean $0\sigma_0\sigma_1\sigma_2\dots \in I^\infty$ and $1\sigma = 1\sigma_0\sigma_1\sigma_2\dots \in I^\infty$.

Define a symmetry function $*$: $I^\infty \rightarrow I^\infty$ by $\sigma^* = \omega$ where $\omega_k = 1 - \sigma_k$ for all k .

Theorem 2. *There exists a unique $\rho \in [1 - b, a]$ such that $\bar{\Omega}^* = \bar{\Omega}$.*

Theorem 1 tells us that $\bar{\Omega}$ is fixed by itineraries of two inverse images of the critical point ρ , and provides the basis for a stable algorithm to determine $\bar{\Omega}$. It relates the address spaces of dynamical systems of the form of W to the beautiful theory of iterated closed relations on compact Hausdorff spaces [3], and hence to the work of Charles Conley.

Theorem 2 is interesting in its own right and also because it has applications in digital imaging, as explained and demonstrated, in the special case of affine maps, in [1]. It enables the construction of parameterized families of nondifferentiable

homeomorphisms on $[0, 1]$, using pairs of overlapping iterated function systems, see Proposition 4. Theorem 2 generalizes results in [1] to nonlinear W_i 's. The proof uses symbolic dynamics in place of the geometrical construction outlined in [1]. The approach and results open up the mathematics underlying [1] and [2].

To tie the present work into [1], note that τ is a section, as defined in [1], for the hyperbolic iterated function system

$$\mathcal{F} := ([0, 1]; W_0^{-1}, W_1^{-1}).$$

Our observations interrelate to, but are more specialized than, the work of Parry [5]. Our point of view is topological rather than measure-theoretic, and our main results appear to be new.

2. BASIC PROPERTIES OF τ

The following list of properties is relatively easy to check. Below the list we elaborate on points 1, 2, and 3.

- (1) W^n is piecewise differentiable and its derivative is uniformly bounded below by d^n ; each, except the leftmost branch of W^n , is defined on an interval of the form $(r, s]$. W_+^n is piecewise differentiable and its derivative is uniformly bounded below by d^n ; each, except for the rightmost branch of W_+^n , is defined on an interval of the form $[r, s)$.
- (2) If (r, s) is the interior of the definition domain of a branch of W^n (and of W_+^n) then $\tau(x)|_n$ is constant on $(r, s]$, $\tau^+(x)|_n$ is constant on $[r, s)$, and $\tau(x)|_n = \tau^+(x)|_n$ for all $x \in (r, s)$.
- (3) The boundary of the definition domain of a branch of W^n is contained in $\{0, 1\} \cup \bigcup_{k=0}^{n-1} W^{-k}(\rho)$; by (1), the length of such a domain is at most d^{-n} .
- (4) The set $\bigcup_{k \in \mathbb{N}} W^{-k}(\rho)$ is dense in $[0, 1]$. This follows from (3).
- (5) $\tau(x) = \tau^+(x)$ unless $x \in \bigcup_{k \in \mathbb{N}} W^{-k}(\rho)$ in which case $\tau(x) \prec \tau^+(x)$.
- (6) Both $\tau(x)$ and $\tau^+(x)$ are strictly increasing functions of $x \in [0, 1]$ and $\tau(x) \preceq \tau^+(x)$. This follows from (4) and (5).
- (7) For all $x \in [0, 1]$, $\tau(x)$ is continuous from the left, $\tau^+(x)$ is continuous from the right. Moreover, for all $x \in (0, 1)$,

$$\tau(x) = \lim_{\varepsilon \rightarrow 0^+} \tau^+(x - \varepsilon) \text{ and } \tau^+(x) = \lim_{\varepsilon \rightarrow 0^+} \tau(x + \varepsilon).$$

These assertions follow from (2), (3) and (4).

- (8) Each $x \in W^{-n}(\rho)$, such that $\tau(x)|_n$ is constant, moves continuously with respect to ρ with positive velocity bounded above by d^{-n} . This follows from (1).
- (9) For $x \in (0, 1) \setminus \bigcup_{k=0}^n W^{-k}(\rho)$, $\tau(x)|_n = \tau^+(x)|_n$ is locally constant with respect to ρ ; moreover, this holds if x depends continuously on ρ . This follows from (2), (3) and (6).
- (10) The symmetry function $*$: $I^\infty \rightarrow I^\infty$ is strictly decreasing and continuous.
- (11) For any $\sigma|_n \in I^n$, $n \in \mathbb{N}$, the set

$$\mathcal{I}(\sigma|_n) := \{x \in [0, 1] : \tau(x)|_n = \sigma|_n \text{ or } \tau^+(x)|_n = \sigma|_n\},$$

is either empty or a non-degenerate compact interval of length at most d^{-n} . This follows from (2), (3) and (6).

(12) The projection $\hat{\pi} : I^\infty \rightarrow [0, 1]$ is well-defined by

$$\hat{\pi}(\sigma) = \sup\{x \in [0, 1] : \tau^+(x) \preceq \sigma\} = \inf\{x \in [0, 1] : \tau(x) \succeq \sigma\}.$$

This follows from (6).

