A Bit Allocation Method for Sparse Source Coding

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Abstract

In this paper, we develop an efficient bit allocation strategy for subband-based image coding systems. More specifically, our objective is to design a new optimization algorithm based on a rate-distortion optimality criterion. To this end, we consider the uniform scalar quantization of a class of mixed distributed sources following a Bernoulli-Generalized Gaussian distribution. This model appears to be particularly well-adapted for image data which have a sparse representation in a wavelet basis. In this context, we propose new approximations of the entropy and the distortion functions by using piecewise affine and exponential forms, respectively. Thanks to these approximations, we reformulate the bit allocation problem as a convex optimization one. Solving the resulting problem allows us to derive the optimal quantization step for each subband. Experimental results show the benefits that can be drawn from the proposed bit allocation method in a typical transform-based coding application.

Index Terms

Bit allocation, sparse sources, generalized Gaussian, lossy source coding, rate-distortion theory, piecewise approximation, convex optimization.

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Part of this work has been presented in [1].

I. INTRODUCTION

In image and video coding systems, it is desired to achieve the best possible image quality for a given bitrate or, conversely, to minimize the bitrate for a given image quality. To this respect, a great attention has been paid to the problem of bit allocation where a given amount of bits must be efficiently distributed among blocks of a DCT-coded image or among subbands of a wavelet-based coder [2], or among frames in a video sequence [3]. The general framework behind the bit allocation strategy is Rate-Distortion (R-D) theory which aims at minimizing the average distortion of the input signal subject to a constraint on the available global bitrate. Since both the rate and the distortion measures in a typical transform coding scheme are controlled by the choice of the quantizers, the major issue is to find the optimal quantization steps for the constrained minimization problem. It is thus necessary to study the rate and distortion functions of the source to be encoded.

Two main classes of methods have been developed to deal with the bit allocation problem: numerical- and analytical-based approaches. Algorithms in the first category aim at empirically estimating the R-D curves and resort to some iterative techniques to find the optimal quantization parameters [3]. For instance, Lagrangian optimization techniques have been well investigated in the literature [4], [5], [6]. In these approaches, the constrained minimization problem is transformed into an unconstrained version by incorporating the constraint into the objective function. In [4], a bit allocation method for completely arbitrary input signals (or blocks) and discrete quantizer sets is considered in the case of independent coding contexts. An extension of this work to subband coding has been proposed in [2]. Another extension to a dependent coding environment has also been considered in [6]. More precisely, the authors describe the R-D Lagrangian cost function in the form of a trellis and use the Viterbi algorithm to find the optimal solution for coders exploiting temporal and spatial dependencies such as MPEG and pyramidal coders. In [7], the bit allocation problem is converted into the graph theoretic problem of finding the shortest path in a directed acyclic graph. Besides, it should be noticed that dynamic programming algorithms [3], [8] and descent algorithms [9], [10] have also been proposed to select the optimal quantization parameters. It is important to note that these numerical methods may be computationally intensive since a large number of R-D operating points must be measured for each subband in order to obtain R-D curves which are both differentiable and convex [11]. In other words, the R-D data are first evaluated for all possible quantization settings. Then, the optimal solution is derived. In order to reduce the complexity, André *et al.* [12] have recently proposed to perform the computation of a few points (i.e. for some possible quantization settings) and interpolate them using spline approximations.

To further overcome the complexity of these numerical methods, alternative approaches which do not require the estimation of R-D curves have also been developed. These approaches provide closed-form expressions of the R-D function by assuming various input distributions and quantizer characteristics. For instance, the performance of optimum scalar quantizers subject to an entropy constraint was investigated through numerical methods [13], [14] for different source probability densities (e.g uniform, Gaussian, Laplacian, Generalized Gaussian) at low resolution (i.e. bitrate). In [15], a parametric representation of the operational R-D function of a scalar quantizer is derived for a uniformly distributed source and a wide class of distortion measures. In [16], a distortion measure based on differential entropy has been introduced for image coding using uniform scalar quantization. In [17], an approach for designing entropy constrained scalar quantizers for exponential and Laplace distributions is proposed and comparisons are made with uniform quantizers. Recently in [18], the asymptotic behavior of a uniform quantizer is characterized at low resolution for a memoryless Gaussian source and a squared error distortion measure. Other studies have also considered the use of Laplace and Generalized Gaussian probability models in modern compression systems [19], [20], [21].

The objective of this paper is to design an efficient bit allocation algorithm in a subband coding context (typically, for wavelet-based coders) by adopting an analytical approach. More precisely, we will consider the uniform scalar quantization of the different subband coefficients resulting from a multiresolution analysis. While the GG model has been extensively employed for modelling wavelet coefficients [22], [23], we adopt in this paper a more general mixture model, referred to as Bernoulli-GG (BGG), which was found to be well-suited for modelling sparse wavelet coefficients [24], [25]. After extending recent approximation formulas of the entropy and the distortion of uniformly quantized BGG sources [26], we propose a piecewise affine (resp. piecewise exponential) form of the entropy (resp. distortion) which allows us to get fine approximations of these functions. Thanks to the proposed approximations, we reformulate the bit allocation problem by making use of convex analysis tools. Following this approach, we finally derive explicit expressions of the optimal quantization parameters.

The remainder of this paper is organized as follows: in Section II, we define the probabilistic model for the considered subband coefficients as well as the quantizer characteristics. We introduce the resulting entropy and distortion functions. In Section III, we provide new piecewise convex approximations of the entropy and the distortion. In Section IV, we reformulate the bit allocation problem as a set of convex optimization problems, for which we derive the optimal solutions. Finally, an application of the proposed method to transform-based image coding is illustrated in Section V and some conclusions are drawn in Section VI.

II. Entropy and distortion of a uniformly quantized BGG source

A. Source and quantization models

First, we consider the problem of coding an input signal by performing a wavelet (or frame-based) decomposition. Let us assume that the source to be quantized is composed of J subbands having n_j coefficients ($j \in \{1,\dots,J\}$) so that $n = \sum_{j=1}^{J} n_j$ is the total number of coefficients. An appropriate model for characterizing the sparsity of the coefficients of the j-th subband is the BGG model whose probability distribution f_j is defined by:

$$\forall \xi \in \mathbb{R}, \qquad f_j(\xi) = (1-\epsilon_j)\delta(\xi) + \epsilon_j \tilde{f}_j(\xi) \tag{1}$$

where $\epsilon_j \in [0,1]$ is a mixture parameter, δ denotes the Dirac distribution (i.e. point mass at 0) and $\tilde{f}_j$ is the probability density function for a GG distribution with shape parameter $\beta_j \in]0,2]$ and scale factor $\omega_j \in]0,+\infty[$:

$$\forall \xi \in \mathbb{R}, \qquad \tilde{f}_j(\xi) = \frac{\beta_j \omega_j^{1/\beta_j}}{2\Gamma(1/\beta_j)} e^{-\omega_j |\xi|^{\beta_j}} \tag{2}$$

where Γ is the gamma function. Recall that the differential entropy of such a GG variable is given [27] as:

$$h_{\beta_j}(\omega_j) = -\int_{-\infty}^{\infty} \tilde{f}_j(\xi) \log_2 \tilde{f}_j(\xi)\, d\xi = \log_2\Big(\frac{2\Gamma(1/\beta_j)}{\beta_j \omega_j^{1/\beta_j}}\Big) + \frac{1}{\beta_j}.$$

Each coefficient $X_{j,s}$ with $s \in \{1,\dots,n_j\}$ in subband $j \in \{1,\dots,J\}$ is quantized before being entropy coded. For this purpose, we assume that, for each subband j, a scalar uniform quantizer with a quantization step q_j and having a deadzone of size $(2\tau_j - 1)q_j$ where $\tau_j > 1/2$ is used [28]. Note that $\tau_j = 1$ corresponds to a deadzone of size q_j. Thus, for every $s \in \{1,\dots,n_j\}$, the output $\overline{X}_{j,s}$ of the quantizer is given by:

$$\overline{X}_{j,s} = r_0 = 0, \qquad \text{if } |X_{j,s}| < \big(\tau_j - \tfrac{1}{2}\big)\, q_j, \qquad \text{where } \tau_j > 1/2$$

and, for all $i \in \mathbb{Z}$, $\overline{X}_{j,s} = r_{i,j}$,

$$\text{(if } (\tau_j + i - \tfrac{3}{2})q_j \le X_{j,s} < (\tau_j + i - \tfrac{1}{2})q_j \text{ and } i \ge 1)$$
$$\text{or (if } (-\tau_j + i + \tfrac{1}{2})q_j < X_{j,s} \le (-\tau_j + i + \tfrac{3}{2})q_j \text{ and } i \le -1),$$

where the reconstruction levels are given by

$$\forall i \ge 1, \qquad r_{i,j} = -r_{-i,j} = (\tau_j + i - 1 + \zeta_j)q_j \tag{3}$$

and $\zeta_j \in [-1/2, 1/2]$ is an "offset" parameter indicating the shift of the reconstruction level with respect to the center of the quantization interval. Note that we will not consider any saturation effect. The most

commonly used quantization rule corresponds to the case when $\zeta_j = 0$ (i.e mid-point reconstruction). For example, this rule constitutes the basic ingredient of many encoding strategies (such as EBCOT [29]) which have been developed in wavelet-based image compression techniques.

Since the objective of the paper is to focus on the bit allocation problem for the quantized coefficients, it is now necessary to study their rate and distortion functions.

B. Entropy and distortion measures

As frequently done in the development of R-D algorithms, we approximate the bitrate of a memoryless source by the zero-order entropy of the quantized coefficients [30], [31]. Thus, by assuming that the random variable $X_{j,s}$ with $j \in \{1,\ldots,J\}$ and $s \in \{1,\ldots,n_j\}$ is distributed according to (1), the entropy of the associated quantized variable $\overline{X}_{j,s}$ with $j \in \{1,\ldots,J\}$ and $s \in \{1,\ldots,n_j\}$ is given by:

$$H_{f_j}(q_j,\epsilon_j) = -\sum_{i=-\infty}^{\infty} \mathsf{p}_{i,j} \log_2 \mathsf{p}_{i,j} \tag{4}$$

where, for every $i \in \mathbb{Z}$, $\mathsf{p}_{i,j} = \mathsf{P}(\overline{X}_{j,s} = r_{i,j})$ represents the probability of occurrence of the $r_{i,j}$ reconstruction level.

We also propose to express the distortion function by using the p_j-th order moment of the quantization error:

$$e_j(q_j,\epsilon_j) = \mathsf{E}[|X_{j,s} - \overline{X}_{j,s}|^{p_j}] \tag{5}$$

where $p_j \geq 1$ is a real exponent. In particular, $p_j = 2$ corresponds to the mean square error criterion whereas $p_j = 1$ corresponds to the mean absolute one. Taking real values of the exponent which depend on j provides flexibility in the choice of the distortion measure.

It is important to note here that close approximation of the entropy and asymptotic expressions of the distortion of a quantized BGG random variable are provided in [26] for both low and high bitrates. However, these approximations have been derived in the case of log-concave distributions (more precisely when $1 \leq \beta_j \leq 2$) and for a quantizer with a deadzone of size q_j (i.e. $\tau_j = 1$). It is worth pointing that in practice, typical values of β_j can be smaller than 1 and the size of the deadzone can be parameterized to have a different value for each subband (as in JPEG2000 Part 2, while in Part 1, a typical deadzone of size $2q_j$ is used) [32]. Therefore, the main approximation results given in [26] need to be extended in this paper by incorporating a nontrivial deadzone in the quantizer and also considering the case when $\beta_j < 1$.

Once the entropy and distortion functions have been defined, the bit allocation problem can be formulated.

