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Banafsheh Farang-Hariri. Geometric local theta correspondence for dual reductive pairs of type II at the Iwahori level. 2013. hal-00796584v1

HAL Id: hal-00796584

<https://hal.science/hal-00796584v1>

Preprint submitted on 4 Mar 2013 (v1), last revised 27 Oct 2013 (v2)

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GEOMETRIC LOCAL THETA CORRESPONDENCE FOR DUAL REDUCTIVE PAIRS OF TYPE II AT THE IWAHORI LEVEL

BANAFSHEH FARANG-HARIRI

ABSTRACT. In this Paper we are interested in the geometric local theta correspondence at the Iwahori level for dual reductive pairs (G, H) of type II over a non-Archimedean field of characteristic $p \neq 2$ in the framework of geometric Langlands program. We consider the geometric version of the $I_H \times I_G$ -invariants of the Weil representation $S^{I_H \times I_G}$ as a bimodule under the action Iwahori-Hecke algebras \mathcal{H}_{I_G} and \mathcal{H}_{I_H} and we give some partial geometric description of the corresponding category under the action of Hecke functors. We also define geometric Jacquet functors for any connected reductive group G at the Iwahori level and we show that they commute with the Hecke action of the \mathcal{H}_{I_L} -subalgebra of \mathcal{H}_{I_G} for some Levi subgroup L .

Keywords—geometric representations theory, local theta correspondence, Iwahori-Hecke algebra, perverse sheaves, affine flag varieties.

Mathematics Subject Classification (2010)— Primary 14D24; 11F27; Secondary; 22E57; 20C08

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1. INTRODUCTION

Let \mathbf{k} be a finite field \mathbb{F}_q of characteristic different from 2, let $F = \mathbf{k}((t))$ and $\mathcal{O} = k[[t]]$. All representations will be assumed to be smooth and will be defined over $\overline{\mathbb{Q}}_\ell$, where ℓ is a prime number different from the characteristic of F . The basic notions of the Howe correspondence from the classical point of view have been presented in [25], see also [16]. Let (G, H) be a split dual reductive pair in some symplectic group $Sp(W)$ over \mathbf{k} and let $\widetilde{Sp}(W)$ be the metaplectic group which is the twofold topological covering of the symplectic group $Sp(W)$. Let \mathcal{S} be the Weil representation of $\widetilde{Sp}(W)$. Assume that the metaplectic cover

$\widetilde{Sp}(W) \rightarrow Sp(W)$ admits a section over $G(F)$ and $H(F)$, then Howe correspondence becomes a correspondence between some classes of representations of $G(F)$ and $H(F)$. It is well-known that Howe correspondence realizes Arthur-Langlands functoriality in some special cases [1], [3], [16] [26], [21], [10]. It is of great interest to understand the geometry underlying the Howe correspondence and establish its analogue in the framework of the geometric Langlands program (see [11], [7], [21], [17], [10]). The unramified geometric Howe correspondence has been studied in [17] and [21]. One of our motivations is to extend the results of [21] to the Iwahori (tamely ramified) case in the geometric setting and complete the description of the Howe correspondence for dual reductive pairs of type II already initiated in [10]. This will be a new step towards proving the relation between Howe correspondence and Arthur-Langlands functoriality conjecture for dual reductive pairs of type II announced in [9, Conjecture 1.2].

In the sequel we will restrict ourselves to the dual reductive pairs of type II. More Precisely, let L_0 (resp. U_0) be a n -dimensional (resp. m -dimensional) \mathbf{k} -vector space with $n \leq m$, and let $G = \mathbf{GL}(L_0)$ and $H = \mathbf{GL}(U_0)$. Denote by $\Pi(F)$ the space $(U_0 \otimes L_0)(F)$ and $\mathcal{S}(\Pi(F))$ the Schwartz space of locally constant functions with compact support on $\Pi(F)$. This space realizes the restriction of the Weil representation to $G(F) \times H(F)$. According to Howe and [22], we know that the Howe correspondence associates to any smooth irreducible representation of $G(F)$ appearing as a quotient of the restriction of the Weil representation a unique smooth irreducible non-zero representation of $H(F)$, denoted by $\theta_{n,m}(\pi)$, such that $\pi \otimes \theta_{n,m}(\pi)$ is a quotient of the restriction of the Weil representation to $G(F) \times H(F)$.

One of the most interesting classes of representations to be considered for the study of the Howe correspondence is the class of tamely ramified representations. A representation of $G(F)$ is said to be tamely ramified if it admits a non zero vector fixed under an Iwahori subgroup I_G of $G(F)$. Let us consider the functor sending any tamely ramified representation V of $G(F)$ to its space of invariants V^{I_G} under I_G . Then, the latter is naturally a module over the Iwahori-Hecke algebra \mathcal{H}_{I_G} . According to [8, Theorem 4.10] this functor is an equivalence of categories between the category of tamely ramified admissible representations of G and the category of finite-dimensional modules over \mathcal{H}_{I_G} . Moreover, this functor is exact over the category of smooth representations of $G(F)$. Hence, in the tamely ramified case, we can interpret the Howe correspondence in the language of modules over Iwahori-Hecke algebras. The space $\mathcal{S}(\Pi(F))^{I_H \times I_G}$ of $I_H \times I_G$ -invariants in the Schwartz space $\mathcal{S}(\Pi(F))$ is naturally a module over each Iwahori-Hecke algebra \mathcal{H}_{I_G} and \mathcal{H}_{I_H} . We would like to understand this module structure by geometric means. The geometric analogue of the Schwartz space $\mathcal{S}(\Pi(F))^{I_H \times I_G}$ and the action of Iwahori-Hecke algebras of G and H on it have been already constructed in [10, §3]. Namely, in the geometric setting the space $\mathcal{S}(\Pi(F))^{I_H \times I_G}$ is the category $P_{I_H \times I_G}(\Pi(F))$ of $I_H \times I_G$ -equivariant perverse sheaves on $\Pi(F)$, its precise definition involves some limit procedure. Denote by $D_{I_H \times I_G}(\Pi(F))$ the derived category of ℓ -adic $I_H \times I_G$ -equivariant sheaves on $\Pi(F)$ constructed in [10]. The action of Iwahori-Hecke algebras is geometrized to an action of Hecke functors. Denote by $\mathcal{F}l_G$ the affine flag variety of G and by $P_{I_G}(\mathcal{F}l_G)$ the category of I_G -equivariant perverse sheaves on $\mathcal{F}l_G$. These Hecke functors define an action of $P_{I_G}(\mathcal{F}l_G)$ and $P_{I_H}(\mathcal{F}l_H)$ on $P_{I_H \times I_G}(\Pi(F))$. This geometrization has actually been done in a more general setting of any dual reductive pair and at the level of derived categories in [10]. While in [10, §7] we gave an explicit description of the bimodule $P_{I_H \times I_G}(\Pi(F))$ in the case of $n = 1$ and $m \geq 1$, in this article, we obtain some partial results towards the description of the category $P_{I_H \times I_G}(\Pi(F))$ as a module over the Hecke functors for \mathbf{GL}_n and \mathbf{GL}_m for any $n \leq m$.

One of the key steps in [10] is the description of the simple objects of the category $P_{I_H \times I_G}(\Pi(F))$. Let us recall this result. Let $S_{n,m}$ denote the set of pairs: a subset $I_s \subset \{1, \dots, m\}$ of n elements and a bijection $s : I_s \rightarrow \{1, \dots, n\}$. For N, r two integers such that $N + r > 0$, let $\Pi_{N,r} = t^{-N}\Pi/t^r\Pi$. Fix a maximal torus T in G and a Borel subgroup B containing T . Denote X_G the lattice of cocharacters of T . For each pair (λ, s) in $X_G \times S_{n,m}$, we have introduced some subvarieties $\Pi_{N,r}^w$ in $\Pi_{N,r}$ for N, r large enough and we obtained the following result in [10, Theorem 6.7]: the simple objects of the category $P_{I_H \times I_G}(\Pi(F))$ are parametrized by $X_G \times S_{n,m}$. For any element $w = (\lambda, s)$ in $X_G \times S_{n,m}$, the irreducible object of $P_{I_H \times I_G}(\Pi(F))$ indexed by w is the intersection cohomology sheaf \mathcal{I}^w of $\Pi_{N,r}^w$ for N, r large enough. We also introduced the objects $\mathcal{I}^{w!}$, which are extensions by zero of the constant perverse sheaf under $\Pi_{N,r}^w \hookrightarrow \Pi_{N,r}$. Our aim is to understand as possible as we can the action of Hecke functors on these simple objects. In this direction we construct a filtration on the category $P_{I_H \times I_G}(\Pi(F))$ which is compatible with the action of Hecke functors. We study some submodules of $P_{I_H \times I_G}(\Pi(F))$ and give a precise description of those under the action of Hecke functors. Particularly we construct the geometric version of the first term of Kudla's filtration and we show that it can be identified with the induced representation from a parabolic subalgebra of \mathcal{H}_{I_H} by geometric means. We also construct the geometric version of Jacquet functors at the Iwahori level and show that they commute with the action of the subalgebra \mathcal{H}_{I_L} of \mathcal{H}_{I_G} for some Levi subgroup L .

Let us briefly discuss how the paper is organized. In section 3, assuming $n \leq m$, we introduce a filtration on $P_{I_H \times I_G}(\Pi(F))$ indexed by \mathbb{Z} and show that it is compatible with the natural grading of $P_{I_G}(\mathcal{F}l_G)$ given by the connected components of $\mathcal{F}l_G$ and by Hecke functors. This filtration on $P_{I_H \times I_G}(\Pi(F))$ is expected to be compatible with the filtration already studied in the conjectural bimodule $K(X)$ in [10, §8] describing the geometric local Arthur-Langlands functoriality at the Iwahori level for some map between corresponding dual Langlands groups.

Consider the element $w = (0, (I, w_0)) \in X_G \times S_{n,m}$, where $I = \{1, \dots, n\}$ and $w_0 : I \rightarrow \{1, \dots, 0\}$ is the longest element of the finite Weyl group of G . We obtain some results on the submodule in $P_{I_H \times I_G}(\Pi(F))$ generated by \mathcal{I}^{w_0} (resp., $\mathcal{I}^{w_0!}$). In subsection 4.1, we consider the case $n = m$. The submodule in $D_{I_H \times I_G}(\Pi(F))$ generated by $\mathcal{I}^{w_0!}$ over $P_{I_G}(\mathcal{F}l_G)$ is free of rank one (and is also preserved by $P_{I_H}(\mathcal{F}l_H)$). We also precise an equivalence of categories $\tilde{\sigma} : P_{I_H}(\mathcal{F}l_H) \xrightarrow{\sim} P_{I_G}(\mathcal{F}l_G)$ which defines at the level of functions an anti-involution of Iwahori-Hecke algebras \mathcal{H}_{I_G} and \mathcal{H}_{I_H} . By means of this equivalence we relate the actions of Hecke functors for H and G on the submodule generated by \mathcal{I}^{w_0} (resp. $\mathcal{I}^{w_0!}$). In subsection 4.2, we assume $n \leq m$ and we consider the submodule Θ in the Grothendieck group $K(D_{I_H \times I_G}(\Pi(F)))$ of $D_{I_H \times I_G}(\Pi(F))$ under the action of $K(P_{I_G}(\mathcal{F}l_G))$ generated by the elements $\mathcal{I}^{\mu!}$, where I_μ runs through all possible subsets of n elements in $\{1, \dots, m\}$. We show that the submodule Θ is free of rank C_m^n over $K(P_{I_G}(\mathcal{F}l_G))$. The elements $\mathcal{I}^{\mu!}$ form a basis of this module over $K(P_{I_G}(\mathcal{F}l_G))$. The submodule Θ is a key object in the proof of the classical Howe correspondence. It is indeed the first term of the Kudla's filtration defined over the Weil representation in [25]. The considerations in this subsection are essentially on the level of Grothendieck groups, we formulate them on the level of derived categories however when this is possible. One may assume that we work over a finite field with pure complexes only. More precisely, let \mathcal{S}_0 be the $\bar{\mathbb{Q}}_\ell$ -subspace of $K(D_{I_H \times I_G}(\Pi(F))) \otimes \bar{\mathbb{Q}}_\ell$ generated by elements of the form $\mathcal{I}^{(w, w_0)!}$, where w runs through \widetilde{W}_G the affine extended Weyl group of G . The module \mathcal{S}_0 is a free module of rank one over \mathcal{H}_{I_G} . We consider the standard Levi subgroup M of H corresponding to the partition $(n, m - n)$ of m and we recall briefly the construction of the subalgebra \mathcal{H}_{I_M} of the Iwahori-Hecke algebra \mathcal{H}_{I_H} and some properties according to [24]. Then we endow \mathcal{S}_0 with a right

action of \mathcal{H}_{I_M} and by parabolic induction we construct an induced module. We obtain the two following results: The space \mathcal{S}_0 is a submodule of $K(D_{I_G \times I_H}(\Pi(F))) \otimes \overline{\mathbb{Q}}_\ell$ for the right action of \mathcal{H}_{I_M} . The adjunction map $\alpha : \mathcal{S}_0 \otimes_{\mathcal{H}_{I_M}} \mathcal{H}_{I_H} \rightarrow \mathcal{S}^{I_H \times I_G}(\Pi(F))$ is injective and its image equals $\Theta \otimes \overline{\mathbb{Q}}_\ell$. In the rest of this subsection we show that the action of the Iwahori-Hecke algebra of the factor \mathbf{GL}_n of M identifies with the action of \mathcal{H}_{I_G} via the anti-involution $\tilde{\sigma}$ defined in Theorem 4.11. The action of the Iwahori-Hecke algebra of \mathbf{GL}_{m-n} is by shifting by $[-\ell(w)]$, where ℓ denotes the length function on \widetilde{W}_G .

At last in section 5, we construct a geometric analogue of Jacquet functors for any connected split reductive group G over \mathbf{k} in the Iwahori case. These functors are a key step in the geometric proof of the Howe correspondence in the unramified case in [21]. Moreover, we prove that they are compatible with the Hecke action of the \mathcal{H}_{I_L} -subalgebra of \mathcal{H}_{I_G} . We also show that they preserve pure perverse sheaves of weight zero.

In Appendix 6 we recall the construction of Hecke functors from [10] and in Appendix 7 we present a complete calculation of the Hecke functor corresponding to H in the special case of objects I^μ .

Acknowledgement: The author would like to thank her advisor Sergey Lysenko for initiating her to this subject and sharing his insights and ideas.

2. NOTATION AND SET UP

Let \mathbf{k} be an algebraically closed field of characteristic $p > 2$ and let $F = \mathbf{k}((t))$ be the field of Laurent series with coefficients in \mathbf{k} and $\mathcal{O} = \mathbf{k}[[t]]$ be its ring of integers. Denote by ℓ a prime number different from p . Let L_0 (resp. U_0) be a n -dimensional (resp. m -dimensional) \mathbf{k} -vector space with $n \leq m$, and let $G = \mathbf{GL}(L_0)$ and $H = \mathbf{GL}(U_0)$. Denote by $\{e_1, \dots, e_n\}$ the standard basis of L_0 and $\{u_1, \dots, u_m\}$ the standard basis of U_0 and $\{u_1^*, \dots, u_m^*\}$ its dual basis. Denote by $\Pi(F)$ the space $(U_0 \otimes L_0)(F)$ and $\mathcal{S}(\Pi(F))$ the Schwartz space of $\overline{\mathbb{Q}}_\ell$ -valued locally constant functions with compact support on $\Pi(F)$. Let T_G (resp. T_H) be the standard maximal torus of diagonal matrices in G (resp. H) and B_G (resp. B_H) be the Borel subgroup of upper triangular matrices containing T_G (resp. T_H). Denote by I_G (resp. I_H) the Iwahori subgroup of $G(F)$ (resp. $H(F)$) corresponding to the standard Borel subgroup B_G (resp. B_H). Denote by $(\check{X}_G, \check{R}, X_G, R, \Delta)$ the root datum associated with (G, T_G, B_G) . Throughout this article we denote by \check{X}_G the characters of T_G and X_G denotes the cocharacter lattice of T_G . The set \check{R} is the set of roots and R is the set of coroots and Δ_G is the basis formed by simple roots. If there is no ambiguity we will omit the subscript G . We denote by W_G the finite Weyl group associated with the root datum $(\check{X}_G, \check{R}, X_G, R)$. Let \widetilde{W}_G be the affine extended Weyl group which is the semi-direct product $W_G \ltimes X_G$. Denote by ℓ the length function on \widetilde{W}_G . Let X_G^+ be the set of dominant elements in X_G .

