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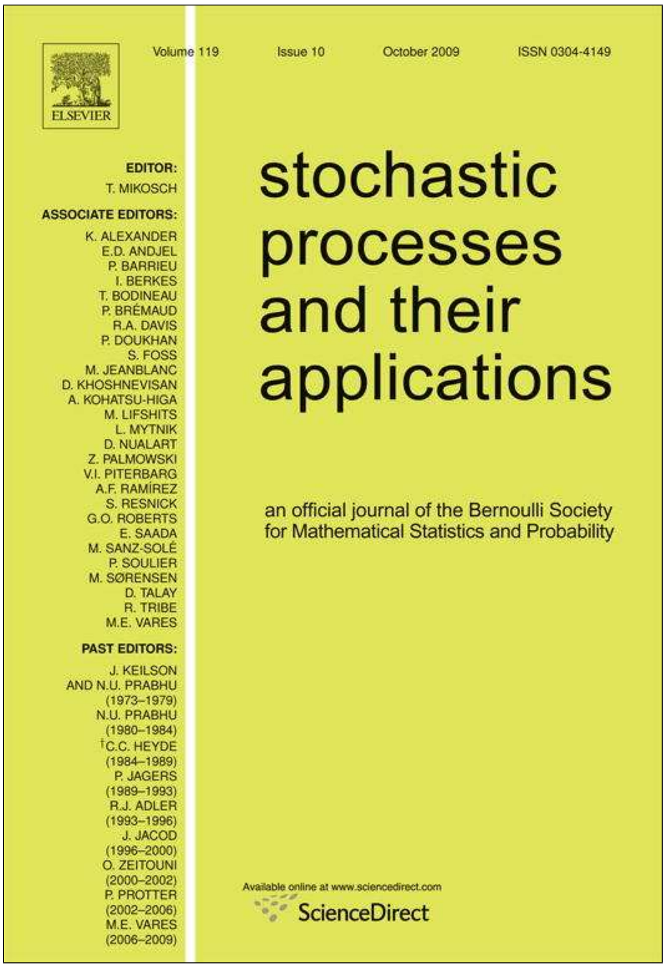
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Optimal stopping with irregular reward functions

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Abstract

We consider optimal stopping problems with finite horizon for one-dimensional diffusions. We assume that the reward function is bounded and Borel-measurable, and we prove that the value function is continuous and can be characterized as the unique solution of a variational inequality in the sense of distributions.

MSC: 60G40; 60J60; 60H10

Keywords: Optimal stopping; One-dimensional diffusions; Irregular reward functions

1. Introduction

In a recent paper with M. Zervos (see [10]), we studied optimal stopping problems with infinite horizon for one-dimensional diffusions. In particular, we proved that, under very general conditions, the value function is the unique solution (in the sense of distributions) of a stationary variational inequality. The purpose of the present paper is to examine optimal stopping problems with finite horizon and bounded Borel-measurable reward functions. We will prove that the value function is continuous and can be characterized as the unique solution (in the sense of distributions) of a suitable variational inequality.

The connection between optimal stopping and variational inequalities goes back to the work of Bensoussan and Lions [3] and Friedman [6]. This approach is very general and applies to multi-dimensional problems, but it requires uniform ellipticity of the diffusion and some regularity of

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the reward function. Note that the techniques of viscosity solutions do not require ellipticity, but generally impose some continuity conditions on the reward function and the coefficients of the diffusion. For our results, we will not need any regularity assumption on the reward function, and we will deal with very general one-dimensional diffusions. On the other hand, our analysis will be limited to one-dimensional situations (cf. Remark 2.3).

The paper is organized as follows. In Section 2, we present our assumptions and the main results. In particular, we give the proper formulation of the variational inequality. In Section 3, we prove the continuity of the value function. In Section 4, we essentially relate the value function to the Snell envelope. Section 5 is devoted to the analytic interpretation of the supermartingale property. The proof that the value function satisfies the variational inequality is given in Section 6. Uniqueness of the solution is proved in Section 7. In the last section, we have gathered a number of auxiliary results, which are classical in somewhat different contexts, but which require some justification under our assumptions. In particular, we derive regularity estimates for the semi-group of one-dimensional diffusions (see Theorem 8.11 and Corollary 8.13), which we have not found in the literature.

2. Assumptions and main results

We consider an open interval $I = (\alpha, \beta)$ (with $-\infty \leq \alpha < \beta \leq +\infty$) and a stochastic differential equation

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t, \qquad X_0 = x \in I, \tag{1}$$

where W is a standard one-dimensional Brownian motion, and $b, \sigma : I \to \mathbb{R}$ are Borel-measurable functions satisfying the following condition.

A1. For all $x \in (\alpha, \beta)$, $\sigma^2(x) > 0$, and $\exists \varepsilon > 0$, $\int_{x-\varepsilon}^{x+\varepsilon} \frac{1+|b(y)|}{\sigma^2(y)}\mathrm{d}y < \infty$.

Under assumption A1, we have existence and uniqueness in law of a weak solution of (1) up to a possible explosion time (cf. [9, Section 5.5C]). In fact, we will also assume that the diffusion is non-explosive. This assumption can be expressed in terms of the so-called scale function $p(x)$ and speed measure $m(\mathrm{d}x)$, defined by

$$p(x) = \int_c^x \exp\left(-2\int_c^y \frac{b(z)}{\sigma^2(z)}\,\mathrm{d}z\right)\mathrm{d}y, \quad \text{for } x \in I, \tag{2}$$

$$m(\mathrm{d}x) = \frac{2}{\sigma^2(x)p'(x)}\,\mathrm{d}x, \tag{3}$$

where c is an arbitrary fixed element of I. The condition for no explosion can now be written as follows, according to Feller's test (see Theorem 5.5.29 in Karatzas and Shreve [9]).

A2. We have $\lim_{x\downarrow\alpha} l(x) = \lim_{x\uparrow\beta} l(x) = \infty$, where

$$l(x) = \int_c^x [p(x) - p(y)]\,m(\mathrm{d}y), \quad \text{for } x \in I.$$

Throughout the paper, assumptions A1 and A2 are in force. A *weak solution* of (1) is defined by a triple $\left[(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}_x), W, X\right]$, where $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}_x)$ is a filtered probability space with the filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions, $W = (W_t)_{t\geq 0}$ is a standard $(\mathcal{F}_t)$-Brownian motion and X is a continuous adapted process satisfying (1). Given such a weak solution, we denote by $(\mathcal{F}_t^0)_{t\geq 0}$ the natural right-continuous filtration of X.

We now introduce an optimal stopping problem with a discounting rate (or interest rate) r. The function $r : I \to \mathbb{R}$ is assumed to be non-negative, Borel-measurable and locally bounded on I. We denote by $\mathcal{T}_t^0$ (resp. $\bar{\mathcal{T}}_t^0$) the set of all stopping times with respect to the filtration $(\mathcal{F}_t^0)_{t\geq 0}$, with values in the interval $[0, t)$ (resp. $[0, t]$). Given a bounded Borel-measurable function f on I, we introduce the functions u_f and v_f defined on $(0, +\infty) \times I$ as follows:

$$u_f(t, x) = \sup_{\tau \in \mathcal{T}_t^0} \mathbb{E}_x \left[e^{-\Lambda_\tau} f(X_\tau) \right], \tag{4}$$

$$v_f(t, x) = \sup_{\tau \in \bar{\mathcal{T}}_t^0} \mathbb{E}_x \left[e^{-\Lambda_\tau} f(X_\tau) \right], \tag{5}$$

where

$$\Lambda_t = \int_0^t r(X_s)\,ds.$$

Note that, due to the fact that we consider stopping times with respect to the natural filtration, the functions u_f and v_f depend only on the law of X, which is uniquely defined under assumptions A1 and A2. On the other hand, let $\mathcal{T}_t$ (resp. $\bar{\mathcal{T}}_t$) be the set of all stopping times with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$, with values in the interval $[0, t)$ (resp. $[0, t]$). If we define by $\bar{u}_f$ (resp. $\bar{v}_f$) the value function where $\mathcal{T}_t^0$ (resp. $\bar{\mathcal{T}}_t^0$) is replaced with $\mathcal{T}_t$ (resp. $\bar{\mathcal{T}}_t$), we have $\bar{u}_f = u_f$ and $\bar{v}_f = v_f$ (see Section 8, Remark 8.7).

We obviously have $u_f \leq v_f$. Our first observation is the following result, the proof of which is quite similar to the one given for infinite horizon problems (see [10], Lemma 7), and is therefore omitted.

Proposition 2.1. *Let $f : I \to \mathbb{R}$ be a bounded Borel-measurable function on I. Denote by $\hat{f}$ the upper semicontinuous envelope of f:*

$$\hat{f}(x) = \limsup_{y \to x} f(y), \quad x \in I.$$

Then we have

$$u_f = u_{\hat{f}}.$$

Our next result concerns the joint continuity of the value function. The following theorem will be proved in Section 3.

Theorem 2.2. *We have $u_f = v_f$ and the function v_f is jointly continuous on $(0, +\infty) \times I$.*

Remark 2.3. The fact that we have a one-dimensional diffusion is essential for the continuity of the value function. Indeed, consider a two-dimensional Brownian motion $(W_t^1, W_t^2)_{t\geq 0}$ and let f be the indicator function of the singleton $\{0\}$. Since Brownian motion starting from $x \neq 0$ will never hit 0 with probability one, we clearly have (with similar notations as above) $u_f = v_f = f$, so that v_f is discontinuous. In fact, crucial to the continuity of the value function is the fact that the diffusion hits any given point close to the initial point with positive probability. Note that the regularization procedure that we develop in Section 5 also depends heavily on the one-dimensional setting.

In order to write the variational inequality satisfied by the value function, we need to introduce the infinitesimal generator $\mathcal{L}_0$ of the diffusion. For a twice continuously differentiable function

u, $\mathcal{L}_0 u$ is defined by

$$\mathcal{L}_0 u(x) = \frac{\sigma^2(x)}{2} u''(x) + b(x)u'(x), \quad x \in I.$$

As should be expected, the variational inequality will involve the operator $-\frac{\partial}{\partial t} + \mathcal{L}$, where the operator $\mathcal{L}$ is defined by

$$\mathcal{L}u(t,x) = \mathcal{L}_0 u(t,x) - r(x)u(t,x), \quad (t,x) \in (0,+\infty) \times I.$$

In fact, in order to be able to apply the operator to possibly non-smooth functions, we will rather consider the operator $\mathcal{A}$, where

$$\mathcal{A} = \frac{2}{\sigma^2 p'} \left(-\frac{\partial}{\partial t} + \mathcal{L} \right).$$

The following proposition is the key to the extension of $\mathcal{A}$ to irregular functions, in the sense of distributions.

Proposition 2.4. *If $u \in C^{1,2}((0,+\infty) \times I)$, for any C^∞ function Φ, with compact support in $(0,+\infty) \times \mathbb{R}$, we have*

$$\iint \mathcal{A}u(t,x)\Phi(t,x)\mathrm{d}t\mathrm{d}x = \iint u(t,x) \left(\frac{\partial \Phi}{\partial t} + \mathcal{L}\Phi \right)(t,x)\, \mathrm{d}t\, m(\mathrm{d}x).$$

Proof. It follows from integration by parts with respect to time and from the definition of the speed measure that

$$-\iint \frac{2}{\sigma^2(x)p'(x)} \frac{\partial u}{\partial t}(t,x)\Phi(t,x)\mathrm{d}t\mathrm{d}x = \iint u(t,x)\frac{\partial \Phi}{\partial t}(t,x)\, \mathrm{d}t\, m(\mathrm{d}x).$$

On the other hand, using the fact that the scale function p satisfies

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{p'} \right) = \frac{2b}{\sigma^2} \frac{1}{p'},$$

we have

$$\mathcal{L}_0 u = \frac{\sigma^2 p'}{2} \left(\frac{1}{p'} \frac{\partial^2 u}{\partial x^2} + \frac{2b}{\sigma^2} \frac{1}{p'} \frac{\partial u}{\partial x} \right) = \frac{\sigma^2 p'}{2} \frac{\partial}{\partial x} \left(\frac{1}{p'} \frac{\partial u}{\partial x} \right).$$

Hence, integrating by parts twice with respect to x,

$$\iint \frac{2}{\sigma^2(x)p'(x)} \mathcal{L}_0 u(t,x)\Phi(t,x)\mathrm{d}t\mathrm{d}x = \iint u(t,x)\mathcal{L}_0\Phi(t,x)\mathrm{d}t m(\mathrm{d}x).$$

The result now follows easily. $\diamond$

In view of Proposition 2.4, it is natural, given a locally bounded measurable function u on $(0,+\infty) \times I$, to define the distribution $\mathcal{A}u$ by setting, for any smooth test function Φ,

$$\langle \mathcal{A}u, \Phi \rangle = \iint u(t,x) \left(\frac{\partial \Phi}{\partial t} + \mathcal{L}\Phi \right)(t,x)\, \mathrm{d}t m(\mathrm{d}x).$$

Remark 2.5. We will also need the distribution $\tilde{\mathcal{A}}u$, defined, for u locally bounded on $(0,T)\times I$ (where T is a fixed positive number) by

$$\langle \tilde{\mathcal{A}}u, \Phi\rangle = \iint u(t,x)\left(-\frac{\partial \Phi}{\partial t} + \mathcal{L}\Phi\right)(t,x)\,\mathrm{d}t m(\mathrm{d}x),$$

for Φ smooth with compact support in $(0,T)\times I$. Note that one can prove, as in Proposition 2.4 that if $u \in C^{1,2}((0,T)\times I)$, $\tilde{\mathcal{A}}u = \frac{2}{\sigma^2 p'}\left(+\frac{\partial}{\partial t} + \mathcal{L}\right)u$.

We can now state our main result. Recall that $\hat{f}$ denotes the upper semicontinuous envelope of f.

Theorem 2.6. *The value function v_f is the only continuous and bounded function on the open set $(0,+\infty)\times I$ satisfying the following conditions.*

1. *On the set $(0,+\infty)\times I$, we have $v \geq f$, and the distribution $\mathcal{A}v$ satisfies $\mathcal{A}v \leq 0$.*
2. *We have $\mathcal{A}v = 0$ on the open set $U := \{(t,x) \in (0,+\infty)\times I \mid v(t,x) > \hat{f}(x)\}$.*
3. *For every $x \in I$, $\lim_{t\downarrow 0} v(t,x) = \hat{f}(x)$.*

3. Continuity of the value function

This section is devoted to the proof of Theorem 2.2. At the end of the section, we also include a proposition concerning the behaviour of the value function for small time (see Proposition 3.4).

The equality $u_f = v_f$ is an easy consequence of the following proposition.

Proposition 3.1. *Let τ be a stopping time with values in $[0,t]$. We have*

$$\mathbb{E}_x\left[\mathrm{e}^{-\Lambda_\tau} f(X_\tau)\right] = \lim_{s\to t, s<t} \mathbb{E}_x\left[\mathrm{e}^{-\Lambda_{\tau\wedge s}} f(X_{\tau\wedge s})\right].$$

Proof. We have

$$\mathbb{E}_x\left[\mathrm{e}^{-\Lambda_{\tau\wedge s}} f(X_{\tau\wedge s})\right] = \mathbb{E}_x\left[\mathrm{e}^{-\Lambda_\tau} f(X_\tau)\mathbf{1}_{\{\tau<s\}}\right] + \mathbb{E}_x\left[\mathrm{e}^{-\Lambda_s} f(X_s)\mathbf{1}_{\{\tau\geq s\}}\right], \tag{6}$$

and

$$\begin{aligned}\mathbb{E}_x\left[\mathrm{e}^{-\Lambda_s} f(X_s)\mathbf{1}_{\{\tau\geq s\}}\right] &= \mathbb{E}_x\left[\mathrm{e}^{-\Lambda_t} f(X_s)\mathbf{1}_{\{\tau=t\}}\right]\\ &\quad + \mathbb{E}_x\left[f(X_s)\left(\mathrm{e}^{-\Lambda_s}\mathbf{1}_{\{\tau\geq s\}} - \mathrm{e}^{-\Lambda_t}\mathbf{1}_{\{\tau=t\}}\right)\right].\end{aligned}$$

By dominated convergence,

$$\lim_{s\to t, s<t} \mathbb{E}_x\left[\mathrm{e}^{-\Lambda_\tau} f(X_\tau)\mathbf{1}_{\{\tau<s\}}\right] = \mathbb{E}_x\left[\mathrm{e}^{-\Lambda_\tau} f(X_\tau)\mathbf{1}_{\{\tau<t\}}\right],$$

and

$$\lim_{s\to t, s<t} \mathbb{E}_x\left[f(X_s)\left(\mathrm{e}^{-\Lambda_s}\mathbf{1}_{\{\tau\geq s\}} - \mathrm{e}^{-\Lambda_t}\mathbf{1}_{\{\tau=t\}}\right) = 0\right].$$

We now want to prove

$$\lim_{s\to t, s<t} \mathbb{E}_x\left|f(X_s) - f(X_t)\right| = 0. \tag{7}$$

This is clearly true if f is continuous. If f is arbitrary, we have

$$\begin{aligned}\mathbb{E}_x\left|f(X_s) - f(X_t)\right| &\leq \mathbb{E}_x\left|f(X_s) - \varphi(X_s)\right| + \mathbb{E}_x\left|\varphi(X_s) - \varphi(X_t)\right|\\ &\quad + \mathbb{E}_x\left|\varphi(X_t) - f(X_t)\right|.\end{aligned}$$

So, in order to prove (7), we need only prove that, given $\varepsilon > 0$, one can find a bounded continuous function φ such that

$$\sup_{t/2 \le s \le t} \mathbb{E}_x |f(X_s) - \varphi(X_s)| \le \varepsilon.$$

From Corollary 8.13 we know that, given $x \in \mathcal{I}$, there exists a constant $C_x > 0$ such that, for all $t > 0$ and $h \ge 0$,

$$P_t h(x) \le C_x \left(1 + \frac{1}{\sqrt{t}}\right) \|h\|_{L^2(m)}. \tag{8}$$

Now assume that $f \in L^2(m)$ (the extension to f bounded on I is straightforward). Given $\varepsilon > 0$, one can find a continuous function φ with compact support such that $\|f - \varphi\|_{L^2(m)} < \varepsilon$, so that (using (8) with $h = |f - \varphi|$)

$$\sup_{t/2 \le s \le t} \mathbb{E}_x |f(X_s) - \varphi(X_s)| \le C_x \left(1 + \frac{\sqrt{2}}{\sqrt{t}}\right) \varepsilon,$$

which completes the proof of (7). $\diamond$

Remark 3.2. The proof of Proposition 3.1 relies on the convergence (in probability) of $f(X_s)$ to $f(X_t)$, when $s \to t$, for f bounded and Borel-measurable. As proved in [4], this is related to the relative weak compactness of the laws of the random variables X_s. The argument we give can be seen as a way of proving this property.