In our case, this problem consists of finding the quantization steps for each subband or, equivalently, the vector $\mathbf{q} = (q_1, q_2, \ldots, q_J) \in [0, +\infty[^J$ minimizing the average distortion

$$D(\mathbf{q}) = \sum_{j=1}^{J} \rho_j e_j(q_j, \epsilon_j) \tag{6}$$

$$\text{where} \quad \forall j \in \{1, \ldots, J\}, \qquad \rho_j \in]0, +\infty[\quad \text{and} \quad \sum_{j=1}^{J} \rho_j = 1,$$

subject to the constraint that the total bitrate is equal to or smaller than a target bitrate $R_{\max}$:

$$H(\mathbf{q}) = \sum_{j=1}^{J} \frac{n_j}{n} H_{f_j}(q_j, \epsilon_j) \leq R_{\max}. \tag{7}$$

Note that, for orthonormal representations, when for every $j \in \{1, \ldots, J\}$, $p_j = 2$ and $\rho_j = n_j/n$, $D(\mathbf{q})$ is also equal to the distortion in the spatial domain. For other scenarios (biorthogonal representations or redundant frames), a good approximation of the distortion in the spatial domain can be obtained in a number of cases by appropriate choices of the constants $(\rho_j)_{1 \leq j \leq J}$ [33]. The degrees of freedom in the choices of the constants $(p_j)_{1 \leq j \leq J}$ and $(\rho_j)_{1 \leq j \leq J}$ can also be exploited in order to define perceptual criteria [34] better fitting the Human Visual System (HVS) characteristics.

III. Approximations of the entropy and of the distortion

The objective of this section is to develop accurate approximations of the entropy and the distortion for a general BGG source model. These approximations will allow us to reformulate the bit allocation problem in a more tractable form.

A. Piecewise affine approximation of the entropy

Let Q_a with $a \in \mathbb{R}_+^*$ be the normalized incomplete Gamma function [35], defined as

$$\forall \xi \in \mathbb{R}, \qquad Q_a(\xi) = \frac{1}{\Gamma(a)} \int_0^{\xi} \theta^{a-1} e^{-\theta} d\theta. \tag{8}$$

As shown in Appendix A, a close approximation of the entropy of a quantized BGG source can be obtained as follows:

Proposition 1: The entropy $H_{f_j}(q_j, \epsilon_j)$ of a quantized BGG random variable distributed according to (1) *can be approximated by*

$$\widehat{H}_{f_j}(q_j, \epsilon_j) = \Phi(\mathsf{p}_{0,j}, \epsilon_j) + \epsilon_j \widehat{H}_{\widetilde{f}_j}(q_j) \tag{9}$$

with $\Phi(\mathsf{p}_{0,j}, \epsilon_j) = -\big(1 - \epsilon_j(1 - \mathsf{p}_{0,j})\big) \log_2 \big(1 - \epsilon_j(1 - \mathsf{p}_{0,j})\big) - \epsilon_j(1 - \mathsf{p}_{0,j}) \log_2 \epsilon_j + \epsilon_j \mathsf{p}_{0,j} \log_2 \mathsf{p}_{0,j},$

and $$\widehat{H}_{\widetilde{f}_j}(q_j) = -\,\mathsf{p}_{0,j}\log_2\mathsf{p}_{0,j} - 2\mathsf{p}_{1,j}\log_2\mathsf{p}_{1,j} + \big(h_{\beta_j}(\omega_j) - \log_2 q_j\big)\Big(1 - Q_{1/\beta_j}\Big(\omega_j\big(\tau_j+\frac{1}{2}\big)^{\beta_j}q_j^{\beta_j}\Big)\Big) + \frac{\omega_j^{1/\beta_j}(\tau_j+\frac{1}{2})q_j}{\Gamma(1/\beta_j)}\,e^{-\omega_j(\tau_j+\frac{1}{2})^{\beta_j}q_j^{\beta_j}}. \tag{10}$$

The error incurred in this approximation is such that

$$0 \le \widehat{H}_{f_j}(q_j,\epsilon_j) - H_{f_j}(q_j,\epsilon_j) \le 2\epsilon_j q_j C(\beta_j,\tau_j)\tilde{f}_j\big((\tau_j+\frac{1}{2})q_j\big),$$

$$\textit{with} \qquad C(\beta_j,\tau_j) = \begin{cases} \left(\frac{2\tau_j+1}{2\tau_j-1}\right)^{1-\beta_j} & \textit{if} \quad \beta_j < 1 \\ \left(\frac{2\tau_j+2}{2\tau_j+1}\right)^{\beta_j-1} & \textit{if} \quad \beta_j \in [1,2]. \end{cases} \tag{11}$$

It is worth pointing out that such an approximation formula may be useful in practice in the sense that it allows us to efficiently compute the entropy for any given set of quantization steps.

Generally, analytical-based R-D algorithms use the standard Bennett formula to obtain a close approximation of the entropy [26], [30]. This high-resolution approximation formula, which is also valid when a quantizer with a deadzone is used and $\beta_j \in (0,2]$, allows us to express the entropy of the j-th subband as an affine function of $l_j = \log_2(q_j)$:

$$H_{f_j}(q_j,\epsilon_j) = H_{\epsilon_j} + \epsilon_j(h_{\beta_j}(\omega_j) - l_j) + o(l_j 2^{l_j}) \tag{12}$$

where $H_{\epsilon_j} = -\epsilon_j\log_2\epsilon_j - (1-\epsilon_j)\log_2(1-\epsilon_j)$ is the entropy of a Bernoulli random variable with parameters $(1-\epsilon_j,\epsilon_j)$.

However, the approximation formula (10) is not tractable for optimization purposes, whereas (12) is only valid at high resolution (i.e. when q_j is small). In order to develop a bit allocation strategy well-adapted for both high and low resolutions, we propose to define a piecewise convex approximation of the entropy function by considering a more flexible function of $\mathbf{l} = (l_1, l_2, \ldots, l_J)$, given by $\sum_{j=1}^{J}\frac{n_j}{n}g_j(l_j)$, where g_j have the following piecewise affine form:

$$\forall j \in \{1,\ldots,J\}, \quad g_j(l_j) = a_j^k l_j + c_j^k \qquad \text{if } l_j^{(h,k-1)} \le l_j \le l_j^{(h,k)} \tag{13}$$

with $k \in \{1,2,\ldots,m^{(h)}\}$ and $m^{(h)}$ is a given parameter corresponding to the considered number of intervals (i.e. the number of segments chosen to approximate the entropy). For every $j \in \{1,\ldots,J\}$, the parameters $(a_j^k)_{1\le k\le m^{(h)}}$ are nonpositive reals, and the parameters $(c_j^k)_{1\le k\le m^{(h)}}$ are real numbers. Note that the superscript h has been used to distinguish between the intervals used for the approximation of the entropy and those later used for the approximation of the distortion.

In practice, we set $l_j^{(h,0)} = -\infty$ and we choose the other points $\left(l_j^{(h,k)}\right)_{1\le k\le m^{(h)}}$ in such a way that the

resulting piecewise affine function constitutes a good approximation of the entropy H_{f_j} of the source. For the first interval, the high resolution approximation (12) can be employed, leading to

$$\forall\ j \in \{1, \ldots, J\}, \qquad a_j^1 = -\epsilon_j \quad \text{and} \quad c_j^1 = H_{\epsilon_j} + \epsilon_j h_{\beta_j}(\omega_j).$$

By considering an arbitrary point $\widetilde{l}_j^{(h,1)}$, we derive (a_j^2, c_j^2) such that g_j on the second interval is tangent to the graph of the entropy function at $\widetilde{l}_j^{(h,1)}$. The lower bound $l_j^{(h,1)}$ of the second interval is then fixed to the abscissis of the intersection of the lines obtained on the first and second intervals. By repeating the process, we deduce the remaining values $l_j^{(h,2)}, l_j^{(h,3)}, \ldots, l_j^{(h,m^{(h)}-1)}$ and the associated affine approximations. Since the entropy must be a nonnegative function, the last interval bound $l_j^{(h,m^{(h)})}$ is found such that $a_j^{m^{(h)}} l_j^{(h,m^{(h)})} + c_j^{m^{(h)}} = 0$. This entails:

$$\forall j \in \{1, \ldots, J\}, \quad g_j(l_j) = 0 \qquad \text{if } l_j \geq l_j^{(h,m^{(h)})}. \tag{14}$$

Fig. 1 illustrates the approximations of the entropy using two intervals ($m^{(h)} = 2$) and four intervals ($m^{(h)} = 4$). As expected, increasing the number of intervals leads to a better approximation of the entropy.

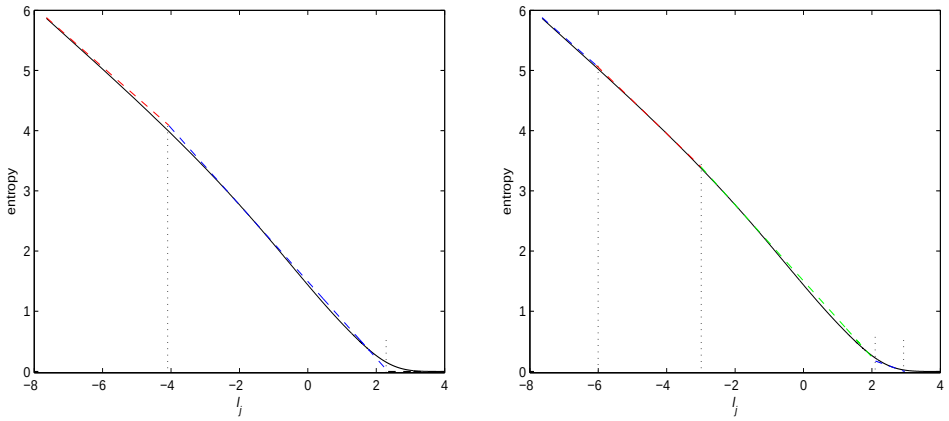

Fig. 1. Approximations g_j (in dashed line) of the entropy H_{f_j} (in solid line) of a uniformly quantized BGG source versus l_j: $m^{(h)} = 2$ (left side), $m^{(h)} = 4$ (right side). The parameters of the BGG source are $\epsilon_j = 0.5$, $\beta_j = 1.2$ and $\omega_j = 1$.

B. Piecewise exponential approximation of the distortion

On the other hand, we show in Appendix B that a good approximation of the distortion of a quantized BGG source can be obtained as follows:

Proposition 2: The distortion $e_j(q_j, \epsilon_j)$ of a quantized BGG random variable distributed according

to (1) *can be approximated by*

$$\widehat{e}_j(q_j,\epsilon_j) = 2\epsilon_j \left(\frac{\omega_j^{-p_j/\beta_j}\Gamma((p_j+1)/\beta_j)}{2\Gamma(1/\beta_j)} Q_{(p_j+1)/\beta_j}\big(\omega_j(\tau_j-\frac{1}{2})^{\beta_j} q_j^{\beta_j}\big) + \int_{(\tau_j-\frac{1}{2})q_j}^{(\tau_j+\frac{1}{2})q_j} |\xi - r_{1,j}|^{p_j} \widetilde{f}_j(\xi) d\xi \right.$$
$$\left. + \frac{\nu_j q_j^{p_j}}{2(p_j+1)}\big(1 - Q_{1/\beta_j}\big(\omega_j(\tau_j+\frac{1}{2})^{\beta_j} q_j^{\beta_j}\big)\big) \right) \tag{15}$$

where the approximation error is such that

$$|e_j(q_j,\epsilon_j) - \widehat{e}_j(q_j,\epsilon_j)| \leq 2\epsilon_j \frac{\nu_j q_j^{p_j+1}}{p_j+1} \widetilde{f}_j\big((\tau_j+\frac{1}{2})q_j\big). \tag{16}$$

Some comments can be made about this result:

- When $q_j \to 0$, the classical high resolution approximation is recovered:

$$e_j(q_j,\epsilon_j) = \epsilon_j \frac{\nu_j}{p_j+1} q_j^{p_j}(1+O(q_j)) \tag{17}$$

  where $\nu_j = (\frac{1}{2}+\zeta_j)^{p_j+1} + (\frac{1}{2}-\zeta_j)^{p_j+1}$.
- When $p_j = 2$ (or more generally when p_j is an even integer), the integral in (15) can be easily expressed by using incomplete Gamma functions.

Similarly to the approximation of the entropy, Proposition 2 will be useful to compute both fast and accurate approximations of the distortion, but the derived expressions remain too intricate for developing efficient bit allocation algorithms.