(13) The projection $\hat{\pi} : I^\infty \rightarrow [0, 1]$ is increasing, by (6); continuous, by (11); and, by (7),

$$\begin{aligned} \hat{\pi}(\tau(x)) &= \hat{\pi}(\tau^+(x)) = x \text{ for all } x \in [0, 1], \\ \tau(\hat{\pi}(\sigma)) &\preceq \sigma \preceq \tau^+(\hat{\pi}(\sigma)) \text{ for all } \sigma \in I^\infty. \end{aligned}$$

(14) Let $S : I^\infty \rightarrow I^\infty$ denote the left-shift map $\sigma_0\sigma_1\sigma_2\dots \mapsto \sigma_1\sigma_2\sigma_3\dots$. For all $\sigma \in I^\infty$ such that $\sigma \preceq \tau(\rho)$ or $\sigma \succeq \tau^+(\rho)$,

$$\hat{\pi}(S(\sigma)) = W(\hat{\pi}(\sigma)).$$

Also $\hat{\pi}(\tau^+(\rho)) = \rho$ and $\hat{\pi}(S(\tau^+(\rho))) = W_1(\rho)$. These statements follow from (7).

Here we elaborate on points (1), (2) and (3). Consider the piecewise continuous function $W^k(x)$, for $k \in \{1, 2, \dots\}$. Its discontinuities are at ρ and, for $k > 1$, other points in $(0, 1)$, each of which can be written in the form $W_{\sigma_0}^{-1} \circ W_{\sigma_2}^{-1} \circ \dots \circ W_{\sigma_{l-1}}^{-1}(\rho)$ for some $\sigma_0\sigma_1\dots\sigma_{l-1} \in \{0, 1\}^l$ for some $l \in \{1, 2, \dots, k-1\}$. We denote these discontinuities, together with the points 0 and 1, by

$$D_{k,0} := 0 < D_{k,1} < D_{k,2} < \dots < D_{k,D(k)-1} < 1 =: D_{k,D(k)},$$

where $D(1) = 3, D(2) = 5 < D(3) < D(4) \dots$. For each $k \geq 1$, one of the $D_{k,j}$'s is equal to ρ . For $k \geq 1$ we have $W^k(x) = W_0^k(x)$ for $x \in [D_{k,0}, D_{k,1}]$ and $W_+^k(x) = W_0^k(x)$ for $x \in [D_{k,0}, D_{k,1}]$. Similarly $W^k(x) = W_1^k(x)$ for all $x \in (D_{k,D(k)-1}, D_{k,D(k)})$ and $W_+^k(x) = W_1^k(x)$ for $x \in [D_{D(k)-1}, D_{D(k)}]$.

For all $x \in (D_{k,l}, D_{k,l+1})$ ($l = 0, 1, \dots, D(k)-1$), $W^k(x) = W_+^k(x) = W_{\theta_k} \circ W_{\theta_{k-1}} \circ \dots \circ W_{\theta_1}(x)$ for some fixed $\theta_1\theta_2\dots\theta_k \in \{0, 1\}^k$. We refer to $\theta_1\theta_2\dots\theta_k$ as the *address of the interval* $(D_{k,l}, D_{k,l+1})$, we say $(D_{k,l}, D_{k,l+1})$ that "has address $\theta_1\theta_2\dots\theta_k$ ", and we write, by slight abuse of notation, $\tau((D_{k,l}, D_{k,l+1})) = \theta_1\theta_2\dots\theta_k$.

Let $k > 1$. Consider two adjacent intervals, $(D_{k,m-1}, D_{k,m}]$ and $(D_{k,m}, D_{k,m+1}]$ for $m \in \{1, 2, \dots, D(k)-1\}$ and $k > 1$. Let the one on the right have address $\theta_0\theta_1\dots\theta_{k-1}$ and the one on the left have address $\eta_0\eta_1\dots\eta_{k-1}$. Then $\eta_0\eta_1\dots\eta_{k-1} \prec \theta_0\theta_1\dots\theta_{k-1}$ and we have

$$\begin{aligned} \tau(x)|_{k-1} &= \eta_0\eta_1\dots\eta_{k-1} \text{ for all } x \in (D_{k,m-1}, D_{k,m}], \\ \tau^+(x)|_{k-1} &= \eta_0\eta_1\dots\eta_{k-1} \text{ for all } x \in [D_{k,m-1}, D_{k,m}), \\ \tau(x)|_{k-1} &= \theta_0\theta_1\dots\theta_{k-1} \text{ for all } x \in (D_{k,m}, D_{k,m+1}], \\ \tau^+(x)|_{k-1} &= \theta_0\theta_1\dots\theta_{k-1} \text{ for all } x \in [D_{k,m}, D_{k,m+1}). \end{aligned}$$

In particular, $\tau(x)|_{k-1}$ and $\tau^+(x)|_{k-1}$ are constant and equal on each of the open intervals $(D_{k,m-1}, D_{k,m})$ and have distinct values at the discontinuity points $\{D_{k,m}\}_{m=1}^{D(k)-1}$.