For any scheme or stack locally of finite type over \mathbf{k} , we denote by $D(S)$ the bounded derived category of constructible $\overline{\mathbb{Q}}_\ell$ -sheaves over S . Write \mathbb{D} for the Verdier duality functor and we denote by $P(S)$ the full subcategory of perverse sheaves in $D(S)$. Denote by $K(P(S))$ the Grothendieck group of the category $P(S)$. Let X be a scheme of finite type over \mathbf{k} . For Z a smooth d -dimensional irreducible locally closed subscheme of X and $i : Z \rightarrow X$ the corresponding immersion, we define the intersection cohomology sheaf (IC-sheaf for short), $\mathrm{IC}(Z)$ as the perverse sheaf $i_{Z!}(\overline{\mathbb{Q}}_\ell)[d]$.

Assume temporary that the ground field \mathbf{k} is the finite field \mathbb{F}_q . Denote by \mathcal{H}_{I_G} the Iwahori-Hecke algebra of G which is the space of locally constant, I_G -bi-invariant compactly supported $\overline{\mathbb{Q}}_\ell$ -valued functions on $G(F)$ endowed with the convolution product. There are two well-known presentations of this algebra by generators

and relations. The first is due to Iwahori-Matsumoto [14] and the second is by Bernstein in [20] and [18]. We will use the second one.

Denote by Gr_G the affine Grassmanian associated with G . The $G(\mathcal{O})$ -orbits on Gr_G are parametrized by W_G -orbits in X_G and for a given λ in X_G , the $G(\mathcal{O})$ -orbit associated to $W_G \cdot \lambda$ is $G(\mathcal{O}) \cdot t^\lambda$ denoted by Gr_G^λ , where t^λ is the image of t under the map $\lambda : F^* \rightarrow G(F)$. The I_G -orbits on Gr_G are parametrized by cocharacters λ in X_G . For any λ in X_G , denote by O^λ the I_G -orbit through $t^\lambda G(\mathcal{O})$ in Gr_G and by $\overline{O^\lambda}$ its closure. Each orbit is an affine space. Denote by $\mathcal{F}l_G$ the affine flag variety associated with G and denote by $P_{I_G}(\mathcal{F}l_G)$ the category of I_G -equivariant perverse sheaves on $\mathcal{F}l_G$. This category is endowed with the geometric convolution denoted by \star , see [12], and we have $K(P_{I_G}(\mathcal{F}l_G)) \otimes \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathcal{H}_{I_G}$.

Let R be a \mathbf{k} -algebra. A complete periodic flag of lattices inside $R((t))^n$ is a flag

$$L_{-1} \subset L_0 \subset L_1 \subset \dots$$

such that each L_i is a lattice in $R((t))^n$, each quotient L_{i+1}/L_i is a locally free R -module of rank one and $L_{n+k} = t^{-1}L_k$ for any k in \mathbb{Z} . For $1 \leq i \leq n$, set

$$\Lambda_{i,R} = (\oplus_{j=1}^i t^{-1}R[[t]]e_j) \oplus (\oplus_{j=i+1}^n R[[t]]e_j).$$

For all i in \mathbb{Z} , we set $\Lambda_{i+n,R} = t^{-1}\Lambda_{i,R}$. This defines the standard complete lattice flag

$$\Lambda_{-1,R} \subset \Lambda_{0,R} \subset \Lambda_{1,R} \subset \dots$$

denoted by $\Lambda_{\bullet,R}$ in $R((t))^n$. For any \mathbf{k} -algebra R , the set $\mathcal{F}l_G(R)$ is naturally in bijection with the set of complete periodic lattice flags in $R((t))^n$ and is an ind-scheme. The affine flag variety decomposes as a disjoint union

$$\mathcal{F}l_G = \bigcup_{w \in \widetilde{W}_G} I_G w I_G / I_G.$$

The closure of each Schubert cell, $I_G w I_G / I_G$ is a union of Schubert cells and the closure relations are given by the Bruhat order:

$$\overline{I_G w I_G / I_G} = \bigcup_{w' \leq w} I_G w' I_G / I_G.$$

For any $w \in \widetilde{W}_G$ we will denote the Schubert cell $I_G w I_G / I_G$ by $\mathcal{F}l_G^w$ which is isomorphic to $\mathbb{A}^{\ell(w)}$. For $w \in \widetilde{W}_G$, denote by j_w the inclusion of $\mathcal{F}l_G^w$ in $\mathcal{F}l_G$, and let $L_w = j_{w!*} \overline{\mathbb{Q}}_\ell[\ell(w)](\ell(w)/2)$, the IC-sheaf of $\mathcal{F}l_G^w$. We write $L_{w!} = j_{w!} \overline{\mathbb{Q}}_\ell[\ell(w)](\ell(w)/2)$ and $L_{w*} = j_{w*} \overline{\mathbb{Q}}_\ell[\ell(w)](\ell(w)/2)$ for the standard and costandard objects. They satisfy $\mathbb{D}(L_{w*}) = L_{w!}$. Remark that in the notation of $L_{w!}$ and L_{w*} we wrote the Tate twists. When working over an algebraically closed field, we will forget the Tate twists.

Assume that \mathbf{k} is finite. To \mathcal{G} in $P_{I_G}(\mathcal{F}l_G)$ we attach a function $[\mathcal{G}] : G(F)/I_G \rightarrow \overline{\mathbb{Q}}_\ell$ given by $[\mathcal{G}](x) = Tr(Fr_x, \mathcal{G}_x)$, for x a point in $G(F)/I_G$ and Fr_x is the geometric Frobenius at x . The function $[\mathcal{G}]$ is an element of the \mathcal{H}_{I_G} . In particular $[L_{w!}] = (-1)^{\ell(w)} q_w^{-1/2} T_w$ and $[L_{w*}] = (-1)^{\ell(w)} q_w^{1/2} T_w^{-1}$, where $q_w = q^{\ell(w)}$. Here T_w denotes the characteristic function of the double coset $I_G w I_G$. The map sending λ to L_{t^λ} , for any λ in X_G^+ extends naturally to a monoidal functor

$$R(T) \rightarrow D_{I_G}(\mathcal{F}l_G).$$

The image of λ under the above functor is usually called a Wakimoto sheaf. There are two conventions for defining the Wakimoto sheaves. The first convention is due to Bezrukavnikov in [2]. We will use the

convention due to Prasad in [24] by letting $\Theta_\lambda = L_{t^\lambda}$ for λ dominant and $\Theta_\lambda = L_{t^{\lambda^*}}$ for λ anti-dominant. In any case Wakimoto sheaves verify the following: $\lambda \in X$, if $\lambda = \lambda_1 - \lambda_2$ where λ_i are dominant for $i = 1, 2$, then $\Theta_\lambda \simeq \Theta_{\lambda_1} \star \Theta_{-\lambda_2}$. According to [2, Theorem 5], these are actually objects of the category $P_{I_G}(\mathcal{F}l_G)$ (a priori they are defined as objects of the triangulated category $D_{I_G}(\mathcal{F}l_G)$).

As mentioned above the space $\mathcal{S}(\Pi(F))^{I_H \times I_G}$ is naturally a module over Iwahori-Hecke algebras \mathcal{H}_{I_G} and \mathcal{H}_{I_H} of G and H . The action is defined by convolution. The geometric analogue of the $(\mathcal{H}_{I_G}, \mathcal{H}_{I_H})$ -bimodule $\mathcal{S}(\Pi(F))^{I_H \times I_G}$ is constructed in [10, §3] which is $P_{I_H \times I_G}(\Pi(F))$ the category of $I_H \times I_G$ -equivariant perverse sheaves on $\Pi(F)$ in the derived category $D_{I_H \times I_G}(\Pi(F))$ under the action of the two Hecke functors:

$$\overleftarrow{H}_G : P_{I_G}(\mathcal{F}l_G) \times D_{I_H \times I_G}(\Pi(F)) \longrightarrow D_{I_H \times I_G}(\Pi(F))$$

and

$$\overleftarrow{H}_H : P_{I_H}(\mathcal{F}l_H) \times D_{I_H \times I_G}(\Pi(F)) \longrightarrow D_{I_H \times I_G}(\Pi(F)).$$

For the sake of the reader, we recall briefly the construction of these Hecke functors in Appendix 6. The goal is to understand these two Hecke functors as much as possible.

For any two integers N, r such that $N + r > 0$, let $\Pi_{N,r} = t^{-N}\Pi/t^r\Pi$. Let U^* be the dual of U . A point v in $\Pi(F)$ may be seen as a \mathcal{O} -linear map $v : U^* \rightarrow L(F)$. For v in $\Pi_{N,r}$, let $U_{v,r} = v(U^*) + t^r L$. Then $U_{v,r}$ is a \mathcal{O} -module in $L(F)$ and may be seen as a point of Gr_G . Let $\check{\omega}_1 = (1, 0, \dots, 0)$ be the highest weight of the standard representation of G , let w_0 be the longest element of the finite Weyl group W_G . For $\lambda \in X_G$ such that for any v in $W_G \cdot \lambda$

$$(2.1) \quad \langle v, \check{\omega}_1 \rangle \leq r \quad \text{and} \quad \langle -v, \check{\omega}_1 \rangle \leq N.$$

let $\Pi_{\lambda,r} \subset \Pi_{N,r}$ be the locally closed subscheme of $v \in \Pi_{N,r}$ such that $U_{v,r}$ lies in $I_G t^\lambda G(\mathcal{O})$. According to [10], the $H(\mathcal{O}) \times I_G$ -orbits on $\Pi_{N,r}$ are parameterized by elements λ in X_G satisfying (2.1) and are exactly $\Pi_{\lambda,r}$. Let $S_{n,m}$ be the set of pairs: a subset $I_s \subset \{1, \dots, m\}$ of n elements and a bijection $s : I_s \rightarrow \{1, \dots, n\}$. Let $w = (\lambda, s) \in X_G \times S_{n,m}$, where $\lambda = (a_1, \dots, a_n)$ and assume $a_i < r$ for all i . Denote by $\Pi_{N,r}^w$ the $I_H \times I_G$ -orbit on $\Pi_{N,r}$ through an element v given by

$$(2.2) \quad \begin{cases} v(u_i^*) = t^{a_{si}} e_{si} & \text{for } i \in I_s; \\ v(u_i^*) = 0 & \text{for } i \notin I_s. \end{cases}$$

The closure of $\Pi_{N,r}^w$ in $\Pi_{N,r}$ will be denoted by $\overline{\Pi}_{N,r}^w$. The simple objects of the category $P_{I_H \times I_G}(\Pi(F))$ are parametrized by $X_G \times S_{n,m}$, [10, Theorem 6.7]. For any element $w = (\lambda, s)$ in $X_G \times S_{n,m}$, the simple object of $P_{I_H \times I_G}(\Pi(F))$ indexed by w is the intersection cohomology sheaf \mathcal{I}^w of $\Pi_{N,r}^w$ for N, r large enough. The object of $P_{I_H \times I_G}(\Pi(F))$ so obtained is independent of N, r , so our notation is unambiguous. We denote $\mathcal{I}^{w!}$ the objects which are extensions by zero of the constant perverse sheaf under $\Pi_{N,r}^w \hookrightarrow \Pi_{N,r}$.

3. FILTRATION AND GRADING

Denote by $\pi_1(G)$ the algebraic fundamental group of G which is formed by the elements of length zero in the affine extended Weyl group of G . The connected components of Gr_G and $\mathcal{F}l_G$ are indexed by $\pi_1(G)$. This yields a following grading

$$\bigoplus_{\theta \in \pi_1(G)} P_{I_G}(\mathcal{F}l_G^\theta) \xrightarrow{\simeq} P_{I_G}(\mathcal{F}l_G).$$

Denote by $\check{\omega}_n$ the character by which the group G acts on $\det(L_0)$, i.e. $\check{\omega}_n = (1, \dots, 1)$. We may identify $\pi_1(G) \xrightarrow{\sim} \mathbb{Z}$ via the map $\theta \mapsto \langle \theta, \check{\omega}_n \rangle$. This grading is compatible with the convolution product on $P_{I_G}(\mathcal{F}l_G)$. There is also a grading $\mathcal{H}_{I_G}^k$, $k \in \mathbb{Z}$ of the Iwahori-Hecke algebra \mathcal{H}_{I_G} , see [20]. Besides the isomorphism between $K(P_{I_G}(\mathcal{F}l_G)) \otimes \overline{\mathbb{Q}_\ell}$ and \mathcal{H}_{I_G} becomes a graded isomorphism.

There exists a filtration on the category $P_{I_H \times I_G}(\Pi(F))$ indexed by \mathbb{Z} . Indeed, for an integer a in \mathbb{Z} , let Filt^a be the full subcategory in $P_{I_H \times I_G}(\Pi(F))$ defined as the Serre subcategory generated by the objects \mathcal{I}^w , where $w = t^\lambda \tau$ are elements of $X_G \times S_{n,m}$ satisfying $\langle \lambda, \check{\omega}_n \rangle \geq a$. We are going to show that this filtration is compatible with the action of $P_{I_G}(\mathcal{F}l_G)$ and the grading on it.

Let $w = t^\lambda \tau$ and $u = t^\mu \nu$ be two elements in $X_G \times S_{n,m}$. The condition that $\Pi_{N,r}^u$ lies in the closure of $\Pi_{N,r}^w$ implies that $\langle \mu, \check{\omega}_n \rangle \geq \langle \lambda, \check{\omega}_n \rangle$. Indeed, for any point v of $\Pi_{N,r}$, the dimension of $U_{v,r}/t^r L$ can only decrease under specialization. For the orbit $\Pi_{N,r}^u$ lying in the closure of $\Pi_{N,r}^w$ the number $\langle \mu, \check{\omega}_n \rangle$ can be arbitrary large. This number depends on r is not uniformly bounded.

Lemma 3.1. *Let $w_1 = t^{\lambda_1} \tau_1$, and $w_2 = t^{\lambda_2} \tau_2$, be two elements in $X_G \times S_{n,m}$. For $i = 1, 2$ choose two integers N_i and r_i such that the following conditions are satisfied : for any*

$$v \in W_G \lambda_i, \langle v, \check{\omega}_1 \rangle \leq r_1, \langle v, \check{\omega}_1 \rangle < r_2, \text{ and } \langle -v, \check{\omega}_1 \rangle \leq N_i.$$

Let v be an element in $\Pi_{N_2, r_2}^{w_2} \subset \Pi_{\lambda_2, r_2}$ and gI_G be an element in $\mathcal{F}l_G^{w_1}$. For $i = 1, 2$ let μ_i be a dominant cocharacter lying in $W_G \lambda_i$. Then there exists a cocharacter μ smaller than or equal to $\mu_1 + \mu_2$ such that gv belongs to Π_{μ, r_1+r_2} .