We thus propose to use a rougher approximation of the distortion. More specifically, we propose to express the global distortion as a function of $\mathbf{l} = (l_1, l_2, \ldots, l_J) = \big(\log_2(q_1), \ldots, \log_2(q_J)\big)$ under the form $\sum_{j=1}^J \rho_j d_j(l_j)$, where d_j has the following piecewise exponential form:

$$\forall\ j \in \{1,\ldots,J\}, \quad d_j(l_j) = \begin{cases} \epsilon_j(\alpha_j^k 2^{l_j\gamma_j^k} + \delta_j^k) & \text{if } l_j^{(d,k-1)} \leq l_j < l_j^{(d,k)} \\ \epsilon_j \omega_j^{-p_j/\beta} \dfrac{\Gamma\big((p_j+1)/\beta_j\big)}{\Gamma(1/\beta_j)} & \text{if } l_j \geq l_j^{(d,m^{(d)})} \end{cases} \tag{18}$$

where $k \in \{1,2,\ldots,m^{(d)}\}$ and $m^{(d)}$ is a given integer corresponding to the number of intervals used in our approximation. For every $j \in \{1,\ldots,J\}$, the parameters $(\alpha_j^k)_{1\leq k\leq m^{(d)}}$ and $(\gamma_j^k)_{1\leq k\leq m^{(d)}}$ are nonnegative reals, and the parameters $(\delta_j^k)_{1\leq k\leq m^{(d)}}$ are real numbers. Similarly to the selection procedure for $\left(l_j^{(h,k)}\right)_{0\leq k\leq m^{(h)}}$, the values of the interval bounds $\left(l_j^{(d,k)}\right)_{0\leq k\leq m^{(d)}}$ (with $l_j^{(d,0)} = -\infty$) are chosen in such a way that $d_j(l_j)$ constitutes a good approximation of $e_j(2^{l_j},\epsilon_j)$. In particular, by taking $\gamma_j^1 = p_j$, $\alpha_j^1 = \frac{\nu_j}{p_j+1}$, and $\delta_j^1 = 0$, we obtain the high bitrate approximation of the distortion (see (17)) on the first interval $[l_j^{(d,0)}, l_j^{(d,1)}]$.

Fig. 2 shows the approximations of the distortion for 2 and 4 intervals. It can be observed that setting $m^{(d)}$

to 2 results in a less precise approximation of the distortion e_j than $m^{(d)} = 4$, especially at low bitrate. It can also be noticed from Figs. 1 and 2 that the chosen approximation interval bounds $l_j^{(h,1)}$, $l_j^{(h,2)}$ and $l_j^{(h,3)}$ for the entropy differ from those $l_j^{(d,1)}$, $l_j^{(d,2)}$ and $l_j^{(d,3)}$ for the distortion. This illustrates the fact that the selection steps for $\left(l_j^{(h,k)}\right)_{1\leq k\leq m^{(h)}}$ and $\left(l_j^{(d,k)}\right)_{1\leq k\leq m^{(d)}}$ should be performed independently in order to obtain good approximations of both the entropy and the distortion functions.

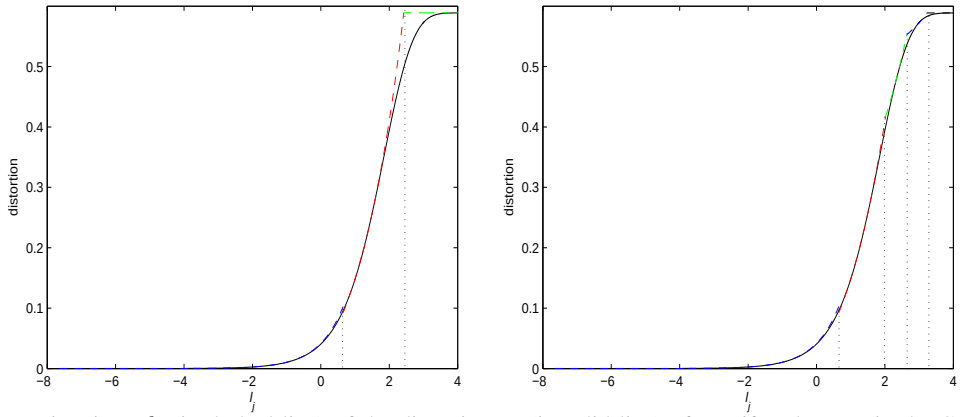

Fig. 2. Approximations d_j (in dashed line) of the distortion e_j (in solid line) of a uniformly quantized BGG source versus l_j: $m^{(d)} = 2$ (left side), $m^{(d)} = 4$ (right side). The parameters of the BGG source are $\epsilon_j = 0.5$, $\beta_j = 1.2$ and $\omega_j = 1$.

IV. Proposed bit allocation method

In this part, we show how the approximations of the entropy and distortion functions proposed in the previous section allow us to solve the bit allocation problem in an efficient manner.

A. Optimization problem

Using the approximations g_j (resp. d_j) of the entropy in (13) (resp. of the distortion in (18)), the bit allocation problem defined at the end of Section II, can be recast as follows:

Problem 1: Find $\tilde{\mathbf{l}}$ minimizing the distortion function

$$\forall\ \mathbf{l} = (l_1, \ldots, l_J) \in \mathbb{R}^J, \qquad D(\mathbf{l}) = \sum_{j=1}^{J} \rho_j d_j(l_j)$$

over the set C defined as

$$C := \{\mathbf{l} = (l_1, \ldots, l_J) \in \mathbb{R}^J \mid \sum_{j=1}^{J} \frac{n_j}{n} g_j(l_j) \leq R_{\max}\}. \tag{19}$$

A major difficulty for solving this problem stems from the fact that the functions g_j and d_j are non-differentiable and non convex. To define the different domains where the optimization is performed, we shall jointly sort the coefficients $\left(l_j^{(h,k)}\right)_{1\leq k\leq m^{(h)}}$ and $\left(l_j^{(d,k)}\right)_{1\leq k\leq m^{(d)}}$ in ascending order for each $j \in$

$\{1,\ldots,J\}$. The resulting sorted coefficients will be denoted by $(l_j^1,\ldots,l_j^m)$ such that $l_j^1 \leq l_j^2 \leq \ldots \leq l_j^m$ where $m \leq m^{(h)} + m^{(d)}$. From the definition of the total bitrate constraint, a necessary condition for $\mathbf{l}$ to belong to C is

$$\forall\ j \in \{1,\ldots,J\}, \quad a_j^1 l_j + c_j^1 \leq n n_j^{-1} R_{\max}. \tag{20}$$

This means that, for every $j \in \{1,\ldots,J\}$, we can set the lower bound l_j^0 of the search interval to

$$l_j^0 = \min(\frac{n n_j^{-1} R_{\max} - c_j^1}{a_j^1}, l_j^1). \tag{21}$$

Moreover, since $g_j(l_j) = 0$ for every $l_j \geq l_j^m$, and d_j is an increasing function of l_j, it is clear that the optimal value of l_j will be lower than or equal to l_j^m. As a result, the problem is equivalent to minimize the distortion over the domain $[l_1^0, l_1^m] \times \ldots \times [l_J^0, l_J^m]$. In order to overcome the problem of the non-differentiability of the functions g_j or d_j at points $\left(l_j^k\right)_{1\leq k<m}$, we propose to subdivide the previous domain into boxes of the form $[l_1^{b_1}, l_1^{b_1+1}] \times \ldots \times [l_J^{b_J}, l_J^{b_J+1}]$ where $\mathbf{b} = (b_1,\ldots,b_J) \in \{0,\ldots,m-1\}^J$. On each box, the entropy and distortion are convex functions. Therefore, this subdivision technique leads to m^J subdomains where a convex optimization problem must be solved.

B. Solution of the bit allocation problem

In the following, we provide a closed form expression of the optimal quantization parameters. Suppose that $\mathbf{P_b} = [l_1^{b_1}, l_1^{b_1+1}] \times \ldots \times [l_J^{b_J}, l_J^{b_J+1}]$ corresponds to a given subdomain and let us denote by $(\mathcal{P}_\mathbf{b})$ the convex minimization problem on this subdomain. For concision purposes, let us introduce the following notation, for every $j \in \{1,\ldots,J\}$,

$$N_j = -\frac{n_j\, a_j^{b_j}}{\gamma_j^{b_j}}, \quad \kappa_j = \frac{n}{N_j}\rho_j \epsilon_j \alpha_j^{b_j} \ln 2, \tag{22}$$

$$\underline{\lambda}_j = \kappa_j 2^{\gamma_j^{b_j} l_j^{b_j}}, \quad \overline{\lambda}_j = \kappa_j 2^{\gamma_j^{b_j} l_j^{b_j+1}}. \tag{23}$$

The solution to the Problem $(\mathcal{P}_\mathbf{b})$ is given below.

Proposition 3:

(i) *If* $\sum_{j=1}^J \frac{n_j}{n} g_j(l_j^{b_j+1}) > R_{\max}$*, then there is no solution.*

(ii) *If* $\sum_{j=1}^J \frac{n_j}{n} g_j(l_j^{b_j}) \leq R_{\max}$*, then the solution is* $\widetilde{\mathbf{l}} = (l_1^{b_1},\ldots,l_J^{b_J})$.

(iii) *Otherwise, the solution is the vector* $\widetilde{\mathbf{l}}_\mathbf{b}$ *defined by*

$$\forall j \in \{1,\ldots,J\},\ \widetilde{l}_{j,\mathbf{b}} = \begin{cases} l_j^{b_j} & \text{if } j \in \mathbb{I} \\ \frac{1}{\gamma_j^{b_j}} \log_2\left(\frac{\widetilde{\lambda}}{\kappa_j}\right) & \text{if } j \in \mathbb{J} \\ l_j^{b_j+1} & \text{if } j \in \mathbb{K} \end{cases} \tag{24}$$

where

$$\widetilde{\lambda}^{N_{\mathbb{J}}} = \frac{2^{(\sum_{j=1}^{J} n_j c_j^{b_j} - nR_{\max})}}{2^{(\sum_{j\in\mathbb{I}} N_j \gamma_j^{b_j} l_j^{b_j} + \sum_{j\in\mathbb{K}} N_j \gamma_j^{b_j} l_j^{b_j+1})}} \prod_{j\in\mathbb{J}} \kappa_j^{N_j} \tag{25}$$

$$N_{\mathbb{J}} = \sum_{j\in\mathbb{J}} N_j \tag{26}$$

$$\mathbb{I} = \{j \in \{1,\ldots,J\} \mid \Phi'(\underline{\lambda}_j) \leq 0\}, \tag{27}$$

$$\mathbb{K} = \{j \in \{1,\ldots,J\} \mid \Phi'(\overline{\lambda}_j) > 0\} \tag{28}$$

$$\mathbb{J} = \{1,\ldots,J\} \setminus (\mathbb{I} \cup \mathbb{K}) \tag{29}$$

$$\forall\ \lambda \in \mathbb{R}_+, \quad \Phi(\lambda) = \lambda\Big(\sum_{j=1}^{J} \frac{n_j}{n} c_j^{b_j} - R_{\max}\Big) - \sum_{j=1}^{J} \varphi_j(\lambda) \tag{30}$$

with $\forall\ j \in \{1,\ldots,J\}$,

$$\varphi_j(\lambda) = \begin{cases} \frac{N_j}{n}(\gamma_j^{b_j} l_j^{b_j} \lambda - \frac{\underline{\lambda}_j}{\ln 2}) - \rho_j \epsilon_j \delta_j^{b_j} & \text{if } \lambda \leq \underline{\lambda}_j \\ \frac{N_j \lambda}{n \ln 2}(\ln(\frac{\lambda}{\kappa_j}) - 1) - \rho_j \epsilon_j \delta_j^{b_j} & \text{if } \underline{\lambda}_j < \lambda < \overline{\lambda}_j \\ \frac{N_j}{n}(\gamma_j^{b_j} l_j^{b_j+1} \lambda - \frac{\overline{\lambda}_j}{\ln 2}) - \rho_j \epsilon_j \delta_j^{b_j} & \text{if } \lambda \geq \overline{\lambda}_j. \end{cases}$$

Proof: See Appendix C.