3. THE STRUCTURES OF Ω , Ω_+ AND $\overline{\Omega}$.

In this section we characterize Ω and Ω_+ as certain inverse limits, and we characterize $\overline{\Omega}$ as an attractor of an iterated closed relation on I^∞ . These inverse limits are natural and they clarify the structures of Ω and Ω_+ . They are implied by the shift invariance of Ω and Ω_+ . Recall that $S : I^\infty \rightarrow I^\infty$ denotes the left-shift map $\sigma_0\sigma_1\sigma_2\dots \mapsto \sigma_1\sigma_2\sigma_3\dots$.

Proposition 1. (i) $\tau(W(x)) = S(\tau(x))$ and $\tau^+(W_+(x)) = S(\tau^+(x))$ for all $x \in [0, 1]$.

(ii) $S(\Omega) = \Omega$ and $S(\Omega_+) = \Omega_+$.

Proof. (i) This follows at once from the definitions of τ and τ^+ . (ii) This follows from (i) together with $W([0, 1]) = W_+([0, 1]) = [0, 1]$. \square

We say that $\Lambda \subset I^\infty$ is *closed from the left* if, whenever $\{x_n\}_{n=0}^\infty$ is a non-decreasing sequence of points in Λ , $\lim x_n \in \Lambda$. We say that $\Lambda \subset I^\infty$ is *closed from the right* if, whenever $\{x_n\}_{n=0}^\infty$ is a non-increasing sequence in Λ , $\lim x_n \in \Lambda$. For $S \subset X$, where $X = I^\infty$ or $[0, 1]$, we write $L(S) = \{\sigma \in X : \text{there is a non-decreasing sequence } \{z_n\}_{n=0}^\infty \subset S \text{ with } \sigma = \lim z_n\}$ to denote the closure of S from the left. Analogously, we define $R(S)$ for the closure of S from the right.

Proposition 2. (i) Ω is closed from the left and Ω_+ is closed from the right;

(ii) $\overline{\Omega} = \overline{\Omega_+} = \Omega \cup \Omega_+ = \overline{\Omega \cap \Omega_+}$

Proof. Proof of (i): By (6) $\tau : [0, 1] \rightarrow I^\infty$ is monotone strictly increasing. By (7) τ is continuous from the left. Let $\{z_n\}_{n=0}^\infty$ be a non-decreasing sequence of points in Ω . Let $y_n = \tau^{-1}(z_n)$. Let $y = \lim y_n \in [0, 1]$. Since τ is continuous from the left, $\Omega \ni \tau(y) = \tau(\lim y_n) = \lim \tau(y_n) = \lim z_n$. It follows that Ω is closed from the left. Similarly, Ω_+ is closed from the right.

Proof of (ii): Let $Q = \{x \in [0, 1] : \tau(x) = \tau^+(x)\}$. Then by (4) $\overline{Q} = [0, 1]$. Also, by (5),

$$\Omega \cap \Omega_+ = \tau([0, 1]) \cap \tau^+([0, 1]) = \tau(Q) = \tau^+(Q).$$

Hence

$$\overline{\Omega \cap \Omega_+} = \overline{\tau(Q)} = \overline{\tau^+(Q)} = \overline{\Omega} = \overline{\Omega_+}.$$

Finally, $\Omega \cup \Omega_+ = L(\tau(Q)) \cup R(\tau^+(Q)) = L(\tau(Q)) \cup R(\tau(Q)) = \overline{\tau(Q)} = \overline{\Omega}$. \square

We define $s_i : I^\infty \rightarrow I^\infty$ by $s_i(\sigma) = i\sigma$ ($i = 0, 1$). Note that both s_0 , and s_1 , are contractions with contractivity $1/2$. We write 2^{I^∞} to denote the set of all subsets of I^∞ . For $\sigma, \omega \in I^\infty$ we define

$$\begin{aligned} [\sigma, \omega] &:= \{\zeta \in I^\infty : \sigma \preceq \zeta \preceq \omega\}, \\ (\sigma, \omega) &:= \{\zeta \in I^\infty : \sigma \prec \zeta \prec \omega\}, \\ (\sigma, \omega] &:= \{\zeta \in I^\infty : \sigma \prec \zeta \preceq \omega\}, \\ [\sigma, \omega) &:= \{\zeta \in I^\infty : \sigma \preceq \zeta \prec \omega\}. \end{aligned}$$

Proposition 3. Let $\alpha = S(\tau(\rho))$ and $\beta = S(\tau^+(\rho))$.