For the proof of the continuity of the value function, we will also need the following lemma.

Lemma 3.3. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P}_x, W, X)$ be a weak solution of (1). For $y \in I$, define*

$$\tau_y = \inf\{t \ge 0 \mid X_t = y\}.$$

We have $\lim_{y\to x} \mathbb{P}_x(\tau_y < \infty) = 1$ and, for all $t > 0$, $\lim_{y\to x} \mathbb{P}_x(\tau_y \ge t) = 0$. We also have $\lim_{y\to x} \mathbb{E}_y(e^{-\Lambda_{\tau_x}}) = 1$ and $\lim_{y\to x} \mathbb{P}_y(\tau_x \ge t) = 0$, for all $t > 0$.

Proof. It is well known that the function $(x, y) \mapsto \mathbb{E}_x(e^{-\tau_y})$ is jointly continuous on $I \times I$ (see for instance [8]). In particular, we have $\lim_{y\to x} \mathbb{E}_x(e^{-\tau_y}) = 1$. We have

$$\mathbb{P}_x(\tau_y \ge t) = \mathbb{P}_x(1 - e^{-\tau_y} \ge 1 - e^{-t}) \le \frac{\mathbb{E}_x(1 - e^{-\tau_y})}{1 - e^{-t}}.$$

Hence, $\lim_{y\to x} \mathbb{P}_x(\tau_y \ge t) = 0$. A similar argument gives $\lim_{y\to x} \mathbb{P}_y(\tau_x \ge t) = 0$. Since $\mathbb{P}_x(\tau_y = \infty) \le \mathbb{P}_x(\tau_y \ge t)$, we have $\lim_{y\to x} \mathbb{P}_x(\tau_y < \infty) = 1$.

On the other hand, we have $\mathbb{E}_y(e^{-\Lambda_{\tau_x}}) \ge \mathbb{E}_y(e^{-\tau_x - \Lambda_{\tau_x}}) = \mathbb{E}_y(e^{-\int_0^{\tau_x}(1+r(X_s))ds})$ and we also have that $(x, y) \mapsto \mathbb{E}_x(e^{-\int_0^{\tau_y}(1+r(X_s))ds})$ is jointly continuous on $I \times I$ (see for instance [8]), so that $\lim_{y\to x} \mathbb{E}_y(e^{-\int_0^{\tau_x}(1+r(X_s))ds}) = 1$. Hence, $\lim_{y\to x} \mathbb{E}_y(e^{-\Lambda_{\tau_x}}) = 1$. $\diamond$

Proof of Theorem 2.2. As mentioned above, the equality $u_f = v_f$ follows easily from Proposition 3.1. On the other hand, since $u_f = u_{\hat{f}} = v_{\hat{f}}$, we also have $v_f = v_{\hat{f}}$. It is known that if f is upper semicontinuous, so is v_f (see [2], Proposition 17 or [5]). It remains to prove that v_f is lower semicontinuous.

Fix $(t, x) \in (0, +\infty) \times I$ and $\tau \in \bar{\mathcal{T}}_t^0$. Since τ is a stopping time of $(\mathcal{F}_t^0)$, we have $\{\tau = 0\} \in \mathcal{F}_0^0$, and we deduce from the zero–one law (cf. Remark 8.4) that $\mathbb{P}_x(\tau = 0) \in \{0, 1\}$.

Suppose $\mathbb{P}_x(\tau = 0) = 1$. We then have $\mathbb{E}_x\left(\mathrm{e}^{-\Lambda_\tau} f(X_\tau)\right) = f(x)$. On the other hand, we have, for any $(s, y) \in (0, +\infty) \times I$, with $M = \sup_{y \in I} |f(y)|$,

$$v_f(s, y) \geq \mathbb{E}_y\left(\mathrm{e}^{-\Lambda_{\tau_x \wedge s}} f(X_{\tau_x \wedge s})\right) \geq \mathbb{E}_y\left(\mathrm{e}^{-\Lambda_{\tau_x \wedge s}} f(X_{\tau_x \wedge s}) \mathbf{1}_{\{\tau_x < s\}}\right) - M\mathbb{P}_y(\tau_x \geq s)$$
$$= f(x)\mathbb{E}_y(\mathrm{e}^{-\Lambda_{\tau_x}} \mathbf{1}_{\{\tau_x < s\}}) - M\mathbb{P}_y(\tau_x \geq s).$$

Hence $v_f(s, y) \geq f(x)\mathbb{E}_y(\mathrm{e}^{-\Lambda_{\tau_x}}) - 2M\mathbb{P}_y(\tau_x \geq s)$. Using Lemma 3.3, we have $\lim_{y \to x} \mathbb{E}_y(\mathrm{e}^{-\Lambda_{\tau_x}}) = 1$ and $\lim_{y \to x} \mathbb{P}_y(\tau_x \geq t/2) = 0$. Hence $f(x) \leq \liminf_{(s,y) \to (t,x)} v_f(s, y)$.

We now assume that $\mathbb{P}_x(\tau = 0) = 0$. From Proposition 3.1, we know that, given $\varepsilon > 0$, there exists $\delta \in (0, t)$ such that, for all $s \in [t - \delta, t]$,

$$\left|\mathbb{E}_x(\mathrm{e}^{-\Lambda_{\tau \wedge s}} f(X_{\tau \wedge s})) - \mathbb{E}_x(\mathrm{e}^{-\Lambda_\tau} f(X_\tau))\right| \leq \varepsilon. \tag{9}$$

Obviously, this inequality is also true for $s \geq t$. Now, we have, for all $(s, y) \in (0, +\infty) \times I$,

$$\mathbb{E}_x\left(\mathrm{e}^{-\Lambda_{\tau \wedge s}} f(X_{\tau \wedge s})\right) = \mathbb{E}_x\left(\mathrm{e}^{-\Lambda_{\tau \wedge s}} f(X_{\tau \wedge s}) \mathbf{1}_{\{\tau_y \leq \tau \wedge s\}}\right)$$
$$+ \mathbb{E}_x\left(\mathrm{e}^{-\Lambda_{\tau \wedge s}} f(X_{\tau \wedge s}) \mathbf{1}_{\{\tau_y > \tau \wedge s\}}\right).$$

We have

$$\left|\mathbb{E}_x\left(\mathrm{e}^{-\Lambda_{\tau \wedge s}} f(X_{\tau \wedge s}) \mathbf{1}_{\{\tau_y > \tau \wedge s\}}\right)\right| \leq M\mathbb{P}_x(\tau_y > \tau \wedge s).$$

On the other hand, with the notation τ^s for $\tau \wedge s$, we have

$$\mathbb{E}_x\left(\mathrm{e}^{-\Lambda_{\tau^s}} f(X_{\tau^s}) \mathbf{1}_{\{\tau_y \leq \tau^s\}}\right) = \mathbb{E}_x\left(\mathrm{e}^{-\Lambda_{\tau_y}} \mathbf{1}_{\{\tau_y \leq \tau^s\}} \mathrm{e}^{-\int_0^{(\tau^s - \tau_y)_+} r(X_\theta)\mathrm{d}\theta} f(X_{\tau_y + (\tau^s - \tau_y)_+})\right)$$
$$\leq \mathbb{E}_x\left(\mathbf{1}_{\{\tau_y < \infty\}} \mathrm{e}^{-\int_0^{(\tau^s - \tau_y)_+} r(X_\theta)\mathrm{d}\theta} f(X_{\tau_y + (\tau^s - \tau_y)_+})\right)$$
$$= \mathbb{P}_x(\tau_y < \infty)\mathbb{E}_x\left(\mathrm{e}^{-\int_0^{(\tau^s - \tau_y)_+} r(X_\theta)\mathrm{d}\theta} f(X_{\tau_y + (\tau^s - \tau_y)_+}) \mid \tau_y < \infty\right).$$

Note that, conditionally on $\{\tau_y < \infty\}$, $\left[(\Omega, \mathcal{F}, (\mathcal{F}_{\tau_y + \theta})_{\theta \geq 0}, \mathbb{P}), (W_{\tau_y + \theta} - W_{\tau_y}), (X_{\tau_y + \theta})\right]$ is a weak solution of the stochastic differential equation with starting point y, and $(\tau^s - \tau_y)_+$ is an $(\mathcal{F}_{\tau_y + \theta})_{\theta \geq 0}$-stopping time. Hence (using Remark 8.7)

$$\mathbb{E}_x\left(\mathrm{e}^{-\Lambda_{\tau^s}} f(X_{\tau^s}) \mathbf{1}_{\{\tau_y \leq \tau^s\}}\right) \leq \mathbb{P}_x(\tau_y < \infty) v_f(s, y).$$

Therefore, we have

$$\mathbb{E}_x\left(\mathrm{e}^{-\Lambda_{\tau \wedge s}} f(X_{\tau \wedge s})\right) \leq \mathbb{P}_x(\tau_y < \infty) v_f(s, y) + M\mathbb{P}_x(\tau_y > \tau \wedge s).$$

Now, take $s \in [t - \delta, +\infty)$. We have, using (9),

$$\mathbb{E}_x\left(\mathrm{e}^{-\Lambda_\tau} f(X_\tau)\right) - \varepsilon \leq \mathbb{E}_x\left(\mathrm{e}^{-\Lambda_{\tau \wedge s}} f(X_{\tau \wedge s})\right)$$
$$\leq \mathbb{P}_x(\tau_y < \infty) v_f(s, y) + M\mathbb{P}_x(\tau_y > \tau \wedge s).$$

It follows from Lemma 3.3 that $\lim_{y\to x}\mathbb{P}_x(\tau_y<\infty)=1$ and $\lim_{(s,y)\to(t,x)}\mathbb{P}_x(\tau_y>\tau\wedge s)=0$. Hence

$$\mathbb{E}_x\left(\mathrm{e}^{-\Lambda_\tau}f(X_\tau)\right)\leq\liminf_{(s,y)\to(t,x)}v_f(s,y).$$

We conclude that v_f is lower semicontinuous. ◇

The following Proposition clarifies the asymptotic behaviour of the value function as time goes to zero.

Proposition 3.4. *We have* $\lim_{t\downarrow 0}v_f(t,x)=\hat{f}(x)$.

Recall that $\hat{f}$ is the upper semicontinuous envelope of f. In view of Proposition 3.4, it is natural to extend the definition of $v_f(t,x)$ at $t=0$ by setting $v_f(0,x)=\hat{f}(x)$.

Proof of Proposition 3.4. Note that $t\mapsto v_f(t,x)$ is clearly non-decreasing, so that the limit exists. Since $v_f=v_{\hat{f}}$, we have $v_f(t,x)\geq\hat{f}(x)$, so that $\lim_{t\to 0}v_f(t,x)\geq\hat{f}(x)$. Now, let $\tau\in\bar{\mathcal{T}}_t^0$. For any $\varepsilon>0$ such that $(x-\varepsilon,x+\varepsilon)\subset I$, we have

$$\mathbb{E}_x\left(\mathrm{e}^{-\Lambda_\tau}f(X_\tau)\right)\leq\sup_{y\in(x-\varepsilon,x+\varepsilon)}f(y)+\sup_I f\,\mathbb{P}_x(\tau_{x-\varepsilon}\wedge\tau_{x+\varepsilon}\leq t).$$

Hence

$$v_f(t,x)\leq\sup_{y\in(x-\varepsilon,x+\varepsilon)}f(y)+\sup_I f\left(\mathbb{P}_x(\tau_{x-\varepsilon}\leq t)+\mathbb{P}_x(\tau_{x+\varepsilon}\leq t)\right).$$

Observe that $\lim_{t\to 0}\mathbb{P}_x(\tau_{x\pm\varepsilon}\leq t)$. Therefore $\lim_{t\to 0}v_f(t,x)\leq\sup_{y\in(x-\varepsilon,x+\varepsilon)}f(y)$, and by making ε go to 0, we get $\lim_{t\downarrow 0}v_f(t,x)\leq\hat{f}(x)$. ◇

4. The value function along the paths

The main result of this section is the following.

Theorem 4.1. *Fix a positive number T and let $f:I\to\mathbb{R}$ be a bounded, non-negative and upper semicontinuous function. For any weak solution $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P}_x,W,X)$ of (1), the process V defined by*

$$V_t=\mathrm{e}^{-\Lambda_t}v_f(T-t,X_t),\quad 0\leq t\leq T,$$

is a supermartingale. Moreover, if $\hat{\tau}=\inf\{t\geq 0\mid v_f(T-t,X_t)=f(X_t)\}$, the process $(V_{t\wedge\hat{\tau}})_{0\leq t\leq T}$ is a martingale.

This result is not surprising. It appears in various forms in the literature (see for instance [5]). However, since, under our assumptions, it does not seem to follow directly from known results, we will give a complete proof.

The proof of Theorem 4.1 will be based on the following lemma.

Lemma 4.2. *Let $f:I\to\mathbb{R}$ be bounded, non-negative and Borel-measurable. For all $T>0$ and for all $\tau\in\bar{\mathcal{T}}_T^0$, we have*

$$\forall x\in I,\quad\mathbb{E}_x\left(\mathrm{e}^{-\Lambda_\tau}v_f(T-\tau,X_\tau)\right)\leq v_f(T,x).$$

Proof. For the proof of this lemma, we will need the strong Markov property. So, we will work with the canonical realization of the process X. More precisely, denote by $(\Omega, \mathcal{F}^0)$ the canonical space, where Ω is the set of all continuous functions on $\mathbb{R}_+$, with values in I, and $\mathcal{F}^0$ is the σ-algebra generated by the finite-dimensional cylinder sets. We endow this space with the right-continuous natural filtration $(\mathcal{F}_t^0)$ of the coordinate mapping process X defined by $X_t(\omega) = \omega(t)$, for $t \geq 0$ and $\omega \in \Omega$. This space supports the family of shift operators $(\theta_t, t \geq 0)$, defined by $\theta_t(\omega) = \omega(t + \cdot)$, for $t \geq 0$ and $\omega \in \Omega$. Given an initial condition $x \in I$, we denote by $\mathbb{P}_x$ the (unique) law of a weak solution of (1). The fact that we have the strong Markov property for the family of probability measures $(\mathbb{P}_x, x \in I)$ on the canonical space follows from the Markov property and the fact that the semi-group preserves continuity (the weak Markov property and the fact that the semi-group preserves continuity are proved in Section 8, and the strong Markov property can be deduced by classical arguments, see [11], chapter III, Section 3).