The above expressions of the quantization parameters, obtained for each subdomain, allow us to determine a finite set of candidate distortion values. Once this has been performed, the subdomain leading to the global minimum distortion value is selected and its resulting quantization steps correspond to the optimal ones. It is worth pointing out that the computation of the quantization parameters as well as their corresponding distortion can be carried out for the subdomains independently of each other. Furthermore, it can be noticed that the maximum number m^J of these evaluations can be reduced by checking Conditions (i) and (ii) in Proposition 3.

V. Experimental results

In this part, we study the performance of the proposed bit allocation method in the context of transform-based coding applications. We employ the 9/7 biorthogonal wavelet transform, retained in the JPEG2000 compression standard. The decomposition is carried out over three resolution levels (i.e. $J = 10$). Note also that the weights $(\rho_j)_{1\leq j\leq J}$ for the different wavelet subbands are computed by using the procedure presented in [36]. Our experiments have been performed for various standard test images with different characteristics. As mentioned before, the first step of our method consists of modelling the resulting

wavelet coefficients. For this purpose, we consider the two following models: the GG one and the more general BGG one.

A. GG-based model

In this case, the parameters β_j and ω_j for each subband are estimated by using the maximum likelihood technique. Afterwards, we compute their corresponding entropy and distortion approximations and deduce their optimal quantization steps using Proposition 3. Figs. 3(a), 3(b) and 3(c) show the influence of the choice of the parameters $m^{(h)}$ and $m^{(d)}$ used for approximating the entropy and distortion functions. The plotted curve using the 'circle' symbols corresponds to the quadratic distortion (i.e. $p_j = 2$) resulting from an uniform scalar quantization of the GG model. The rate-distortion curve plotted using the 'star' symbol is obtained by performing a similar quantization of the wavelet coefficients of the image with the derived optimal quantizers. More precisely, we consider the cases $m^{(h)} = m^{(d)} = 2$, $m^{(h)} = m^{(d)} = 3$ and $m^{(h)} = m^{(d)} = 4$. It can be noticed that the difference between the plots corresponding to the theoretical GG source model and the image wavelet coefficients is reduced when the number of segments increases. In addition, one can observe from Fig. 3(d) that the image rate-distortion curves behave similarly when 3 or 4 approximating intervals are used. Based on this observation (which was confirmed by tests performed on other images), it can be concluded that there is no need to increase the number of segments, and therefore, it is sufficient in practice to use 3 or 4 intervals to approximate the entropy and distortion functions. Finally, we propose to compare the proposed bit allocation method with state-of-the-art methods based on Lagrangian optimization techniques [4]. More precisely, we consider the improved version of these methods, proposed recently in [12], where a spline interpolation method for rate-distortion curves is introduced. Fig. 4 shows the variations of the PSNR curves versus the entropy for different images. It can be observed that our method outperforms the state-of-the-art method by 0.2-1.2 dB. While the deadzone parameter τ_j is set to 1 in Fig. 4, Figs. 5(a) and 5(b) illustrate the performance of our method when the size of the deadzone is increased ($\tau_j = 2$). Thus, it can be noticed that the proposed method achieves a significant improvement compared with the state-of-the-art method.

B. BGG-based model

Although the GG model is well adapted to a large class of natural images, we have observed that this model is not the best suited for the class of images with *flat regions* separated by smooth contours. Examples of such images include cartoon ones and depth maps. To confirm this, we illustrate in Figs. 6(a)

and 6(b) the histogram of the diagonal detail wavelet subband of the "cartoon" image at the first resolution level as well as the distribution used for modelling its coefficients. To find the best model, we propose to use a statistical goodness-of-fit test such as the Kolmogorov-Smirnov (KS) test which is based on the comparison of the cumulative distribution functions (cdf) [37]. Figs. 6(c) and 6(d) display these functions for both models with their resulting KS measure. Hence, it can be noticed that the cdf associated with the BGG model is very close to the cdf associated with the subband wavelet coefficients. This illustrates the fact that the BGG model is more appropriate than the GG one for modelling very *sparse* representations. Based on this model, we have also employed the proposed bit allocation method for this class of images. Compared with the improved version of the Lagrangian based optimization technique [12], Figs. 7(a) and 7(b) show that the proposed method achieves an improvement of about 0.3-1 dB. In Fig. 7, the deadzone parameter τ_j is set to 1. Figs. 8(a) and 8(b) illustrate the performance of our method when the size of the deadzone is equal to $3q_j$ ($\tau_j = 2$). It can be concluded that the proposed method outperforms the state-of-the-art method in all these experiments.

Finally, in order to measure the relative gain of the proposed method, we used the Bjontegaard metric [38]. The results are illustrated in Table I for low and high bitrates corresponding respectively to the four bitrate points $\{0.1, 0.2, 0.3, 0.4\}$ and $\{0.7, 0.8, 0.9, 1\}$ bpp. Table I gives the gain of our method compared with the improved version of the Lagrangian based optimization technique [12]. Note that a bitrate saving with respect to the reference method corresponds to negative values. It can be observed that the proposed approach outperforms the state-of-the-art method by about -10% and 0.7 dB in terms of bitrate saving and PSNR. All these results, obtained with different images, confirm the effectiveness of the considered probabilistic models and of the proposed bit allocation method.

VI. Conclusion

In this work, we have proposed to reformulate the bit allocation problem as a set of convex programming problems which can be dealt with in parallel. For this purpose, we have first proposed new piecewise convex approximations of the entropy and the distortion functions. Then, we have derived explicit expressions of the optimal quantization parameters which are valid in a given subdomain. This study has been carried out by considering two probabilistic models: the well-known GG model and its more general BGG form, which is particularly well-adapted for very *sparse* sources. Finally, we have illustrated through experimental results the benefits which can be drawn from the application of the proposed technique in the context of transform-based coding application.

APPENDIX A
APPROXIMATION OF THE ENTROPY

We recall that the entropy of a quantized BGG random variable distributed according to (1) is given by [26]:

$$H_{f_j}(q_j,\epsilon_j) = \Phi(\mathsf{p}_{0,j},\epsilon_j) + \epsilon_j H_{\widetilde{f}_j}(q_j) \tag{31}$$

$$\text{with} \quad H_{\widetilde{f}_j}(q_j) = -\mathsf{p}_{0,j}\log_2\mathsf{p}_{0,j} - 2\sum_{i=1}^{\infty}\mathsf{p}_{i,j}\log_2\mathsf{p}_{i,j} \tag{32}$$

is the entropy of a quantized GG random variable with probability density function $\widetilde{f}_j$, where the probability of the zero level is

$$\mathsf{p}_{0,j} = 2\int_0^{q_j(\tau_j-\frac{1}{2})}\widetilde{f}_j(\xi)d\xi = Q_{1/\beta_j}\Big(\omega_j(\tau_j-\frac{1}{2})^{\beta_j}q_j^{\beta_j}\Big) \tag{33}$$

and the probability $\mathsf{p}_{i,j}$ of the $r_{i,j}$ reconstruction level, $i\geq 1$, is

$$\int_{(\tau_j+i-\frac{3}{2})q_j}^{(\tau_j+i-\frac{1}{2})q_j}\widetilde{f}_j(\xi)d\xi = \frac{1}{2}\Big(Q_{1/\beta_j}\Big(\omega_j\big((\tau_j+i-\frac{1}{2})q_j\big)^{\beta_j}\Big) - Q_{1/\beta_j}\Big(\omega_j\big((\tau_j+i-\frac{3}{2})q_j\big)^{\beta_j}\Big)\Big).$$

In the following, in order to prove the desired result, it is sufficient to show that the following approximation formula of the discrete entropy of a quantized GG random variable holds:

$$H_{\widetilde{f}_j}(q_j) = \widehat{H}_{\widetilde{f}_j}(q_j) + \Delta \tag{34}$$

$$\text{where} \quad 0\leq\Delta\leq 2q_jC(\beta_j,\tau_j)\tilde{f}_j\big((\tau_j+\frac{1}{2})q_j\big). \tag{35}$$

Note that the case $\beta_j\in[1,2]$ was addressed in [26] for a quantizer with a deadzone of size q_j (i.e. $\tau_j=1$). Let us now proceed to the general case.

Since $\widetilde{f}_j$ is a decreasing function on $\mathbb{R}_+$, we have, for all $i>0$,

$$q_j\widetilde{f}_j\big((\tau_j+i-1/2)q_j\big) \leq \mathsf{p}_{i,j} \leq q_j\widetilde{f}_j\big((\tau_j+i-3/2)q_j\big)$$

By noticing that

$$-\mathsf{p}_{i,j}\log_2\mathsf{p}_{i,j} + \int_{(\tau_j+i-3/2)q_j}^{(\tau_j+i-1/2)q_j}\widetilde{f}_j(\xi)\log_2\widetilde{f}_j(\xi)\,d\xi = \int_{(\tau_j+i-3/2)q_j}^{(\tau_j+i-1/2)q_j}\widetilde{f}_j(\xi)(\log_2\widetilde{f}_j(\xi)-\log_2\mathsf{p}_{i,j})\,d\xi \tag{36}$$

we get the inequality:

$$\begin{aligned}-\mathsf{p}_{i,j}\log_2\mathsf{p}_{i,j} + \int_{(\tau_j+i-3/2)q_j}^{(\tau_j+i-1/2)q_j}\widetilde{f}_j(\xi)\log_2\widetilde{f}_j(\xi)\,d\xi \leq \int_{(\tau_j+i-3/2)q_j}^{(\tau_j+i-1/2)q_j}\widetilde{f}_j(\xi)\Big(\log_2\widetilde{f}_j(\xi)-\\ \log_2\widetilde{f}_j\big((\tau_j+i-1/2)q_j\big)-\log_2 q_j\Big)\,d\xi.\end{aligned} \tag{37}$$

On the other hand, from the positivity of the Kullback-Leibler divergence,

$$\int_{(\tau_j+i-3/2)q_j}^{(\tau_j+i-1/2)q_j} \frac{\widetilde{f}_j(\xi)}{\mathsf{p}_{i,j}} \log_2\Big(\frac{\widetilde{f}_j(\xi)/\mathsf{p}_{i,j}}{1/q_j}\Big)\, d\xi \geq 0 \tag{38}$$

After developing (38) and using (37), we obtain for all $i \geq 1$

$$\begin{aligned} 0 \leq & -\mathsf{p}_{i,j}\log_2 \mathsf{p}_{i,j} + \int_{(\tau_j+i-3/2)q_j}^{(\tau_j+i-1/2)q_j} \widetilde{f}_j(\xi)\log_2 \widetilde{f}_j(\xi)\, d\xi + \log_2 q_j \int_{(\tau_j+i-3/2)q_j}^{(\tau_j+i-1/2)q_j} \widetilde{f}_j(\xi)\, d\xi \\ & \leq \Big(\log_2 \widetilde{f}_j((\tau_j+i-3/2)q_j) - \log_2 \widetilde{f}_j\big((\tau_j+i-1/2)q_j\big)\Big) \int_{(\tau_j+i-3/2)q_j}^{(\tau_j+i-1/2)q_j} \widetilde{f}_j(\xi)\, d\xi \\ & = \omega_j q_j^{\beta_j}\big((\tau_j+i-1/2)^{\beta_j} - (\tau_j+i-3/2)^{\beta_j}\big) \int_{(\tau_j+i-3/2)q_j}^{(\tau_j+i-1/2)q_j} \widetilde{f}_j(\xi)\, d\xi. \end{aligned} \tag{39}$$

Now, two cases shall be considered:

• If $\beta_j < 1$, then, for every $i \geq 1$,

$$(\tau_j+i-1/2)^{\beta_j} - (\tau_j+i-3/2)^{\beta_j} \leq \beta_j(\tau_j+i-3/2)^{\beta_j-1},$$

where the upper bound follows from the fact that $\xi \mapsto \xi^{\beta_j}$ is a concave function when $\beta_j < 1$. In this case, we have

$$\begin{aligned} \omega_j q_j^{\beta_j}\big((\tau_j+i-1/2)^{\beta_j} - (\tau_j+i-3/2)^{\beta_j}\big) \int_{(\tau_j+i-3/2)q_j}^{((\tau_j+i-1/2)q_j} \widetilde{f}_j(\xi)\, d\xi \leq \beta_j\omega_j q_j\Big(q_j(\tau_j+i-\frac{1}{2}) - q_j\Big)^{\beta_j-1} \\ \times \int_{(\tau_j+i-3/2)q_j}^{(\tau_j+i-1/2)q_j} \widetilde{f}_j(\xi)\, d\xi, \end{aligned}$$

$$\begin{aligned} \text{and} \qquad 0 \leq & -\mathsf{p}_{i,j}\log_2 \mathsf{p}_{i,j} + \int_{(\tau_j+i-3/2)q_j}^{(\tau_j+i-1/2)q_j} \widetilde{f}_j(\xi)\log_2 \widetilde{f}_j(\xi)\, d\xi + \log_2 q_j \int_{(\tau_j+i-3/2)q_j}^{(\tau_j+i-1/2)q_j} \widetilde{f}_j(\xi)\, d\xi \\ & \leq \beta_j\omega_j q_j \int_{(\tau_j+i-3/2)q_j}^{(\tau_j+i-1/2)q_j} (\xi-q_j)^{\beta_j-1}\widetilde{f}_j(\xi)\, d\xi. \end{aligned}$$

It can be deduced that

$$0 \leq -\sum_{i=2}^{+\infty} \mathsf{p}_{i,j}\log_2 \mathsf{p}_{i,j} + \int_{(\tau_j+\frac{1}{2})q_j}^{+\infty} \widetilde{f}_j(\xi)\log_2 \widetilde{f}_j(\xi) d\xi + \log_2 q_j \int_{(\tau_j+\frac{1}{2})q_j}^{+\infty} \widetilde{f}_j(\xi) d\xi \leq I_1 \tag{40}$$

$$\text{where} \qquad I_1 = \beta_j\omega_j q_j \int_{(\tau_j+\frac{1}{2})q_j}^{+\infty} (\xi-q_j)^{\beta_j-1}\widetilde{f}_j(\xi)\, d\xi.$$

Since $\xi \geq (\tau_j+1/2)q_j \Leftrightarrow \xi - q_j \geq (2\tau_j-1)\xi/(2\tau_j+1)$, it can be concluded that

$$\begin{aligned} I_1 \leq & \beta_j\omega_j q_j\Big(\frac{2\tau_j-1}{2\tau_j+1}\Big)^{\beta_j-1} \int_{(\tau_j+\frac{1}{2})q_j}^{+\infty} \xi^{\beta_j-1}\widetilde{f}_j(\xi)\, d\xi \\ & = \frac{\beta_j\omega_j^{1/\beta_j}q_j}{2\Gamma(1/\beta_j)}\Big(\frac{2\tau_j-1}{2\tau_j+1}\Big)^{\beta_j-1} e^{-\omega_j(\tau_j+\frac{1}{2})^{\beta_j}q_j^{\beta_j}}. \end{aligned} \tag{41}$$

• If $\beta_j \in [1,2]$ then, for every $i \geq 1$,

$$(\tau_j+i-1/2)^{\beta_j}-(\tau_j+i-3/2)^{\beta_j} = (\tau_j+i-1)^{\beta_j}\left((1+\frac{1}{2(\tau_j+i-1)})^{\beta_j}-(1-\frac{1}{2(\tau_j+i-1)})^{\beta_j}\right)$$
$$\leq \beta_j(\tau_j+i-1)^{\beta_j-1}. \tag{42}$$

Consequently,

$$\omega_j q_j^{\beta_j}\big((\tau_j+i-1/2)^{\beta_j}-(\tau_j+i-3/2)^{\beta_j}\big)\int_{(\tau_j+i-\frac{3}{2})q_j}^{(\tau_j+i-\frac{1}{2})q_j}\widetilde{f}_j(\xi)\,d\xi \leq \beta_j\omega_j q_j\left(q_j(\tau_j+i-\frac{3}{2})+\frac{q_j}{2}\right)^{\beta_j-1}$$
$$\times\int_{(\tau_j+i-\frac{3}{2})q_j}^{(\tau_j+i-\frac{1}{2})q_j}\widetilde{f}_j(\xi)\,d\xi$$

and $$0 \leq -\mathsf{p}_{i,j}\log_2\mathsf{p}_{i,j}+\int_{(\tau_j+i-\frac{3}{2})q_j}^{(\tau_j+i-\frac{1}{2})q_j}\widetilde{f}_j(\xi)\ln\widetilde{f}_j(\xi)\,d\xi+\log_2 q_j\int_{(\tau_j+i-\frac{3}{2})q_j}^{(\tau_j+i-\frac{1}{2})q_j}\widetilde{f}_j(\xi)\,d\xi$$
$$\leq \beta_j\omega_j q_j\int_{(\tau_j+i-\frac{3}{2})q_j}^{(\tau_j+i-\frac{1}{2})q_j}(\xi+\frac{q_j}{2})^{\beta_j-1}\widetilde{f}_j(\xi)\,d\xi.$$

Thus,

$$0 \leq -\sum_{i=2}^{+\infty}\mathsf{p}_{i,j}\log_2\mathsf{p}_{i,j}+\int_{(\tau_j+\frac{1}{2})q_j}^{+\infty}\widetilde{f}_j(\xi)\log_2\widetilde{f}_j(\xi)d\xi+\log_2 q_j\int_{(\tau_j+\frac{1}{2})q_j}^{+\infty}\widetilde{f}_j(\xi)d\xi \leq I_2 \tag{43}$$

$$\text{where}\qquad I_2=\beta_j\omega_j q_j\int_{(\tau_j+\frac{1}{2})q_j}^{+\infty}(\xi+\frac{q_j}{2})^{\beta_j-1}\widetilde{f}_j(\xi)\,d\xi.$$

Since $\xi \geq (\tau_j+1/2)q_j \Leftrightarrow \xi+\frac{q_j}{2} \leq (2\tau_j+2)\xi/(2\tau_j+1)$, it can be concluded that

$$I_2 \leq \beta_j\omega_j q_j\Big(\frac{2\tau_j+2}{2\tau_j+1}\Big)^{\beta_j-1}\int_{(\tau_j+\frac{1}{2})q_j}^{+\infty}\xi^{\beta_j-1}\widetilde{f}_j(\xi)\,d\xi=\frac{\beta_j\omega_j^{1/\beta_j}q_j}{2\Gamma(1/\beta_j)}\Big(\frac{2\tau_j+2}{2\tau_j+1}\Big)^{\beta_j-1}e^{-\omega_j(\tau_j+\frac{1}{2})^{\beta_j}q_j^{\beta_j}}. \tag{44}$$

By combining (40) and (41) (resp. (43) and (44)) when $\beta_j<1$ (resp. $\beta_j\in[1,2]$), we get the following result:

$$0 \leq -\sum_{i=2}^{+\infty}\mathsf{p}_{i,j}\log_2\mathsf{p}_{i,j}+\int_{(\tau_j+\frac{1}{2})q_j}^{+\infty}\widetilde{f}_j(\xi)\log_2\widetilde{f}_j(\xi)d\xi+\log_2 q_j\int_{(\tau_j+\frac{1}{2})q_j}^{+\infty}\widetilde{f}_j(\xi)d\xi$$
$$\leq \frac{\beta_j\omega_j^{1/\beta_j}q_j}{2\Gamma(1/\beta_j)}C(\beta_j,\tau_j)e^{-\omega_j(\tau_j+\frac{1}{2})^{\beta_j}q_j^{\beta_j}} \tag{45}$$

where $C(\beta_j,\tau_j)$ is given by (11). Furthermore, it can be checked [26] that we have:

$$2\int_{(\tau_j+\frac{1}{2})q_j}^{+\infty}\widetilde{f}_j(\xi)d\xi=1-Q_{1/\beta_j}\Big(\omega_j\big(\tau_j+\frac{1}{2}\big)^{\beta_j}q_j^{\beta_j}\Big) \tag{46}$$

and $$2\int_{(\tau_j+\frac{1}{2})q_j}^{+\infty} \widetilde{f}_j(\xi)\log_2 \widetilde{f}_j(\xi)d\xi = -h_{\beta_j}(\omega_j)\Big(1-Q_{1/\beta_j}\Big(\omega_j\big(\tau_j+\frac{1}{2}\big)^{\beta_j}q_j^{\beta_j}\Big)\Big) - \frac{\omega_j^{1/\beta_j}(\tau_j+\frac{1}{2})q_j}{\Gamma(1/\beta_j)}\, e^{-\omega_j(\tau_j+\frac{1}{2})^{\beta_j}q_j^{\beta_j}}. \quad (47)$$

Therefore, the approximation formula of the entropy of the quantized GG random variable, given by (34)-(35), follows from (32), (45)-(47). Finally, the approximation formula for the discrete entropy of the quantized BGG random variable can be easily deduced from (31).

Concerning the high bitrate approximation of the entropy, it can be firstly noticed that $\Delta = O(q_j)$. We further know [39, p.891] that for all $a > 0$,

$$Q_a(\xi) = O(\xi^a), \qquad \text{as} \quad \xi \to 0. \quad (48)$$

Therefore, when $q_j \to 0$, we have

$$H_{\widetilde{f}_j}(q_j) = h_{\beta_j}(\omega_j) - \log_2 q_j + O(q_j). \quad (49)$$

Moreover, according to (33) and (48), we get

$$\begin{aligned}\Phi(\mathsf{p}_{0,j},\epsilon_j) &= -\epsilon_j\log_2\epsilon_j - (1-\epsilon_j)\log_2(1-\epsilon_j) + \epsilon_j\mathbf{1}_{(0,1)}(\epsilon_j)\frac{\beta_j\omega_j^{1/\beta_j}q_j}{2\Gamma(1/\beta_j)}\log_2(\omega_j^{1/\beta_j}q_j) + O(q_j)\\ &= H_{\epsilon_j} + O(q_j\log_2 q_j) \end{aligned} \quad (50)$$

where $\mathbf{1}_{(0,1)}$ is the characteristic function of the interval $(0,1)$. Consequently, a high resolution approximation of the entropy of a quantized BGG random variable is given by (12).

APPENDIX B
APPROXIMATION OF THE DISTORTION

If $X_{j,s}$ is distributed according to (1), the distortion evaluated through the p_j-th order moment of the quantization error is given by [26]:

$$e_j(q_j,\epsilon_j) = 2\epsilon_j\Big(\int_0^{(\tau_j-\frac{1}{2})q_j}\xi^{p_j}\widetilde{f}_j(\xi)d\xi + \sum_{i=1}^{+\infty}\int_{(\tau_j+i-\frac{1}{2})q_j}^{(\tau_j+i-\frac{1}{2})q_j}|\xi-r_{i,j}|^{p_j}\widetilde{f}_j(\xi)d\xi\Big).$$

By noticing that

$$\int_0^{(\tau_j-\frac{1}{2})q_j}\xi^{p_j}\widetilde{f}_j(\xi)d\xi = \frac{\omega_j^{-p_j/\beta_j}\Gamma((p_j+1)/\beta_j)}{2\Gamma(1/\beta_j)}Q_{(p_j+1)/\beta_j}\big(\omega_j(\tau_j-\frac{1}{2})^{\beta_j}q_j^{\beta_j}\big) \quad (51)$$

the approximation error can be expressed as

$$\begin{aligned} &e_j(q_j,\epsilon_j) - \widehat{e}_j(q_j,\epsilon_j)\\ &= 2\epsilon_j\Big(\sum_{i=2}^{+\infty}\int_{(\tau_j+i-\frac{1}{2})q_j}^{(\tau_j+i-\frac{1}{2})q_j}|\xi-r_{i,j}|^{p_j}\widetilde{f}_j(\xi)d\xi - \frac{\nu_j q_j^{p_j}}{2(p_j+1)}\Big(1-Q_{1/\beta_j}\Big(\omega_j\big((\tau_j+\frac{1}{2})q_j\big)^{\beta_j}\Big)\Big)\Big). \end{aligned} \quad (52)$$