(i) $\Omega = \bigcap_{k \in \mathbb{N}} \Psi^k([\overline{0}, \overline{1}])$ where $\Psi : 2^{I^\infty} \rightarrow 2^{I^\infty}$ is defined by

$$2^{I^\infty} \ni \Lambda \mapsto s_0(\Lambda \cap [\overline{0}, \alpha]) \cup s_1(\Lambda \cap (\beta, \overline{1})).$$

(ii) $\Omega_+ = \bigcap_{k \in \mathbb{N}} \Psi_+^k([\bar{0}, \bar{1}])$ where $\Psi_+ : 2^{I^\infty} \rightarrow 2^{I^\infty}$ is defined by

$$2^{I^\infty} \ni \Lambda \mapsto s_0(\Lambda \cap [\bar{0}, \alpha]) \cup s_1(\Lambda \cap [\beta, \bar{1}]).$$

(iii) $\bar{\Omega} = \bar{\Omega}_+ = \bigcap_{k \in \mathbb{N}} \bar{\Psi}^k([\bar{0}, \bar{1}])$ where $\bar{\Psi} : 2^{I^\infty} \rightarrow 2^{I^\infty}$ is defined by

$$2^{I^\infty} \ni \Lambda \mapsto s_0(\Lambda \cap [\bar{0}, \alpha]) \cup s_1(\Lambda \cap [\beta, \bar{1}]).$$

Proof. Proof of (i): Let $S|_\Omega : \Omega \rightarrow \Omega$ denote the domain and range restricted shift map. It is readily found that the branches of $S|_\Omega^{-1} : \Omega \rightarrow \Omega$ are $s_0|_\Omega : [\bar{0}, \alpha] \cap \Omega \rightarrow \Omega$ where

$$s_0|_\Omega(\sigma) = s_0(\sigma) = 0\sigma \text{ for all } \sigma \in [\bar{0}, \alpha] \cap \Omega,$$

and $s_1|_\Omega : (\beta, \bar{1}) \cap \Omega \rightarrow \Omega$ where

$$s_1|_\Omega(\sigma) = s_1(\sigma) = 1\sigma \text{ for all } \sigma \in (\beta, \bar{1}) \cap \Omega.$$

(Note that $\alpha_0 = 1$, $\beta_0 = 0$ and $\beta \prec \alpha$.) It follows that

$$S|_\Omega^{-1}(\Lambda) = s_0(\Lambda \cap [\bar{0}, \alpha]) \cup s_1(\Lambda \cap (\beta, \bar{1})) = \Psi(\Lambda)$$

for all $\Lambda \subset \Omega$. Since $\Omega \subset [\bar{0}, \bar{1}]$ it follows that

$$\Omega = S|_\Omega^{-1}(\Omega) = \Psi(\Omega) \subset \Psi([\bar{0}, \bar{1}]).$$

Also, since $\Psi([\bar{0}, \bar{1}]) \subset [\bar{0}, \bar{1}]$ it follows that $\{\Psi^k([\bar{0}, \bar{1}])\}$ is a decreasing (nested) sequence of sets, each of which contains Ω ; hence

$$\Omega \subset \bigcap_{k \in \mathbb{N}} \Psi^k([\bar{0}, \bar{1}]).$$

It remains to prove that $\Omega \supset \bigcap_{k \in \mathbb{N}} \Psi^k([\bar{0}, \bar{1}])$. We note that $s_0([\bar{0}, \alpha]) = [\bar{0}, \tau(\rho)]$ and $s_1((\beta, \bar{1})) = (\tau^+(\rho), \bar{1})$, from which it follows that

$$(3.1) \quad \bigcap_{k \in \mathbb{N}} \Psi^k([\bar{0}, \bar{1}]) = \bigcap_{k \in \mathbb{N}} \{\sigma \in I^\infty : S^k(\sigma) \in [\bar{0}, \tau(\rho)] \cup (\tau^+(\rho), \bar{1}]\}.$$

Let $\omega \in \bigcap_{k \in \mathbb{N}} \Psi^k([\bar{0}, \bar{1}])$. Suppose $\omega \notin \Omega$. Let

$$\omega_- = \sup\{\sigma \in \Omega : \sigma \preceq \omega\} \text{ and } \omega_+ = \inf\{\sigma \in \Omega : \omega \preceq \sigma\},$$

so that

$$\omega_- \preceq \omega \preceq \omega_+.$$

But $\omega_- \in \Omega$ (since Ω is closed from the left), so

$$\omega_- \prec \omega \preceq \omega_+.$$

Note that, since $\inf\{\sigma \in \Omega : \omega \preceq \sigma\} = \inf\{\sigma \in \Omega_+ : \omega \preceq \sigma\}$, and Ω_+ is closed from the right, we have $\omega_+ \in \Omega_+$. Let $K = \min\{k \in \mathbb{N} : (\omega_-)_k \neq (\omega_+)_k\}$. Then $S^K(\omega_-) \prec S^K(\omega) \preceq S^K(\omega_+)$ and we must have $S^K(\omega_-) = \tau(\rho)$ and $S^K(\omega_+) = \tau^+(\rho)$. So

$$\tau(\rho) \prec S^K(\omega) \preceq \tau^+(\rho),$$

therefore $\omega \notin \{\sigma \in I^\infty : S^K(\sigma) \in [\bar{0}, \tau(\rho)] \cup (\tau^+(\rho), \bar{1}]\}$ which, because of (3.1), contradicts our assumption that $\omega \in \bigcap_{k \in \mathbb{N}} \Psi^k([\bar{0}, \bar{1}])$. Hence $\omega \in \Omega$ and we have

$$\Omega \supset \bigcap_{k \in \mathbb{N}} \Psi^k([\bar{0}, \bar{1}]).$$

This completes the proof of (i).