For $(t, x) \in [0, T] \times I$, set $U(t, x) = v_f(T - t, x)$. Note that, since f is upper semicontinuous, $U(T, x) = f(x)$. Given a finite subset F of the interval I, we denote by $\mathcal{T}_T^F$ the set of all stopping times in $\bar{\mathcal{T}}_T^0$, such that, on the set $\{\tau < T\}$, X_τ takes its values in F. We will first prove that

$$\forall \tau \in \mathcal{T}_T^F, \quad \mathbb{E}_x \left(e^{-\Lambda_\tau} U(\tau, X_\tau) \right) \leq U(0, x). \tag{10}$$

Suppose $F = \{a_1, \ldots, a_n\}$, with $a_1 < \cdots < a_n$, and let $\tau \in \mathcal{T}_T^F$. Note that, since X_T has a density (this is an immediate consequence of Corollary 8.13), we have $\mathbb{P}_x(X_T \in F) = 0$, so that, with probability 1,

$$e^{-\Lambda_\tau} U(\tau, X_\tau) = e^{-\Lambda_T} U(T, X_T) \mathbf{1}_{\{\tau = T\}} + \sum_{i=1}^{n} e^{-\Lambda_\tau} U(\tau, a_i) \mathbf{1}_{\{X_\tau = a_i\}}.$$

Let $\rho = (\rho_0 = 0 < \rho_1 < \cdots < \rho_{m-1} < \rho_m = T)$ be a subdivision of the interval $[0, T]$. For $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$, define

$$A_{ij} = \{\rho_{j-1} \leq \tau < \rho_j\} \cap \{X_\tau = a_i\}.$$

We have (using $U(T, \cdot) = f$)

$$\begin{aligned} e^{-\Lambda_\tau} U(\tau, X_\tau) &= e^{-\Lambda_T} f(X_T) \mathbf{1}_{\{\tau = T\}} + \sum_{i=1}^{n} \sum_{j=1}^{m} e^{-\Lambda_\tau} U(\tau, a_i) \mathbf{1}_{A_{ij}} \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m-1} e^{-\Lambda_\tau} U(\rho_j, a_i) \mathbf{1}_{A_{ij}} + e^{-\Lambda_\tau} U(\tau, X_\tau) \mathbf{1}_{\{\rho_{m-1} \leq \tau < T\}} + e^{-\Lambda_T} f(X_T) \mathbf{1}_{\{\tau = T\}} \\ &\quad + \sum_{i=1}^{n} \sum_{j=1}^{m-1} e^{-\Lambda_\tau} \left(U(\tau, a_i) - U(\rho_j, a_i) \right) \mathbf{1}_{A_{ij}}. \end{aligned} \tag{11}$$

We have

$$U(\rho_j, a_i) = u_f(T - \rho_j, a_i) = \sup_{\tau \in \mathcal{T}_{T-\rho_j}^0} \mathbb{E}_{a_i} \left(e^{-\Lambda \tau} f(X_\tau) \right).$$

Fix $\varepsilon > 0$ and denote by τ_{ij} a stopping time in $\mathcal{T}_{T-\rho_j}^0$, such that

$$U(\rho_j, a_i) \leq \mathbb{E}_{a_i} \left(e^{-\Lambda \tau_{ij}} f(X_{\tau_{ij}}) \right) + \varepsilon.$$

Let

$$\tilde{\tau} = T\mathbf{1}_{\{\rho_{m-1}\le\tau\le T\}} + \mathbf{1}_{\{0\le\tau<\rho_{m-1}\}}\left(\tau + \sum_{i=1}^{n}\sum_{j=1}^{m-1}\mathbf{1}_{A_{ij}}\tau_{ij}\circ\theta_\tau\right).$$

This clearly defines a stopping time with values in $[0, T]$. Therefore

$$\begin{aligned} U(0,x) &\ge \mathbb{E}_x\left(\mathrm{e}^{-\Lambda_{\tilde{\tau}}} f(X_{\tilde{\tau}})\right)\\ &= \mathbb{E}_x\left(\mathrm{e}^{-\Lambda_T} f(X_T)\mathbf{1}_{\{\rho_{m-1}\le\tau\le T\}}\right) + \sum_{i=1}^{n}\sum_{j=1}^{m-1}\mathbb{E}_x\left(\mathbf{1}_{A_{ij}}\mathrm{e}^{-\Lambda_{\tau+\tau_{ij}\circ\theta_\tau}} f(X_{\tau+\tau_{ij}\circ\theta_\tau})\right). \end{aligned}$$

Using the strong Markov property and $A_{ij}\in\mathcal{F}^0_\tau$, we have

$$\begin{aligned} \mathbb{E}_x\left(\mathbf{1}_{A_{ij}}\mathrm{e}^{-\Lambda_{\tau+\tau_{ij}\circ\theta_\tau}} f(X_{\tau+\tau_{ij}\circ\theta_\tau})\right) &= \mathbb{E}_x\left[\mathbf{1}_{A_{ij}}\mathbb{E}\left(\mathrm{e}^{-\Lambda_{\tau+\tau_{ij}\circ\theta_\tau}} f(X_{\tau+\tau_{ij}\circ\theta_\tau}) \mid \mathcal{F}^0_\tau\right)\right]\\ &= \mathbb{E}_x\left[\mathbf{1}_{A_{ij}}\mathrm{e}^{-\Lambda_\tau}\mathbb{E}_{a_i}\left(\mathrm{e}^{-\Lambda_{\tau_{ij}}} f(X_{\tau_{ij}})\right)\right]\\ &\ge \mathbb{E}_x\left(\mathbf{1}_{A_{ij}}\mathrm{e}^{-\Lambda_\tau}\right)\left(U(\rho_j, a_i)-\varepsilon\right). \end{aligned}$$

Hence, using $r\ge 0$,

$$\begin{aligned} U(0,x) &\ge \mathbb{E}_x\left(\mathrm{e}^{-\Lambda_T} f(X_T)\mathbf{1}_{\{\rho_{m-1}\le\tau\le T\}}\right) + \mathbb{E}_x\left(\sum_{i=1}^{n}\sum_{j=1}^{m-1}\mathbf{1}_{A_{ij}}\mathrm{e}^{-\Lambda_\tau}\left(U(\rho_j, a_i)-\varepsilon\right)\right)\\ &\ge \mathbb{E}_x\left(\mathrm{e}^{-\Lambda_T} f(X_T)\mathbf{1}_{\{\rho_{m-1}\le\tau\le T\}}\right) + \mathbb{E}_x\left(\sum_{i=1}^{n}\sum_{j=1}^{m-1}\mathbf{1}_{A_{ij}}\mathrm{e}^{-\Lambda_\tau}U(\rho_j, a_i)\right) - \varepsilon. \end{aligned}$$

It follows from (11) that

$$\begin{aligned} \sum_{i=1}^{n}\sum_{j=1}^{m-1}\mathbf{1}_{A_{ij}}\mathrm{e}^{-\Lambda_\tau}U(\rho_j, a_i) &= \mathrm{e}^{-\Lambda_\tau}U(\tau, X_\tau) - \mathrm{e}^{-\Lambda_\tau}U(\tau, X_\tau)\mathbf{1}_{\{\rho_{m-1}\le\tau\le T\}}\\ &\quad - \sum_{i=1}^{n}\sum_{j=1}^{m-1}\mathrm{e}^{-\Lambda_\tau}\left(U(\tau, a_i) - U(\rho_j, a_i)\right)\mathbf{1}_{A_{ij}}. \end{aligned}$$

Hence

$$\begin{aligned} U(0,x) &\ge \mathbb{E}_x\left(\mathrm{e}^{-\Lambda_\tau}U(\tau, X_\tau)\right) + \mathbb{E}_x\left[\left(\mathrm{e}^{-\Lambda_T} f(X_T) - \mathrm{e}^{-\Lambda_\tau}U(\tau, X_\tau)\right)\mathbf{1}_{\{\rho_{m-1}\le\tau<T\}}\right]\\ &\quad - \mathbb{E}_x\left(\sum_{i=1}^{n}\sum_{j=1}^{m-1}\mathrm{e}^{-\Lambda_\tau}\left(U(\tau, a_i) - U(\rho_j, a_i)\right)\mathbf{1}_{A_{ij}}\right) - \varepsilon. \end{aligned}$$

Let $|\rho| = \max_{1\le j\le m}|\rho_j - \rho_{j-1}|$. By passing to the limit as $|\rho|\to 0$, we get, using the continuity of U on $[0, T)\times I$, $U(0,x)\ge\mathbb{E}_x\left(\mathrm{e}^{-\Lambda_\tau}U(\tau, X_\tau)\right)-\varepsilon$, and, since ε is arbitrary, (10) is proved.

Now, suppose $\tau\in\bar{\mathcal{T}}^0_T$, and denote by $(a_n)_{n\ge1}$ a dense sequence of elements in I. Set $F_n = \{a_1,\dots,a_n\}$ and

$$\tau_n = \inf\{t\ge\tau \mid X_t\in F_n\}\wedge T.$$

We have $\tau_n\in\mathcal{T}^{F_n}$, so that, according to (10), $\mathbb{E}_x\left(\mathrm{e}^{-\Lambda_{\tau_n}}U(\tau_n, X_{\tau_n})\right)\le U(0,x)$. On the other hand, the sequence $(\tau_n)_{n\ge1}$ is non-increasing and $\lim_{n\to\infty}\tau_n = \tau$. Indeed, if we denote the limit by τ_∞, we clearly have $\tau_\infty \geq \tau$, and, if the inequality were strict, X would be constant on the interval $[\tau, \tau_\infty)$, which, with probability one, cannot happen, since, in natural scale, X is a time changed Brownian motion. Since

$$\mathbb{E}_x\left(\mathrm{e}^{-\Lambda_{\tau_n}}U(\tau_n, X_{\tau_n}) - \mathrm{e}^{-\Lambda_\tau}U(\tau, X_\tau)\right)$$
$$= \mathbb{E}_x\left[\left(\mathrm{e}^{-\Lambda_{\tau_n}}U(\tau_n, X_{\tau_n}) - \mathrm{e}^{-\Lambda_\tau}U(\tau, X_\tau)\right)\mathbf{1}_{\{\tau<T\}}\right],$$

we have, by dominated convergence and the continuity of U on $[0, T) \times I$,

$$\lim_{n\to\infty}\mathbb{E}_x\left(\mathrm{e}^{-\Lambda_{\tau_n}}U(\tau_n, X_{\tau_n})\right) = \mathbb{E}_x\left(\mathrm{e}^{-\Lambda_\tau}U(\tau, X_\tau)\right),$$

which completes the proof of the lemma. $\diamond$

Proof of Theorem 4.1. We easily deduce from Lemma 4.2 and the Markov property that V is a supermartingale. Note that, due to the continuity of v_f on $(0, T] \times I$, $\lim_{t\to t_0} V_t = V_{t_0}$ for $t_0 < T$. Introducing the Doob–Meyer decomposition of V, we have, with probability one,

$$V_t = M_t - A_t, \quad 0 \leq t \leq T,$$

where M is a martingale and A is a non-decreasing process with $A_0 = 0$.

Now, let $(\tau_j)_{j\geq 1}$ be a sequence of stopping times in $\mathcal{T}_T^0$, such that $\lim_{j\to\infty}\mathbb{E}_x\left(\mathrm{e}^{-\Lambda_{\tau_j}}f(X_{\tau_j})\right)$ $= v_f(T, x)$. We have $V_{\tau_j} \geq \mathrm{e}^{-\Lambda_{\tau_j}}f(X_{\tau_j})$. Therefore

$$\mathbb{E}_x\left(\mathrm{e}^{-\Lambda_{\tau_j}}f(X_{\tau_j})\right) \leq \mathbb{E}_x\left(V_{\tau_j}\right) = \mathbb{E}_x(M_{\tau_j}) - \mathbb{E}_x(A_{\tau_j}) = \mathbb{E}_x(M_0) - \mathbb{E}_x(A_{\tau_j}).$$

On the other hand, $\mathbb{E}_x(V_0) = \mathbb{E}_x(M_0) = v_f(T, x) = \lim_{j\to\infty}\mathbb{E}_x\left(\mathrm{e}^{-\Lambda_{\tau_j}}f(X_{\tau_j})\right)$. Therefore, we have $\lim_{j\to\infty}\mathbb{E}_x\left(\mathrm{e}^{-\Lambda_{\tau_j}}\left(v_f(T-\tau_j, X_{\tau_j}) - f(X_{\tau_j})\right)\right) = 0$ and $\lim_{j\to\infty}\mathbb{E}_x\left(A_{\tau_j}\right) = 0$. By extracting a subsequence, we can assume that, with probability one, $\lim_{j\to\infty} A_{\tau_j} = 0$ and $\lim_{j\to\infty}\mathrm{e}^{-\Lambda_{\tau_j}}\left(v_f(T-\tau_j, X_{\tau_j}) - f(X_{\tau_j})\right) = 0$, so that $A_{\limsup\tau_j} = 0$ and $\hat{\tau} \leq \liminf\tau_j$. Hence $A_{\hat{\tau}} = 0$ and $V_{t\wedge\hat{\tau}} = M_{t\wedge\hat{\tau}}$ a.s., which proves that $(V_{t\wedge\hat{\tau}})_{0\leq t\leq T}$ is a martingale. $\diamond$

5. Analytic interpretation of the supermartingale property

We first introduce some notations. For $t \geq 0$ and $q > 0$ and for $f : I \to \mathbb{R}$ bounded and Borel-measurable, define the functions $P_t^\Lambda f$ and U_q^Λ by

$$P_t^\Lambda f(x) = \mathbb{E}_x\left(\mathrm{e}^{-\Lambda_t}f(X_t)\right) \quad \text{and} \quad U_q^\Lambda f(x) = \mathbb{E}_x\left(\int_0^\infty \mathrm{e}^{-qs-\Lambda_s}f(X_s)\mathrm{d}s\right).$$

It is easy to prove that

$$P_t^\Lambda U_q^\Lambda f(x) = U_q^\Lambda P_t^\Lambda f(x) = \mathbb{E}_x\left(\int_0^\infty \mathrm{e}^{-qs-\int_0^{t+s}r(X_\theta)\mathrm{d}\theta}f(X_{t+s})\mathrm{d}s\right).$$

Theorem 5.1. *Let $F : (t, x) \mapsto F(t, x)$ be a continuous and bounded function on $[0, T) \times I$ such that, for all $s, t \in [0, T)$ with $0 \leq s \leq t$, $P_{t-s}^\Lambda F(t, \cdot) \leq F(s, \cdot)$. Then, the distribution $\tilde{\mathcal{A}}F$ (defined in Remark 2.5) satisfies $\tilde{\mathcal{A}}F \leq 0$ in the open set $(0, T) \times I$.*

For the proof of Theorem 5.1, we will need to approximate F by more regular functions. Given a time interval $[t_1, t_2]$, with $0 \leq t_1 < t_2$, we will denote by $\mathcal{W}([t_1, t_2] \times I)$ the set of all continuously differentiable functions F on $[t_1, t_2] \times I$ such that for all $t \in [t_1, t_2]$, the partial derivative $F'_x(t, \cdot)$ is absolutely continuous and its derivative $F''_{xx}(t, \cdot)$ satisfies the following condition, for every compact subset K of I:

$$\int_K \sup_{t \in [t_1, t_2]} |F''_{xx}(t, x)| \mathrm{d}x < \infty \quad \text{and} \quad \lim_{\delta \to 0} \int_K \sup_{|t-s| \leq \delta} |F''_{xx}(t, x) - F''_{xx}(s, x)| \mathrm{d}x = 0. \tag{12}$$

For functions in $\mathcal{W}([0, T] \times I)$, we have the following version of Itô's formula.

Proposition 5.2. *If $F \in \mathcal{W}([0, T] \times I)$ and if $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}_x, W, X)$ is a weak solution of (1), we have, with probability one, for $t \in [0, T]$,*

$$F(t, X_t) = F(0, X_0) + \int_0^t F'_t(s, X_s)\mathrm{d}s + \int_0^t F'_x(s, X_s)\mathrm{d}X_s + \frac{1}{2}\int_0^t F''_{xx}(s, X_s)\mathrm{d}\langle X, X\rangle_s.$$

Proof. Let $\Delta = (t_0 = 0 < t_1 < \cdots < t_n = t)$ be a subdivision of the interval $[0, t]$. We have

$$F(t, X_t) - F(0, X_0) = \sum_{i=1}^n F(t_i, X_{t_i}) - F(t_{i-1}, X_{t_i}) + \sum_{i=1}^n F(t_{i-1}, X_{t_i}) - F(t_{i-1}, X_{t_{i-1}}).$$

Note that $F(t_i, X_{t_i}) - F(t_{i-1}, X_{t_i}) = \int_{t_{i-1}}^{t_i} F'_t(s, X_{t_i})\mathrm{d}s$. Applying the generalized Itô formula to the function $F(t_{i-1}, \cdot)$, we have

$$F(t_{i-1}, X_{t_i}) - F(t_{i-1}, X_{t_{i-1}}) = \int_{t_{i-1}}^{t_i} F'_x(t_{i-1}, X_s)\mathrm{d}X_s + \frac{1}{2}\int_{t_{i-1}}^{t_i} F''_{xx}(t_{i-1}, X_s)\mathrm{d}\langle X, X\rangle_s.$$

Hence

$$F(t, X_t) - F(0, X_0) = A_\Delta + B_\Delta + \frac{1}{2}C_\Delta,$$

where

$$A_\Delta = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} F'_t(s, X_{t_i})\mathrm{d}s, \qquad B_\Delta = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} F'_x(t_{i-1}, X_s)\mathrm{d}X_s$$

and

$$C_\Delta = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} F''_{xx}(t_{i-1}, X_s)\mathrm{d}\langle X, X\rangle_s.$$

If we let the mesh size $|\Delta| = \sup_{1 \leq i \leq n} |t_i - t_{i-1}|$ go to zero, we have

$$A_\Delta + B_\Delta \to \int_0^t F'_t(s, X_s)\mathrm{d}s + \int_0^t F'_x(s, X_s)\mathrm{d}X_s$$

in probability. Therefore, it suffices to prove that $C_\Delta \to \int_0^t F''_{xx}(s, X_s)\mathrm{d}\langle X, X\rangle_s$ in probability. We have, using the local time L_t^a of X and the occupation times formula,

$$\begin{aligned}\left|C_\Delta - \int_0^t F''_{xx}(s, X_s)\mathrm{d}\langle X, X\rangle_s\right| &\leq \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left|F''_{xx}(t_{i-1}, X_s) - F''_{xx}(s, X_s)\right| \mathrm{d}\langle X, X\rangle_s \\ &= \int L_t^a \sup_{|\theta-\theta'| \leq |\Delta|} \left|F''_{xx}(\theta, a) - F''_{xx}(\theta', a)\right| \mathrm{d}a.\end{aligned}$$

The local time $a \mapsto L_t^a$ is locally bounded and vanishes outside the compact set $X([0,t])$, so that, using (12), we have $\lim_{|\Delta|\to 0} C_\Delta = 0$ almost surely. ◇