First, for every $i \geq 1$, we have

$$\widetilde{f}_j\big((\tau_j+i-\frac{1}{2})q_j\big)\int_{(\tau_j+i-\frac{3}{2})q_j}^{(\tau_j+i-\frac{1}{2})q_j}|\xi-r_{i,j}|^{p_j}d\xi \leq \int_{(\tau_j+i-\frac{3}{2})q_j}^{(\tau_j+i-\frac{1}{2})q_j}|\xi-r_{i,j}|^{p_j}\widetilde{f}_j(\xi)d\xi$$
$$\leq \widetilde{f}_j\big((\tau_j+i-\frac{3}{2})q_j\big)\int_{(\tau_j+i-\frac{3}{2})q_j}^{(\tau_j+i-\frac{1}{2})q_j}|\xi-r_{i,j}|^{p_j}d\xi \quad (53)$$

$$\text{with} \qquad \int_{(\tau_j+i-\frac{3}{2})q_j}^{(\tau_j+i-\frac{1}{2})q_j}|\xi-r_{i,j}|^{p_j}d\xi = \frac{\nu_j q_j^{p_j+1}}{p_j+1}. \quad (54)$$

In addition, we have the following inequalities:

$$\int_{(\tau_j+i-\frac{1}{2})q_j}^{(\tau_j+i+\frac{1}{2})q_j}\widetilde{f}_j(\xi)d\xi \leq q_j\widetilde{f}_j\big((\tau_j+i-\frac{1}{2})q_j\big) \quad (55)$$

and, for every $i \geq 2$,

$$q_j\widetilde{f}_j\big((\tau_j+i-\frac{3}{2})q_j\big) \leq \int_{(\tau_j+i-\frac{5}{2})q_j}^{(\tau_j+i-\frac{3}{2})q_j}\widetilde{f}_j(\xi)d\xi. \quad (56)$$

We deduce from (53), (54), (55), (56) and (51) that

$$\frac{\nu_j q_j^{p_j}}{2(p_j+1)}\Big(1-Q_{1/\beta_j}\big(\omega_j(\tau_j+\frac{1}{2})^{\beta_j}q_j^{\beta_j}\big)\Big)-\frac{\nu_j q_j^{p_j+1}}{p_j+1}\widetilde{f}_j\big((\tau_j+\frac{1}{2})q_j\big) \leq \sum_{i=2}^{+\infty}\int_{(\tau_j+i-\frac{3}{2})q_j}^{(\tau_j+i-\frac{1}{2})q_j}|\xi-r_{i,j}|^{p_j}\widetilde{f}_j(\xi)d\xi$$
$$\leq \frac{\nu_j q_j^{p_j}}{2(p_j+1)}\Big(1-Q_{1/\beta_j}\big(\omega_j(\tau_j+\frac{1}{2})^{\beta_j}q_j^{\beta_j}\big)\Big)+\frac{\nu_j q_j^{p_j+1}}{p_j+1}\widetilde{f}_j\big((\tau_j+\frac{1}{2})q_j\big). \quad (57)$$

Therefore, the approximation error satisfies

$$-2\epsilon_j\frac{\nu_j q_j^{p_j+1}}{p_j+1}\widetilde{f}_j\big((\tau_j+\frac{1}{2})q_j\big) \leq e_j(q_j,\epsilon_j)-\widehat{e}_j(q_j,\epsilon_j) \leq 2\epsilon_j\frac{\nu_j q_j^{p_j+1}}{p_j+1}\widetilde{f}_j\big((\tau_j+\frac{1}{2})q_j\big) \quad (58)$$

which yields the desired approximation of the distortion.

Let us now focus on the expression of the distortion at high bitrate. When $q_j \to 0$, according to (48), the first term in the left hand side of (15) is such that

$$Q_{(p_j+1)/\beta_j}\big(\omega_j(\tau_j-\frac{1}{2})^{\beta_j}q_j^{\beta_j}\big) = O(q_j^{p_j+1}). \quad (59)$$

Moreover, using (53) and (54), we obtain

$$\frac{\nu_j q_j^{p_j+1}}{p_j+1}\widetilde{f}_j\big((\tau_j+\frac{1}{2})q_j\big) \leq \int_{(\tau_j-\frac{1}{2})q_j}^{(\tau_j+\frac{1}{2})q_j}|\xi-r_{1,j}|^{p_j}\widetilde{f}_j(\xi)d\xi \leq \frac{\nu_j q_j^{p_j+1}}{p_j+1}\widetilde{f}_j\big((\tau_j-\frac{1}{2})q_j\big) \quad (60)$$

which shows that

$$\int_{(\tau_j-\frac{1}{2})q_j}^{(\tau_j+\frac{1}{2})q_j}|\xi-r_{1,j}|^{p_j}\widetilde{f}_j(\xi)d\xi = O(q_j^{p_j+1}). \quad (61)$$

In addition, we have

$$\frac{\nu_j q_j^{p_j}}{2(p_j+1)}\Big(1-Q_{1/\beta_j}\big(\omega_j(\tau_j+\frac{1}{2})^{\beta_j}q_j^{\beta_j}\big)\Big)=\frac{\nu_j q_j^{p_j}}{2(p_j+1)}\big(1+O(q_j)\big). \tag{62}$$

Since (16) shows that $e_j(q_j,\epsilon_j)-\widehat{e}_j(q_j,\epsilon_j)=O(q_j^{p_j+1})$, it can be deduced from (59), (61) and (62) that (17) holds.

APPENDIX C
SOLUTION OF THE BIT ALLOCATION PROBLEM

For simplicity, for every $j\in\{1,\ldots,J\}$, we will drop the index k in the variables a_j^k, c_j^k, γ_j^k, α_j^k, and δ_j^k, which are used in (13) and (18).

As $g_j(l_j)$ is a decreasing function of l_j for every $j\in\{1,\ldots,J\}$, it is clear that, if $\sum_{j=1}^J \frac{n_j}{n}g_j(l_j^{b_j+1})>R_{\max}\Leftrightarrow\sum_{j=1}^J\frac{n_j}{n}(a_jl_j^{b_j+1}+c_j)>R_{\max}$, then Problem $(\mathcal{P}_{\mathbf{b}})$ admits no solution since $C\cap([l_1^{b_1},l_1^{b_1+1}]\times\cdots\times[l_J^{b_J},l_J^{b_J+1}])$ is empty. Another particular case is when

$$\sum_{j=1}^J\frac{n_j}{n}(a_jl_j^{b_j}+c_j)\leq R_{\max} \tag{63}$$

Since, for every $j\in\{1,\ldots,J\}$, d_j is an increasing function, the solution to $(\mathcal{P}_{\mathbf{b}})$ is obviously $\widetilde{\mathbf{l}}_{\mathbf{b}}=(l_1^{b_1},\ldots,l_J^{b_J})$.

In the following, we will discard these two trivial cases by assuming that

$$\sum_{j=1}^J\frac{n_j}{n}(a_jl_j^{b_j}+c_j)>R_{\max}$$
$$\text{and}\quad\sum_{j=1}^J\frac{n_j}{n}(a_jl_j^{b_j+1}+c_j)\leq R_{\max}. \tag{64}$$

Under these assumptions, since $(l_1^{b_1+1},\ldots,l_J^{b_J+1})\in C\cap([l_1^{b_1},l_1^{b_1+1}]\times\cdots[l_J^{b_J},l_J^{b_J+1}])$, the intersection set is nonempty and the problem $(\mathcal{P}_{\mathbf{b}})$ has a solution $\widetilde{\mathbf{l}}_{\mathbf{b}}$. In order to find this solution, we will apply the Fenchel-Rockafellar duality theorem [40].

Theorem 1: Let f and g be two lower-semicontinuous convex functions from $\mathbb{R}^J$ to $]-\infty,+\infty]$. Then, provided that $\mathrm{dom}(f)\cap\mathrm{dom}(g)$ is nonempty, we have

$$\inf_{\mathbf{l}\in\mathbb{R}^J}(f(\mathbf{l})+g(\mathbf{l}))=\max_{\mathbf{l}^*\in\mathbb{R}^J}(-g^*(-\mathbf{l}^*)-f^*(\mathbf{l}^*)), \tag{65}$$

where f^* (resp. g^*) is the convex conjugate of f (resp. g).[1]

[1]Recall that $\mathrm{dom}(f)=\{\mathbf{l}\in\mathbb{R}^J|f(\mathbf{l})<+\infty\}$ and f^* is defined as: $\forall\,\mathbf{l}^*\in\mathbb{R}^J$, $f^*(\mathbf{l}^*)=\sup_{\mathbf{l}\in\mathbb{R}^J}(\mathbf{l}^\top\mathbf{l}^*-f(\mathbf{l}))$.

In our case, we take $g = \iota_C$ where ι_C is the indicator function[2] of the closed convex set C defined by (19). Taking $\mathbf{l} \in C$ is equivalent to take $\mathbf{l} \in \mathbb{R}^J$ such that

$$\mathbf{e}^\top \mathbf{l} \geq \sum_{j=1}^{J} \frac{n_j}{n} c_j - R_{\max}, \qquad \text{where} \quad \mathbf{e} = -\frac{1}{n}(n_1 a_1, \ldots, n_J a_J) \tag{66}$$

Thus, the conjugate of g satisfies

$$\forall\ \mathbf{l}^* \in \mathbb{R}^J, \qquad g^*(\mathbf{l}^*) = \sup_{\mathbf{l} \in C} \mathbf{l}^\top \mathbf{l}^* = \sup_{\mathbf{l} \in C} (\lambda \mathbf{l}^\top \mathbf{e} + \mathbf{l}^\top \mathbf{l}^*_\perp), \tag{67}$$

where $\mathbf{l}^*_\perp$ belongs to $\mathrm{Vect}\{\mathbf{e}\}^\perp$, the orthogonal subspace of $\mathbf{e}$, and $\lambda \in \mathbb{R}$. From (66), we see that if $\mathbf{l}^*_\perp \neq \mathbf{0}$, $g^*(\mathbf{l}^*) = +\infty$. Furthermore, if $\mathbf{l}^* = \lambda \mathbf{e}$ with $\lambda > 0$, the supremum over $\mathbf{l}$ of $\mathbf{l}^\top \mathbf{e}$ is infinite. Finally, we obtain for all $\mathbf{l}^* \in \mathbb{R}^J$

$$g^*(\mathbf{l}^*) = \begin{cases} \lambda \Big(\displaystyle\sum_{j=1}^{J} \frac{n_j}{n} c_j - R_{\max} \Big) & \text{if } \mathbf{l}^* = \lambda \mathbf{e} \text{ with } \lambda \leq 0 \\ +\infty & \text{else.} \end{cases}$$

On the other hand, we take, for every $\mathbf{l} \in \mathbb{R}^J$, $f(\mathbf{l}) = D(\mathbf{l}) + \iota_{\mathbf{P_b}}(\mathbf{l})$, where $\mathbf{P_b}$ is the box defined at the beginning of Section IV-B. Thus, f can be rewritten as

$$\forall\ \mathbf{l} \in \mathbb{R}^J, \quad f(\mathbf{l}) = \sum_{j=1}^{J} \phi_j(l_j) \tag{68}$$

where, for every $j \in \{1, \ldots, J\}$,

$$\forall\ l_j \in \mathbb{R}, \qquad \phi_j(l_j) = \rho_j \epsilon_j (\alpha_j 2^{\gamma_j l_j} + \delta_j) + \iota_{[l_j^{b_j}, l_j^{b_j+1}]}(l_j). \tag{69}$$

Using the separability of the convex conjugate of f, we get

$$\forall\ \mathbf{l}^* = (l_1^*, \ldots, l_J^*) \in \mathbb{R}^J \qquad f^*(\mathbf{l}^*) = \sum_{j=1}^{J} \phi_j^*(l_j^*). \tag{70}$$

For any given $j \in \{1, \ldots, J\}$ and $l_j^* \in \mathbb{R}$, let us define

$$\forall\ l_j \in \mathbb{R}, \qquad \psi_j(l_j) = l_j l_j^* - \rho_j \epsilon_j (\alpha_j 2^{\gamma_j l_j} + \delta_j). \tag{71}$$

We can write

$$\phi_j^*(l_j^*) = \sup_{l_j^{b_j} \leq l_j \leq l_j^{b_j+1}} \psi_j(l_j). \tag{72}$$

Furthermore,

$$\forall\ l_j \in \mathbb{R}, \qquad \psi_j'(l_j) = l_j^* - \ln(2) \rho_j \epsilon_j \gamma_j \alpha_j 2^{\gamma_j l_j} = l_j^* - \frac{\kappa_j N_j \gamma_j}{n} 2^{\gamma_j l_j}.$$

[2]The indicator function of C is defined as: $\forall\ x \in \mathbb{R}^J$, $\iota_C(x) = 0$ if $x \in C$; $+\infty$ otherwise.