Proof of (ii): similar to the proof of (i), with the role of $[\bar{0}, \tau(\rho)]$ played by $[\bar{0}, \tau(\rho))$ and the role of $(\tau^+(\rho), \bar{1}]$ played by $[\tau^+(\rho), \bar{1}]$.

Proof of (iii): similar to the proofs of (i) and (ii). \square

It is helpful to note that the addresses α and β in Proposition 3 obey

$$\begin{aligned}\alpha &= \tau(W_0(\rho)), \beta = \tau(W_1(\rho)), \\ \tau(\rho) &= 0\alpha = 01\alpha_1\alpha_2\dots \text{ and } \tau^+(\rho) = 0\beta = 10\beta_1\beta_2\dots\end{aligned}$$

Let $M > 0$ be such that $D_{k,M+1} = \rho$. It follows from the discussion at the end of Section 2 that $\tau((D_{k,M}, \rho)) = \tau^+((D_{k,M}, \rho)) = 01\alpha_1\alpha_2\dots\alpha_{k-2}$ and $\tau((\rho, D_{k,M+2})) = \tau^+((\rho, D_{k,M+2})) = 10\beta_1\beta_2\dots\beta_{k-2}$.

Corollary 1. *Let $k \geq 1$, $\alpha = \tau(W_0(\rho))$, $\beta = \tau(W_1(\rho))$, and let $M > 0$ be such that $D_{k,M+1} = \rho$. The set of addresses $\{\tau((D_{k,l}, D_{k,l+1}))\}_{l=0}^{D(k)-1}$ is uniquely determined by $\alpha|_{k-1}$ and $\beta|_{k-1}$. For some n_1, n_2 such that $0 \leq n_1 < M < n_2 \leq D(k) - 1$, $\tau((D_{k,n_1}, D_{k,n_1+1})) = \beta_0\beta_1\dots\beta_{k-2}\beta_{k-1}$ and $\tau((D_{k,n_2}, D_{k,n_2+1})) = \alpha_0\alpha_1\dots\alpha_{k-2}\alpha_{k-1}$. The set of addresses $\{\tau((D_{k,l}, D_{k,l+1})) : l \in \{0, 1, \dots, D(k) - 1\}, l \neq n_1, l \neq n_2\}$ are uniquely determined by $\alpha|_{k-2}$ and $\beta|_{k-2}$; for example, $\tau((D_{k,M}, \rho)) = 0\alpha_0\alpha_1\dots\alpha_{k-2}$, and $\tau((\rho, D_{k,M+2})) = 1\beta_0\beta_1\dots\beta_{k-2}$.*

Proof. It follows from Proposition 3 that the set of addresses at level k , namely $\{\tau((D_{k,l}, D_{k,l+1}))\}_{l=0}^{D(k)-1}$, is invariant under the following operation: put a "0" in front of each address that is less than or equal to α , then truncate back to length k ; take the union of the resulting set of addresses with the set of addresses obtained by: put a "1" in front of each address that is greater than or equal to β , and drop the last digit. \square

4. SYMMETRY OF $\bar{\Omega}$ AND A CONSEQUENT HOMEOMORPHISM OF $[0, 1]$

Lemma 1. $\bar{\Omega} = \{\sigma \in I^\infty : \text{for all } k \in \mathbb{N}, \sigma_k = 0 \Rightarrow S^k(\sigma) \preceq \tau(\rho) \text{ and } \sigma_0 = 1 \Rightarrow \tau^+(\rho) \preceq S^k(\sigma)\}$.

Proof. This is an immediate consequence of Proposition 3. \square

Corollary 2. $\bar{\Omega}$ is symmetric if and only if $\alpha = \beta^*$ (or equivalently $\tau(\rho) = (\tau^+(\rho))^*$).

Lemma 2. *The maps $\tau(\rho)$ and $\tau^+(\rho)$ are strictly increasing as functions of $\rho \in [a, b]$ to I^∞ .*

Proof. Note that $\tau(\rho)$ depends both implicitly and explicitly on ρ . Let $1 - b \leq \rho < \rho' \leq a$ be such that $\tau(\rho) \succeq \tau(\rho')$. Observe that $\tau(\rho)|_0 = \tau(\rho')|_0$.

Assume first that there is a largest $n > 0$ such that $\tau(\rho)|_n = \tau(\rho')|_n := \theta_0\theta_1\dots\theta_n$. Then $\tau(\rho) = \theta_0\theta_1\dots\theta_n1\dots$ and $\tau^+(\rho) = \theta_0\theta_1\dots\theta_n0\dots$, which implies

$$(4.1) \quad W_\rho^n(\rho) \geq \rho \text{ and } W_{\rho'}^n(\rho') \leq \rho'.$$

(We write $W = W_\rho$ when we want to note the dependence on ρ .) We may assume that $\tau(\rho)|_n$ is constant on $[\rho, \rho']$ for otherwise we can restrict to a smaller interval with a strictly smaller value of n . As a consequence, at every iteration, we apply the same branch W_0 or W_1 to W_ξ to compute $g(\xi) := W_\xi^n(\xi)$ for all $\xi \in [\rho, \rho']$. Therefore g is continuous with derivative at least $d^n > 1$, which contradicts (4.1).