Proposition 5.3. *Let $h : (t,x) \mapsto h(t,x)$ be continuous and bounded on $[t_1,t_2]\times I$ (where $0 \le t_1 < t_2$) with a partial derivative $\partial h/\partial t$ continuous and bounded on $[t_1,t_2]\times I$. Fix a positive number q and, for each $t \in [t_1,t_2]$, let $F(t,\cdot) = U_q^\Lambda h(t,\cdot)$. Then, the function F is in $\mathcal{W}([t_1,t_2]\times I)$.*

Proof. We have

$$F(t,x) = \mathbb{E}_x\left(\int_0^\infty \mathrm{e}^{-qs-\Lambda_s} h(t,X_s)\mathrm{d}s\right),$$

and, by differentiating under the integral,

$$F_t'(t,x) = \mathbb{E}_x\left(\int_0^\infty \mathrm{e}^{-qs-\Lambda_s} h_t'(t,X_s)\mathrm{d}s\right).$$

It is now clear that F_t' is bounded on $[t_1,t_2]\times I$. On the other hand, we also know (cf. for instance [8]) that $F(t,\cdot)$ is the unique bounded solution of the ordinary differential equation $u'' + \frac{2b}{\sigma^2}u' - \frac{2(r+q)}{\sigma^2}u + \frac{2h(t,\cdot)}{\sigma^2} = 0$. This means that $F(t,\cdot)$ is continuously differentiable, that its derivative is absolutely continuous and that we have

$$F_{xx}''(t,x) + \frac{2b(x)}{\sigma^2(x)}F_x'(t,x) - \frac{2(r(x)+q)}{\sigma^2(x)}F(t,x) + \frac{2h(t,x)}{\sigma^2(x)} = 0, \quad \mathrm{d}x\text{-a.e.} \tag{13}$$

We also have the following representation

$$F(t,x) = \phi(x)\int_\alpha^x \psi(y)h(t,y)m(\mathrm{d}y) + \psi(x)\int_x^\beta \phi(y)h(t,y)m(\mathrm{d}y), \quad x \in I,$$

where ϕ and ψ are the fundamental increasing and decreasing solutions of the homogeneous ODE $u'' + \frac{2b}{\sigma^2}u' - \frac{2(r+q)}{\sigma^2}u = 0$. The partial derivative with respect to x is then given by

$$F_x'(t,x) = \phi'(x)\int_\alpha^x \psi(y)h(t,y)m(\mathrm{d}y) + \psi'(x)\int_x^\beta \phi(y)h(t,y)m(\mathrm{d}y), \quad x \in I,$$

and the time derivative by

$$F_t'(t,x) = \phi(x)\int_\alpha^x \psi(y)h_t'(t,y)m(\mathrm{d}y) + \psi(x)\int_x^\beta \phi(y)h_t'(t,y)m(\mathrm{d}y), \quad x \in I.$$

It is now clear that F is C^1 on $[t_1,t_2]\times I$. Moreover, it follows from (13) that

$$F_{xx}''(t,x) = -\frac{2b(x)}{\sigma^2(x)}F_x'(t,x) + \frac{2(r(x)+q)}{\sigma^2(x)}F(t,x) - \frac{2h(t,x)}{\sigma^2(x)}$$

$\mathrm{d}x$-almost everywhere, so that we have $F_{xx}''(t,x) = \sum_{i=1}^3 \varphi_i(x)F_i(t,x)$, where φ_1, φ_2 and φ_3 are locally integrable, and F_1, F_2, F_3 are continuous on $[t_1,t_2]\times I$. The condition (12) is now easy to check and we conclude that $F \in \mathcal{W}([t_1,t_2]\times I)$. ◇

We are now in a position to construct a suitable approximation procedure for a function satisfying the assumptions of Theorem 5.1.

Lemma 5.4. *Under the assumptions of Theorem 5.1, for any decreasing sequence of positive numbers $(\varepsilon_j)_{j\geq 1}$, with $\varepsilon_j \in (0,T)$ and $\lim_{j\to\infty}\varepsilon_j = 0$, one can construct a sequence of functions $(F_j)_{j\geq 1}$ satisfying the following conditions.*

1. *For each $j \geq 1$, $F_j \in \mathcal{W}([\varepsilon_j, T-\varepsilon_j]\times I)$, and $\sup_{(t,x)\in[\varepsilon_j,T-\varepsilon_j]\times I}|F_j(t,x)| \leq \|F\|_\infty$.*
2. *For each $j \geq 1$, and for all $t \in [\varepsilon_j, T-\varepsilon_j]$, we have $(\partial F_j/\partial t)(t,x) + \mathcal{L}F_j(t,x) \leq 0$ $\mathrm{d}x$-almost everywhere and $(\partial F_j/\partial t) + \mathcal{L}F_j$ is bounded on $[\varepsilon_j, T-\varepsilon_j]\times I$.*
3. *For all $(t,x) \in (0,T)\times I$, $\lim_{j\to\infty} F_j(t,x) = F(t,x)$.*

Before proving the lemma we will prove Theorem 5.1.

Proof of Theorem 5.1. Take F_j as in Lemma 5.4 and let $U_j = (\varepsilon_j, T-\varepsilon_j)\times I$. Given a test function with compact support in $(0,T)\times I$, for j large enough, the support of Φ lies in U_j and we easily deduce from the definition of the distribution $\tilde{\mathcal{A}}F$ and the regularity properties of F_j

$$\begin{aligned}\langle \tilde{\mathcal{A}}F_j, \Phi\rangle &= \iint F_j(t,x)\left(-\frac{\partial \Phi}{\partial t} + \mathcal{L}\Phi\right)(t,x)\,\mathrm{d}t\, m(\mathrm{d}x) \\ &= \iint_{U_j} \frac{2}{\sigma^2(x)p'(x)}\left(\frac{\partial F_j}{\partial t} + \mathcal{L}F_j\right)(t,x)\Phi(t,x)\mathrm{d}t\mathrm{d}x \leq 0, \quad \text{if } \Phi \geq 0, \end{aligned} \tag{14}$$

because $(\partial F_j/\partial t) + \mathcal{L}F_j \leq 0$ a.e. on U_j. On the other hand we have, using (14) and the convergence of F_j to F, $\lim_{j\to\infty}\langle \tilde{\mathcal{A}}F_j, \Phi\rangle = \langle \tilde{\mathcal{A}}F, \Phi\rangle$. Hence $\langle \tilde{\mathcal{A}}F, \Phi\rangle \leq 0$ for $\Phi \geq 0$. ◇

Proof of Lemma 5.4. For each positive integer j, let ρ_j be a non-negative C^∞ function with support in the interval $(0,\varepsilon_j)$, such that $\int_0^{\varepsilon_j}\rho_j(s)\mathrm{d}s = 1$. For $t \in [0,T)$ and $\theta \in [0, T-t)$, we have

$$P_\theta^\Lambda F(t+\theta,\cdot) \leq F(t,\cdot).$$

Therefore, if $t \in [\varepsilon_j, T-\varepsilon_j]$ and $s \in (0,\varepsilon_j)$, we have, for all $\theta \in [0,\varepsilon_j]$, $P_\theta^\Lambda F(t-s+\theta,\cdot) \leq F(t-s,\cdot)$, hence

$$\forall t \in [\varepsilon_j, T-\varepsilon_j], \forall \theta \in [0,\varepsilon_j], \quad \int P_\theta^\Lambda F(t-s+\theta,\cdot)\rho_j(s)\mathrm{d}s \leq \int F(t-s,\cdot)\rho_j(s)\mathrm{d}s.$$

Now, for $t \in [\varepsilon_j, T-\varepsilon_j]$ and $x \in I$, let

$$F^{\rho_j}(t,x) = \int F(t-s,x)\rho_j(s)\mathrm{d}s.$$

The function F^{ρ_j} satisfies $P_\theta^\Lambda F^{\rho_j}(t+\theta,\cdot) \leq F^{\rho_j}(t,\cdot)$ for $t \in [\varepsilon_j, T-\varepsilon_j]$ and $\theta \in [0,\varepsilon_j]$. It is continuous on $[\varepsilon_j, T-\varepsilon_j]\times I$, and admits a partial derivative $\partial F^{\rho_j}/\partial t$, which is also continuous on $[\varepsilon_j, T-\varepsilon_j]\times I$. Now, let $(q_j)_{j\geq 1}$ be a sequence of positive real numbers satisfying $\lim_{j\to\infty} q_j = +\infty$. Define

$$F_j(t,\cdot) = q_j U_{q_j}^\Lambda F^{\rho_j}(t,\cdot).$$

It follows from Proposition 5.3 that $F_j \in \mathcal{W}([\varepsilon_j, T-\varepsilon_j]\times I)$. Note that, since $\rho_j \geq 0$ and $\int \rho_j(s)\mathrm{d}s = 1$, we have $|F^{\rho_j}(t,x)| \leq \|F\|_\infty$, and $|F_j(t,x)| \leq \|F\|_\infty$, because qU_q^Λ is a contraction on L^∞.

Since $P_\theta^\Lambda U_q^\Lambda = U_q^\Lambda P_\theta^\Lambda$ and $f \geq 0 \Rightarrow U_q^\Lambda f \geq 0$, we also have $P_\theta^\Lambda F_j(t+\theta,\cdot) \leq F_j(t,\cdot)$, for $t \in [\varepsilon_j, T-\varepsilon_j]$ and $\theta \in [0,\varepsilon_j]$.

Now, fix $t \in [\varepsilon_j, T-\varepsilon_j)$ and note that the function $(\theta, x) \mapsto F_j(t+\theta, x)$ is in the space $\mathcal{W}\left([0, T-\varepsilon_j-t] \times I\right)$, so that, using Proposition 5.2 and the stochastic differential equation satisfied by X, we have, for θ close to 0,

$$\mathrm{e}^{-\Lambda_\theta} F_j(t+\theta, X_\theta) = F_j(t, X_0) + \int_0^\theta \mathrm{e}^{-\Lambda_s} \left(\frac{\partial F_j}{\partial t} + \mathcal{L} F_j\right)(t+s, X_s)\mathrm{d}s + M_t, \tag{15}$$

with $M_t = \int_0^\theta \mathrm{e}^{-\Lambda_s} \frac{\partial F_j}{\partial x}(t+s, X_s)\sigma(X_s)\mathrm{d}W_s$. Observe that $\mathcal{L}F_j = q_j\left(F_j - F^{\rho_j}\right)$, so that $\mathcal{L}F_j$ is bounded. Since F_j and $\partial F_j/\partial t$ are also bounded, the process (M_t) in (15) is a martingale and, by taking expectations, we get

$$P_\theta^\Lambda F_j(t+\theta, x) - F_j(t, x) = \mathbb{E}_x\left(\int_0^\theta \mathrm{e}^{-\Lambda_s}\left(\frac{\partial F_j}{\partial t} + \mathcal{L}F_j\right)(t+s, X_s)\mathrm{d}s\right).$$

Since $P_\theta^\Lambda F_j(t+\theta, \cdot) \le F_j(t, \cdot)$, we deduce that $\mathbb{E}_x\left(\int_0^\theta \mathrm{e}^{-\Lambda_s}\left(\frac{\partial F_j}{\partial t} + \mathcal{L}F_j\right)(t+s, X_s)\mathrm{d}s\right) \le 0$. Note that, by construction, $\partial F_j/\partial t$ is continuous and $\mathcal{L}F_j = q_j\left(F_j - F^{\rho_j}\right)$, so that $\mathcal{L}F_j$ is continuous as well. Now, divide by θ and let $\theta \to 0$ to conclude that $\left(\frac{\partial F_j}{\partial t} + \mathcal{L}F_j\right)(t, x) \le 0$.

We now prove that $\lim_{j\to\infty} F_j(t, x) = F(t, x)$. Fix $(t, x) \in (0, T) \times I$. For j large enough, we have $\varepsilon_j < t < T - \varepsilon_j$ and $F_j - F = q_j U_{q_j}^\Lambda F^{\rho_j} - q_j U_{q_j}^\Lambda F + q_j U_{q_j}^\Lambda F - F$. Now, by an obvious change of variable,

$$(q_j U_{q_j}^\Lambda F^{\rho_j} - q_j U_{q_j}^\Lambda F)(t, x) = \mathbb{E}_x\left(\int_0^\infty \mathrm{e}^{-s-\Lambda_{s/q_j}}\left(F^{\rho_j}(t, X_{s/q_j}) - F(t, X_{s/q_j})\right)\mathrm{d}s\right).$$

Due to the continuity of F, we have, for any compact subset K of I,

$$\lim_{j\to\infty} \sup_{x\in K} |F^{\rho_j}(t, x) - F(t, x)| = 0.$$

Therefore, $\lim_{j\to\infty}\left(F^{\rho_j}(t, X_{s/q_j}) - F(t, X_{s/q_j})\right) = 0$ for all $s \ge 0$ a.s., so that, by dominated convergence $\lim_{j\to\infty}(q_j U_{q_j}^\Lambda F^{\rho_j} - q_j U_{q_j}^\Lambda F)(t, x) = 0$. We also easily have $\lim_{j\to\infty} q_j U_{q_j}^\Lambda F(t, x) = F(t, x)$, which completes the proof. ◇

6. The value function solves the variational inequality

The proof that the value function solves the variational inequality is based on Theorem 5.1 and the following result.

Theorem 6.1. *Let $F : [0, T) \times I \to \mathbb{R}$ satisfy the assumptions of Theorem 5.1, and let U be an open subset of $(0, T) \times I$ such that*

$$\forall (t, x) \in U,\ \forall \theta \in \mathcal{T}_{T-t}^0, \quad \mathbb{E}_x\left(\mathrm{e}^{-\Lambda_{\theta\wedge\tau_U^t}} F\left(t+\theta\wedge\tau_U^t, X_{\theta\wedge\tau_U^t}\right)\right) = F(t, x),$$

where $\tau_U^t = \inf\{s \ge 0 \mid (t+s, X_s) \notin U\}$.

Then, the distribution $\tilde{\mathcal{A}}F$ is null in the open set U.

Before proving Theorem 6.1, we will show that the value function v_f satisfies the three conditions in Theorem 2.6. Without loss of generality, we can assume that f is upper semicontinuous. Note that the proof of the third condition ($\lim_{t\downarrow 0} v_f(t, x) = f(x)$) follows from Proposition 3.4. For the first condition, the inequality $v_f \ge f$ is trivial, so we need to prove that

$\mathcal{A}v_f \leq 0$ on $(0, +\infty) \times I$. It suffices to prove this property on the set $(0, T) \times I$ for all $T > 0$. For $t \in [0, T)$, let $F(t, x) = v_f(T-t, x)$. We know from Theorem 4.1 that, for all $T > 0$, the process V, defined by $V_t = \mathrm{e}^{-\Lambda_t} v_f(T-t, X_t)$, is a supermartingale, so that $P^{\Lambda}_{\theta} v_f(T-\theta, x) \leq v_f(T, x)$, for $\theta \in [0, T]$. Therefore, we have $P^{\Lambda}_{s} v_f(t-s, x) \leq v_f(t, x)$, for all s, t with $0 \leq s \leq t$. Apply this with $T - t$ instead of t and θ instead of s to get $P^{\Lambda}_{\theta} F(t+\theta, \cdot) \leq F(t, \cdot)$, where $F(t, x) = v_f(T-t, x)$, which means that F satisfies the assumptions of Theorem 5.1. Hence $\tilde{\mathcal{A}}F \leq 0$, which gives $\mathcal{A}v \leq 0$.

We also know from Theorem 4.1 that $(V_{t\wedge\hat{\tau}})$ is a martingale, where $\hat{\tau} = \inf\{t \geq 0 \mid v_f(T-t, X_t) = f(X_t)\}$, so that for all $\theta \in \mathcal{T}^0_T$, $\mathbb{E}_x\left(\mathrm{e}^{-\Lambda_{\theta\wedge\hat{\tau}}} v_f(T-\theta\wedge\hat{\tau}, X_{\theta\wedge\hat{\tau}})\right) = v_f(T, x)$. Applying this with $T - t$ instead of T, we obtain that $F(t, x) = v_f(T-t, x)$ satisfies the assumptions of Theorem 6.1, with $U = \{(t, x) \in (0, T) \times I \mid F(t, x) > f(x)\}$ (U is open because f is upper semicontinuous and F is continuous). Therefore, we have $\tilde{\mathcal{A}}F = 0$ on U, so that $\mathcal{A}v_f = 0$ on the set $\{v_f > f\}$, and we have established that v_f satisfies the three conditions of Theorem 2.6.