Proof. The lattice $U_{v, r_2} = v(U^*) + t^{r_2} L$ lies in $O^{\lambda_2} \subset Gr_G^{\mu_2}$. Thus $g(U_{v, r_2})$ lies in gO^{λ_2} . Since $gG(O) \in O^{\lambda_1} \subset Gr_G^{\mu_1}$, we have

$$I_G t^{\lambda_1} G(O) t^{\lambda_2} G(O) / G(O) \subset \overline{Gr}_G^{\mu_1 + \mu_2}$$

and this implies the assertion. \square

Proposition 3.2. *Let $w_1 = t^{\lambda_1} \tau_1$, and $w_2 = t^{\lambda_2} \tau_2$, be two elements in $X_G \times S_{n,m}$. For $i = 1, 2$ choose two integers N_i and r_i such that the following conditions are satisfied : for any*

$$v \in W_G \lambda_i, \langle v, \check{\omega}_1 \rangle \leq r_1, \langle v, \check{\omega}_1 \rangle < r_2, \text{ and } \langle -v, \check{\omega}_1 \rangle \leq N_i.$$

Then $\overleftarrow{H}_G(L_{w_1}, \mathcal{I}^{w_2})$ lies in Filt^d with $d = \langle \lambda_1 + \lambda_2, \check{\omega}_n \rangle$.

Proof. The sheaf \mathcal{I}^{w_2} is the IC-sheaf of the orbit $\Pi_{N_2, r_2}^{w_2}$ which is a subspace of Π_{λ_2, r_2} . In the notation of Appendix 6, we have $\overline{\mathcal{F}l}_G^{w_1} \subset_{r_1, N_1} \mathcal{F}l_G$. Choose $r \geq r_1 + r_2$ and $s \geq N_2 + r_2$. The space $\Pi_{N_2, r_2} \overline{\mathcal{F}l}_G^{w_1}$ is the scheme classifying pairs (v, gI_G) , where gI_G is in $\overline{\mathcal{F}l}_G^{w_1}$ and v is in $t^{-N_2} g\Pi/t^r \Pi$. We have the following diagram

$$\Pi_{N_1+N_2, r} \xleftarrow{\pi} \Pi_{N_2, r_2} \overline{\mathcal{F}l}_G^{w_1} \xrightarrow{\text{act}_{q,s}} K_s \setminus (\Pi_{N_2, r_2}),$$

where π is the projection sending (v, gI_G) to v . Let $\mathcal{I}^{w_2} \tilde{\boxtimes} L_{w_1}$ be the twisted exterior product of \mathcal{I}^{w_2} and L_{w_1} over $\Pi_{N_2, r_2} \overline{\mathcal{F}l}_G^{w_1}$ which is normalized to be perverse. Then by definition

$$\overleftarrow{H}_G(L_{w_1}, \mathcal{I}^{w_2}) \xrightarrow{\sim} \pi_1(\mathcal{I}^{w_2} \tilde{\boxtimes} L_{w_1}).$$

In our case $\mathcal{I}^{w_2} \boxtimes L_{w_1}$ is the IC-sheaf of $\text{act}_{q,s}^{-1}(\overline{\Pi}_{N_2, r_2}^{w_2})$. For a point v in $\Pi_{N_1+N_2, r}$, let μ be in X_G such that $U_{v,r}$ lies in O^μ . The part of the fibre of the map π over v that contributes to $\pi_1(\mathcal{I}^{w_2} \boxtimes L_{w_1})$ is

$$\{gI_G \in \overline{\mathcal{F}l_G^{w_1}} | g^{-1}v \in \overline{\Pi}_{N_2, r_2}^{w_2}\}.$$

The latter scheme is empty unless $\langle \mu, \check{\omega}_n \rangle \geq \langle \lambda_1 + \lambda_2, \check{\omega}_n \rangle$. It follows that $\overleftarrow{H}_G(L_{w_1}, \mathcal{I}^{w_2})$ lies in Filt^d with $d = \langle \lambda_1 + \lambda_2, \check{\omega}_n \rangle$. \square

Theorem 3.3. *Assume $n \leq m$. The filtration Filt^d on $P_{I_H \times I_G}(\Pi(F))$ is compatible with the grading on $P_{I_G}(\mathcal{F}l_G)$ defined by the connected components. Namely set $w_1 = t^{\lambda_1} \tau$ with τ in $S_{n,m}$, λ_1 in X_G , and let $m_1 = \langle \lambda_1, \check{\omega}_n \rangle$. Then $\overleftarrow{H}_G(L_{w_1}, \cdot)$ sends an irreducible object of Filt^d to a direct sum of shifted objects of Filt^{d+m_1} .*

Proof. We use the notation of Lemma 3.1. For gI_G in $\overline{\mathcal{F}l_G^{w_1}}$, let $L' = gL$ and equip L' with the flag $L'_i = gL_i$ for $i = 1, \dots, n$. Let v be the map from U^* to $t^{-N_2} L' / t^r L$ such that its composition with

$$t^{-N_2} L' / t^r L \longrightarrow t^{-N_2} L' / t^{r_2} L'$$

lies in the closure of the orbit $(U_{N_2, r_2} \otimes L')^{w_2}$. The latter scheme is the corresponding orbit on $U_{N_2, r_2} \otimes L'$. The relative dimension formula gives us

$$\dim(L, L') + \dim(L', v(U^*) + t^{r_2} L') = \dim(L, v(U^*) + t^r L').$$

Moreover, we have

$$\dim(L', v(U^* + t^{r_2} L')) \geq \langle \lambda_2, \check{\omega}_n \rangle$$

and

$$\dim(L, L') = \langle \lambda_1, \check{\omega}_n \rangle.$$

This leads to $\dim(L, v(U^* + t^{r_2} L')) \geq \langle \lambda_1 + \lambda_2, \check{\omega}_n \rangle$. On the other hand we have $t^r L \subset t^{r_2} L'$ so $\dim(L, v(U^*) + t^r L)$ can not be strictly smaller than $\dim(L, v(U^*) + t^{r_2} L')$. \square

As a consequence of this proposition $K(P_{I_H \times I_G}(\Pi(F)))$ is a filtered module over \mathcal{H}_{I_G} , so that graded part $\oplus_{d \in \mathbb{Z}} \text{Filt}^d / \text{Filt}^{d+1}$ is a left \mathcal{H}_{I_G} -module.

4. KUDLA'S FILTRATION AND SOME SUB-MODULES

4.1. Case $n = m$. We will assume $n = m$ in this subsection. Denote by w_0 the longest element of W_G . Let \mathcal{I}^{w_0} be the IC-sheaf of the orbit $\Pi_{0,1}^{w_0}$. Let $\Pi_{\mathcal{I}^{w_0}}$ be subscheme of $\text{Hom}(U_0^*, L_0)$ consisting of elements v such that v sends $\text{Vect}(u_n^*, \dots, u_{n-i+1}^*)$ to $L_i = \text{Vect}(e_1, \dots, e_i)$, for $i = 1, \dots, n$. Note that $\Pi_{\mathcal{I}^{w_0}}$ is an affine space, the closure of $\Pi_{0,1}^{w_0}$. Thus \mathcal{I}^{w_0} is the constant perverse sheaf on $\Pi_{\mathcal{I}^{w_0}}$. For any w in the affine extended Weyl group \widetilde{W}_G , let us describe $\overleftarrow{H}_G(L_w, \mathcal{I}^{w_0})$. Let $w = t^\lambda \tau$, where $\tau \in W_G$ and $\lambda \in X_G$. Let N, r be two integers with $N + r \geq 0$ such that the following condition is verified: for any v in $W_G \cdot \lambda$ we have

$$(4.1) \quad \langle v, \check{\omega}_1 \rangle < r \text{ and } \langle -v, \check{\omega}_1 \rangle \leq N.$$

For any element gI_G of $\overline{\mathcal{F}l_G^w}$, we put $L' = gL$ and we equip L' / tL' with the flag $L'_i = gL_i$, for $i = 1, \dots, n$. Let $\Pi_{\mathcal{I}^{w_0}, r} \overline{\mathcal{F}l_G^w}$ be the scheme classifying pairs (v, gI_G) , where gI_G is in $\overline{\mathcal{F}l_G^w}$ and v is a map $U^* \rightarrow L' / tL'$ such that the induced map

$$(4.2) \quad \bar{v} : U^* / tU^* \longrightarrow L' / tL'$$

sends $\text{Vect}(u_n^*, \dots, u_{n-i}^*)$ to L'_{i+1} , for $i = 0, \dots, n-1$. Let

$$\pi : \Pi_{\mathcal{I}^{w_0}, r} \widetilde{\mathcal{F}} \mathcal{I}_G^w \longrightarrow \Pi_{N, r}$$

be the map sending (v, gI_G) to v . The second projection $pr : \Pi_{\mathcal{I}, r} \widetilde{\mathcal{F}} \mathcal{I}_G^w \longrightarrow \widetilde{\mathcal{F}} \mathcal{I}_G^w$ is a locally trivial fibration with fibres isomorphic to the affine space. Let $\overline{\mathcal{Q}}_\ell \boxtimes L_w$ be the perverse sheaf $pr^* L_w[\dim. \text{rel}(pr)]$. Then by definition

$$\overleftarrow{H}_G(L_w, \mathcal{I}^{w_0}) \xrightarrow{\sim} \pi_!(\overline{\mathcal{Q}}_\ell \boxtimes L_w).$$

Remark that the condition (4.1) initially appears in the construction of the irreducible objects \mathcal{I}^w in $P_{I_H \times I_G}(\Pi(F))$ in [10].

Lemma 4.3. *For any $w \in \widetilde{W}_G$ the perverse sheaf \mathcal{I}^{ww_0} appears in $\overleftarrow{H}_G(L_w, \mathcal{I})$ with multiplicity one.*

Proof. Consider the open subscheme V in $\Pi_{\mathcal{I}, r} \widetilde{\mathcal{F}} \mathcal{I}_G^w$ given by the condition that $gI \in \mathcal{F} \mathcal{I}_G^w$, and the map (4.2) is surjective. Clearly, $\pi(V)$ is contained in $\Pi_{N, r}^{ww_0}$. So, π can be viewed as a map

$$\pi : \Pi_{\mathcal{I}, r} \widetilde{\mathcal{F}} \mathcal{I}_G^w \longrightarrow \overline{\Pi}_{N, r}^{ww_0}$$

The restriction of the complex $\overleftarrow{H}_G(L_w, \mathcal{I}^{w_0})$ to $\Pi_{\lambda, r} \subset \Pi_{N, r}$ identifies with \mathcal{I}^{ww_0} . \square

Proposition 4.4. *For any w in \widetilde{W}_G there is a canonical isomorphism $\overleftarrow{H}_G(L_{w!}, \mathcal{I}^{w_0!}) \xrightarrow{\sim} \mathcal{I}^{ww_0!}$.*

Proof. Let $\Pi_{\mathcal{I}^{w_0}, r}^0 \widetilde{\mathcal{F}} \mathcal{I}_G^w \subset \Pi_{\mathcal{I}^{w_0}, r} \widetilde{\mathcal{F}} \mathcal{I}_G^w$ be the open subscheme of pairs (v, gI_G) such that gI_G is in $\mathcal{F} \mathcal{I}_G^w$ and the map \bar{v} in (4.2) is an isomorphism. For any points (v, gI_G) in this subscheme the map \bar{v} is an isomorphism between $\text{Vect}(u_1^*, \dots, u_{n-i}^*)$ and L'_{i+1} , for $i = 0, \dots, n-1$. Let

$$\pi^0 : \Pi_{\mathcal{I}^{w_0}, r}^0 \widetilde{\mathcal{F}} \mathcal{I}_G^w \longrightarrow \Pi_{N, r}$$

be the restriction of π to $\Pi_{\mathcal{I}^{w_0}, r}^0 \widetilde{\mathcal{F}} \mathcal{I}_G^w$. Thus by definition we have

$$\overleftarrow{H}_G(L_{w!}, \mathcal{I}^{w_0!}) \xrightarrow{\sim} \pi_0^!(\overline{\mathcal{Q}}_\ell \boxtimes L_w)$$

The image of π^0 is equal to $\Pi_{N, r}^{ww_0}$ and the map π^0 is an isomorphism onto its image. \square

Definition 4.5. *For λ in X_G and τ in W_G , let $w \longrightarrow \bar{w}$ be the map $\widetilde{W}_G \rightarrow \widetilde{W}_G$ defined by*

$$\bar{t}^\lambda \tau \longrightarrow t^{\tau^{-1}(\lambda)} \tau^{-1}$$

This is an anti-automorphism of \widetilde{W}_G . Note that $\bar{w}_0 = w_0$.

The following analogue of Proposition 4.4 for H instead of G is proved similarly.

Proposition 4.6. *For w in \widetilde{W}_H , the complex $\overleftarrow{H}_H(L_{w!}, \mathcal{I}^{w_0!})$ is canonically isomorphic to $\overline{\mathcal{I}^{ww_0!}}$. The sheaf $\overline{\mathcal{I}^{ww_0}}$ occurs in $\overleftarrow{H}_H(L_w, \mathcal{I}^{w_0})$ with multiplicity one. \square*

The following example shows that $\overleftarrow{H}_G(L_w, \mathcal{I}^{w_0})$ is not always irreducible and gives us some interesting objects in $D_{I_H \times I_G}(\Pi(F))$.

Corollary 4.7. *Let $1 \leq i < n$. Let w be the transposition $(i, i+1)$ in W_G , $\lambda = (0, \dots, 0, 1, 0, \dots, 0)$, where 1 appears on the i^{th} position, and $w' = t^\lambda w_0$. Then*

$$\overleftarrow{H}_G(L_w, \mathcal{I}^{w_0}) \xrightarrow{\sim} \mathcal{I}^{ww_0} \oplus \mathcal{I}^{w'}.$$

Proof. The variety $\overline{\mathcal{F}l_G^w}$ classifies lattices L'_0 endowed with a complete flag of lattices $L'_{-1} \subset L'_0 \subset L'_1 \subset \dots$ such that $L'_{i+n} = t^{-1}L'_i$ for all i , and $L'_j = L_j$ unless $j = i \bmod n$. Here L_j is the standard flag on $L(F)$. So, $\overline{\mathcal{F}l_G^w}$ identifies with the projective space of lines in L_{i+1}/L_{i-1} . Let Y_i be the closed subscheme of $\Pi_{0,1}$ given by $v(W_j) \subset L_j$ for $j \neq i$. Here $\{W_j\}$ is the flag on U_0^* preserved by B_H . Note that Y_i is an affine space. Define a closed subscheme Y'_i of Y_i consisting of elements v of Y_i such that $v(L_i) \subset L_{i-1}$. Then Y'_i is also an affine space.

Let $\Pi_I \times \overline{\mathcal{F}l_G^w}$ be the scheme classifying pairs (v, gI_G) , where gI_G is in $\overline{\mathcal{F}l_G^w}$ and $v : U_0^* \rightarrow L_0$ such that $v(W_j) \subset gL_j$, for all $1 \leq j \leq n$. We have the diagram

$$Y_i \xleftarrow{\pi} \Pi_I \times \overline{\mathcal{F}l_G^w} \xrightarrow{pr} \overline{\mathcal{F}l_G^w}.$$

By definition of the Hecke operators one has

$$\overleftarrow{H}_G(L_w, \mathcal{I}^{w_0}) = \pi_!(\overline{\mathbb{Q}}_\ell \boxtimes L_w).$$

For a point v in $Y_i \setminus Y'_i$ the fibre of the map π over v is reduced to a point and the map π is an isomorphism over $Y_i \setminus Y'_i$. The restriction of $\overleftarrow{H}_G(L_w, \mathcal{I}^{w_0})$ to $Y_i \setminus Y'_i$ is isomorphic to $\text{IC}(Y_i) = \overline{\mathbb{Q}}_\ell[\dim(Y_i)]$. On the other hand, the space Y_i identifies with $\Pi_{N,r}^{w_0}$. The fibre of π over a point v of Y'_i is isomorphic to \mathbb{P}^1 . Since Y'_i is an affine space of codimension 2 in Y_i , we are done. \square

Proposition 3.3 in the special case of \mathcal{I}^{w_0} yields the following.