The only remaining possibility is that $\tau(\rho) = \tau(\rho')$. We may assume that $\tau(\rho)$ is constant on $[\rho, \rho']$, otherwise we can reduce the problem to the previous case. This would mean that for arbitrarily large n , the image of the interval $[\rho, \rho']$ under g is at an interval of size least $d^n(\rho' - \rho)$, a contradiction.

Essentially the same argument, with the role of τ played by τ^+ and the role of played by W_+ , proves that $\tau^+(\rho)$ is strictly increasing as a function of $\rho \in [1 - b, a]$ to I^∞ . \square

Corollary 3. *The map $\rho \mapsto \tau(\rho)$ is left continuous and the map $\rho \mapsto \tau^+(\rho)$ is right continuous.*

Proof. Fix a parameter ρ_0 and let $\varepsilon > 0$. Then by (7) there is $x < \rho_0$ which is not a preimage of ρ_0 for any order and such that

$$d(\tau_{\rho_0}^+(x), \tau_{\rho_0}(\rho_0)) < \frac{\varepsilon}{2}.$$

By (9), for any $n \in \mathbb{N}$ there exists $\delta > 0$ such that the prefix $\tau_\rho^+(x)|_n$ is constant when $\rho \in (\rho_0 - \delta, \rho_0 + \delta)$. Let n be such that $2^{-n} < \varepsilon$, and let $\rho > x$ and $\rho \in (\rho_0 - \delta, \rho_0)$. We have that $\tau_\rho^+(x) \prec \tau_\rho^+(\rho)$ and

$$d(\tau_\rho^+(x), \tau_\rho^+(\rho)) < \frac{\varepsilon}{2}.$$

Combining the two inequalities we obtain

$$d(\tau_\rho^+(x), \tau_{\rho_0}(\rho_0)) < \varepsilon,$$

and by Lemma 2 we also have

$$\tau_\rho^+(x) \prec \tau_\rho(\rho) \prec \tau_{\rho_0}(\rho_0).$$

The distance d has the property that if $\sigma \prec \zeta \prec \sigma'$ then $d(\sigma, \zeta) \leq d(\sigma, \sigma')$ and $d(\zeta, \sigma') \leq d(\sigma, \sigma')$. This shows that $\rho \mapsto \tau(\rho)$ is left continuous. The right continuity of $\rho \mapsto \tau^+(\rho)$ admits an analogous proof. \square

As a consequence of Corollary 2, Lemma 2 and (10), we obtain the unicity of ρ for which $\bar{\Omega}$ is symmetric. As a consequence of Corollary 2, Lemma 2 and (10), we obtain the unicity of ρ for which $\bar{\Omega}$ is symmetric.

Corollary 4. *There is at most one $\rho \in [1 - b, a]$ such that $\bar{\Omega} = \bar{\Omega}^*$.*

Proof of Theorem 2. By Lemma 2 and (10), we may define

$$\rho_0 := \sup\{\rho \in [1 - b, a] : \tau(\rho) \preceq \tau^+(\rho)^*\} = \inf\{\rho \in [1 - b, a] : \tau(\rho)^* \preceq \tau^+(\rho)\}.$$

Assume $\tau(\rho_0) \prec \tau^+(\rho_0)^*$. It is straightforward to check $1 - b < \rho_0 < a$.

There is a largest $n \geq 2$ such that $\tau(\rho_0)|_n = \tau^+(\rho_0)^*|_n =: \eta = 01\dots$.

Observe that $\tau(\rho_0) = 0\tau(W_0(\rho_0))$ and $\tau^+(\rho_0) = 1\tau^+(W_1(\rho_0))$. If neither $W_0(\rho_0)$ nor $W_1(\rho_0)$ belongs to $\{0, 1\} \cup \bigcup_{k=0}^{n-1} W^{-k}(\rho_0)$, then by (9) both $\tau(\rho)|_{n+1}$ and $\tau^+(\rho)|_{n+1}$ are constant on a neighborhood of ρ_0 which contradicts the definition of ρ_0 .

Let us consider the projection $\hat{\pi}(\tau^+(W_1(\rho_0))^*)$. If $\hat{\pi}(\tau^+(W_1(\rho_0))^*) > W_0(\rho_0)$ then by the continuity of W_0 , of $\hat{\pi}$ by (13), of $\rho \mapsto \tau_\rho^+(\rho)$ (Corollary 3) there is a $\rho > \rho_0$ such that $\hat{\pi}_\rho(\tau_\rho^+(W_1(\rho_0))^*) > W_0(\rho)$. By (6) and (13) this implies $\tau_\rho(\rho) \prec \tau_\rho^+(\rho)^*$, which again contradicts the definition of ρ_0 .