Proof of Theorem 6.1. Fix (t_0, x_0) in U. We will prove that the distribution $\tilde{\mathcal{A}}F$ vanishes in a neighborhood of (t_0, x_0). Let ε be a positive number such that $(t_0-2\varepsilon, t_0+2\varepsilon) \times (x_0-\varepsilon, x_0+\varepsilon) \subset U$. Let

$$\tau_\varepsilon = \inf\{s \geq 0 \mid X_s \notin (x_0-\varepsilon, x_0+\varepsilon)\}.$$

For $(t, y) \in (t_0-\varepsilon, t_0+\varepsilon) \times (x_0-\varepsilon, x_0+\varepsilon)$, we have $\varepsilon \wedge \tau_\varepsilon \leq \tau^t_U$. Without loss of generality, we assume that $\varepsilon < (T-t_0)/2$, so that $t < t_0 + \varepsilon \Rightarrow T - t > \varepsilon$ and the stopping time $\varepsilon \wedge \tau_\varepsilon$ is in $\mathcal{T}^0_{T-t}$. We then have, according to the assumptions of Theorem 6.1,

$$\forall (t, y) \in (t_0-\varepsilon, t_0+\varepsilon) \times (x_0-\varepsilon, x_0+\varepsilon),$$
$$\mathbb{E}_y\left(\mathrm{e}^{-\Lambda_{\varepsilon\wedge\tau_\varepsilon}} F(t+\varepsilon\wedge\tau_\varepsilon, X_{\varepsilon\wedge\tau_\varepsilon})\right) = F(t, y).$$

Since F satisfies the assumptions of Theorem 5.1, we can take an approximating sequence $(F_j)_{j\geq 1}$ as in Lemma 5.4. For j large enough, we have $(t_0-\varepsilon, t_0+2\varepsilon) \subset (\varepsilon_j, T-\varepsilon_j)$. Let

$$\alpha_j(t, y) = \mathbb{E}_y\left(\mathrm{e}^{-\Lambda_{\varepsilon\wedge\tau_\varepsilon}} F_j(t+\varepsilon\wedge\tau_\varepsilon, X_{\varepsilon\wedge\tau_\varepsilon})\right) - F_j(t, y).$$

Denote $V_\varepsilon = (t_0-\varepsilon, t_0+\varepsilon) \times (x_0-\varepsilon, x_0+\varepsilon)$. For $(t, y) \in V_\varepsilon$, we have

$$\lim_{j\to\infty} \alpha_j(t, y) = \mathbb{E}_y\left(\mathrm{e}^{-\Lambda_{\varepsilon\wedge\tau_\varepsilon}} F(t+\varepsilon\wedge\tau_\varepsilon, X_{\varepsilon\wedge\tau_\varepsilon})\right) - F(t, y) = 0.$$

On the other hand, we have, using Proposition 5.2,

$$\alpha_j(t, y) = \mathbb{E}_y\left(\int_0^{\varepsilon\wedge\tau_\varepsilon} \mathrm{e}^{-\Lambda_s}\left(\frac{\partial F_j}{\partial t} + \mathcal{L}F_j\right)(t+s, X_s)\mathrm{d}s\right).$$

Now, let ψ be the unique continuous function on $[x_0-\varepsilon, x_0+\varepsilon]$ satisfying

$$\psi(x_0-\varepsilon) = \psi(x_0+\varepsilon) = 0 \quad \text{and} \quad \mathcal{L}\psi + 1 = 0 \quad \text{a.e. on } (x_0-\varepsilon, x_0+\varepsilon).$$

We have, for $y \in [x_0-\varepsilon, x_0+\varepsilon]$, $\psi(y) = \mathbb{E}_y\left(\int_0^{\tau_\varepsilon} \mathrm{e}^{-\Lambda_s}\mathrm{d}s\right)$, so that, for all $y \in (x_0-\varepsilon, x_0+\varepsilon)$, $\psi(y) > 0$. By dominated convergence, we have

$$\forall t \in (t_0-\varepsilon, t_0+\varepsilon), \qquad \lim_{j\to+\infty} \int_{x_0-\varepsilon}^{x_0+\varepsilon} m(\mathrm{d}y)\psi(y)\alpha_j(t, y) = 0.$$

On the other hand,

$$\begin{aligned}
&\int_{x_0-\varepsilon}^{x_0+\varepsilon} m(\mathrm{d}y)\psi(y)\alpha_j(t,y)\\
&\quad = \int_{x_0-\varepsilon}^{x_0+\varepsilon} m(\mathrm{d}y)\psi(y)\mathbb{E}_y\left(\int_0^{\varepsilon\wedge\tau_\varepsilon} \mathrm{e}^{-\Lambda_s}\left(\frac{\partial F_j}{\partial t}+\mathcal{L}F_j\right)(t+s,X_s)\mathrm{d}s\right)\\
&\quad = \int_0^{\varepsilon}\mathrm{d}s\int_{x_0-\varepsilon}^{x_0+\varepsilon} m(\mathrm{d}y)\psi(y)\mathbb{E}_y\left(\mathrm{e}^{-\Lambda_s}\left(\frac{\partial F_j}{\partial t}+\mathcal{L}F_j\right)(t+s,X_s)\mathbf{1}_{\{s<\tau_\varepsilon\}}\right).
\end{aligned}$$

The process X is symmetric with respect to the speed measure (cf. Proposition 8.9). Using the symmetry of the killed process at the exit time of $(x_0-\varepsilon, x_0+\varepsilon)$ (cf. [7], Lemmas 4.1.2 and 4.1.3), we deduce

$$\begin{aligned}
&\int_{x_0-\varepsilon}^{x_0+\varepsilon} m(\mathrm{d}y)\psi(y)\alpha_j(t,y)\\
&\quad = \int_0^{\varepsilon}\mathrm{d}s\int_{x_0-\varepsilon}^{x_0+\varepsilon} m(\mathrm{d}y)\left(\frac{\partial F_j}{\partial t}+\mathcal{L}F_j\right)(t+s,y)P_s^{\varepsilon}\psi(y),
\end{aligned}\tag{16}$$

where we use the notation $P_s^{\varepsilon}\psi(y)=\mathbb{E}_y\left(\mathrm{e}^{-\Lambda_s}\psi(X_s)\mathbf{1}_{\{s<\tau_\varepsilon\}}\right)$, for $y\in(x_0-\varepsilon,x_0+\varepsilon)$.

Note that

$$\begin{aligned}
P_s^{\varepsilon}\psi(y) &= \psi(y)+\mathbb{E}_y\left(\int_0^{s\wedge\tau_\varepsilon}\mathrm{e}^{-\Lambda_\theta}\mathcal{L}\psi(X_\theta)\mathrm{d}\theta\right)\\
&= \psi(y)-\mathbb{E}_y\left(\int_0^{s\wedge\tau_\varepsilon}\mathrm{e}^{-\Lambda_\theta}\mathrm{d}\theta\right)\geq \psi(y)-s.
\end{aligned}$$

Let K be a compact subset of $(x_0-\varepsilon,x_0+\varepsilon)$ and $\delta=\inf_{y\in K}\psi(y)$. Note that, since ψ is positive on $(x_0-\varepsilon,x_0+\varepsilon)$, $\delta>0$. For $y\in K$ and $s\in[0,\delta/2]$, we have $P_s^{\varepsilon}\psi(y)\geq\delta/2$. Hence (recall that $(\partial F_j/\partial t)+\mathcal{L}F_j\leq 0$)

$$\begin{aligned}
&\int_0^{\frac{\delta\wedge\varepsilon}{2}}\mathrm{d}s\int_K m(\mathrm{d}y)\left(\frac{\partial F_j}{\partial t}+\mathcal{L}F_j\right)(t+s,y)P_s^{\varepsilon}\psi(y)\\
&\quad\leq\frac{\delta}{2}\int_0^{\frac{\delta\wedge\varepsilon}{2}}\mathrm{d}s\int_K m(\mathrm{d}y)\left(\frac{\partial F_j}{\partial t}+\mathcal{L}F_j\right)(t+s,y).
\end{aligned}$$

Now take $t=t_0-\delta'$, with $\delta'=(\delta\wedge\varepsilon)/4$. Going back to (16), we have

$$-\int_{x_0-\varepsilon}^{x_0+\varepsilon} m(\mathrm{d}y)\psi(y)\alpha_j(t_0-\delta',y)\geq-\frac{\delta}{2}\int_{t_0-\delta'}^{t_0+\delta'}\mathrm{d}s\int_K m(\mathrm{d}y)\left(\frac{\partial F_j}{\partial t}+\mathcal{L}F_j\right)(s,y).$$

Now, if Φ is a smooth test function with support in $[t_0-\delta',t_0+\delta']\times K$, we have

$$\langle\tilde{\mathcal{A}}F_j,\Phi\rangle=\int_{t_0-\delta'}^{t_0+\delta'}\mathrm{d}s\int_K m(\mathrm{d}y)\left(\frac{\partial F_j}{\partial t}+\mathcal{L}F_j\right)(s,y)\Phi(s,y),$$

so that, for $\Phi\geq 0$, we have $\lim_{j\to\infty}\langle\tilde{\mathcal{A}}F_j,\Phi\rangle=0$. Hence $\langle\tilde{\mathcal{A}}F,\Phi\rangle=0$, which proves that $\tilde{\mathcal{A}}F$ is null in a neighborhood of (t_0,x_0). Since (t_0,x_0) is arbitrary in U, we conclude that $\tilde{\mathcal{A}}F=0$ on U. $\diamond$

7. Uniqueness

The proof of uniqueness in Theorem 2.6 will be based on essentially two steps: the first step is to relate the condition $\mathcal{A}u \leq 0$ to the supermartingale property (cf. Theorem 7.1). The second step is to relate the condition $\mathcal{A}u = 0$ to the martingale property: this will be done in Theorem 7.4.

Theorem 7.1. *Suppose $F : [0, T) \times I \to \mathbb{R}$ is bounded and continuous and satisfies $\tilde{\mathcal{A}}F \leq 0$ on $(0, T) \times I$. Then we have, for all $s \in [0, T)$ and $t \in [0, T - s)$,*

$$\mathbb{E}_x \left(\mathrm{e}^{-\Lambda_t} F(s + t, X_t) \right) \leq F(s, x).$$

Given an open subset $\mathcal{O}$ of $\mathbb{R}$ or $\mathbb{R}^2$, we denote by $\mathcal{D}(\mathcal{O})$ the set of all C^∞ functions with compact support in $\mathcal{O}$, and by $\mathcal{D}_+(\mathcal{O})$ the set of all non-negative functions in $\mathcal{D}(\mathcal{O})$.

Lemma 7.2. *Let J be an open subinterval of I and μ a Radon measure on J. A continuous function $F : J \to \mathbb{R}$ satisfies the equation*

$$\frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{1}{p'} \frac{\mathrm{d}F}{\mathrm{d}x} \right) = \mu \tag{17}$$

in the sense of distributions if and only if F has the following form

$$F(x) = \int_d^x p'(y) M(y) \mathrm{d}y + k p(x) + l, \quad x \in J,$$

for some $d \in J$ and constants $k, l \in \mathbb{R}$, where $M(y) = \mu((d, y])$ for $y \geq d$ and $M(y) = -\mu((y, d])$ for $y < d$.

Proof. First note that the meaning of (17) in the sense of distributions is that for all $\psi \in \mathcal{D}(J)$, $\int_J F(x) \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{1}{p'} \frac{\mathrm{d}\psi}{\mathrm{d}x} \right)(x) \mathrm{d}x = \int_J \psi(x) \mu(\mathrm{d}x)$.

We will first prove that the function $F_0 : J \to \mathbb{R}$, defined by

$$F_0(x) = \int_d^x p'(y) M(y) \mathrm{d}y, \quad x \in J,$$

is a solution of (17). Note that

$$F_0(x) = \int_d^x p'(y) \mu((d, y]) \mathrm{d}y \quad \text{if } x \geq d \quad \text{and}$$

$$F_0(x) = \int_x^d p'(y) \mu((y, d]) \mathrm{d}y \quad \text{if } x < d,$$

so that

$$F_0(x) = \int_J \int_J \left(\mathbf{1}_{\{d < z \leq y \leq x\}} + \mathbf{1}_{\{x \leq y < z \leq d\}} \right) p'(y) \mathrm{d}y \mu(\mathrm{d}z).$$

From this expression, we easily derive that, for $\psi \in \mathcal{D}(J)$,

$$\int_J F_0(x) \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{1}{p'} \frac{\mathrm{d}\psi}{\mathrm{d}x} \right)(x) \mathrm{d}x = \int_J \psi(z) \mu(\mathrm{d}z),$$

which proves that F_0 solves (17). To complete the proof of the lemma, it suffices to prove that a continuous function F satisfies the homogeneous equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{p'}\frac{\mathrm{d}F}{\mathrm{d}x}\right)=0 \tag{18}$$

if and only if $F = kp + l$ for some constants k and l. It is clear that functions of that form satisfy (18). Conversely, suppose F satisfies (18). Note that we cannot directly state that F'/p' is constant, because the definition of the distribution F'/p' is not clear if p' is not C^∞. So, we have to start from the fact that for all $\psi \in \mathcal{D}(J)$,

$$\int_J F(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{p'}\frac{\mathrm{d}\psi}{\mathrm{d}x}\right)(x)\mathrm{d}x = 0. \tag{19}$$

Fix an open relatively compact subinterval $J_1 = (\alpha_1, \beta_1)$ of J and a function $u \in \mathcal{D}(J)$ such that $u = 1$ on $K = [\alpha_1, \beta_1]$. Since the second derivative of p in the sense of distributions is locally integrable, we can construct a sequence $(p_n)_{n\geq 1}$ of C^∞ functions on J such that (p_n) (resp. (p_n')) converges uniformly to p (resp. p') on K and $\lim_{n\to\infty}\int_K |p_n''(x) - p''(x)|\mathrm{d}x = 0$.

Now, given $g \in \mathcal{D}(J_1)$, let $\psi_n(x) = u(x)\int_{\alpha_1}^x p_n'(y)g(y)\mathrm{d}y$. We have $\psi_n \in \mathcal{D}(J)$. Therefore, $\int_J F(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{p'}\frac{\mathrm{d}\psi_n}{\mathrm{d}x}\right)(x)\mathrm{d}x = 0$. We have, using the fact that $u = 1$ in a neighborhood of the support of g,

$$\begin{aligned}\psi_n'(x) &= u'(x)\int_{\alpha_1}^x p_n'(y)g(y)\mathrm{d}y + u(x)p_n'(x)g(x)\\ &= u'(x)\int_{\alpha_1}^x p_n'(y)g(y)\mathrm{d}y + p_n'(x)g(x).\end{aligned}$$

Note that, for $x \leq \beta_1$, $u'(x)\int_{\alpha_1}^x p_n'(y)g(y)\mathrm{d}y = 0$, and, for $x > \beta_1$, $u'(x)\int_{\alpha_1}^x p_n'(y)g(y)\mathrm{d}y = u'(x)\int_J p_n'(y)g(y)\mathrm{d}y$. Hence, introducing a function $\bar{u} \in \mathcal{D}(J)$ such that $\bar{u}(x) = u'(x)$ for $x > \beta_1$ and $\bar{u}(x) = 0$ for $x \leq \beta_1$,

$$\psi_n'(x) = \bar{u}(x)\int_J p_n'(y)g(y)\mathrm{d}y + p_n'(x)g(x),$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\psi_n'}{p'}\right)(x) = {}& \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\bar{u}}{p'}\right)(x)\int_J p_n'(y)g(y)\mathrm{d}y\\ &+\frac{p_n'(x)}{p'(x)}g'(x) + \frac{p_n''p' - p_n'p''}{p'^2}(x)g(x).\end{aligned}$$

We deduce from $\int_J F(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{p'}\frac{\mathrm{d}\psi_n}{\mathrm{d}x}\right)(x)\mathrm{d}x = 0$ that

$$k\int_J p_n'(y)g(y)\mathrm{d}y + \int_J F(x)\left(\frac{p_n'(x)}{p'(x)}g'(x) + \frac{p_n''p' - p_n'p''}{p'^2}(x)g(x)\right)(x)\mathrm{d}x = 0,$$

where $k = \int_J F(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\bar{u}}{p'}\right)(x)\mathrm{d}x$. By taking the limit as $n \to \infty$, we derive, for all $g \in \mathcal{D}(J_1)$,

$$k\int_J p'(y)g(y)\mathrm{d}y + \int_J F(x)g'(x)\mathrm{d}x = 0,$$

which means that $F' = kp'$ (in the sense of distributions) on J_1. Since J_1 is an arbitrary relatively compact open subinterval of the interval J, we have $F' = kp'$ on J, so that $F - kp$ is constant. ⋄

Remark 7.3. Take $J = I$ in Lemma 7.2 and suppose F solves (17). From the representation of F, we easily deduce that the derivative of F is given by $F'(x) = p'M + kp'$, so that the second derivative of F in the sense of distributions is a Radon measure (which means that F is the difference of two convex functions on I). This measure is defined by

$$F''(\mathrm{d}a) = p''(a)M(a)\mathrm{d}a + p'(a)\mu(\mathrm{d}a) + kp''(a)\mathrm{d}a,$$

so that, using $p'' = -2(b/\sigma^2)p'$, $F''(\mathrm{d}a) = p'(a)\mu(\mathrm{d}a) - 2(b/\sigma^2)(a)F'(a)\mathrm{d}a$. Given any weak solution X of (1), we can apply the Itô–Tanaka formula and write

$$F(X_t) = F(X_0) + \int_0^t F'(X_s)\sigma(X_s)\mathrm{d}W_s + \int_0^t b(X_s)F'(X_s)\mathrm{d}s + \frac{1}{2}\int L_a^t F''(\mathrm{d}a).$$

Using the occupation times formula, we have

$$\int_0^t b(X_s)F'(X_s)\mathrm{d}s = \int L_a^t \frac{b(a)}{\sigma^2(a)} F'(a)\mathrm{d}a,$$

where L is the local time of X. Therefore

$$\begin{aligned} F(X_t) &= F(X_0) + \int_0^t F'(X_s)\sigma(X_s)\mathrm{d}W_s + \frac{1}{2}\int L_a^t \left(F''(\mathrm{d}a) + 2\frac{b(a)}{\sigma^2(a)}F'(a)\mathrm{d}a\right) \\ &= F(X_0) + \int_0^t F'(X_s)\sigma(X_s)\mathrm{d}W_s + \frac{1}{2}\int L_a^t p'(a)\mu(\mathrm{d}a). \end{aligned}$$