Corollary 4.8. *Let $w = t^\lambda \tau$, where λ in X_G and τ is in W_G . Then if $d = \langle \lambda, \check{\omega}_n \rangle$, there exists K in Filt^{d+1} such that $\overleftarrow{H}_G(L_w, \mathcal{I}^{w_0}), \xrightarrow{\sim} \mathcal{I}^{w_0} \oplus K$. \square*

It also follows that for $n = m$, the space $\oplus_{d \in \mathbb{Z}} \text{Filt}^d / \text{Filt}^{d+1}$ is a free module of rank one over \mathcal{H}_{I_G} generated by \mathcal{I}^{w_0} . Note that the homomorphism of \mathcal{H}_{I_G} -algebras $\mathcal{H}_{I_G} \rightarrow \mathcal{S}(\Pi(F))^{I_H \times I_G}$ sending \mathcal{S} to $\overleftarrow{H}_G(\mathcal{S}, \mathcal{I}^{w_0!})$ is injective. The submodule generated by $\mathcal{I}^{w_0!}$ is a free module of rank one over each one of the Iwahori-Hecke algebra \mathcal{H}_{I_G} and \mathcal{H}_{I_H} .

We may stratify $\Pi_{N,r}$ in a slightly different way. Let θ be any element of $\pi_1(G)$ and let λ be a lift of θ in X_G satisfying condition (4.1). We define a locally closed subscheme $\Pi_{N,r}^\theta$ of $\Pi_{N,r}$ as follows:

$$v : U^* \rightarrow t^{-N}L/t^rL \text{ such that } \dim(U_{v,r}/t^rL) = \dim(t^\lambda L/t^rL).$$

This definition is in fact independent of the lift λ .

For a given $w = t^\lambda \tau$ in \widetilde{W}_G , let θ be the image of λ in $\pi_1(G)$. Let $\widetilde{\mathcal{I}}^w$ be the extension by zero of $\mathcal{I}^w|_{\Pi_{N,r}^\theta}$ on $\Pi_{N,r}$. Then we have the following result:

Proposition 4.9. *For any w in \widetilde{W}_G , we have two canonical isomorphism*

$$\overleftarrow{H}_G(L_w, \mathcal{I}^{w_0!}) \xrightarrow{\sim} \widetilde{\mathcal{I}}^{w_0} \quad \overleftarrow{H}_H(L_w, \mathcal{I}^{w_0!}) \xrightarrow{\sim} \widetilde{\mathcal{I}}^{w_0 \bar{w}},$$

where the anti-involution \bar{w} is defined in Definition 4.5.

Proof. We show the assertion for \overleftarrow{H}_G , the case of \overleftarrow{H}_H may be proved similarly. As in Proposition 4.4, consider the open subscheme $\Pi_{\mathcal{I}^{w_0, r}}^0 \widetilde{\mathcal{F}} \mathcal{I}_G^w$ inside $\Pi_{\mathcal{I}^{w_0, r}} \overleftarrow{\mathcal{F}} \mathcal{I}_G^w$ given by the additional condition that the map

$$\bar{v} : U^* / tU^* \rightarrow L' / tL'$$

is an isomorphism. Thus the restriction

$$\pi^0 : \Pi_{\mathcal{I}^{w_0, r}}^0 \widetilde{\mathcal{F}} \mathcal{I}_G^w \rightarrow \Pi_{N, r}$$

of the map π is locally a closed immersion. Therefore by definition $\overleftarrow{H}_G(L_w, \mathcal{I}^{w_0!}) \xrightarrow{\sim} \pi_1^0(\overline{\mathbb{Q}}_\ell \boxtimes L_w) \xrightarrow{\sim} \tilde{\mathcal{I}}^{ww_0}$. \square

The map from $G(F)$ to $G(F)$ sending g an element of $G(F)$ to g^{-1} induces an equivalence of categories

$$\star^\sharp : P_{I_G}(\mathcal{F}l_G) \xrightarrow{\sim} P_{I_G}(\mathcal{F}l_G).$$

The similar equivalence of categories holds for $P_{I_H}(\mathcal{F}l_H)$. Hence for w in the affine extended Weyl group, we have canonical isomorphisms

$$\star^\sharp(L_w) \xrightarrow{\sim} L_{w^{-1}}, \quad \star^\sharp(L_{w!}) \xrightarrow{\sim} L_{w^{-1}!}, \quad \star^\sharp(L_{w*}) \xrightarrow{\sim} L_{w^{-1}*}.$$

At the level of Iwahori-Hecke algebras $\star^\sharp : \mathcal{H}_{I_G} \rightarrow \mathcal{H}_{I_G}$ is an anti-isomorphism of algebras.

Definition 4.10. Assume that $n \leq m$. For any \mathcal{T} in $P_{I_H}(\mathcal{F}l_H)$ and \mathcal{K} in $P_{I_H \times I_G}(\Pi(F))$, we define the right action functor $\overrightarrow{H}_H(\mathcal{T}, \mathcal{K})$ of $P_{I_H}(\mathcal{F}l_H)$ on $P_{I_H \times I_G}(\Pi(F))$:

$$\overrightarrow{H}_H(\mathcal{T}, \mathcal{K}) = \overleftarrow{H}_H(\star^\sharp(\mathcal{T}), \mathcal{K}).$$

Theorem 4.11. Remind that $n = m$. There exists an equivalence of categories

$$(4.12) \quad \begin{aligned} \sigma : P_{I_G}(\mathcal{F}l_G) &\xrightarrow{\sim} P_{I_H}(\mathcal{F}l_H) \\ L_w &\longrightarrow L_{w_0 \bar{w} w_0}, \end{aligned}$$

Additionally it verifies the following properties: for any w and w' in \widetilde{W}_G we have

$$\overleftarrow{H}_G(L_w, \mathcal{I}^{w_0!}) \xrightarrow{\sim} \overleftarrow{H}_H(\sigma(L_w), \mathcal{I}^{w_0!})$$

and

$$\sigma(L_w \star L_{w'}) = \sigma(L_{w'}) \star \sigma(L_w),$$

where \star is the convolution product in $P_{I_G}(\mathcal{F}l_G)$.

Proof. The anti-automorphism defined over \widetilde{W}_G in Definition 4.5 sending any w to \bar{w} may be extended to an anti-automorphism of the group $G(F)$ itself. It suffices to take the morphism sending any g an element of $G(F)$ to its transpose ${}^t g$. Denote by σ the anti-involution defined over $G(F)$ sending g to $w_0 {}^t g w_0$. This anti-involution preserves the Iwahori subgroup I_G and induces an equivalence of categories (still denoted by σ):

$$\sigma : P_{I_G}(\mathcal{F}l_G) \xrightarrow{\sim} P_{I_G}(\mathcal{F}l_G).$$

The assertion follows from Propositions 4.4 and 4.6 and Proposition 4.9. \square

The two anti-isomorphisms σ and \star^\sharp defined above commute and their composition is an algebra isomorphism. We will denote this composition by $\bar{\sigma}$, i.e. for any g in $G(F)$ $\bar{\sigma}(g) = w_0 {}^t g^{-1} w_0$.

4.2. Sub-modules Θ and S_0 . We assume in this section that $n \leq m$. We let the affine extended Weyl group \widetilde{W}_G of G act on the set $X_G \times S_{n,m}$ in the following way:

Definition 4.13. Let $w = t^{\lambda_1} \tau_1$ be an element of \widetilde{W}_G and (λ, s) in $X_G \times S_{n,m}$ then we define a left action:

$$w \cdot (\lambda, s) = (\lambda_1 + \tau_1(\lambda), \tau_1 s),$$

where $\tau_1 s$ is the composition $I_s \xrightarrow{s} \{1, \dots, n\} \xrightarrow{\tau_1} \{1, \dots, n\}$.

We will consider the affine extended Weyl group \widetilde{W}_G as a subset of $X_G \times S_{n,m}$. More precisely to a given $w = t^\lambda \tau$ we associate the element (λ, τ) in $X_G \times S_{n,m}$ with $I_\tau = \{1, \dots, n\}$. Let $I_{w_0} = \{1, \dots, n\}$ be a subset of $\{1, \dots, m\}$ and $w_0 : I_{w_0} \rightarrow \{1, \dots, n\}$ be the longest element of the Weyl group W_G . By the above convention the element w_0 becomes the element $(0, w_0)$ in $X_G \times S_{n,m}$.

For any strictly decreasing map ν from $\{1, \dots, n\}$ to $\{1, \dots, m\}$, denote by I_μ the image of ν and denote by $\mu : I_\mu \rightarrow \{1, \dots, n\}$ the inverse of ν . Thus μ can be viewed as an element of $X_G \times S_{n,m}$ by assuming that the corresponding term on X_G vanishes. Let $\overline{\Pi}_{0,1}^\mu$ be the closure of $I_H \times I_G$ -orbit $\Pi_{0,1}^\mu$ in $\Pi_{N,r}$, $\overline{\Pi}_{0,1}^\mu$ is an affine space. Denote by \mathcal{I}^μ the IC-sheaf of $\Pi_{0,1}^\mu$, it is the constant perverse sheaf on its support.

Denote by $U_1 \subset U_2 \subset \dots \subset U_m = U_0$ the standard flag on U/tU . We consider $\Pi_{0,1}$ the space of maps $\nu : L^* \rightarrow U/tU$ such that the domain and the range are both equipped with a flag preserved by ν . Thus $\overline{\Pi}_{0,1}^\mu$ is the space of maps $\nu : L^* \rightarrow U/tU$ such that $\nu(e_i^*)$ lies in $U_{\nu(i)}$ for all $i = 1, \dots, n$. In other terms the map ν sends $\text{Vect}(e_n^*, \dots, e_{n-i}^*)$ to $U_{\nu(n-i)}$ for all $i = 0, \dots, n-1$. An element ν lies in $\overline{\Pi}_{0,1}^\mu$ if additionally the map sending $\text{Vect}(e_n^*, \dots, e_{n-i}^*)$ to $U_{\nu(n-i)}/U_{\nu(n-i)-1}$ is non-zero for all $i = 1, \dots, n$. We may also consider the element ν in $\Pi_{0,1}$ as a map from U^* to L/tL , so ν lies in $\overline{\Pi}_{0,1}^\mu$ if and only if ν sends $\text{Vect}(u_m^*, \dots, u_{1+\nu(j)}^*)$ to L_{j-1} for all $j = 1, \dots, n$. Moreover, the map ν lies in $\overline{\Pi}_{0,1}^\mu$ if in addition $\nu(u_{\nu(j)}^*) \notin L_{j-1}$ for all $j = 1, \dots, n$. Let $w = t^\lambda \tau$ be an element of \widetilde{W}_G . Choose two integers N and r with $N + r > 0$ such that for any ν in W_G the following condition is satisfied (condition (4.1)):

$$\langle \nu, \check{\omega}_n \rangle < r \quad \text{and} \quad \langle -\nu, \check{\omega}_n \rangle \leq N.$$

For a point gL_G in $\overline{\mathcal{F}}_G^v$, we set $L' = gL$ and equip L'/tL' with the complete flag $L'_i = gL_i$ for $i = 1, \dots, n$. Here $(L_1 \subset \dots \subset L_n = L/tL)$ is the complete flag on L/tL preserved by B_G . Let $\overline{\Pi}_r^\mu \times \overline{\mathcal{F}}_G^v$ be the scheme classifying pairs (ν, gL_G) , where gL_G is in $\overline{\mathcal{F}}_G^v$, and $\nu : U^* \rightarrow L'/tL'$ such that the induced map

$$\bar{\nu} : U^*/tU^* \rightarrow L'/tL'$$

sends $\text{Vect}(u_m^*, \dots, u_{\nu(j)+1}^*)$ to L'_{j-1} for all $j = 1, \dots, n$. We have a proper map

$$\pi : \overline{\Pi}_r^\mu \times \overline{\mathcal{F}}_G^v \rightarrow \Pi_{N,r}$$

sending any element (ν, gL_G) to ν . By definition of \overleftarrow{H}_G ,

$$\overleftarrow{H}_G(L_w, \mathcal{I}^\mu) \xrightarrow{\sim} \pi_! (\overline{\mathbb{Q}}_\ell \boxtimes L_w),$$

where $\overline{\mathbb{Q}}_\ell \boxtimes L_w$ is normalized to be perverse.

Proposition 4.14. Let w be an element of \widetilde{W}_G . Then $\overleftarrow{H}_G(L_{w^1}, \mathcal{I}^{\mu^1})$ is canonically isomorphic to $\mathcal{I}^{w \cdot \mu^1}$.

Proof. Let $w = t^\lambda \tau$ with $\lambda = (a_1, \dots, a_n)$ in X_G and τ in W_G . Let $\Pi_r^\mu \tilde{\times} \mathcal{F} l_G^w$ be the open subscheme of $\overline{\Pi}_r^\mu \tilde{\times} \overline{\mathcal{F} l_G^w}$ given by the additional condition that $g l_G$ lies in $\mathcal{F} l_G^w$ and the map $\bar{v} : \text{Vect}(u_m^*, \dots, u_{v(j)}^*) \rightarrow L_j'$ is surjective for $j = 1, \dots, n$. Denote by π^0 the restriction of π to this open subscheme. The image of π^0 consist of the $I_H \times I_G$ -orbit on $\Pi_{N,r}$ through v such that $v(u_{v(j)}^*) = t^{a_{v(j)}} e_{\tau(j)}$ for all $j = 1, \dots, n$ and $v(u_k^*) = 0$ for $k \in I_\mu$. Therefore the image of the map π^0 is $\Pi_{N,r}^{w,\mu}$ and π^0 is an isomorphism onto its image. Thus

$$\overleftarrow{H}_G(L_{w!}, \mathcal{I}^{\mu!}) \xrightarrow{\sim} \mathcal{I}^{(w,\mu)!}.$$

□

Definition 4.15. Let Θ be $K(P_{I_G}(\mathcal{F} l_G))$ -module in $K(D_{I_H \times I_G}(\Pi(F)))$ generated by the elements $\mathcal{I}^{\mu!}$, where I_μ runs through all possible subsets of n elements in $\{1, \dots, m\}$.

It is understood that for each such subset I_μ there is a unique strictly decreasing map $\mu : I_\mu \rightarrow \{1, \dots, n\}$, so we may view μ as the element $(0, \mu)$ in $X_G \times S_{n,m}$ as above.

The subspace $\Theta \otimes \overline{\mathcal{Q}}_\ell \subset K(D_{I_H \times I_G}(\Pi(F))) \otimes \overline{\mathcal{Q}}_\ell$ is different from the group $K(D_{I_H \times I_G}(\Pi(F))) \otimes \overline{\mathcal{Q}}_\ell$. For example, each function from $\Theta \otimes \overline{\mathcal{Q}}_\ell$ vanishes at 0 in $\Pi(F)$. Actually, Θ is the geometrization of the first term of the filtration of Kudla on $K(D_{I_H \times I_G}(\Pi(F))) \otimes \overline{\mathcal{Q}}_\ell$ introduced in [25].

Our calculation yields the following generalization:

Proposition 4.16. The module Θ is free module of rank C_m^n over $K(P_{I_G}(\mathcal{F} l_G))$. The elements $\mathcal{I}^{\mu!}$, where I_μ runs through all possible subsets of n elements in $\{1, \dots, m\}$, form a basis of this module over $K(P_{I_G}(\mathcal{F} l_G))$.

Our purpose now is to show that Θ is a submodule with respect to the right action of $K(P_{I_H}(\mathcal{F} l_H)) \otimes \overline{\mathcal{Q}}_\ell$ on $K(D_{I_H \times I_G}(\Pi(F)))$ and identify Θ as the induced representation from a parabolic subalgebra. The considerations are essentially on the level of Grothendieck groups (we formulate them on the level of derived categories however when this is possible). Let us simply denote $K(D_{I_H \times I_G}(\Pi(F))) \otimes \overline{\mathcal{Q}}_\ell$ by \mathcal{S} . Let \mathcal{S}_0 be the $\overline{\mathcal{Q}}_\ell$ -subspace of \mathcal{S} generated by the elements $\mathcal{I}^{(w,w_0)!}$, where w runs through \widetilde{W}_G .