Thus, if $l_j^* \leq 0$, then $\psi_j'(l_j) < 0$ and $\phi_j^*(l_j^*) = \psi_j(l_j^{b_j})$. In turn, if $l_j^* > 0$, then it can be checked that $\psi_j'(l_j) < 0$ if and only if

$$l_j > \frac{1}{\gamma_j} \log_2 \Big(\frac{n l_j^*}{\kappa_j N_j \gamma_j} \Big). \tag{73}$$

Three cases have then to be considered:

(i) If $l_j^{b_j} \geq \frac{1}{\gamma_j} \log_2 \Big(\frac{n l_j^*}{\kappa_j N_j \gamma_j} \Big)$ which is equivalent to $2^{\gamma_j l_j^{b_j}} \geq \frac{n l_j^*}{\kappa_j N_j \gamma_j}$ then, for every $l_j \geq l_j^{b_j}$, $\psi_j'(l_j) < 0$ and

$$\phi_j^*(l_j^*) = \psi_j(l_j^{b_j}) = l_j^* l_j^{b_j} - \rho_j \epsilon_j (\alpha_j 2^{\gamma_j l_j^{b_j}} + \delta_j). \tag{74}$$

(ii) Similarly, if $\frac{n l_j^*}{\kappa_j N_j \gamma_j} \geq 2^{\gamma_j l_j^{b_j+1}}$ then, for every $l_j \in [l_j^{b_j}, l_j^{b_j+1}]$, $\psi_j'(l_j) > 0$ and

$$\phi_j^*(l_j^*) = \psi_j(l_j^{b_j+1}) = l_j^* l_j^{b_j+1} - \rho_j \epsilon_j (\alpha_j 2^{\gamma_j l_j^{b_j+1}} + \delta_j). \tag{75}$$

(iii) Otherwise, if $2^{\gamma_j l_j^{b_j}} < \frac{n l_j^*}{\kappa_j N_j \gamma_j} < 2^{\gamma_j l_j^{b_j+1}}$ then,

$$\begin{aligned} \phi_j^*(l_j^*) &= \psi_j\Big(\frac{1}{\gamma_j} \log_2 \Big(\frac{n l_j^*}{\kappa_j N_j \gamma_j} \Big) \Big) \\ &= \frac{l_j^*}{\gamma_j \ln 2} \Big(\ln \Big(\frac{n l_j^*}{\kappa_j N_j \gamma_j} \Big) - 1 \Big) - \rho_j \epsilon_j \delta_j. \end{aligned} \tag{76}$$

Now, by recalling that $\mathrm{dom}(g) = \{-\lambda \mathbf{e}, \lambda \geq 0\}$, the dual problem can be reexpressed as

$$\max_{\mathbf{l}^* \in \mathbb{R}^J} (-g^*(-\mathbf{l}^*) - f^*(\mathbf{l}^*)) = \max_{-\mathbf{l}^* \in \mathrm{dom}(g)} (-g^*(-\mathbf{l}^*) - f^*(\mathbf{l}^*)) = \max_{\lambda \geq 0} \Phi(\lambda) \tag{77}$$

where

$$\forall \lambda \in \mathbb{R}_+, \qquad \Phi(\lambda) = \lambda \Big(\sum_{j=1}^J \frac{n_j}{n} c_j - R_{\max} \Big) - \sum_{j=1}^J \phi_j^* \Big(-\lambda \frac{n_j}{n} a_j \Big).$$

According to (74)-(76) and the notation introduced in (22) and (23), Φ is the function defined in (30). The derivative of this function is given by

$$\forall \lambda \in \mathbb{R}_+, \qquad \Phi'(\lambda) = \sum_{j=1}^J \frac{n_j}{n} c_j - R_{\max} - \sum_{j=1}^J \varphi_j'(\lambda) \tag{78}$$

where φ_j' corresponds to the derivative of the function φ_j defined in Proposition 3. Thus, it can be checked that, for every $\lambda \in \mathbb{R}_+^*$, we have $\Phi''(\lambda) \leq 0$. The inequality being strict if and only if $\min_{1 \leq j \leq J} \underline{\lambda}_j < \lambda < \max_{1 \leq j \leq J} \overline{\lambda}_j$, Φ is strictly concave on this interval. In addition, if $\lambda \leq \min_{1 \leq j \leq J} \underline{\lambda}_j$, then

$$\Phi'(\lambda) = \sum_{j=1}^J \frac{n_j}{n} (a_j l_j^{b_j} + c_j) - R_{\max} > 0 \tag{79}$$

and, if $\lambda \geq \max_{1 \leq j \leq J} \overline{\lambda}_j$, then

$$\Phi'(\lambda) = \sum_{j=1}^J \frac{n_j}{n} (a_j l_j^{b_j+1} + c_j) - R_{\max} \leq 0 \tag{80}$$

where the Assumptions given by (64) have been used.

As Φ' is strictly decreasing on $[\min_{1\leq j\leq J}\underline{\lambda}_j, \max_{1\leq j\leq J}\overline{\lambda}_j]$, we deduce that there exists a unique value $\widetilde{\lambda}$ in this interval such that $\Phi'(\widetilde{\lambda})=0$. Thus, $\widetilde{\lambda}$ corresponds to the maximizer of Φ over $\mathbb{R}_+$. From the definitions of the sets in (27), (28) and (29), we get:

$$\forall j \in \mathbb{I}, \qquad \widetilde{\lambda} \leq \underline{\lambda}_j \tag{81}$$

$$\forall j \in \mathbb{J}, \qquad \underline{\lambda}_j < \widetilde{\lambda} \leq \overline{\lambda}_j \tag{82}$$

$$\forall j \in \mathbb{K}, \qquad \widetilde{\lambda} > \overline{\lambda}_j. \tag{83}$$

Finally, it can be deduced from (78) that $\Phi'(\widetilde{\lambda})=0$ implies

$$\sum_{j\in\mathbb{J}} N_j \log_2\left(\frac{\widetilde{\lambda}}{\kappa_j}\right) + \sum_{j\in\mathbb{I}} N_j\gamma_j l_j^{b_j} + \sum_{j\in\mathbb{K}} N_j\gamma_j l_j^{b_j+1} = \sum_{j=1}^{J} n_j c_j - nR_{\max} \tag{84}$$

which yields the expression of $\widetilde{\lambda}$ in (25).

Furthermore, the optimal value $\widetilde{\mathbf{l}}_{\mathbf{b}} = (\widetilde{l}_{1,\mathbf{b}},\ldots,\widetilde{l}_{J,\mathbf{b}})$ of $\mathbf{l}$ is given by the critical point of f. This means that, for every $j\in\{1,\ldots,J\}$, $\widetilde{l}_{j,\mathbf{b}}$ is the maximizer of ψ_j over $[l_j^{b_j}, l_j^{b_j+1}]$ when $l_j^* = -\widetilde{\lambda} n_j a_j/n$. Therefore, we get the optimal values $\widetilde{l}_{j,\mathbf{b}}$ given by (24).

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TABLE I

THE AVERAGE PSNR DIFFERENCES AND THE BITRATE SAVING AT LOW AND HIGH BITRATES. THE GAIN OF THE PROPOSED APPROACH W.R.T THE STATE-OF-THE-ART METHOD [12].

	bitrate saving (in %)		PSNR gain (in dB)	
Images	low	high	low	high
einst	-8.20	-19.42	0.15	0.65
marseille	-6.96	-13.86	0.24	0.68
straw	-16.41	-19.53	0.51	0.77
elaine	-0.45	-16.37	0.02	0.69
castle	-7.78	-9.02	0.40	0.68
cartoon	-5.16	-4.27	0.36	0.63

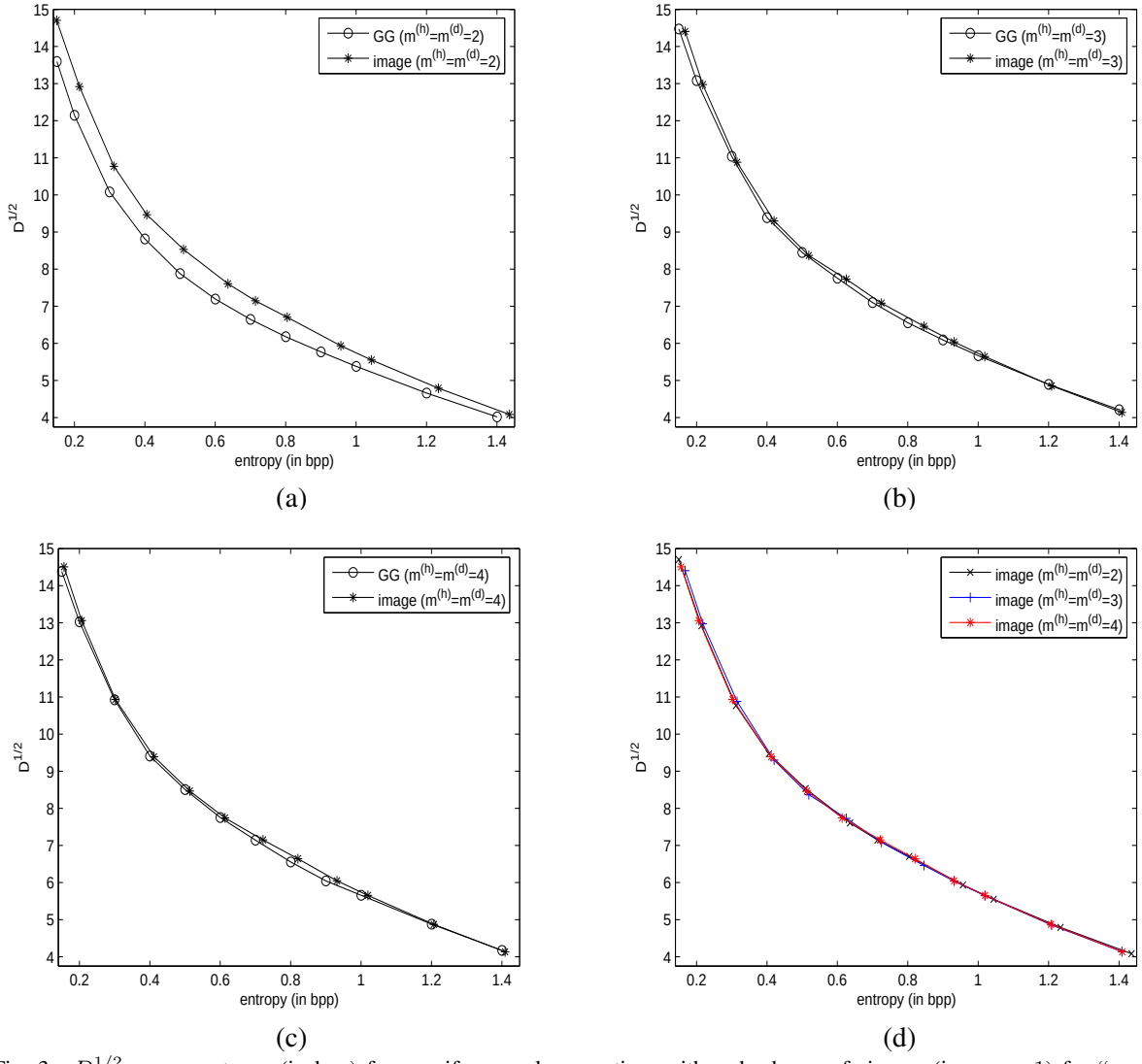


Fig. 3. $D^{1/2}$ versus entropy (in bpp) for a uniform scalar quantizer with a deadzone of size q_j (i.e. $\tau_j = 1$) for "marseille" image: influence of the number of intervals.