Proof of Theorem 7.1. We first regularize F with respect to time. Fix $\varepsilon \in (0, T)$ and $\rho \in \mathcal{D}((0, \varepsilon))$, with $\rho \geq 0$ and $\int_0^\varepsilon \rho(s)\mathrm{d}s = 1$. For $t \in (\varepsilon, T)$ and $x \in I$, let

$$F^\rho(t, x) = \int_0^T \rho(t - s)F(s, x)\mathrm{d}s.$$

Note that F^ρ is continuous and bounded on $(\varepsilon, T) \times I$ and its time derivative (given by $(\partial F^\rho/\partial t)(t, x) = \int_0^T \rho'(t - s)F(s, x)\mathrm{d}s$) is also continuous and bounded on $(\varepsilon, T) \times I$. We easily deduce from $\rho \geq 0$ and $\tilde{\mathcal{A}}F \leq 0$ on $(0, T) \times I$ that $\tilde{\mathcal{A}}F^\rho \leq 0$ on $(\varepsilon, T) \times I$. In particular, if $\phi \in \mathcal{D}_+((\varepsilon, T))$ and $\psi \in \mathcal{D}_+(I)$, we have

$$-\int \mathrm{d}t \int m(\mathrm{d}x)F^\rho(t, x)\phi'(t)\psi(x) + \int \mathrm{d}t \int m(\mathrm{d}x)F^\rho(t, x)\phi(t)\mathcal{L}\psi(x) \leq 0.$$

An integration by parts with respect to time gives

$$-\int \mathrm{d}t \int m(\mathrm{d}x)F^\rho(t, x)\phi'(t)\psi(x) = \int \mathrm{d}t \int m(\mathrm{d}x)\frac{\partial F^\rho}{\partial t}(t, x)\phi(t)\psi(x).$$

Using the continuity of F^ρ and $\partial F^\rho/\partial t$, we deduce that for all $t \in (0, T - \varepsilon)$ and for all $\psi \in \mathcal{D}_+(I)$,

$$\int m(\mathrm{d}x)\frac{\partial F^\rho}{\partial t}(t, x)\psi(x) + \int m(\mathrm{d}x)F^\rho(t, x)\mathcal{L}\psi(x) \leq 0,$$

or, equivalently,

$$\int m(\mathrm{d}x)\left(\frac{\partial F^{\rho}}{\partial t}-rF^{\rho}\right)(t,x)\psi(x)+\int \mathrm{d}x\, F^{\rho}(t,x)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\psi'}{p'}\right)(x)\leq 0. \tag{20}$$

This means that, for each $t\in(0,T-\varepsilon)$, the distribution $\left(\frac{\partial F^{\rho}}{\partial t}-rF^{\rho}\right)(t,\cdot)\mathrm{d}m+\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{(d/\mathrm{d}x)F(t,\cdot)}{p'}\right)$ is a non-positive Radon measure on I. Using Lemma 7.2 and Remark 7.3, we have, for each $t\in(0,T-\varepsilon)$, and for $s\geq 0$,

$$\begin{aligned}F^{\rho}(t,X_s) = {} & F^{\rho}(t,X_0)+\int_0^s\frac{\partial F^{\rho}}{\partial x}(t,X_\theta)\sigma(X_\theta)\mathrm{d}W_\theta\\ & +\frac{1}{2}\int L_a^s p'(a)\frac{\partial}{\partial x}\left(\frac{(\partial F^{\rho}/\partial x)}{p'}\right)(t,da).\end{aligned}$$

It follows from (20) that

$$\left(\frac{\partial}{\partial x}\frac{(\partial F^{\rho}/\partial x)}{p'}\right)(t,da)\leq\frac{2}{\sigma^2(a)p'(a)}\left(r(a)F^{\rho}(t,a)-\frac{\partial F^{\rho}}{\partial t}(t,a)\right)\mathrm{d}a.$$

Therefore, for any two s_1,s_2, with $0\leq s_1\leq s_2$,

$$\begin{aligned}F^{\rho}(t,X_{s_2})-F^{\rho}(t,X_{s_1})\leq{} & \int_{s_1}^{s_2}\frac{\partial F^{\rho}}{\partial x}(t,X_\theta)\sigma(X_\theta)\mathrm{d}W_\theta\\ & +\int_{s_1}^{s_2}\left(r(X_s)F^{\rho}(t,X_s)-\frac{\partial F^{\rho}}{\partial t}(t,X_s)\right)\mathrm{d}s.\end{aligned}\tag{21}$$

Note that this inequality is valid for all $t\in(0,T-\varepsilon)$. We are now in a position to prove that if $s\in(0,T-\varepsilon)$ and $t\in[0,T-\varepsilon-s)$,

$$\mathbb{E}_x\left(\mathrm{e}^{-\Lambda_t}F(s+t,X_t)\right)\leq F(s,x). \tag{22}$$

Denote by τ a stopping time such that the random variable $\int_0^\tau r(X_\theta)\mathrm{d}\theta$ is bounded. We have, for $s\in(0,T-\varepsilon)$ and $t\in[0,T-\varepsilon-s)$

$$\mathrm{e}^{-\Lambda_{t\wedge\tau}}F^{\rho}(s+t\wedge\tau,X_{t\wedge\tau})-F^{\rho}(s,X_0)=\sum_{i=1}^{n}U_i-U_{i-1},$$

where $U_i=\mathrm{e}^{-\Lambda_{\tau_i^n}}F^{\rho}(s+\tau_i^n,X_{\tau_i^n})$ and $\tau_i^n=\tau\wedge(it/n)$. Note that

$$U_i-U_{i-1}=\mathrm{e}^{-\Lambda_{\tau_i^n}}F^{\rho}(s+\tau_i^n,X_{\tau_i^n})-\mathrm{e}^{-\Lambda_{\tau_{i-1}^n}}F^{\rho}(s+\tau_{i-1}^n,X_{\tau_i^n})+\mathrm{e}^{-\Lambda_{\tau_{i-1}^n}}\Delta_i,$$

where

$$\Delta_i=F^{\rho}(s+\tau_{i-1}^n,X_{\tau_i^n})-F^{\rho}(s+\tau_{i-1}^n,X_{\tau_{i-1}^n}).$$

Hence

$$U_i-U_{i-1}=\int_{\tau_{i-1}^n}^{\tau_i^n}\mathrm{e}^{-\Lambda_\theta}\left(\frac{\partial F^{\rho}}{\partial t}(s+\theta,X_{\tau_i^n})-r(X_\theta)F^{\rho}(s+\theta,X_{\tau_i^n})\right)\mathrm{d}\theta+\mathrm{e}^{-\Lambda_{\tau_{i-1}^n}}\Delta_i.$$

It follows from (21) (applied with $t = s + \tau_{i-1}^n$, $s_1 = \tau_{i-1}^n$, and $s_2 = \tau_i^n$), that

$$\begin{aligned}\Delta_i \leq &\int_{\tau_{i-1}^n}^{\tau_i^n} \frac{\partial F^\rho}{\partial x}(s + \tau_{i-1}^n, X_\theta)\sigma(X_\theta)\mathrm{d}W_\theta \\ &+ \int_{\tau_{i-1}^n}^{\tau_i^n} \left(r(X_\theta)F^\rho(s+\tau_{i-1}^n, X_\theta) - \frac{\partial F^\rho}{\partial t}(s+\tau_{i-1}^n, X_\theta)\right)\mathrm{d}\theta.\end{aligned}$$

Using the fact that F^ρ, $\frac{\partial F^\rho}{\partial t}$ and the random variable $\int_0^\tau r(X_\theta)\mathrm{d}\theta$ are bounded, we easily derive from this inequality that

$$\mathbb{E}\left(\Delta_i \mid \mathcal{F}_{\tau_{i-1}^n}\right) \leq \mathbb{E}\left(\int_{\tau_{i-1}^n}^{\tau_i^n}\left(r(X_\theta)F^\rho(s+\tau_{i-1}^n, X_\theta) - \frac{\partial F^\rho}{\partial t}(s+\tau_{i-1}^n, X_\theta)\right)\mathrm{d}\theta \mid \mathcal{F}_{\tau_{i-1}^n}\right).$$

Introduce the function $G : [0, T-\varepsilon) \times I \times I \to \mathbb{R}$ defined by

$$G(\theta, x, y) = \frac{\partial F^\rho}{\partial t}(\theta, x) - r(y)F^\rho(\theta, x).$$

We now have, by conditioning with respect to $\mathcal{F}_{\tau_{i-1}^n}$,

$$\begin{aligned}\mathbb{E}_x(U_i - U_{i-1}) &= \mathbb{E}_x\left(\int_{\tau_{i-1}^n}^{\tau_i^n} \mathrm{e}^{-\Lambda_\theta} G(s+\theta, X_{\tau_i^n}, X_\theta)\mathrm{d}\theta\right) + \mathbb{E}_x\left(\mathrm{e}^{-\Lambda_{\tau_{i-1}^n}}\Delta_i\right) \\ &\leq \mathbb{E}_x\left(\int_{\tau_{i-1}^n}^{\tau_i^n} \mathrm{e}^{-\Lambda_\theta} G(s+\theta, X_{\tau_i^n}, X_\theta)\mathrm{d}\theta\right) - \mathbb{E}_x\left(\mathrm{e}^{-\Lambda_{\tau_{i-1}^n}}\int_{\tau_{i-1}^n}^{\tau_i^n} G(s+\tau_{i-1}^n, X_\theta, X_\theta)\mathrm{d}\theta\right) \\ &= \mathbb{E}_x\left(\int_{\tau_{i-1}^n}^{\tau_i^n}\left(\mathrm{e}^{-\Lambda_\theta} G(s+\theta, X_{\tau_i^n}, X_\theta) - \mathrm{e}^{-\Lambda_{\tau_{i-1}^n}} G(s+\tau_{i-1}^n, X_\theta, X_\theta)\right)\mathrm{d}\theta\right).\end{aligned}$$

It follows that

$$\begin{aligned}&\mathbb{E}_x\left(\mathrm{e}^{-\Lambda_{t\wedge\tau}} F^\rho(s+t\wedge\tau, X_{t\wedge\tau}) - F^\rho(s, X_0)\right) \\ &\quad\leq \mathbb{E}_x\left(\sum_{i=1}^n \int_{\tau_{i-1}^n}^{\tau_i^n}\left(\mathrm{e}^{-\Lambda_\theta} G(s+\theta, X_{\tau_i^n}, X_\theta) - \mathrm{e}^{-\Lambda_{\tau_{i-1}^n}} G(s+\tau_{i-1}^n, X_\theta, X_\theta)\right)\mathrm{d}\theta\right).\end{aligned}$$

Due to the continuity of F^ρ, $\partial F^\rho/\partial t$ and to the boundedness of the random variable $\int_0^\tau r(X_\theta)\mathrm{d}\theta$, the right-hand side clearly goes to 0 as $n \to \infty$. Hence

$$\mathbb{E}_x\left(\mathrm{e}^{-\Lambda_{t\wedge\tau}} F^\rho(s+t\wedge\tau, X_{t\wedge\tau})\right) \leq F(s, x).$$

Since this is true for any stopping time such that $\int_0^\tau r(X_\theta)\mathrm{d}\theta$ is bounded and r is locally bounded, we get (22). By taking a sequence $(\varepsilon_j)_{j\geq 1}$ of positive numbers such that $\lim_{j\to\infty}\varepsilon_j$ and $\rho_j \in \mathcal{D}_+((0, \varepsilon_j))$ such that $\int_0^{\varepsilon_j} \rho_j(s)\mathrm{d}s = 1$, we have

$$\mathbb{E}_x\left(\mathrm{e}^{-\Lambda_{t\wedge\tau}} F^{\rho_j}(s+t\wedge\tau, X_{t\wedge\tau})\right) \leq F^{\rho_j}(s, x)$$

and $\lim_{j\to\infty} F^{\rho_j} = F$ on $(0, T) \times I$ and we obtain, in the limit as $j \to \infty$, the inequality of Theorem 7.1 for $s \in (0, T)$, $t \in (0, T-s)$, and also for $s \in [0, T)$ by continuity. ◇

Theorem 7.4. *Suppose $F : [0, T) \times I \to \mathbb{R}$ is bounded and continuous and satisfies $\tilde{\mathcal{A}}F = 0$ on an open subset U of $(0, T) \times I$. Then, for all $(t, x) \in U$ and for any stopping time θ with values in $[0, T - t)$, we have*

$$\mathbb{E}_x\left(\mathrm{e}^{-\Lambda_{\theta \wedge \tau_U^t}} F(t + \theta \wedge \tau_U^t, X_{\theta \wedge \tau_U^t})\right) = F(t, x),$$

where $\tau_U^t = \inf\{s \geq 0 \mid (t + s, X_s) \notin U\}$.

Proof. We introduce a distance d on $[0, T] \times \mathbb{R}$ by setting $d(\xi_1, \xi_2) = \max\{|t_1 - t_2|, |x_1 - x_2|\}$, where $\xi_i = (t_i, x_i)$. For $\varepsilon > 0$, define

$$U_\varepsilon = \{\xi \in U \mid d(\xi, U^c) > \varepsilon\}.$$

We assume ε small enough so that $U_\varepsilon \neq \emptyset$. Now, given $\rho \in \mathcal{D}_+((0, \varepsilon))$, with $\int_0^\varepsilon \rho(s)\mathrm{d}s = 1$, we denote by F^ρ the function defined on U_ε by

$$F^\rho(t, x) = \int_0^T F(t - s, x)\rho(s)\mathrm{d}s, \quad (t, x) \in U_\varepsilon.$$

Note that the function F^ρ is bounded and continuous on U_ε and that its time derivative $\partial F^\rho/\partial t$ is also bounded and continuous on U_ε. It is easy to check that $\tilde{\mathcal{A}}F^\rho = 0$ in U_ε.

Fix $(t_0, x_0) \in U_\varepsilon$ and $\delta > 0$ such that $(t_0 - \delta, t_0 + \delta) \times (x_0 - \delta, x_0 + \delta) \subset U_\varepsilon$. We will use the notation $V_\delta = (t_0 - \delta, t_0 + \delta) \times (x_0 - \delta, x_0 + \delta)$. We have, for all $\varphi \in \mathcal{D}((t_0 - \delta, t_0 + \delta))$, $\psi \in \mathcal{D}((x_0 - \delta, x_0 + \delta))$,

$$-\int \mathrm{d}t \int m(\mathrm{d}x)F^\rho(t, x)\varphi'(t)\psi(x) + \int \mathrm{d}t \int m(\mathrm{d}x)F^\rho(t, x)\mathcal{L}\psi(x) = 0.$$

This implies (as in the proof of Theorem 7.1) that, for all $t \in (t_0 - \delta, t_0 + \delta)$ and for all $\psi \in \mathcal{D}((x_0 - \delta, x_0 + \delta))$

$$\int F^\rho(t, x)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\psi'}{p'}\right) + \int \left(\frac{2}{\sigma^2 p'}(x)\frac{\partial F^\rho}{\partial t}(t, x) - \frac{2r}{\sigma^2 p'}(x)F^\rho(t, x)\right)\psi(x)\mathrm{d}x = 0.$$

This means that, for each $t \in (t_0 - \delta, t_0 + \delta)$, $F^\rho(t, \cdot)$ solves the equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F^\rho/\partial x}{p'}\right) = j(t, \cdot),$$

on the interval $(x_0 - \delta, x_0 + \delta)$, where

$$j(t, x) = \frac{2r}{\sigma^2 p'}(x)F^\rho(t, x) - \frac{2}{\sigma^2 p'}(x)\frac{\partial F^\rho}{\partial t}(t, x).$$

Using Lemma 7.2, we have

$$F^\rho(t, x) = \int_{x_0}^x p'(y)M(t, y)\mathrm{d}y + k(t)p(x) + l(t),$$

where $M(t, y) = \int_{x_0}^y j(t, z)\mathrm{d}z$. We deduce from the continuity of F^ρ and $\partial F^\rho/\partial t$ that M is jointly continuous, so that $t \mapsto k(t)$ and $t \mapsto l(t)$ must be continuous. We also have, for $(t, x) \in V_\delta$,

$$\frac{\partial F^\rho}{\partial x}(t, x) = p'(x)M(t, x) + k(t)p'(x),$$

and

$$\frac{\partial^2 F^\rho}{\partial x^2}(t,x) = \frac{2r}{\sigma^2}(x)F^\rho(t,x) - \frac{2b}{\sigma^2}(x)\frac{\partial F^\rho}{\partial x}(t,x) - \frac{2}{\sigma^2}(x)\frac{\partial F^\rho}{\partial t}(t,x)$$

dx-almost everywhere. Note that, on V_δ, we have $\mathcal{L}F^\rho = 0$ and

$$\frac{\partial^2 F^\rho}{\partial x^2}(t,x) = \sum_{i=1}^{3} \varphi_i(x)\Phi_i(t,x), \tag{23}$$

where Φ_i is continuous on V_δ and φ_i is locally integrable on I.