Denote by M a standard Levi subgroup of H , and by W_M the corresponding finite Weyl group. $I_M = M(F) \cap I_H$. Denote by T_w the characteristic function of the double coset IwI for any w in \widetilde{W}_H . The algebra \mathcal{H}_{I_M} is the subalgebra of \mathcal{H}_{I_H} generated by $(T_w)_{w \in W_M}$, and by the Bernstein functions $(\theta_\lambda)_{\lambda \in X_H}$. Remind that our convention for the Wakimoto objects is the one in [24], already mentioned in §2. According to [24, § 5.4], each coset $W_M \backslash W_H$ has a unique element of minimal length. Let ${}^M W_H$ be the set of such elements. If Δ_M denotes the simple roots of M then

$${}^M W_H = \{w \in W_H | w(\check{\alpha}) > 0 \text{ for each } \check{\alpha} \text{ in } \Delta_M\}.$$

Any w in W_H can be written as $w'' w'$, where w'' and w' are respective elements of W_M and ${}^M W_H$ satisfying $\ell(w) = \ell(w'') + \ell(w')$. Therefore T_w equals $T_{w''} T_{w'}$. We recall that \mathcal{H}_{I_H} is a free module over \mathcal{H}_{I_M} generated by $\{T_{w'} | w' \text{ in } {}^M W_H\}$. We are going to prove the two following results:

Theorem 4.17. The space \mathcal{S}_0 is a submodule of \mathcal{S} for the right action of \mathcal{H}_{I_M} .

The inclusion of \mathcal{S}_0 in \mathcal{S} is a homomorphism of right \mathcal{H}_{I_M} -modules and left \mathcal{H}_{I_G} -modules. By adjunction, we get a morphism

$$\alpha : \mathcal{S}_0 \otimes_{\mathcal{H}_{I_M}} \mathcal{H}_{I_H} \rightarrow \mathcal{S}$$

of right \mathcal{H}_{I_H} -modules and left \mathcal{H}_{I_G} -modules.

Theorem 4.18. *The map $\alpha : \mathcal{S}_0 \otimes_{\mathcal{H}_{I_M}} \mathcal{H}_{I_H} \rightarrow \mathcal{S}$ is injective, and its image equals $\Theta \otimes \overline{\mathcal{Q}}_\ell$.*

The rest of the section is devoted to the proof of these two theorems. The following lemma proves that \mathcal{S}_0 is a free module of rank one over \mathcal{H}_{I_G} .

Lemma 4.19. *For any element w in \widetilde{W}_G , we have*

$$\overleftarrow{H}_G(L_{w!}, \mathcal{I}^{w_0!}) \xrightarrow{\sim} \mathcal{I}^{(w, w_0)!}.$$

Proof. For a point gI_G in $\overline{\mathcal{F}l}_G^w$, let $L' = gL$ and equip L'/tL' with the flag $L'_i = g(L_i)$, for $1 \leq i \leq n$. Let $\Pi_{\mathcal{I}^{w_0}, r} \widetilde{\mathcal{F}l}_G^w$ be the scheme classifying pairs (v, gI_G) , where gI_G is in $\overline{\mathcal{F}l}_G^w$, and v is a map from U^* to L'/tL' such that the induced map

$$\bar{v} : U^*/tU^* \rightarrow L'/tL'$$

sends u_m^*, \dots, u_{n+1}^* to zero and $\text{Vect}(u_n^*, \dots, u_{n-i}^*)$ to L'_{i+1} for $i = 0, \dots, n-1$. Let

$$(4.20) \quad \pi : \Pi_{\mathcal{I}^{w_0}, r} \widetilde{\mathcal{F}l}_G^w \rightarrow \Pi_{N, r}$$

be the proper map sending a couple (v, gI_G) to v . By definition we have $\overleftarrow{H}_G(L_w, \mathcal{I}^{w_0}) \xrightarrow{\sim} \pi_!(\overline{\mathcal{Q}}_\ell \boxtimes L_w)$. Let $\Pi_{\mathcal{I}^{w_0}, r}^0 \widetilde{\mathcal{F}l}_G^w$ be the open subscheme of $\Pi_{\mathcal{I}^{w_0}, r} \widetilde{\mathcal{F}l}_G^w$ consisting of pairs (v, I_G) such that gI_G is in $\mathcal{F}l_G^w$, and the map $\bar{v} : \text{Vect}(u_n^*, \dots, u_{n-i}^*) \rightarrow L'_{i+1}$ is an isomorphism for $i = 0, \dots, n-1$. Thus $\overleftarrow{H}_G(L_{w!}, \mathcal{I}^{w_0!}) \xrightarrow{\sim} \pi_!^0(\overline{\mathcal{Q}}_\ell \boxtimes L_w)$, where $\pi^0 : \Pi_{\mathcal{I}^{w_0}, r}^0 \widetilde{\mathcal{F}l}_G^w \rightarrow \Pi_{N, r}$ is the restriction of π . The image of π^0 equals $\Pi_{N, r}^{w, w_0}$ and π^0 is an isomorphism onto its image. Thus we have $\overleftarrow{H}_G(L_{w!}, \mathcal{I}^{w_0!}) \xrightarrow{\sim} \mathcal{I}^{(w, w_0)!}$. \square

Now let $w = t^l \tau$ be an element of \widetilde{W}_H . The cocharacter λ in X_H is of the form (a_1, \dots, a_m) with a_i in \mathbb{Z} . Choose two integers N, r such that $-N \leq a_i < r$ for all i . Denote by $U_1 \subset U_2 \subset \dots \subset U_m$ the standard flag over U/tU . We define the scheme $\Pi_{\mathcal{I}^{w_0}, r} \times \overline{\mathcal{F}l}_H^w$ in the same way we did for G . For any point hI_H in $\overline{\mathcal{F}l}_H^w$, we put $U' = hU$ and equip U'/tU' with the complete flag $U'_i = hU_i$. Then $\Pi_{\mathcal{I}^{w_0}, r} \times \overline{\mathcal{F}l}_H^w$ is the scheme classifying pairs (v, hI_H) , where hI_H is in $\overline{\mathcal{F}l}_H^w$ and v is a map from L^* to U'/tU' such that the induced map

$$\bar{v} : L^*/tL^* \rightarrow U'/tU'$$

sends $\text{Vect}(e_n^*, \dots, e_{n-i}^*)$ to U'_{i+1} for all $i = 1, \dots, n-1$. Let π be the projection

$$\pi : \Pi_{\mathcal{I}^{w_0}, r} \times \overline{\mathcal{F}l}_H^w \rightarrow \Pi_{N, r}$$

then by definition we obtain $\overleftarrow{H}_H(L_w, \mathcal{I}^{w_0}) \xrightarrow{\sim} \pi_!(\overline{\mathcal{Q}}_\ell \boxtimes L_w)$. Let $\Pi_{\mathcal{I}^{w_0}, r}^0 \times \mathcal{F}l_H^w$ be the open subscheme of $\Pi_{\mathcal{I}^{w_0}, r} \times \overline{\mathcal{F}l}_H^w$ defined by the additional condition that the above map $\bar{v} : L^*/tL^* \rightarrow U'_n$ above is an isomorphism. Let

$$(4.21) \quad \pi^0 : \Pi_{\mathcal{I}^{w_0}, r}^0 \times \mathcal{F}l_H^w \rightarrow \Pi_{N, r}$$

be the restriction of π . Then we have

$$\overleftarrow{H}_H(L_{w!}, \mathcal{I}^{w_0!}) \xrightarrow{\sim} \pi_!^0(\overline{\mathcal{Q}}_\ell \boxtimes L_w).$$

Lemma 4.22. *Assume that $\lambda = (0, \dots, 0, a_{n+1}, \dots, a_m)$ and that the coefficients a_i are non negative. If τ is a permutation acting trivially on $\{1, \dots, n\}$ and permuting $\{n+1, \dots, m\}$, we have*

$$\overleftarrow{H}_H(L_{w!}, \mathcal{I}^{w_0!}) \xrightarrow{\sim} \mathcal{I}^{w_0!} [n\langle \lambda, \check{\omega}_m \rangle - \ell(w)].$$

Proof. In this case, the image of the map π^0 (4.21) is exactly the orbit $\Pi_{0,r}^{w_0}$ and the fibre of π^0 is an affine space. We need to compute the dimension of the fibres. Note that $\Pi_{0,r}^{w_0}$ has dimension $(r-1)mn + \frac{n(n+1)}{2}$. The scheme $\Pi_{\mathcal{I}^{w_0,r}}^0 \times_{\mathcal{F}_H} \ell_H^w$ is of dimension $\ell(w) - n\langle \lambda, \check{\omega}_m \rangle + (r-1)nm + \frac{n(n+1)}{2}$. This implies that the dimension of the fibre of π^0 equals $\ell(w) - n\langle \lambda, \check{\omega}_m \rangle$. This yields the result. \square

Proposition 4.23. *Let τ be in W_H . Then*

$$\overleftarrow{H}_H(L_{\tau!}, \mathcal{I}^{w_0!}) \xrightarrow{\sim} \mathcal{I}^{\nu!}[r],$$

where $\nu = (0, w_0\tau^{-1})$ is an element of $X_H \times S_{n,m}$ and r is the dimension of the fibre of the map π^0 in (4.21).

Before proving this proposition we will need the following lemma:

Lemma 4.24. *Let $U_1 \subset \cdots \subset U_m = U_0$ be a complete flag on U_0 . Consider a partial flag*

$$V_1 \subset V_2 \subset \cdots \subset V_n \subset U_0$$

inside U_0 . Let τ be a reflection in the finite Weyl group W_H of H . Denote by Y the variety of complete flags

$$V'_1 \subset \cdots \subset V'_m$$

which are in relative position τ with respect to the standard complete flag $U_1 \subset \cdots \subset U_m$ such that $V'_i = V_i$ for all $i = 1, \dots, n$. Then the variety Y is isomorphic to a finite dimensional affine space.

Proof. The stabilizer of the complete flag $U_1 \subset \cdots \subset U_m = U_0$ in H is the Borel subgroup B_H . For $i = 1, \dots, m$ fix a basis u_i of U_0 such that $U_i = \text{Vect}(u_1, \dots, u_i)$. The variety H/B_H is identified with the complete flags in U_0 . Given a vector subspace V of U_0 of dimension k , we associate to this subspace a subset $I(V)$ of k elements in $\{1, \dots, m\}$ defined by $I(V) = \{1 \leq i \leq m \mid \dim(V \cap U_i) > \dim(V \cap U_{i-1})\}$. Thus, for w in W_H , the orbit $B_H w B_H / B_H$ is the variety of complete flags

$$U'_1 \subset \cdots \subset U'_m$$

on U_0 such that $1 \leq i \leq m$ we have $I(U'_i) = \{w(1), \dots, w(i)\}$.

Let \mathcal{V}_n be a flag $V_1 \subset \cdots \subset V_n \subset U_0$ such that $\dim(V_i) = i$. The space Y is the variety of complete flags $V'_1 \subset \cdots \subset V'_m$ lying in the orbit $B_H w B_H / B_H$ and satisfying $V'_i = V_i$ for all $1 \leq i \leq n$. In order that the space Y be non-empty, we must have $I(V_n) = \{w(1), \dots, w(n)\}$. Assume that this is true. Given a subset of k elements I_k in $\{1, \dots, m\}$, denote by Z_{I_k} the variety of subspaces V of U_0 such that $I(V) = I_k$ (in particular we have $\dim(V) = k$). Given another subset I_{k+1} of $k+1$ elements containing I_k , let $Z_{I_k, I_{k+1}}$ be the variety of pairs $(V \subset V')$, where V lies in Z_{I_k} and V' lies in $Z_{I_{k+1}}$. Denote by π the projection from $Z_{I_k, I_{k+1}}$ onto Z_{I_k} sending $(V \subset V')$ to V . Let us prove that the map π is B_H -equivariant affine fibration.

For V in Z_{I_k} denote by \overline{U}_i the image of U_i under the map $U_0 \rightarrow U_0/V$. Then $\overline{U}_i = \overline{U}_{i-1}$ if and only if i lies in I_k . Denote by s the single element of $I_{k+1} - I_k$. The fibre of the map π identifies with the variety of 1-dimensional subspaces $V'/V \subset U_0/V$ such that V'/V is a subset of \overline{U}_s and V'/V is not contained in \overline{U}_{s-1} . This fibre is affine and since the space Z_{I_k} is B_H -homogeneous, the map π is a B_H -equivariant affine fibration.

For $r \geq n$ denote by Y_r the variety of flags $V_1 \subset \cdots \subset V_n \subset V'_{n+1} \subset \cdots \subset V'_r$ such that $I(V_i) = \{\tau(1), \dots, \tau(i)\}$ for $n \leq i \leq r$. We have the forgetful maps

$$Y_m \xrightarrow{f_m} Y_{m-1} \xrightarrow{f_{m-1}} \cdots \xrightarrow{f_{n+1}} Y_n = \text{Spec}(\mathbf{k}).$$

Any of the map f_i above is obtained by a base change from the map π for a suitable pair $(I_k \subset I_{k+1})$. The fibre of a map f_r depends only on V'_{r-1} and not on the smaller V'_j for $j \leq r-2$. As any affine fibration over an affine space is trivial this leads to the result and Y is an affine space of dimension r for some $r \geq n$. \square

Proof of Proposition 4.23. Let us precise the definition of ν , $\nu = (0, w_0\tau^{-1})$ is an element of $X_H \times S_{n,m}$, where the set $I_{w_0\tau^{-1}}$ is the set $\tau(\{1, \dots, n\})$ and $w_0\tau^{-1} : I_{w_0\tau^{-1}} \rightarrow \{1, \dots, n\}$ is the corresponding bijection. Consider the map

$$\pi^0 : \Pi_{I^{w_0,1}}^0 \times \mathcal{F}I_H^r \longrightarrow \Pi_{0,1}.$$

The fibres of the map π^0 are affine spaces according to Lemma 4.24. We denote by r their dimension. Since the image of π^0 is the orbit $\Pi_{0,1}^v$. We obtain that $\overleftarrow{H}_H(L_{\tau^!}, I^{w_0!}) \xrightarrow{\sim} I^{\nu^!}[r]$. \square

Remark 4.25. *If the permutation τ is a actually a permutation of $\{1, \dots, n\}$ and acts trivially on $\{n+1, \dots, m\}$ then the shift in the above formula disappears and the map π^0 will be an isomorphism onto its image $\Pi_{0,1}^v$.*

Let M be the standard Levi subgroup in H corresponding to the partition $(n, m-n)$ of m . Then M is of the form $M_1 \times M_2$, where $M_1 \xrightarrow{\sim} \mathbf{GL}_n$ and $M_2 \xrightarrow{\sim} \mathbf{GL}_{m-n}$. Write \mathcal{H}_{I_M} for the Iwahori Hecke algebra associated to M viewed as subalgebra of \mathcal{H}_H . We have naturally $\mathcal{H}_{I_M} \xrightarrow{\sim} \mathcal{H}_{I_{M_1}} \otimes_{\mathbb{Q}_\ell} \mathcal{H}_{I_{M_2}}$. We will denote by X_{M_i} the coweight lattice of M_i , for $i = 1, 2$. The space \mathcal{S}_0 is not a \mathcal{H}_{I_M} -submodule for the natural left action of \mathcal{H}_{I_M} on $\mathcal{S}(\Pi(F))^{I_H \times I_G}$. For instance, let $\lambda = (1, \dots, 1, 0, \dots, 0)$ where 1 appears n times the complex $\overleftarrow{H}_H(L_{\tau^!}, I^{w_0!})$ doesn't occur in \mathcal{S}_0 . We will consider this right action and will show that \mathcal{S}_0 is a right \mathcal{H}_{I_M} -module under this right action. Remind that the right action of \mathcal{H}_{I_M} commutes with the left action of \mathcal{H}_{I_G} .