As $\hat{\pi}$ is increasing, (13) and $\tau(W_0(\rho_0)) \prec \tau^+(W_1(\rho_0))^*$, we have $\hat{\pi}(\tau^+(W_1(\rho_0))^*) = W_0(\rho_0)$. Let $0 < m < n$ be minimal such that $W^m \circ W_0(\rho_0) = \rho_0$ or $W^m \circ W_1(\rho_0) = \rho_0$. We may apply (14) m times and obtain

$$(4.2) \quad W^m \circ W_0(\rho_0) = \hat{\pi}(S^m(\tau^+(W_1(\rho_0))^*)) = \hat{\pi}(\tau^+(W^m \circ W_1(\rho_0))^*).$$

As $\tau^+(\rho_0) = 1\dots$, if $W^m \circ W_1(\rho_0) = \rho_0$ then we have

$$\tau(\rho_0) \prec \tau^+(\rho_0)^* = \tau^+(W^m \circ W_1(\rho_0))^* \prec \tau^+(\rho_0),$$

which by (6) and equation (4.2) implies $W^m \circ W_0(\rho_0) = \rho_0$. Therefore $\tau(\rho_0) = \tau^+(\rho_0)^*$ as both are periodic of period $m + 1$ and have the same prefix of length $n > m$, a contradiction.

If $W^m \circ W_1(\rho_0) \neq \rho_0$ then $W^m \circ W_0(\rho_0) = \rho_0$ thus by (13), (10) and equality (4.2) we obtain

$$\tau(\rho_0) \prec \tau^+(\rho_0)^* \preceq \tau^+(W^m \circ W_1(\rho_0)) := \sigma'.$$

By (6), this means that $\rho_0 \leq W^m \circ W_1(\rho_0)$ so in fact

$$\rho_0 < W^m \circ W_1(\rho_0).$$

As $W^{m+1}(\rho_0) = \rho_0$, $\tau(\rho_0) = \kappa\kappa\kappa\dots := \kappa^\infty$ where $\kappa = \tau(\rho_0)|_{m+1} = \tau^+(\rho_0)^*|_{m+1}$, as $m + 1 \leq n$. We can write $\tau^+(\rho_0) = \kappa^*\sigma'$ therefore $\kappa^*\sigma' \prec \sigma'$ by (6) and the previous inequality. By induction we get $\kappa^{*\infty} \prec \sigma'$ so

$$\tau^+(\rho_0)^* = \kappa(\sigma'^*) \prec \kappa^\infty = \tau(\rho_0),$$

a contradiction.

The case $\tau(\rho_0) \succ \tau^+(\rho_0)^*$ is analagous by the symmetric definition of ρ_0 , therefore $\bar{\Omega}_{\rho_0}$ is symmetric. \square

Proposition 4. *If $\bar{\Omega} = \bar{\Omega}^*$ then the map $h : [0, 1] \rightarrow [0, 1]$ defined by $h(x) = \hat{\pi}(\tau(x)^*)$ is a homeomorphism and $h \circ \hat{\pi} = \hat{\pi} \circ \sigma^*$ on I^∞ .*

Proof. First by Corollary 2, we have $\tau(\rho) = \tau^+(\rho)^*$ and points x for which $\tau(x) \neq \tau^+(x)$ are exactly preimages of ρ . In this case, there is $n \geq 0$ such that $\tau(x)$ and $\tau^+(x)$ have the same initial prefix $\kappa := \tau(x)|_n = \tau^+(x)|_n$, and $\tau(x) = \kappa\tau(\rho)$, $\tau^+(x) = \kappa\tau^+(\rho)$. Therefore, by (13), for all $x \in [0, 1]$, we have

$$\tau(h(x)) = \tau^+(x)^* \text{ and } \tau^+(h(x)) = \tau(x)^*,$$

thus $h \circ h(x) = x$. By (6), (10) and (13), h is also decreasing. Therefore $h : [0, 1] \rightarrow [0, 1]$ is a homeomorphism.

Let $\sigma \in I^\infty$ and $x = \hat{\pi}(\sigma)$. By (13) we have $\tau(x) \preceq \sigma \preceq \tau^+(x)$. As $\bar{\Omega} = \bar{\Omega}^*$, by Proposition 2, we obtain that there exists $y \in [0, 1]$ such that $\tau(x)^* = \tau^+(y)$. By Lemma 1 and Corollary 2 we also have that $\tau^+(x)^* = \tau(y)$. We may compute $h \circ \hat{\pi}(\sigma) = h(x) = \hat{\pi}(\tau(x)^*) = \hat{\pi}(\tau(y)) = y$, which is also equal to $\hat{\pi}(\sigma^*)$ as $\tau(y) \preceq \sigma^* \preceq \tau^+(y)$. \square

5. ITERATED CLOSED RELATIONS AND CONLEY DECOMPOSITION FOR ITINERARIES OF W

Theorem 1 follows from Proposition 3, but some extra language is needed. In explaining this language we describe the Conley-McGehee-Wiandt decomposition theorem, [3, Theorem 13.1].