Now, let V be a relatively compact open subset of U_ε. One can find a finite number of points (t_j, x_j) and positive numbers δ_j, $j = 1, \ldots, N$, such that $\bar{V} \subset \bigcup_{j=1}^N V_j \subset U_\varepsilon$, where $\bar{V}$ is the closure of V and $V_j = (t_j - \delta_j, t_j + \delta_j) \times (x_j - \delta_j, x_j + \delta_j)$. Now, let (α_j) be a partition of unity associated with the V_j's, that is a sequence of functions $\alpha_j \in \mathcal{D}(V_j)$, with $0 \le \alpha_j \le 1$ and $\sum_{j=1}^N \alpha_j = 1$ on $\bar{V}$. Let

$$\tilde{F}^\rho(t,x) = \sum_{j=1}^{N} \alpha_j(t,x)F^\rho(t,x), \quad (t,x) \in (0,T) \times I.$$

We have $F^\rho = \tilde{F}^\rho$ on $\bar{V}$. On the other hand, since F^ρ is C^1 on each V_j, with a second derivative with respect to x of the form (23), with Φ_i continuous on V_j and φ_i locally integrable on I, we have $\tilde{F}^\rho \in \mathcal{W}([\varepsilon, T-\varepsilon] \times I)$ (see the beginning of Section 5 for the definition of the space $\mathcal{W}$). We can apply Proposition 5.2 and write, for $\varepsilon < t < T - \varepsilon$, $s \in [0, T - t - \varepsilon)$,

$$\begin{aligned} e^{-\Lambda_s}\tilde{F}^\rho(t+s, X_s) &= \tilde{F}^\rho(t, X_0) + \int_0^s e^{-\Lambda_a}\frac{\partial \tilde{F}^\rho}{\partial x}(t+a, X_a)\sigma(X_a)dW_a \\ &\quad + \int_0^s e^{-\Lambda_a}\left(\frac{\partial \tilde{F}^\rho}{\partial t} + \mathcal{L}\tilde{F}^\rho\right)(t+a, X_a)da. \end{aligned}$$

Now, let $\tau_V^t = \inf\{s \ge 0 \mid (t+s, X_s) \notin V\}$. Observe that, since $(\partial\tilde{F}^\rho/\partial t) + \mathcal{L}\tilde{F}^\rho = 0$ on V, we have

$$e^{-\Lambda_{s\wedge\tau_V^t}}\tilde{F}^\rho(t + s\wedge\tau_V^t, X_{s\wedge\tau_V^t}) = \tilde{F}^\rho(t, X_0) + \int_0^{s\wedge\tau_V^t} e^{-\Lambda_a}\frac{\partial\tilde{F}^\rho}{\partial x}(t+a, X_a)\sigma(X_a)dW_a,$$

so that, for any stopping time θ with values in $[0, T - t - \varepsilon)$, we have

$$\mathbb{E}_x\left(e^{-\Lambda_{\theta\wedge\tau_V^t}}F^\rho(t + \theta\wedge\tau_V^t, X_{\theta\wedge\tau_V^t})\right) = F^\rho(t,x).$$

Since V is an arbitrary relatively compact open subset of U_ε, we can replace V by U_ε in the above equality, and by taking the limit as ε goes to 0, the proof of Theorem 7.1 is easily completed. ◇

We can now prove the following verification theorem, from which uniqueness in Theorem 2.6 follows easily.

Theorem 7.5. *Let f be a bounded Borel-measurable function on I and $\hat{f}$ its upper semi-continuous envelope. Let $T > 0$. Suppose $F : [0,T) \times I \to \mathbb{R}$ is a continuous and bounded function satisfying the following conditions:*

1. *On $(0,T) \times I$, we have $F \ge f$ and the distribution $\tilde{\mathcal{A}}F$ satisfies $\tilde{\mathcal{A}}F \le 0$.*
2. *On the open set $U = \{(t,x) \in (0,T) \times I \mid F(t,x) > \hat{f}(x)\}$, we have $\tilde{\mathcal{A}}F = 0$.*
3. *For all $x \in I$, $\lim_{t\uparrow T} F(t,x) = \hat{f}(x)$.*

Then we have

$$\forall (t,x) \in [0,T) \times I, \quad F(t,x) = u_f(T-t,x).$$

Lemma 7.6. *Under the assumptions of Theorem 7.5, we have, for $s \in [0,T)$ and $x \in I$, $\lim_{t\to T-s} \mathbb{E}_x \left| F(s+t,X_t) - \hat{f}(X_T) \right| = 0$.*

Proof. We prove the result for $s=0$ (the argument is the same for $s>0$). We have

$$\mathbb{E}_x \left| F(t,X_t) - \hat{f}(X_T) \right| \le \mathbb{E}_x \left| F(t,X_t) - \hat{f}(X_t) \right| + \mathbb{E}_x \left| \hat{f}(X_t) - \hat{f}(X_T) \right|.$$

As we have seen in the proof of Proposition 3.1 (cf. (7)), $\lim_{t\to T} \mathbb{E}_x \left| \hat{f}(X_t) - \hat{f}(X_T) \right| = 0$. On the other hand, if K is a compact subinterval of I, we have

$$\begin{aligned}\mathbb{E}_x &\left| F(t,X_t) - \hat{f}(X_t) \right| \\ &\le \mathbb{E}_x \mathbf{1}_{\{X_t \notin K\}} \left| F(t,X_t) - \hat{f}(X_t) \right| + \mathbb{E}_x \mathbf{1}_{\{X_t \in K\}} \left| F(t,X_t) - \hat{f}(X_t) \right|.\end{aligned}$$

Using Corollary 8.13 and the boundedness of F and f, we have, for some $C_x > 0$,

$$\mathbb{E}_x \mathbf{1}_{\{X_t \in K\}} \left| F(t,X_t) - \hat{f}(X_t) \right| \le C_x \left(1 + \frac{1}{\sqrt{t}}\right) \|\mathbf{1}_K (F(t,\cdot) - \hat{f})\|_{L^2(m)}.$$

Since $\lim_{t\to T} F(t,y) = \hat{f}(y)$ for all $y \in I$, we have, by dominated convergence,

$$\lim_{t\to T} \|\mathbf{1}_K (F(t,\cdot) - \hat{f})\|_{L^2(m)} = 0.$$

Moreover

$$\mathbb{E}_x \mathbf{1}_{\{X_t \notin K\}} \left| F(t,X_t) - \hat{f}(X_t) \right| \le C\mathbb{P}_x(\{\exists s \in [0,T] \mid X_s \notin K\}),$$

with $C = \|F\|_\infty + \|f\|_\infty$. Therefore

$$\limsup_{t\to T} \mathbb{E}_x \left| F(t,X_t) - \hat{f}(X_T) \right| \le C\mathbb{P}_x(\{\exists s \in [0,T] \mid X_s \notin K\}).$$

The right-hand side can be made arbitrarily small by choosing K large enough. $\diamond$

Proof of Theorem 7.5. According to Theorem 7.1, the condition $\tilde{\mathcal{A}}F \le 0$ implies

$$\forall s \in [0,T), \forall t \in [0,T-s), \quad \mathbb{E}_x \left(\mathrm{e}^{-\Lambda_t} F(s+t,X_t) \right) \le F(s,x).$$

We easily deduce from this estimate, combined with the Markov property, that, given $s_0 \in [0,T)$, the process $(V_t)_{0\le t<T-s_0}$, defined by $V_t = \mathrm{e}^{-\Lambda_t} F(s_0+t,X_t)$ is a (bounded) supermartingale. Therefore, for all $\tau \in \mathcal{T}^0_{T-s_0}$, $\mathbb{E}_x \left(\mathrm{e}^{-\Lambda_\tau} F(s_0+\tau,X_\tau) \right) \le F(s_0,x)$, and, since $F \ge f$, $\mathbb{E}_x \left(\mathrm{e}^{-\Lambda_\tau} f(X_\tau) \right) \le F(s_0,x)$. Hence

$$u_f(T-s_0,x) \le F(s_0,x).$$

Now define the stopping time

$$\tau = \inf\{t \ge 0 \mid (s_0+t,X_t) \notin U\} = \inf\{t \ge 0 \mid F(s_0+t,X_t) = \hat{f}(X_t)\},$$

with the convention $\inf \emptyset = T - s_0$. Since $\tilde{\mathcal{A}}F = 0$ in U, we deduce from Theorem 7.4 that, for all $\varepsilon > 0$, we have (with the notation $T_\varepsilon = T - s_0 - \varepsilon$), $\mathbb{E}_x \left(\mathrm{e}^{-\Lambda_{\tau\wedge T_\varepsilon}} F(s_0+\tau, X_{\tau\wedge T_\varepsilon}) \right) = F(s_0,x)$, so that $\mathbb{E}_x \left(V_{\tau\wedge T_\varepsilon} \right) = F(s_0,x)$.

Note that, since $(V_t)_{0\leq t<T-s_0}$ is a bounded supermartingale, the limit $\lim_{t\to T-s_0} V_t$ exists almost surely, and we deduce from Lemma 7.6 that $\lim_{t\to T-s_0} V_t = \mathrm{e}^{-\Lambda_T}\hat{f}(X_T)$. Hence $\lim_{\varepsilon\to 0} V_{\tau\wedge T_\varepsilon} = \mathrm{e}^{-\Lambda_\tau}\hat{f}(X_\tau)$ a.s. and, by dominated convergence, $\mathbb{E}_x\left(\mathrm{e}^{-\Lambda_\tau}\hat{f}(X_\tau)\right) = F(s_0, x)$. Hence $F(s_0, x) \leq v_{\hat{f}}(T - s_0, x) = u_f(T - s_0, x)$. $\diamond$

8. Auxiliary results

8.1. Resolvent and semi-group

Proposition 8.1. *Let $h : I \to \mathbb{R}$ be a bounded Borel-measurable function on I and ρ a positive number. Denote by $U_\rho h$ the unique bounded solution of the ordinary differential equation*

$$\frac{1}{2}\sigma^2(x)u''(x) + b(x)u'(x) - \rho u(x) + h(x) = 0. \tag{24}$$

For any weak solution $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}_x, W, X)$ of (1), we have, for all $t \geq 0$,

$$U_\rho h(X_t) = \mathbb{E}\left(\int_0^{+\infty} \mathrm{e}^{-\rho s} h(X_{t+s})\mathrm{d}s \mid \mathcal{F}_t\right) \quad \textit{a.s.}$$

Proof. By a solution of (24), we mean a continuously differentiable function u, with an absolutely continuous derivative u', such that (24) holds $\mathrm{d}x$ almost everywhere. We refer to [8] for the existence and uniqueness of a bounded solution of (24). If u is this solution, we have, using the generalized Itô formula,

$$\mathrm{e}^{-\rho t}u(X_t) = u(X_0) + \int_0^t \mathrm{e}^{-\rho s}u'(X_s)\sigma(X_s)\mathrm{d}W_s - \int_0^t \mathrm{e}^{-\rho s}h(X_s)\mathrm{d}s.$$

Since u and h are bounded, the stochastic integral on the right-hand side of this equality is a true martingale, so that, for $0 \leq t < t'$, we have

$$\mathrm{e}^{-\rho t}u(X_t) = \mathbb{E}\left(\int_t^{t'} \mathrm{e}^{-\rho s}h(X_s)\mathrm{d}s \mid \mathcal{F}_t\right) + \mathbb{E}\left(\mathrm{e}^{-\rho t'}u(X_{t'}) \mid \mathcal{F}_t\right).$$

By letting t' go to infinity, we derive

$$u(X_t) = \mathbb{E}\left(\int_t^{+\infty} \mathrm{e}^{-\rho(s-t)}h(X_s)\mathrm{d}s \mid \mathcal{F}_t\right) = \mathbb{E}\left(\int_0^{+\infty} \mathrm{e}^{-\rho s}h(X_{t+s})\mathrm{d}s \mid \mathcal{F}_t\right) \quad \text{a.s.} \quad \diamond$$

We now define, for a Borel-measurable and bounded $f : I \to \mathbb{R}$,

$$P_t f(x) = \mathbb{E}_x f(X_t),$$

where X is a weak solution of (1). The following proposition relates P_t to the operators U_ρ, $\rho > 0$.

Proposition 8.2. *For $\rho > 0$, let $V_\rho = \rho U_\rho$, where the operator U_ρ is defined in Proposition 8.1. For $t > 0$ and for any bounded and continuous function $f : I \to \mathbb{R}$, we have*

$$\forall x \in I, \quad P_t f(x) = \lim_{n\to+\infty} \left(V_{n/t}\right)^n f(x).$$

Moreover, for any weak solution $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}, W, X)$ of (1), we have, for $t, t' > 0$,

$$P_t f(X_{t'}) = \mathbb{E}\left(f(X_{t+t'}) \mid \mathcal{F}_{t'}\right) \quad a.s. \tag{25}$$

Proof. We first observe that if $\rho_1,\ldots,\rho_n$ are n positive numbers, we have

$$U_{\rho_1}\ldots U_{\rho_n} f(X_t) = \int_{\mathbb{R}^n_+} \mathrm{d}s_1\ldots \mathrm{d}s_n \mathrm{e}^{-(\rho_1 s_1+\cdots+\rho_n s_n)}\mathbb{E}\left(f(X_{t+s_1+\cdots+s_n}) \mid \mathcal{F}_t\right) \quad \text{a.s.},$$

as follows from Proposition 8.1 and a straightforward induction. We deduce thereof that, for $t, t' > 0$,

$$\left(V_{n/t}\right)^n f(X_{t'}) = \int_{\mathbb{R}^n_+} \mathrm{e}^{-t_1/t}\frac{\mathrm{d}t_1}{t}\ldots \mathrm{e}^{-t_n/t}\frac{\mathrm{d}t_n}{t}\mathbb{E}\left(f\left(X_{t'+\frac{t_1+\cdots+t_n}{n}}\right) \mid \mathcal{F}_{t'}\right). \tag{26}$$

Take $t' = 0$, so that $\left(V_{n/t}\right)^n f(x) = \mathbb{E}P_{\frac{T_1+\cdots T_n}{n}} f(x)$, where $T_1,\ldots,T_n$ are independent exponential variables with mean t, and note that, if f is continuous, $t \mapsto P_t f(x)$ is continuous. Therefore, by the law of large numbers $\lim_{n\to\infty}\left(V_{n/t}\right)^n f(x) = P_t f(x)$. Going back to (26), and taking limits as n goes to infinity, we get (25). $\diamond$

Remark 8.3. By taking expectations in (25), we have the semi-group property: $P_{t+t'} f = P_{t'} P_t f$, for f bounded and continuous, and, by a monotone class argument, for f bounded and Borel-measurable.

Remark 8.4. By a monotone class argument, (25) can be extended to all Borel-measurable and bounded functions f. We also deduce from this property that for any Borel subset A of the space of all continuous functions on I, we have $\mathbb{P}(X \in A \mid \mathcal{F}_t) = \mathbb{P}(X \in A \mid \mathcal{F}^X_t)$ a.s., where $\mathcal{F}^X_t$ is the σ-algebra generated by the random variables X_s, $0 \leq s \leq t$. Since the filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions, it follows that the completion of the filtration $(\mathcal{F}^X_t)_{t\geq 0}$ is right-continuous, so that $\mathcal{F}^X_t$ and $\mathcal{F}^0_t$ coincide, up to negligible events. As a consequence, we have the zero–one law: $\mathbb{P}_x(A) \in \{0, 1\}$ for all $A \in \mathcal{F}^0_0$.

8.2. Randomized stopping times

Definition 8.5. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space. A *randomized* $(\mathcal{F}_t)$-stopping time is a mapping $T : \Omega \times [0, 1] \to [0, +\infty]$, which is a stopping time on the filtered probability space $(\Omega \times [0, 1], \mathcal{F} \otimes \mathcal{B}, (\mathcal{F}_t \otimes \mathcal{B})_{t\geq 0}, \mathbb{P} \otimes \lambda)$, where $\mathcal{B}$ is the Borel σ-algebra on $[0, 1]$, and λ is the Lebesgue measure on $[0, 1]$.

Proposition 8.6. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}_x, W, X)$ be a weak solution of (1). Denote by $(\bar{\mathcal{F}}^0_t)_{t\geq 0}$ the completion of the natural filtration of X. For any $(\mathcal{F}_t)$-stopping time T, there exists a randomized $(\bar{\mathcal{F}}^0_t)$-stopping time $\hat{T}$, such that the pairs (T, X) and $(\hat{T}, \hat{X})$ have identical laws, where $\hat{X}$ is the process defined on $\Omega \times [0, 1]$ by $\hat{X}_t(\omega, u) = X_t(\omega)$.*

Proof. This is a classical result. A proof can be found in [1]. Since this reference is not easily accessible, we sketch the proof. First, note that, for $t \geq 0$, the event $\{T \leq t\}$ is conditionally independent of $\mathcal{F}^X_\infty$, given $\mathcal{F}^X_t$ (where $\mathcal{F}^X_\infty$ is the σ-algebra generated by all the random variables X_t, $t \geq 0$). Indeed, if A is a Borel-measurable subset of $C(\mathbb{R}^+; I)$, we have, with probability one,

$$\mathbb{E}\left(\mathbf{1}_{\{T\leq t\}}\mathbf{1}_{\{X\in A\}} \mid \mathcal{F}^X_t\right) = \mathbb{E}\left(\mathbf{1}_{\{T\leq t\}} \mid \mathcal{F}^X_t\right)\mathbb{E}\left(\mathbf{1}_{\{X\in A\}} \mid \mathcal{F}^X_t\right),$$

as can be seen using $\mathbb{P}(X \in A \mid \mathcal{F}_t) = \mathbb{P}(X \in A \mid \mathcal{F}^X_t)$ (cf. Remark 8.4).