Lemma 4.26. *For τ a simple reflection in the finite Weyl group W_M of M , $\overrightarrow{H}_H(L_{\tau^!}, I^{w_0!})$ lies in \mathcal{S}_0 .*

Proof. According to Lemma 4.23, we have

$$(4.27) \quad \overrightarrow{H}_H(L_{\tau^!}, I^{w_0!}) \xrightarrow{\sim} \overleftarrow{H}_H(L_{\tau^!}, I^{w_0!}) \xrightarrow{\sim} I^{\nu^!}[r],$$

where $\nu = (0, w_0\tau)$ is viewed as an element of $X_H \times S_{n,m}$. Thus $\overrightarrow{H}_H(L_{\tau^!}, I^{w_0!})$ occurs in \mathcal{S}_0 . \square

Lemma 4.28. *Let $\omega = (1, \dots, 1)$ be in X_H , $\mu_1 = (a_1, \dots, a_n)$ be in X_{M_1} and $\mu_2 = (a_{n+1}, \dots, a_m)$ be in X_{M_2} . Then let λ be the coweight $\mu_1 + \mu_2$ in X_H and assume that if $m \geq i > n \geq j \geq 1$ then $a_i \geq a_j$. we also fix r, N two integers such that $-N \leq a_i < r$ for all i . Let v be a \mathcal{O} -linear map from L^* to $t^\lambda U / t^r U$ such that for $0 \leq i < n$, the induced map*

$$\bar{v} : L^* / tL^* \longrightarrow t^\lambda U / t^{\lambda+\omega} U$$

sends $\text{Vect}(e_n^, \dots, e_{n-i}^*)$ isomorphically onto $\text{Vect}(t^{a_1} u_1, \dots, t^{a_{i+1}} u_{i+1})$. Denote by v the element $(w_0(\mu_1), w_0)$ in $X_G \times S_{n,m}$. Then v is an element of the orbit $\Pi_{N,r}^v$.*

Proof. Let $U_1 = \mathcal{O}u_1 \oplus \dots \oplus \mathcal{O}u_n$ and $U_2 = \mathcal{O}u_{n+1} \oplus \dots \oplus \mathcal{O}u_m$. Then the map v can be written as a pair (v_1, v_2) , where $v_i : L^* \rightarrow t^{\mu_i} U_i / t^r U_i$ for $i = 1, 2$. Let $N_G \subset B_G$ be unipotent radical of the standard Borel subgroup of G . Acting by a suitable element of $N_G \subset I_G$ on v , one may assume that $v_1(e_i^*) = t^{a_{w_0(i)}} u_{w_0(i)}$ modulo $t^{\mu_1 + \omega} U_1$.

Furthermore consider the groups

$$I_1 = \{g \in \mathbf{GL}(U_1) \mid g = \text{id mod } t\}$$

and

$$I_{G,0} = \{g \in \mathbf{GL}(L) \mid g = \text{id mod } t\}.$$

Acting by a suitable element of $I_{G,0} \times I_1$ we may assume that $v_1(e_i^*) = t^{a_{w_0(i)}} u_{w_0(i)}$. This implies that \bar{v}_2 vanishes. Now viewing v as a map from L^* to $t^{-N}L$, we observe that r can be replaced by $1 + \min\{a_{n+1}, \dots, a_m\}$. Hence v is an element of $\Pi_{N,r}^*$. \square

Lemma 4.29. *Let $\lambda = (a_1, \dots, a_n, 0, \dots, 0)$ be a anti-dominant cocharacter in X_H , in particular all a_i 's are non positive. Then we have a canonical isomorphism*

$$\overleftarrow{H}_H(L_{r^!}, \mathcal{I}^{w_0!}) \xrightarrow{\sim} \mathcal{I}^{\mu!}[\langle \lambda, 2\check{\rho}_G - 2\check{\rho}_H \rangle + (n-m)\langle \lambda, \check{\omega}_m \rangle],$$

where $\mu = (w_0(\lambda), w_0)$ in $X_G \times S_{n,m}$ and w_0 is the longest element of the finite Weyl group of $M_1 \xrightarrow{\sim} G$

Proof. Consider the map

$$\pi^0 : \Pi_{\mathcal{I}^{w_0}, r}^0 \tilde{\times} \mathcal{F} \ell_H^w \longrightarrow \Pi_{N,r}$$

defined in (4.20). By applying Lemma 4.28 to $w = t^\lambda$, we see that the image of π^0 is the $I_H \times I_G$ -orbit on $\Pi_{N,r}$ passing through the map v from L^* to $t^{-N}U/t^rU$ given by $v(e_i^*) = t^{a_{w_0(i)}} u_{w_0(i)}$ for $1 \leq i \leq n$. This orbit corresponds to element $\mu = (w_0(\lambda), w_0)$ in $X_G \times S_{n,m}$. Restricting π^0 to its image we get

$$(4.30) \quad \pi_H^0 : \Pi_{\mathcal{I}^{w_0}, r}^0 \tilde{\times} \mathcal{F} \ell_H^{t^\lambda} \longrightarrow \Pi_{N,r}^\mu$$

whose fibres are affine spaces. One has $\dim(\mathcal{F} \ell_H^{t^\lambda}) = \langle \lambda, 2\check{\rho}_H \rangle$. For any point hI_H in $\mathcal{F} \ell_H^{t^\lambda}$, let $U' = hU$. Then

$$\dim(\mathrm{Hom}_{\mathcal{O}}(L^*, t^{\lambda+\omega}U/t^rU)) = nm(r-1) - n\langle \lambda, \check{\omega}_m \rangle.$$

Thus the affine space of maps from L^* to hU/t^rU sending $\mathrm{Vect}(e_n^*, \dots, e_{n-i}^*)$ to U'_{i+1} for $i = 1, \dots, n-1$ is of dimension $\frac{n^2+n}{2} + nm(r-1) - n\langle \lambda, \check{\omega}_m \rangle$. Finally

$$\dim(\Pi_{\mathcal{I}^{w_0}, r}^0 \tilde{\times} \mathcal{F} \ell_H^{t^\lambda}) = \langle \lambda, 2\check{\rho}_H \rangle + \frac{n^2+n}{2} + nm(r-1) - n\langle \lambda, \check{\omega}_m \rangle.$$

Moreover, we have the following isomorphism

$$\pi_G^0 : \Pi_{\mathcal{I}^{w_0}, r}^0 \tilde{\times} \mathcal{F} \ell_G^{t^{w_0(\lambda)}} \xrightarrow{\sim} \Pi_{N,r}^\mu.$$

By using $\dim(\mathcal{F} \ell_G^{t^{w_0(\lambda)}}) = \langle \lambda, 2\check{\rho}_G \rangle$, we get that the dimension of $\Pi_{N,r}^\mu$ equals

$$\frac{n^2+n}{2} + nm(r-1) - m\langle \lambda, \check{\omega}_m \rangle + \langle \lambda, 2\check{\rho}_G \rangle.$$

and hence the dimension of the fibres of the map (4.30) equals

$$\langle \lambda, 2(\check{\rho}_G - \check{\rho}_H) \rangle + (m-n)\langle \lambda, \check{\omega}_m \rangle$$

which allows us to calculate the announced shift in the Lemma. Additionally this proves that $\overleftarrow{H}_H(L_{r^!}, \mathcal{I}^{w_0!})$ lies in \mathcal{S}_0 . \square

Remind the following two properties due to [2],

(1) If $w_1, w_2 \in \widetilde{W}_G$ verify $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$ then we have a canonical isomorphism

$$(4.31) \quad L_{w_1*} \star L_{w_2*} \xrightarrow{\sim} L_{w_1 w_2*}.$$

Under the same assumption, and by duality the same result is true for $L_{w!}$.

(2) Denote by e the identity element of \widetilde{W}_G then for any $w \in \widetilde{W}_G$, we have

$$(4.32) \quad L_{w!} \star L_{w^{-1}*} \xrightarrow{\sim} L_{w^{-1}*} \star L_{w!} \xrightarrow{\sim} L_e.$$

Hence the perverse sheaf $L_{w!}$ is an invertible object of $D_{I_G}(\mathcal{F}l_G)$.

Proposition 4.33. *Let $\lambda = (a_1, \dots, a_n, 0, \dots, 0)$ be an anti dominant cocharacter in X_H , in particular all a_i 's are non-positive. Then $\overleftarrow{H}_H(L_{T^{-\lambda*}}, \mathcal{I}^{w_0!})$ occurs in \mathcal{S}_0 .*

Proof. According to Lemma 4.29, we have

$$\overleftarrow{H}_H(L_{T^{\lambda!}}, \mathcal{I}^{w_0}) \xrightarrow{\sim} \mathcal{I}^{\mu!}[d],$$

where the shift d equals $[\langle \lambda, 2\check{\rho}_G - 2\check{\rho}_H \rangle + (n-m)\langle \lambda, \check{\omega}_m \rangle]$. Moreover according to (4.32), we have $L_{T^{-\lambda*}} \star L_{T^{\lambda!}}$ is isomorphic to L_e where e is the identity element in the finite Weyl group of M_1 . Combining these two isomorphisms we obtain

$$(4.34) \quad \begin{aligned} \mathcal{I}^{w_0!} &\xrightarrow{\sim} \overleftarrow{H}_H(L_{T^{-\lambda*}} \star L_{T^{\lambda!}}, \mathcal{I}^{w_0!}) \xrightarrow{\sim} \overleftarrow{H}_H(L_{T^{-\lambda*}}, \mathcal{I}^{\mu!})[d] \\ &\xrightarrow{\sim} \overleftarrow{H}_H(L_{T^{-\lambda*}}, \overleftarrow{H}_G(L_{T^{w_0(\lambda)!}}, \mathcal{I}^{w_0!}))[d] \\ &\xrightarrow{\sim} \overleftarrow{H}_G(L_{T^{w_0(\lambda)!}}, \overleftarrow{H}_H(L_{T^{-\lambda*}}, \mathcal{I}^{w_0}))[d], \end{aligned}$$

where the third isomorphism is due to Lemma 4.19 and the last one is due the fact that the actions of H and G commute. Applying $\overleftarrow{H}_G(L_{T^{-w_0(\lambda)*}, \cdot})$ to both sides of (4.34), we obtain

$$(4.35) \quad \overleftarrow{H}_G(L_{T^{-w_0(\lambda)*}, \mathcal{I}^{w_0!})} \xrightarrow{\sim} \overleftarrow{H}_H(L_{T^{-\lambda*}}, \mathcal{I}^{w_0})[d].$$

Since \mathcal{S}_0 is a left \mathcal{H}_{I_G} -module, the left hand side of (4.35) lies in \mathcal{S}_0 . Thus so does the right hand side. \square

Lemma 4.36. *Let $\lambda = (0, \dots, 0, a_{n+1}, \dots, a_m)$ be a dominant cocharacter of M . If $a_{n+1} \geq \dots \geq a_m \geq 0$ then*

$$\overleftarrow{H}_H(L_{T^{-\lambda*}}, \mathcal{I}^{w_0!}) \xrightarrow{\sim} \mathcal{I}^{w_0!}[\langle \lambda, 2\check{\rho}_H \rangle - n\langle \lambda, \check{\omega}_m \rangle].$$

Proof. Lemma 4.22 applied to $w = t^\lambda$ (τ being the identity) gives us

$$\overleftarrow{H}_H(L_{T^{\lambda!}}, \mathcal{I}^{w_0}) \xrightarrow{\sim} \mathcal{I}^{w_0}[n\langle \lambda, \check{\omega}_m \rangle - \langle \lambda, 2\check{\rho}_H \rangle].$$

This implies the assertion for $L_{T^{-\lambda*}}$. \square

Proposition 4.37. *Let λ be a dominant cocharacter in X_H that can be written as the sum of two cocharacters λ_1 and λ_2 in X_{M_1} and X_{M_2} respectively. If $\nu = (-w_0(\lambda_1), w_0)$ then*

$$\overrightarrow{H}_H(L_{T^{\nu!}}, \mathcal{I}^{w_0!}) \xrightarrow{\sim} \overleftarrow{H}_H(L_{T^{-\lambda!}}, \mathcal{I}^{w_0!}) \xrightarrow{\sim} \mathcal{I}^{\nu!}[\langle \lambda_1, 2\check{\rho}_G \rangle - \langle \lambda, 2\check{\rho}_H \rangle + \langle m\lambda_1 - n\lambda, \check{\omega}_m \rangle],$$

where we identify M_1 with G and hence $\check{\rho}_{M_1}$ with $\check{\rho}_G$. Thus $\overrightarrow{H}_H(L_{T^{\nu!}}, \mathcal{I}^{w_0!})$ occurs in \mathcal{S}_0 .

Proof. Set $-\lambda = (a_1, \dots, a_m)$ and choose two integers r, N such that $-N \leq a_i < r$ for all i . By Lemma 4.28 the map

$$\pi^0 : \Pi_{I^{w_0}, r}^0 \tilde{\times} \mathcal{F} \ell_H^{t^{-\lambda}} \longrightarrow \Pi_{N, r}$$

factors through $\Pi_{N, r}^\nu$ by a map π_H^0 , where ν is equal $(-w_0(\lambda_1), w_0)$. The dimension of $\Pi_{I^{w_0}, r}^0 \tilde{\times} \mathcal{F} \ell_H^{t^{-\lambda}}$ equals $nm(r-1) + n\langle \lambda, \check{\omega}_m \rangle + \frac{n^2+n}{2} + \langle \lambda, 2\check{\rho}_H \rangle$. We have the isomorphism

$$\Pi_{\mathcal{I}^{w_0, r}}^0 \tilde{\times} \mathcal{F} \ell_H^{\ell^{-w_0(\lambda_1)}} \xrightarrow{\sim} \Pi_{N, r}^v$$

and this allows us to calculate the dimension of $I_H \times I_G$ -orbit $\Pi_{N, r}^v$. Namely

$$\dim(\mathcal{F} \ell_G^{\ell^{-w_0(\lambda_1)}}) = \langle -w_0(\lambda_1), 2\check{\rho}_G \rangle = \langle \lambda_1, 2\check{\rho}_G \rangle.$$

Remind that w_0 is longest element of the finite Weyl group of $M_1 \xrightarrow{\sim} G$. This yields

$$\dim(\Pi_{n, r}^v) = nm(r-1) + m\langle \lambda_1, \check{\omega}_m \rangle + \frac{n^2 + n}{2} + \langle \lambda_1, 2\check{\rho}_G \rangle.$$

So the dimension of a fibre of the map π_H^0 is $\langle \lambda, 2\check{\rho}_H \rangle - \langle \lambda_1, 2\check{\rho}_G \rangle + \langle n\lambda - m\lambda_1, \check{\omega}_m \rangle$. This justifies the shift in the formula announced above and the assertion follows. \square

Remark 4.38. In Lemma 4.37 if λ_2 equals 0 then the corresponding map π_H^0 is an isomorphism and the shift in the above formula disappears.

Proposition 4.39. For any λ in X_H^+ , the complex $\overleftarrow{H}_H(L_{t^{\lambda_*}}, \mathcal{I}^{w_0})[d]$ occurs in \mathcal{S}_0 . Besides for a dominant cocharacter μ in X_H^+ then the complex $\overleftarrow{H}_H(L_{t^{-\mu!}} \star L_{t^{\lambda_*}}, \mathcal{I}^{w_0})$ occurs in \mathcal{S}_0 as well.