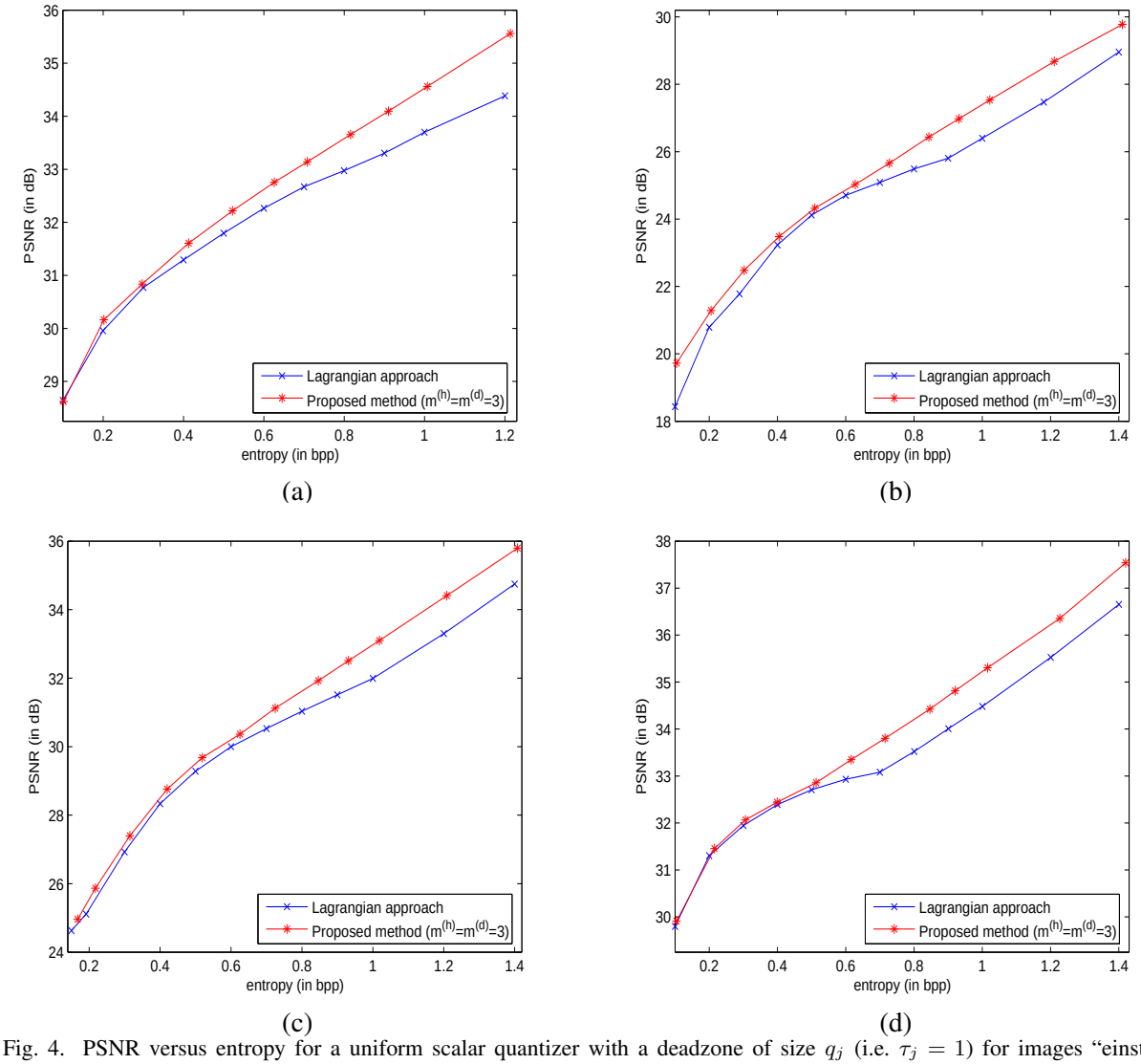
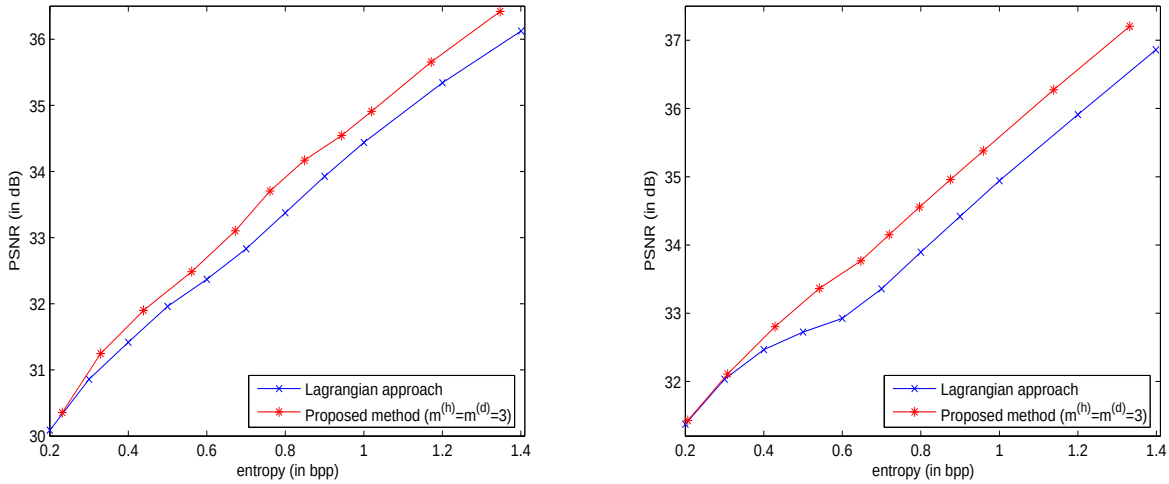
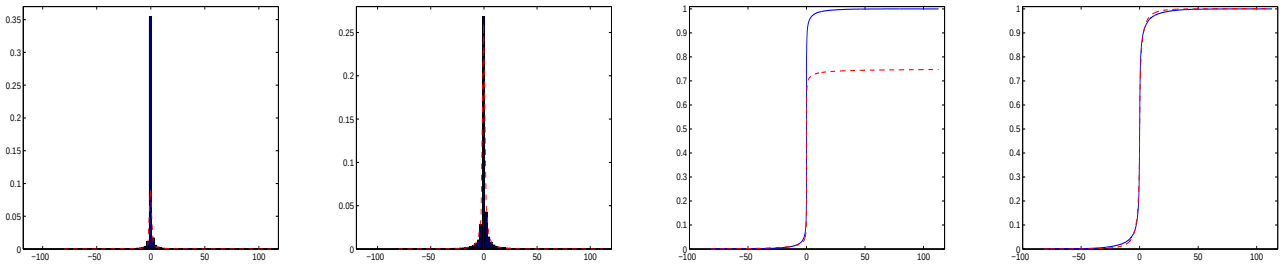


Fig. 4. PSNR versus entropy for a uniform scalar quantizer with a deadzone of size q_j (i.e. $\tau_j = 1$) for images "einst" (a), "straw" (b), "marseille" (c) and "elaine" (d): performance of the proposed approach vs the Lagrangian one.



(a) (b)

Fig. 5. PSNR versus entropy for a uniform scalar quantizer with a deadzone of size $3q_j$ (i.e. $\tau_j = 2$) for images "einst" (left side) and "elaine" (right side): performance of the proposed approach vs the Lagrangian one.



(a) $\beta_j = 0.05$ (b) $\beta_j = 0.34$, $\epsilon_j = 0.40$ (c) KS=0.53 (d) KS=0.06

Fig. 6. Modelling the distribution of the diagonal detail wavelet coefficient of the "cartoon" image using (a) GG model (b) BGG model. The cumulative distribution function using (c) GG model (d) BGG model. The curve plotted in solid (resp. dashed) line is associated with the subband wavelet coefficients (resp. theoretical model).

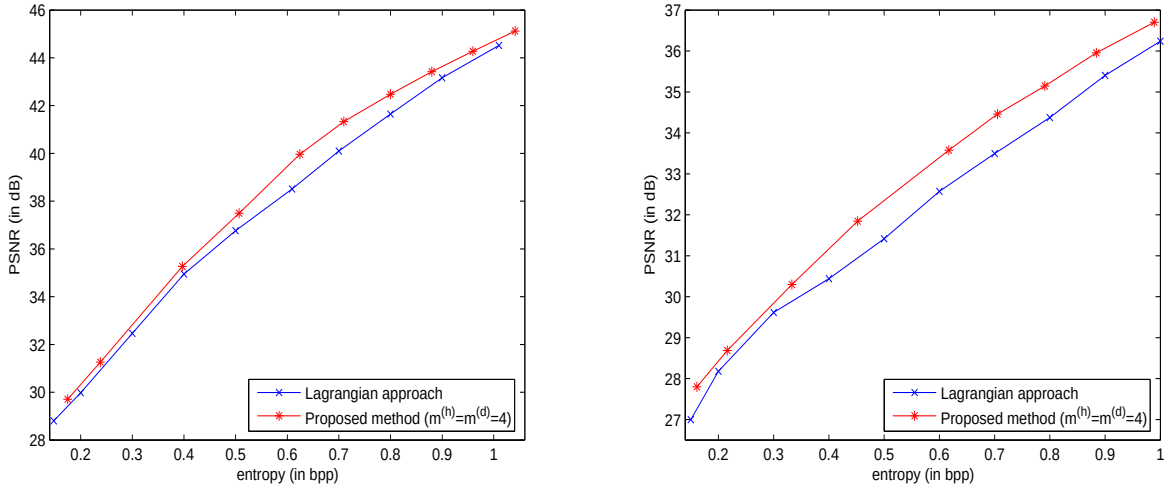


Fig. 7. PSNR versus entropy for a uniform scalar quantizer with a deadzone of size q_j (i.e. $\tau_j = 1$) for images "cartoon" (a) and "castle" (b): performance of the proposed approach vs the Lagrangian one.

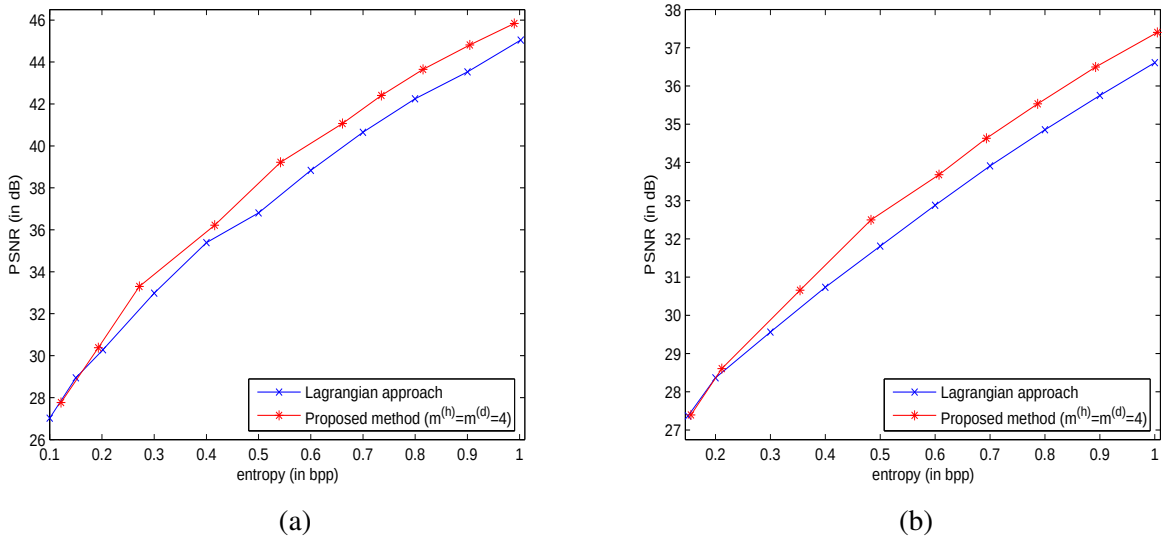


Fig. 8. PSNR versus entropy for a uniform scalar quantizer with a deadzone of size $3q_j$ (i.e. $\tau_j = 2$) for images "cartoon" (a) and "castle" (b): performance of the proposed approach vs the Lagrangian one.