Now, if $0 \le s \le t$, we have

$$\mathbb{E}\left(\mathbb{E}\left(\mathbf{1}_{\{T\le t\}}|\mathcal{F}_t^X\right) \mid \mathcal{F}_s^X\right) \ge \mathbb{E}\left(\mathbb{E}\left(\mathbf{1}_{\{T\le s\}}|\mathcal{F}_t^X\right) \mid \mathcal{F}_s^X\right) = \mathbb{E}\left(\mathbf{1}_{\{T\le s\}} \mid \mathcal{F}_s^X\right).$$

This proves that the process $\left[\mathbb{E}\left(\mathbf{1}_{\{T\le t\}}|\mathcal{F}_t^X\right)\right]_{t\ge 0}$ is a submartingale with respect to the filtration $(\bar{\mathcal{F}}_t^0)_{t\ge 0}$. Since its expectation (equal to $\mathbb{P}(T \le t)$) is a right-continuous function of t, this process has a right-continuous modification, which we denote by A_t. The process A is in fact non-decreasing. Indeed, we have $A_t = \mathbb{E}\left(\mathbf{1}_{\{T\le t\}}|\mathcal{F}_t^X\right) \ge \mathbb{E}\left(\mathbf{1}_{\{T\le s\}}|\mathcal{F}_t^X\right)$ a.s., and $\mathbb{E}\left(\mathbf{1}_{\{T\le s\}}|\mathcal{F}_t^X\right) = \mathbb{E}\left(\mathbf{1}_{\{T\le s\}}|\mathcal{F}_s^X\right)$ a.s., because $\{T \le s\}$ is conditionally independent of $\mathcal{F}_t^X$ given $\mathcal{F}_s^X$. Now, define, for $(\omega, u) \in \hat{\Omega} = \Omega \times [0, 1]$, $\hat{T}(\omega, u) = \inf\{t \ge 0 | A_t(\omega) \ge u\}$ and $U(\omega, u) = u$. Note that we can embed the space Ω into $\hat{\Omega}$, and that the random variable U is independent of $\mathcal{F}$, when $(\hat{\Omega}, \mathcal{F} \otimes \mathcal{B})$ is endowed with the probability $\mathbb{P} \otimes \lambda$. We have $\{\hat{T} \le t\} = \{A_t \ge U\} \in \bar{\mathcal{F}}_t^0 \otimes \mathcal{B}$ and $\mathbb{E}\left(\mathbf{1}_{\{\hat{T}\le t\}} \mid \bar{\mathcal{F}}_t^0\right) = A_t$, because U is independent of $\bar{\mathcal{F}}_t^0$.

Hence, for $t \ge 0$ and for any Borel-measurable subset B of $C(\mathbb{R}^+; I)$,

$$\begin{aligned}\mathbb{E}\left(\mathbf{1}_{\{\hat{T}\le t\}}\mathbf{1}_{\{\hat{X}\in B\}}\right) &= \mathbb{E}\left(A_t \mathbf{1}_{\{X\in B\}}\right)\\ &= \mathbb{E}\left(\mathbb{E}\left(\mathbf{1}_{\{T\le t\}}\mathbf{1}_{\{X\in B\}}|\mathcal{F}_t^X\right)\right) = \mathbb{E}\left(\mathbf{1}_{\{T\le t\}}\mathbf{1}_{\{X\in B\}}\right),\end{aligned}$$

where we have used the fact that $\{T \le t\}$ and X are conditionally independent given $\mathcal{F}_t^X$. $\diamond$

Remark 8.7. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P}, W, X)$ be a weak solution of (1) and let T be an $(\mathcal{F}_t)$-stopping time, and $\hat{T}$ be a randomized $(\bar{\mathcal{F}}_t^0)$-stopping time as in Proposition 8.6. For a fixed $u \in [0, 1]$, let $T_u = \hat{T}(\cdot, u)$. Clearly, T_u is an $(\bar{\mathcal{F}}_t^0)$-stopping time and, therefore is almost surely equal to an $(\mathcal{F}_t^0)$-stopping time. Moreover, if T takes on values in $[0, t]$ (resp. $[0, t)$), the same is true for T_u with probability 1. It follows that, for any Borel-measurable and bounded function $f : I \to \mathbb{R}$, we have $\mathbb{E}_x(f(X_{T_u})) \le u_f(t, x)$ (resp. $\mathbb{E}_x(f(X_{T_u})) \le v_f(t, x)$) if $T \in \mathcal{T}_t$ (resp. $T \in \bar{\mathcal{T}}_t$). Since $\mathbb{E}_x f(X_T) = \int_0^1 \mathbb{E}_x(f(X_{T_u}))\mathrm{d}u$, we conclude that $\sup_{T\in\mathcal{T}_t} \mathbb{E}_x(f(X_T)) \le u_f(t, x)$ and $\sup_{T\in\bar{\mathcal{T}}_t} \mathbb{E}_x(f(X_T)) \le v_f(t, x)$.

8.3. Symmetry with respect to the speed measure

Proposition 8.8. *Let $g, h : I \to \mathbb{R}$ be bounded Borel-measurable, which vanish in the complement of a compact subset of I. We have, for all $\rho > 0$,*

$$\int_\alpha^\beta g(x)U_\rho h(x)m(\mathrm{d}x) = \int_\alpha^\beta h(x)U_\rho g(x)m(\mathrm{d}x).$$

Proof. It is well known (cf. [8]) that the ordinary differential equation $\frac{1}{2}\sigma^2(x)u''(x) + b(x)u'(x) - \rho u(x) = 0$ admits two fundamental positive solutions ϕ_ρ and ψ_ρ, with ϕ_ρ strictly decreasing and ψ_ρ strictly increasing and that, with a suitable normalization of ϕ_ρ and ψ_ρ, $U_\rho h$ can be represented as follows:

$$U_\rho h(x) = \phi_\rho(x)\int_\alpha^x \psi_\rho(y)h(y)m(\mathrm{d}y) + \psi_\rho(x)\int_x^\beta \phi_\rho(y)h(y)m(\mathrm{d}y), \quad x \in I.$$

An equivalent formulation of this representation is

$$U_\rho h(x) = \int_\alpha^\beta h(y) u_\rho(x, y) m(\mathrm{d}y), \quad \text{where } u_\rho(x, y) = \begin{cases} \phi_\rho(x)\psi_\rho(y), & \text{if } x \geq y, \\ \phi_\rho(y)\psi_\rho(x), & \text{if } x < y. \end{cases}$$

Note that u_ρ is continuous and positive on $I \times I$, and that $u_\rho(x, y) = u_\rho(y, x)$. The proposition follows easily from this symmetry property. $\diamond$

Proposition 8.9. *For $t > 0$, the operator P_t can be extended as an operator mapping $L^2(m)$ into itself and we have, for $f, g \in L^2(m)$,*

$$\langle P_t f, g \rangle_{L^2(m)} = \langle f, P_t g \rangle_{L^2(m)},$$

where $\langle \cdot, \cdot \rangle_{L^2(m)}$ denotes the inner product on the Hilbert space $L^2(m)$.

This can be deduced from Proposition 8.8 by standard arguments. We omit the proof.

8.4. Regularity estimates

Proposition 8.10. *We have the following estimate, for h non-negative, bounded and square integrable with respect to the speed measure, and for all $\rho > 0$.*

$$\int_\alpha^\beta \left(\frac{\mathrm{d}}{\mathrm{d}x} (U_\rho h)(x) \right)^2 \frac{\mathrm{d}x}{p'(x)} \leq \langle h, U_\rho h \rangle_{L^2(m)} - \rho \| U_\rho h \|^2_{L^2(m)}.$$

Proof. Fix $\rho > 0$ and, for simpler notation, set $u = U_\rho h$. The differential equation satisfied by u can be rewritten as follows

$$\frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{u'}{p'} \right) - \rho u \frac{2}{\sigma^2 p'} + h \frac{2}{\sigma^2 p'} = 0.$$

Multiplying by u and integrating from a_1 to b_1, where $\alpha < a_1 < b_1 < \beta$, we get

$$\int_{a_1}^{b_1} \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{u'}{p'} \right)(x) u(x) \mathrm{d}x = \int_{a_1}^{b_1} \left(\rho u^2(x) - h(x)u(x) \right) m(\mathrm{d}x).$$

By integration by parts, we have

$$\int_{a_1}^{b_1} \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{u'}{p'} \right)(x) u(x) \mathrm{d}x = \frac{u'(b_1)u(b_1)}{p'(b_1)} - \frac{u'(a_1)u(a_1)}{p'(a_1)} - \int_{a_1}^{b_1} \frac{(u'(x))^2}{p'(x)} \mathrm{d}x.$$

Hence

$$\int_{a_1}^{b_1} (u'(x))^2 \frac{\mathrm{d}x}{p'(x)} = \int_{a_1}^{b_1} (hu - \rho u^2) \mathrm{d}m + \frac{u'(b_1)u(b_1)}{p'(b_1)} - \frac{u'(a_1)u(a_1)}{p'(a_1)}.$$

We know that we can write, for $x \in I$,

$$u(x) = \phi(x) \int_\alpha^x h(y)\psi(y)m(\mathrm{d}y) + \psi(x) \int_x^\beta h(y)\phi(y)m(\mathrm{d}y),$$

where ϕ is positive and strictly decreasing and ψ is positive and strictly increasing. Now assume that h is null outside a subinterval $[\alpha_0, \beta_0]$, with $\alpha < \alpha_0 < \beta_0 < \beta$. Then we have, for $a_1 < \alpha_0$,

$$u(a_1) = \psi(a_1) \int_{\alpha_0}^{\beta_0} h(y)\phi(y)m(\mathrm{d}y) \quad \text{and} \quad u'(a_1) = \psi'(a_1) \int_{\alpha_0}^{\beta_0} h(y)\phi(y)m(\mathrm{d}y),$$

so that $u(a_1) \geq 0$ and $u'(a_1) \geq 0$, and, by a similar argument, for $b_1 > \beta_0$, $u(b_1) \geq 0$ and $u'(b_1) \leq 0$. Hence $\int_{a_1}^{b_1} (u'(x))^2 \frac{\mathrm{d}x}{p'(x)} \leq \int_{a_1}^{b_1} (hu - \rho u^2)\mathrm{d}m$. Note that, with our assumptions on h, we have $h \in L^2(m)$ and $u \in L^2(m)$ and, by making $a_1 \to \alpha$ and $b_1 \to \beta$, we obtain

$$\int_\alpha^\beta (u'(x))^2 \frac{\mathrm{d}x}{p'(x)} \leq \int_\alpha^\beta (hu - \rho u^2)\mathrm{d}m.$$

We have assumed that h was null outside a subinterval $[\alpha_0, \beta_0]$. For an arbitrary bounded, Borel-measurable and non-negative h, we can approximate h by an increasing sequence of functions h_n which have compact supports, and extend the inequality by approximation. $\diamond$

From the estimate for the resolvent given by Proposition 8.10, we can derive the following estimate for the semi-group P_t (where $P_t f(x) = \mathbb{E}_x f(X_t)$). This estimate seems to follow from a formal integration by parts (see Remark 8.12), but, for a complete justification, we found it easier to deduce it from Proposition 8.10.

Theorem 8.11. *If h is non-negative, square integrable with respect to the speed measure, the function $x \mapsto P_t h(x)$ is absolutely continuous on (α, β) and its derivative satisfies*

$$\int_\alpha^\beta \left(\frac{\mathrm{d}}{\mathrm{d}x}(P_t h)(x)\right)^2 \frac{2\mathrm{d}x}{p'(x)} \leq \frac{1}{t}\|h\|^2_{L^2(m)}. \tag{27}$$

Proof. We first observe that the set $L^2_+(m)$ of all non-negative functions in $L^2(m)$ is stable under U_ρ. By iterating the estimate of Proposition 8.10, we get, for $h \in L^2_+(m)$ and for any positive integer n, $\int_\alpha^\beta \left(\frac{\mathrm{d}}{\mathrm{d}x} U_\rho^n h(x)\right)^2 \frac{\mathrm{d}x}{p'(x)} \leq \langle U_\rho^{n-1}h - \rho U_\rho^n h, U_\rho^n h\rangle_{L^2(m)}$. In terms of the operator $V_\rho = \rho U_\rho$, we obtain

$$\int_\alpha^\beta \left(\frac{\mathrm{d}}{\mathrm{d}x} V_\rho^n h(x)\right)^2 \frac{\mathrm{d}x}{p'(x)} \leq \rho\langle V_\rho^{n-1}h - V_\rho^n h, V_\rho^n h\rangle_{L^2(m)}. \tag{28}$$

We deduce from Remark 8.3 and Proposition 8.9 that the family of operators $(P_t)_{t\geq 0}$ defines a strongly continuous and symmetric semi-group on the space $L^2(m)$. Therefore, it admits a spectral representation $P_t = \int_{[0,+\infty)} \mathrm{e}^{-t\lambda}\mathrm{d}E_\lambda$, and the operator V_ρ can be represented as $V_\rho = \int_{[0,+\infty)} \frac{\rho}{\rho+\lambda}\mathrm{d}E_\lambda$. With this representation, we have

$$\begin{aligned}\rho\langle V_\rho^{n-1}h - V_\rho^n h, V_\rho^n h\rangle_{L^2(m)} &= \rho\langle h - V_\rho h, V_\rho^{2n-1}h\rangle_{L^2(m)}\\ &= \int_{[0,+\infty)} \frac{\lambda\rho}{\rho+\lambda}\left(\frac{\rho}{\rho+\lambda}\right)^{2n-1} \langle \mathrm{d}E_\lambda h, h\rangle_{L^2(m)}.\end{aligned}$$

Now, take $\rho = n/t$. We then have $\frac{\lambda\rho}{\rho+\lambda}\left(\frac{\rho}{\rho+\lambda}\right)^{2n-1} \leq \frac{1}{t}\frac{n}{2n-1}$. Hence $\frac{n}{t}\langle h - V_{n/t}h, V_{n/t}^{2n-1}h\rangle_{L^2(m)} \leq \frac{1}{t}\frac{n}{2n-1}\|h\|^2_{L^2(m)}$, and, going back to (28),

$$\limsup_{n\to\infty} \int_\alpha^\beta \left(\frac{\mathrm{d}}{\mathrm{d}x} V_{n/t}^n h(x)\right)^2 \frac{2\mathrm{d}x}{p'(x)} \leq \frac{1}{t}\|h\|^2_{L^2(m)}. \tag{29}$$

Now, fix $t > 0$ and assume h is continuous and bounded. For $n \geq 1$, Let $v_n = V_{n/t}^n h$. We know that v_n is continuously differentiable on I and that, for $x \in I$, $\lim_{n\to\infty} v_n(x) = P_t h(x)$

(cf. Proposition 8.2). It follows from (29) that the sequence $(v_n')_{n\geq 1}$ is bounded in $L^2(\mathrm{d}x/p')$. Therefore, $P_t h$ must be absolutely continuous and its derivative satisfies (27). We have proved (27) for h continuous and bounded. The extension to $h \in L^2_+(m)$ follows from a straightforward approximation argument. $\diamond$

Remark 8.12. Using the expression $\mathcal{L}_0 u = \frac{\sigma^2 p'}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{p'}\frac{\mathrm{d}u}{\mathrm{d}x}\right)$ for the generator of the semi-group, we have $\langle \mathcal{L}_0 u, u\rangle_{L^2(m)} = \int_\alpha^\beta \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{p'}\frac{\mathrm{d}u}{\mathrm{d}x}\right) u\mathrm{d}x$, so that, at least for a function u with compact support, $\int_\alpha^\beta \frac{1}{p'}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 \mathrm{d}x = -\langle \mathcal{L}_0 u, u\rangle_{L^2(m)}$. By applying this with $u = P_t h$, we would be able to deduce (27) from the classical estimate $\|\mathcal{L}_0 P_t h\|_{L^2(m)} \leq (C/t)\|h\|_{L^2(m)}$. However, the justification of the integration by parts in the case $u = P_t h$ does not seem completely obvious.

Corollary 8.13. *If $h : I \to \mathbb{R}$ is non-negative and square integrable with respect to the speed measure, we have, for $\alpha < \alpha_0 < \beta_0 < \beta$, $x \in [\alpha_0, \beta_0]$ and $t > 0$,*

$$P_t h(x) \leq \|h\|_{L^2(m)}\left(\frac{1}{\sqrt{m([\alpha_0,\beta_0])}} + \frac{\sqrt{p(\beta_0)-p(\alpha_0)}}{\sqrt{t}}\right).$$

Proof. Set $g = \frac{1}{m([\alpha_0,\beta_0])}\mathbf{1}_{[\alpha_0,\beta_0]}$. Since $\int g\mathrm{d}m = 1$, we have, for $x \in [\alpha_0, \beta_0]$,

$$\begin{aligned} P_t h(x) &= \int P_t h(y) g(y) m(\mathrm{d}y) + \int (P_t h(x) - P_t h(y))\, g(y) m(\mathrm{d}y) \\ &= \int h(y) P_t g(y) m(\mathrm{d}y) + \int_{\alpha_0}^{\beta_0}\left(\int_y^x P_t' h(z)\mathrm{d}z\right) g(y) m(\mathrm{d}y) \\ &\leq \|h\|_{L^2(m)}\|g\|_{L^2(m)} + \int_{\alpha_0}^{\beta_0}\left|P_t' h(z)\right|\mathrm{d}z, \end{aligned}$$

where we have used the symmetry of P_t with respect to m and the notation $P_t' h$ for $\frac{\mathrm{d}}{\mathrm{d}x}(P_t h)$. By writing $P_t' h$ as the product $(P_t' h/\sqrt{p'}) \times \sqrt{p'}$, using the Cauchy–Schwarz inequality and Theorem 8.11, we easily conclude. $\diamond$

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