Proof. The equality (4.32) combining this with Proposition 4.37, we get :

$$\begin{aligned} \mathcal{I}^{w_0!} &\xrightarrow{\sim} \overleftarrow{H}_H(L_{t^{\lambda_*}} \star L_{t^{-\lambda!}}, \mathcal{I}^{w_0!}) \xrightarrow{\sim} \overleftarrow{H}_H(L_{t^{\lambda_*}}, \mathcal{I}^v)[d] \\ &\xrightarrow{\sim} \overleftarrow{H}_H(L_{t^{\lambda_*}}, \overleftarrow{H}_G(L_{t^{-w_0(\lambda!)}}, \mathcal{I}^{w_0!}))[d] \\ (4.40) \quad &\xrightarrow{\sim} \overleftarrow{H}_G(L_{t^{-w_0(\lambda!)}}, \overleftarrow{H}_H(L_{t^{\lambda_*}}, \mathcal{I}^{w_0!}))[d], . \end{aligned}$$

The shift d is also the one defined in Proposition 4.37. The third isomorphism is due to Lemma 4.19 and the fourth holds by using the commutativity of the action of G and H . Applying $\overleftarrow{H}_G(L_{t^{w_0(\lambda!)}}, \cdot)$ to both sides, we get

$$\overleftarrow{H}_G(L_{t^{w_0(\lambda!)}}, \mathcal{I}^{w_0!}) \xrightarrow{\sim} \overleftarrow{H}_H(L_{t^{\lambda_*}}, \mathcal{I}^{w_0!})[d].$$

This complex occurs in \mathcal{S}_0 . Now consider the isomorphism

$$\begin{aligned} \overleftarrow{H}_H(L_{t^{-\mu!}} \star L_{t^{\lambda_*}}, \mathcal{I}^{w_0!}) &\xrightarrow{\sim} \overleftarrow{H}_H(L_{t^{-\mu!}}, \overleftarrow{H}_G(L_{t^{w_0(\lambda!)}}, \mathcal{I}^{w_0}))[-d] \\ (4.41) \quad &\xrightarrow{\sim} \overleftarrow{H}_G(L_{t^{w_0(\lambda!)}}, \overleftarrow{H}_H(L_{t^{-\mu!}}, \mathcal{I}^{w_0}))[-d]. \end{aligned}$$

By Lemma 4.37, the complex $\overleftarrow{H}_H(L_{t^{-\mu!}}, \mathcal{I}^{w_0!})$ occurs in \mathcal{S}_0 . Since \mathcal{S}_0 is a \mathcal{H}_{I_G} -module we can apply the functor $\overleftarrow{H}_G(L_{t^{w_0(\lambda!)}}, \cdot)$ to $\overleftarrow{H}_H(L_{t^{-\mu!}}, \mathcal{I}^{w_0!})$. Then the result will still occur in \mathcal{S}_0 . \square

Proof. of Theorem 4.17: The assertion follows from (4.27), Lemmas 4.29 and 4.36, Propositions 4.37, 4.33 and 4.39. \square

Proof. of Theorem 4.18 : Lemma 4.23 and Proposition 4.16 imply that the image of the map α is exactly $\Theta \otimes \overline{\mathbb{Q}}_\ell$. More precisely, if τ runs through ${}^M W_H$ the elements $L_{\tau!}$ form a basis of the left \mathcal{H}_{I_M} -module \mathcal{H}_{I_H} . Hence for w and τ runs through \widetilde{W}_G and ${}^M W_H$ respectively, the objects

$$(4.42) \quad \overrightarrow{H}_H(L_{\tau!}, \overleftarrow{H}_G(L_{w!}, \mathcal{I}^{w_0!}))$$

form a basis of $\mathcal{S}_0 \otimes_{\mathcal{H}_{I_M}} \mathcal{H}_{I_H}$ over $\overline{\mathbb{Q}_\ell}$. An element v in W_H lies in ${}^M W_H$ if and only if v is strictly increasing on $\{1, \dots, n\}$ and on $\{n+1, \dots, m\}$. For τ in ${}^M W_H$ let $\mu = w_0 \tau$ and $I_\mu = \tau^{-1}(\{1, \dots, n\})$. Consider μ as a map from I_μ to $\{1, \dots, n\}$ and so as an element of $X_G \times S_{n,m}$. The map $\tau^{-1} w_0 : \{1, \dots, n\} \rightarrow I_\mu$ is strictly decreasing because $\tau^{-1} : \{1, \dots, n\} \rightarrow I_\mu$ is strictly increasing. According to Lemma 4.23 we have

$$\vec{H}_H(L_{\tau!}, \mathcal{I}^{w_0!}) \xrightarrow{\sim} \mathcal{I}^{\mu!}[r].$$

By Proposition 4.14 we have $\overleftarrow{H}_G(L_{w!}, \mathcal{I}^\mu) \xrightarrow{\sim} \mathcal{I}^{w \cdot \mu}[d']$ for some d' and hence the image of (4.42) under the map α is $\mathcal{I}^{w \cdot \mu!}[d'']$ for some shift d'' . \square

The Iwahori-Hecke algebra \mathcal{H}_{I_M} identifies canonically with $\mathcal{H}_{I_{M_1}} \otimes_{\overline{\mathbb{Q}_\ell}} \mathcal{H}_{I_{M_2}}$. The right action of $\mathcal{H}_{I_{M_1}}$ and $\mathcal{H}_{I_{M_2}}$ on \mathcal{S}_0 commute with each other. We are now going to define the action of the Wakimoto sheaves on $\mathcal{I}^{w_0!}$.

Lemma 4.43. *We have the following isomorphisms:*

(1) For any λ in X_{M_2}

$$\vec{H}_H(L_{r!}, \mathcal{I}^{w_0!}) \xrightarrow{\sim} \overleftarrow{H}_H(L_{r^{-\lambda!}}, \mathcal{I}^{w_0!}) \xrightarrow{\sim} \mathcal{I}^{w_0!}[-\langle \lambda, 2\check{\rho}_{M_2} \rangle].$$

(2) For λ in X_{M_2}

$$(4.44) \quad \vec{H}_H(L_{r^{-\lambda!}}, \mathcal{I}^{w_0!}) \xrightarrow{\sim} \mathcal{I}^{w_0!}[\langle \lambda, 2\check{\rho}_{M_2} \rangle].$$

(3) For any λ in X_{M_2} ,

$$\vec{H}_H(\Theta_\lambda, \mathcal{I}^{w_0!}) \xrightarrow{\sim} \mathcal{I}^{w_0!}[-\langle \lambda, 2\check{\rho}_{M_2} \rangle],$$

where Θ_λ is the Wakimoto sheaf associated to λ .

(4) For w in \widetilde{W}_G and λ in X_{M_2} ,

$$\vec{H}_H(\Theta_\lambda, \mathcal{I}^{(w, w_0)!}) \xrightarrow{\sim} \mathcal{I}^{(w, w_0)!}[-\langle \lambda, \check{\rho}_{M_2} \rangle].$$

Proof. The first formula is obtained by applying Lemma 4.37 to the case where $\lambda_1 = 0$. The second one is obtained from the first and from the isomorphism (4.32). The third one holds by definition of Θ_λ using the two first isomorphisms. Finally we prove the fourth one:

$$(4.45) \quad \begin{aligned} \vec{H}_H(\Theta_\lambda, \mathcal{I}^{(w, w_0)!}) &\xrightarrow{\sim} \vec{H}_H(\Theta_\lambda, \overleftarrow{H}_G(L_{w!}, \mathcal{I}^{w_0!})) \xrightarrow{\sim} \overleftarrow{H}_G(L_{w!}, \vec{H}_H(\Theta_\lambda, \mathcal{I}^{w_0!})) \\ &\xrightarrow{\sim} \overleftarrow{H}_G(L_{w!}, \mathcal{I}^{w_0}[-\langle \lambda, 2\check{\rho}_{M_2} \rangle]) \xrightarrow{\sim} \mathcal{I}^{(w, w_0)!}[-\langle \lambda, 2\check{\rho}_{M_2} \rangle]. \end{aligned}$$

\square

Corollary 4.46. *For any object K in \mathcal{S}_0 and any λ in X_{M_2} we have*

$$\vec{H}_H(\Theta_\lambda, K) \xrightarrow{\sim} K[-\langle \lambda, 2\check{\rho}_{M_2} \rangle].$$

Proposition 4.47. *For any w in the finite Weyl group of M_2 we have*

$$\vec{H}_H(L_{w!}, \mathcal{I}^{w_0!}) \xrightarrow{\sim} \overleftarrow{H}_H(L_{w^{-1}}, \mathcal{I}^{w_0!}) \xrightarrow{\sim} \mathcal{I}^{w_0!}[-\ell(w)].$$

Thus, at the level of the functions, the Iwahori-Hecke algebra $\mathcal{H}_{I_{M_2}}$ acts on \mathcal{S}_0 by the character corresponding to the trivial representation of $M_2(F) \simeq \mathbf{GL}_{m-n}(F)$. Moreover,

$$\vec{H}_H(\Theta_\lambda \star L_\tau, K) \xrightarrow{\sim} K[-\langle \lambda, 2\check{\rho}_{M_2} \rangle - \ell(\tau)].$$

Proof. At the level of functions, for any w in \widetilde{W}_G the character of \mathcal{H}_{I_G} corresponding to the trivial representation sends T_w , the characteristic function of the double coset $I_G w I_G$, to $q^{\ell(w)}$. In our geometric setting, for any w in \widetilde{W}_G this character becomes the functor $\overleftarrow{H}_G(L_{w!}, \cdot)$ sending K in \mathcal{S}_0 to $K[-\ell(w)]$. Remind that the object $L_{w!}$ corresponds to $q^{-\ell(w)/2} T_w$. \square

Remind that M_1 is identified with G . Now let us analyse the structure of \mathcal{S}_0 as a right $\mathcal{H}_{I_{M_1}}$ -module and its relation with the left \mathcal{H}_{I_G} -module structure.

Lemma 4.48.

(1) For any τ in the finite Weyl group of M_1 ,

$$\vec{H}_H(L_{\tau!}, \mathcal{I}^{w_0}) \xrightarrow{\sim} \mathcal{I}^{w_0 \tau!}.$$

(2) For any λ in $X_{M_1}^+$,

$$\vec{H}_H(L_{r^{\lambda!}}, \mathcal{I}^{w_0}) \xrightarrow{\sim} \mathcal{I}^{w_0 r^{-\lambda!}} \quad \text{and} \quad \overleftarrow{H}_G(L_{r^{w_0(\lambda)_*}}, \mathcal{I}^{w_0!}) \xrightarrow{\sim} \vec{H}_H(L_{r^{-\lambda_*}}, \mathcal{I}^{w_0!}).$$

(3) For any λ in X_{M_1} ,

$$\vec{H}_H(\Theta_\lambda, \mathcal{I}^{w_0!}) \xrightarrow{\sim} \overleftarrow{H}_G(\Theta_{-w_0(\lambda)}, \mathcal{I}^{w_0!}).$$

(4) For any τ in the finite Weyl group of M_1 and any λ in X_{M_1} ,

$$\vec{H}_H(\Theta_\lambda \star L_{\tau!}, \mathcal{I}^{w_0}) \xrightarrow{\sim} \overleftarrow{H}_G(\Theta_{-w_0 \lambda} \star L_{w_0 \tau w_0!}, \mathcal{I}^{w_0!}).$$

Proof. The first isomorphism is obtained from Proposition 4.23. The second one is a consequence of (4.2). For the third one, choose λ_1 and λ_2 on $X_H^+ \cap X_{M_1}$ such that $\lambda = \lambda_1 - \lambda_2$. By definition of Wakimoto sheaves, $\Theta_\lambda = L_{r^{-\lambda_2_*}} \star L_{r^{\lambda_1!}}$ in $\mathcal{H}_{I_{M_1}}$. The isomorphism in 2) yields that

$$\begin{aligned} \vec{H}_H(L_{r^{\lambda_1!}}, \vec{H}_H(L_{r^{-\lambda_2_*}}, \mathcal{I}^{w_0!})) &\xrightarrow{\sim} \vec{H}_H(L_{r^{\lambda_1!}}, \overleftarrow{H}_G(L_{r^{w_0(\lambda_2)_*}}, \mathcal{I}^{w_0!})) \\ &\xrightarrow{\sim} \overleftarrow{H}_G(L_{r^{w_0(\lambda_2)_*}}, \vec{H}_H(L_{r^{\lambda_1!}}, \mathcal{I}^{w_0!})) \\ &\xrightarrow{\sim} \overleftarrow{H}_G(L_{r^{w_0(\lambda_2)!}}, \overleftarrow{H}_G(L_{r^{-w_0(\lambda_2)!}, \mathcal{I}^{w_0!}})). \end{aligned} \tag{4.49}$$

The element $-w_0 \lambda_2$ is dominant if λ_2 is dominant. This implies the third assertion. The fourth isomorphism is obtained formally in the following way:

$$\begin{aligned} \vec{H}_H(\Theta_\lambda \star L_{\tau!}, \mathcal{I}^{w_0!}) &\xrightarrow{\sim} \vec{H}_H(L_{\tau!}, \overleftarrow{H}_G(\Theta_{-w_0(\lambda)!}, \mathcal{I}^{w_0!})) \\ &\xrightarrow{\sim} \overleftarrow{H}_G(\Theta_{-w_0(\lambda)}, \vec{H}_H(L_{\tau!}, \mathcal{I}^{w_0!})) \\ &\xrightarrow{\sim} \overleftarrow{H}_G(\Theta_{-w_0 \lambda}, \overleftarrow{H}_G(L_{w_0 \tau w_0!}, \mathcal{I}^{w_0!})) \\ &\xrightarrow{\sim} \overleftarrow{H}_G(\Theta_{-w_0 \lambda} \star L_{w_0 \tau w_0!}, \mathcal{I}^{w_0!}). \end{aligned} \tag{4.50}$$

\square

Corollary 4.51. *The subspace \mathcal{S}_0 is a free right \mathcal{H}_{M_1} -module of rank one generated by $\mathcal{I}^{w_0!}$.*

Proof. The assertion follows from Lemma 4.48 and the fact that if λ and τ runs through X_{M_1} and W_{M_1} respectively the elements $\Theta_\lambda \star L_{\tau!}$ form a basis of \mathcal{H}_{M_1} . \square

Combining Lemma 4.48 with Corollary 4.51 we obtain the following proposition:

Proposition 4.52. *There exists an equivalence of categories*

$$\tilde{\sigma} : P_{I_{M_1}}(\mathcal{F}l_{M_1}) \xrightarrow{\sim} P_{I_G}(\mathcal{F}l_G)$$

such that for any w in W_{M_1} , $\tilde{\sigma}$ sends L_w to $L_{w_0\bar{w}^{-1}w_0}$, (\bar{w} is the anti-involution defined in Definition definir!). Additionally for any \mathcal{T} in $P_{I_{M_1}}(\mathcal{F}l_{M_1})$ we have

$$\vec{H}_H(\mathcal{T}, \mathcal{I}^{w_0!}) \xrightarrow{\sim} \overleftarrow{H}_G(\tilde{\sigma}(\mathcal{T}), \mathcal{I}^{w_0!}).$$

At last, For any λ a cocharacter of M_1 , we have

$$\tilde{\sigma}(\Theta_\lambda \star L_{\tau!}) \xrightarrow{\sim} \Theta_{-w_0\lambda} \star L_{w_0\tau w_0!}.$$

In the case $n = m$, the anti-isomorphism reduces to Proposition 4.11.

5. WEAK GEOMETRIC ANALOGUE OF JACQUET FUNCTORS AND COMPATIBILITY WITH HECKE FUNCTORS

In this section we place ourselves in a more general setting. Let G be a split reductive connected group over \mathbf{k} , T be the maximal standard torus of G and B be the standard Borel subgroup B in G containing T . Denote by I_G the corresponding Iwahori subgroup. Let P be a parabolic subgroup of G containing B and U its unipotent radical. Let L be the Levi subgroup of P isomorphic to P/U . Let M_0 be a faithful representation of G , and let $M = M_0 \otimes_{\mathbf{k}} (\mathcal{O})$.