For X a compact Hausdorff space, let 2^X be the subsets of X . A *relation* r on X is simply a subset of $X \times X$. A relation r on X is called a *closed relation* if r is a closed of $X \times X$. For example the set $r \subset I^\infty \times I^\infty$ defined in Theorem 1, namely

$$r = \{(0\sigma, \sigma) \in I^\infty \times I^\infty : \sigma \preceq \alpha\} \cup \{(1\sigma, \sigma) \in I^\infty \times I^\infty : \beta \preceq \sigma\},$$

is a closed relation. Following [3], a relation $r \in 2^X$ provides a mapping $r : 2^X \rightarrow 2^X$ defined by

$$r(C) = \{y \in X : (x, y) \in r \text{ for some } x \in C\}.$$

Notice that the image of a nonempty set may be empty. Iterated relations are defined by $r^0 = X \times X$ and, for all $k \in \mathbb{N}$,

$$r^{k+1} = r \circ r^k = \{(x, z) : (x, y) \in r, (y, z) \in r^k \text{ for some } y \in X\}.$$

The *omega limit set* of $C \subset X$ under a closed relation $r \subset X \times X$ is

$$\omega(C) = \bigcap \mathfrak{K}(C)$$

where

$$\mathfrak{K}(C) = \{D \text{ is a closed subset of } X : r(D) \cup r^n(C) \subset D \text{ for some } n \in \mathbb{N}\}.$$

By definition, an *attractor* of a closed relation r is a closed set A such that the following two conditions hold:

- (i) $r(A) = A$;
- (ii) there is a closed neighborhood $\overline{\mathcal{N}}(A)$ of A such that $\omega(C) \subset A$ for all $C \subset \overline{\mathcal{N}}(A)$.

The basin $\mathcal{B}(A)$ of an attractor A for a closed relation r on a compact Hausdorff space X is the union of all open sets $O \subset X$ such that $\omega(C) \subset A$ for all $C \subset O$.

Given an attractor A for a closed relation r on a compact Hausdorff space X , there exists a corresponding *attractor block*, namely a closed set $E \subset X$ such that E contains both A and $r(E)$ in its interior, and $A = \omega(E)$. Also, there exists a unique *dual repeller* $A^* = X \setminus \mathcal{B}(A)$. This repeller is an attractor for the transpose relation $r^* = \{(y, x) : (x, y) \in r\}$. The set of connecting orbits associated with the attractor/repeller pair A, A^* is $\mathcal{C}(A) = X \setminus (A \cup A^*)$.

If r is a closed relation on a compact Hausdorff space X , then $x \in X$ is called *chain-recurrent* for r if for every closed neighborhood f of r , x is periodic for f (i.e. there exists a finite sequence of points $\{x_n\}_{n=0}^{p-1} \subset X$ such that $x_0 = x$, $(x_{p-1}, x_0) \in f$ and $(x_{n-1}, x_n) \in f$ for $n = 1, 2, \dots, p-1$). The chain recurrent set \mathcal{R} for r is the union of all the points that are chain recurrent for r . A transitive component of \mathcal{R} is a member of the equivalence class on \mathcal{R} defined by $x \sim y$ when for every closed neighborhood f of r there is an orbit from x to y under f (i.e. there exists a finite sequence of points $\{x_n\}_{n=0}^{p-1} \subset \mathcal{R}$ such that $x_0 = x$, $x_{p-1} = y$, and $(x_n, x_{n+1}) \in f$ for all $n \in \{0, 1, \dots, p-1\}$.)

Theorem 3 (Conley-McGehee-Wiandt). *If r is a closed relation on a compact Hausdorff space X , then*

$$\mathcal{R} = \bigcup_{A \in \mathcal{U}} \mathcal{C}(A)$$

where \mathcal{R} is the chain-recurrent set and \mathcal{U} is the set of attractors.

Proof of Theorem 1. This follows at once from Proposition 3 together with Theorem 3, but see [3]. \square

We note the following. $\overline{\Omega}$ can be embedded in $[0, 1] \subset \mathbb{R}$ using the (continuous and surjective) coding map $\pi : I^\infty \rightarrow [0, 1]$ associated with the iterated function system $([0, 1]; x \mapsto x/2, x \mapsto (1+x)/2)$. This coding map π is defined, for all σ , by

$$\pi(\sigma) = \sum_{k \in \mathbb{N}} \frac{\sigma_k}{2^{k+1}}.$$

π provides a homeomorphism between $\overline{\Omega}$ and $\pi(\overline{\Omega})$. The point $\sigma \in \overline{\Omega}$ is uniquely and unambiguously represented by the binary real number $0.\sigma$. In the representation provided by π , the map $\overline{\Psi} : 2^{I^\infty} \rightarrow 2^{I^\infty}$ becomes the action of the iterated closed relation $\tilde{r} \subset [0, 1] \times [0, 1] \subset \mathbb{R}^2$ defined by

$$\tilde{r} := \{(x, x/2) : x \in [0, \pi(\alpha)]\} \cup \{(x, (x+1)/2) : x \in [\pi(\beta), 1]\}$$

on subsets of $[0, 1]$. It follows from Proposition 3 (iii) that $\pi(\overline{\Omega})$ is the maximal attractor, as defined in [3], of \tilde{r} . The corresponding dual repeller is the empty set. It is also easy to see that $\{0\}$ and $\{1\}$ are the only other attractors, with corresponding dual repellers $[\pi(\alpha), 1]$ and $[0, \pi(\beta)]$ respectively. It follows from Theorem 3 that the chain recurrent set of \tilde{r} is $\{0, 1\} \cup (\pi(\overline{\Omega}) \cap (\pi(\beta), \pi(\alpha)))$.

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