In the classical setting, an important tool is the Jacquet module $\mathcal{S}(M(F))_{U(F)}$ of coinvariants with respect to $U(F)$. We will define a weak analogue of Jacquet functors in the geometric setting. Let V_0 be a P -stable subspace of M_0 endowed with a trivial action of U . Set $V = V_0 \otimes_{\mathbf{k}} (\mathcal{O})$, we have a surjective map of $L(F)$ -representations

$$\mathcal{S}(M(F))_{U(F)} \longrightarrow \mathcal{S}(V(F))$$

given by restriction under the inclusion $V(F) \hookrightarrow M(F)$. We will geometrize the composition

$$\mathcal{S}(M(F)) \longrightarrow \mathcal{S}(M(F))_{U(F)} \longrightarrow \mathcal{S}(V(F)),$$

and we will show that in the Iwahori case, geometric Jacquet functors commute with the action the \mathcal{H}_{I_L} -subalgebra of \mathcal{H}_{I_G} .

Let I_P be the preimage of the Borel subgroup B under the map $P(\mathcal{O}) \rightarrow P$. Denote by B_L the image of B in L . It is a Borel subgroup of L . In the same way consider the map $L(\mathcal{O}) \rightarrow L$ and denote by I_L the preimage of B_L under this map in $L(\mathcal{O})$. Hence I_L is an Iwahori subgroup of $L(F)$. Finally we have a diagram

$$I_L \longleftarrow I_P \hookrightarrow I_G,$$

where the first map is induced by the natural projection $P(\mathcal{O}) \rightarrow L(\mathcal{O})$. According to [10], the categories $D_{I_G}(M(F))$ and $D_{I_L}(V(F))$ are well defined. We are going to define the following functors

$$J_P^*, J_P^! : D_{I_G}(M(F)) \longrightarrow D_{I_L}(V(F)).$$

Let N, r be two integers such that $N+r \geq 0$. Set $V_{N,r} = t^{-N}V/t^rV$. Denote by $i_{N,r}$ the natural closed embedding of $V_{N,r}$ in $M_{N,r}$. For any $s \geq 0$, let K_s be the quotient of I_G by the kernel of the map $GO \rightarrow G(O/t^sO)$. Let $I_{P,s}$ denote the image of I_P under the inclusion

$$I_P \hookrightarrow P(O) \longrightarrow P(O/t^sO).$$

Similarly let $I_{L,s}$ be the image of I_L under $L(O) \rightarrow L(O/t^sO)$. We have the following diagram

$$\begin{array}{ccccc} L(O/t^sO) & \longleftarrow & P(O/t^sO) & \longrightarrow & G(O/t^sO) \\ \uparrow & & \uparrow & & \uparrow \\ I_{L,s} & \longleftarrow & I_{P,s} & \longrightarrow & K_s. \end{array}$$

For $s \geq N+r$, we obtain a digram of stack quotients

$$\begin{array}{ccccc} I_{P,s} \backslash V_{N,r} & \xrightarrow{i_{N,r}} & I_{P,s} \backslash M_{N,r} & \xrightarrow{p} & K_s \backslash M_{N,r} \\ \downarrow q & & & & \\ I_{L,s} \backslash V_{N,r} & & & & \end{array}$$

where p comes from the closed inclusion $I_{P,s} \hookrightarrow I_{G,s}$. Set a equal to $\dim M_0 - \dim V_0$.

Proposition 5.1. *There exists two well-defined functors*

$$(5.2) \quad J_P^*, J_P^! : D_{I_G}(M(F)) \longrightarrow D_{I_L}(V(F)).$$

Proof. For any $s \geq N+r$, we have the following functors:

$$J_{P,N,r}^*, J_{P,N,r}^! : D_{K_s}(M_{N,r}) \longrightarrow D_{I_{L,s}}(V_{N,r})$$

defined by

$$\begin{aligned} q^* \circ J_{P,N,r}^*[\dim .\text{rel}(q)] &= (i_{N,r})^* p^*[\dim .\text{rel}(p) - ra] \\ q^* \circ J_{P,N,r}^![\dim .\text{rel}(q)] &= (i_{N,r})^! p^*[\dim .\text{rel}(p) + ra]. \end{aligned}$$

The sequence

$$1 \longrightarrow U(O/t^sO) \longrightarrow I_{P,s} \longrightarrow I_{L,s} \longrightarrow 1$$

is exact. Hence the functor

$$q^*[\dim .\text{rel}(q)] : D_{I_{L,s}}(V_{N,r}) \longrightarrow D_{I_{P,s}}(V_{N,r})$$

is an equivalence of categories and exact for perverse t -structure. The functors $J_{P,N,r}^*$ and $J_{P,N,r}^!$ are well-defined. They are compatible with the transition functors in the ind-system of categories defining $D_{I_G}(M(F))$ and $D_{I_L}(L(F))$ defined in [10]. By the taking the inductive 2-limit, we obtain the two well-defined functors J_P^* and $J_P^!$ (5.2) which do not depend on the choice of a section of $P \rightarrow P/U$. The Verdier duality functor \mathbb{D} exchanges J_P^* and $J_P^!$, i.e. we have canonically

$$\mathbb{D} \circ J_P^* \xrightarrow{\sim} J_P^! \circ \mathbb{D}.$$

□

As in the case of the affine flag variety, we can define the \mathbf{k} -space quotient $P(F)/I_P$ and define $\mathcal{F}l_P$ to be the sheaf associated to this presheaf in fpqc-topology. It is an ind-scheme. Let X be a projective smooth connected curve over the field \mathbf{k} . Let x be a closed point in X and $X^* = X - \{x\}$. Denote by \mathcal{O}_x the completion of the local ring of X at x and by F_x its field of fractions. We choose a local coordinate at the point x , denoted by t and we may identify $\mathcal{O}_x = \mathbf{k}[[t]]$ and $F_x = \mathbf{k}((t))$. Let $D = \text{Spec}(\mathbf{k}[[t]])$ and $D^* = \text{Spec}(\mathbf{k}((t)))$. Then $\mathcal{F}l_P$ classifies $(\mathcal{F}_P, \beta, \epsilon)$, where \mathcal{F}_P is a P -torsor on D , the map β is a trivialization of \mathcal{F}_P over D^* , and ϵ is a reduction of $\mathcal{F}_P|_x$ to a B -torsor. We have the diagram

$$(5.3) \quad \mathcal{F}l_L \xleftarrow{t_L} \mathcal{F}l_P \xrightarrow{t_P} \mathcal{F}l_G,$$

where t_P (resp. t_L) is given by extension of scalars with respect to $P \hookrightarrow G$ (resp. $P \rightarrow L$).

Let $\mathcal{F}l_{P,G}$ be the $P(F)$ -orbit through 1 in $\mathcal{F}l_G$ viewed as an ind-subscheme with a reduced scheme structure. The reduced ind-scheme $\mathcal{F}l_{P,\text{red}}$ gives a stratification of $\mathcal{F}l_{P,G}$.

Lemma 5.4. *There exists a well-defined geometric restriction functor*

$$\text{gRes} : D_{I_G}(\mathcal{F}l_G) \longrightarrow D_{I_L}(\mathcal{F}l_L).$$

Proof. For $s_1, s_2 \geq 0$, let ${}_{s_1, s_2}P(F) = P(F) \cap {}_{s_1, s_2}G(F)$, and ${}_{s_1, s_2}\mathcal{F}l_P = {}_{s_1, s_2}P(F)/I_P$, where

$${}_{s_1, s_2}G(F) := \{g \in G(F) \mid t^{s_1}V \subset gV \subset t^{-s_2}V\}.$$

The ind-scheme ${}_{s_1, s_2}\mathcal{F}l_P$ is a closed subscheme of $\mathcal{F}l_P$. Similarly we define

$${}_{s_1, s_2}L(F) := \{g \in L(F) \mid t^{s_1}V \subset gV \subset t^{-s_2}V\},$$

and ${}_{s_1, s_2}\mathcal{F}l_L = {}_{s_1, s_2}L(F)/I_L$. Thus the map t_L in (5.3) induces a morphism (denoted again by t_L) from ${}_{s_1, s_2}\mathcal{F}l_P$ to ${}_{s_1, s_2}\mathcal{F}l_L$. For $s \geq s_1 + s_2 + 1$ we have a diagram of stack quotients

$$\begin{array}{ccc} I_{L,s} \backslash ({}_{s_1, s_2}\mathcal{F}l_L) & \xleftarrow{q_L} & I_{P,s} \backslash ({}_{s_1, s_2}\mathcal{F}l_L) \\ & & \uparrow t_L \\ & & I_{P,s} \backslash ({}_{s_1, s_2}\mathcal{F}l_P) \\ & & \downarrow t_P \\ K_s \backslash ({}_{s_1, s_2}\mathcal{F}l_G) & \xleftarrow{\xi} & I_{P,s} \backslash ({}_{s_1, s_2}\mathcal{F}l_G) \end{array}$$

Moreover, the functor

$$q_L^*[\dim .\text{rel}(q_L)] : D_{I_{L,s}}({}_{s_1, s_2}\mathcal{F}l_L) \longrightarrow D_{I_{P,s}}({}_{s_1, s_2}\mathcal{F}l_L)$$

is an equivalence of categories and exact for perverse t -structure. For any perverse sheaf K extension by zero from ${}_{s_1, s_2}\mathcal{F}l_G$ to $\mathcal{F}l_G$, we may define $\text{gRes}(K)$ by the isomorphism

$$q_L^* \text{gRes}(K)[\dim .\text{rel}(q_L)] \xrightarrow{\sim} (t_L)_! t_P^* K[\dim .\text{rel}(\xi)].$$

□

Lemma 5.5. *For any dominant cocharacter λ of G , we have*

$$\mathrm{gRes}(L_{r^!}) \xrightarrow{\sim} L_{r^!}[-\langle \lambda, 2(\check{\rho}_G - \check{\rho}_L) \rangle],$$

where $\check{\rho}_G$ (resp. $\check{\rho}_L$) denote the half sum of positive roots of G (resp. positive roots of L).

Proof. Let U_B be the unipotent radical of B . The space $\mathcal{F}\ell_G^\lambda$ is the $U_B(O)$ -orbit through $t^\lambda I_G$ on $\mathcal{F}\ell_G$. Thus $L_{r^!}$ is the extension by zero from a connected component of $\mathcal{F}\ell_P$. The map $U_B t^\lambda I_P / I_P \rightarrow \mathcal{F}\ell_L^\lambda$ is a trivial affine fibration with affine fibre of dimension $\langle \lambda, 2(\check{\rho}_G - \check{\rho}_L) \rangle$ and the result follows. \square

Lemma 5.6. *For any w in the finite Weyl group of L , we have*

$$\mathrm{gRes}(L_w) \xrightarrow{\sim} L_w, \quad \mathrm{gRes}(L_{w^!}) \xrightarrow{\sim} L_{w^!}, \quad \mathrm{gRes}(L_{w^*}) \xrightarrow{\sim} L_{w^*}.$$

Proof. The P -orbit through I_P gives a natural closed subscheme $L/B_L \xrightarrow{\sim} P/B \hookrightarrow \mathcal{F}l_P$. For any w in the finite Weyl group of L , the double coset BwB is contained in P . Thus $L_{w^!}$ initially defined over $\mathcal{F}\ell_G$ is actually an extension by zero from a connected component of $\mathcal{F}\ell_P$. Hence $\mathrm{gRes}(L_{w^!}) \xrightarrow{\sim} L_{w^!}$. Moreover, \overline{BwB}/B in G/B actually lies in P/B , so $\mathrm{gRes}(L_w) \xrightarrow{\sim} L_w$. The same result holds for L_{w^*} . \square

The following proposition will show that the functors $J_p^!, J_p^*$ commute with the action of the subalgebra \mathcal{H}_{I_L} of \mathcal{H}_{I_G} . Remind that the subalgebra structure of \mathcal{H}_{I_L} has been defined in [§4.2].

Theorem 5.7. *Let \mathcal{T} be a perverse sheaf in $P_{I_G}(\mathcal{F}\ell_G)$ which is the extension by zero from a connected component of $\mathcal{F}\ell_P$, and \mathcal{K} be in $D_{I_G}(M(F))$ then we have*

$$J_p^* \overleftarrow{H}_G(\mathcal{T}, \mathcal{K}) \xrightarrow{\sim} \overleftarrow{H}_L(\mathrm{gRes}(\mathcal{T}), J_p^* \mathcal{K})[\langle \lambda, \check{\nu} - \check{\mu} \rangle],$$

where λ is the cocharacter whose image in $\pi_1(L)$ is θ , $\check{\nu}$ is the character by which L acts on $\det(V_0)$ and $\check{\mu}$ is the character by which G acts on $\det(M_0)$.

Proof. The connected components of $\mathcal{F}\ell_P$ are indexed by $\pi_1(L)$. For θ in $\pi_1(L)$, denote by $\mathcal{F}\ell_P^\theta$ for the corresponding connected component which is the preimage of $\mathcal{F}\ell_L^\theta$ under the map t_L defined in (5.3).

Let s_1 and s_2 be two non negative integers and \mathcal{T} be the extension by zero from ${}_{s_1, s_2} \mathcal{F}\ell_P^\theta = {}_{s_1, s_2} \mathcal{F}l_P \cap \mathcal{F}\ell_P^\theta$. For $N + r \geq 0$ and $s \geq \max\{N + r, s_1 + s_2 + 1\}$ consider the diagram

$$\begin{array}{ccccc}
V_{N,r} \times_{s_1, s_2} P(F) & \xrightarrow{act} & V_{N+s_1, r-s_1} & & \\
\downarrow q_P & & \downarrow q_U & & \\
V_{N,r} \xleftarrow{pr} V_{N,r} \times_{s_1, s_2} \mathcal{F}l_P^\theta & \xrightarrow{act_{q,P}} & I_{P,s} \setminus V_{N+s_1, r-s_1} & & \\
\downarrow i_{N,r} & & \downarrow i_{N+s_1, r-s_1} & & \\
M_{N,r} \xleftarrow{pr} M_{N,r} \times_{s_1, s_2} \mathcal{F}l_P^\theta & \xrightarrow{act_{q,P}} & I_{P,s} \setminus M_{N+s_1, r-s_1} & & \\
\downarrow & & \downarrow p & & \\
M_{N,r} \xleftarrow{pr} M_{N,r} \times_{s_1, s_2} \mathcal{F}l_G^\theta & \xrightarrow{act_q} & K_s \setminus M_{N+s_1, r-s_1} & &
\end{array}$$

where the map act sends (m, p) to $p^{-1}m$, the map q_P sends (m, p) to (m, pI_P) , and q_U is the stack quotient under the action of $I_{P,s}$.

Moreover, the second line of this diagram fits in the following diagram

$$\begin{array}{ccc}
V_{N,r} \times_{s_1, s_2} \mathcal{F}l_P^\theta & \xrightarrow{act_{q,P}} & I_{P,s} \setminus V_{N+s_1, r-s_1} \\
\downarrow id \times t_L & & \downarrow q \\
V_{N,r} \times_{s_1, s_2} \mathcal{F}l_L^\theta & \xrightarrow{act_{q,P}} & I_{L,s} \setminus V_{N+s_1, r-s_1}
\end{array}$$

At the level of reduced ind-schemes the map $_{s_1, s_2} \mathcal{F}l_P^\theta \rightarrow \mathcal{F}l_G$ is a locally closed embedding, thus the perverse sheaf \mathcal{T} may be viewed as a complex over $_{s_1, s_2} \mathcal{F}l_P^\theta$. For a given \mathcal{K} and large enough N, r , by definition, up to a shift independent of \mathcal{K} and \mathcal{T} we have

$$\overleftarrow{H}_G(\mathcal{T}, \mathcal{K}) \xrightarrow{\sim} pr_1(act_{q,P}^*(\mathcal{K}) \otimes pr_2^*(\mathcal{T})),$$

where $pr : K_s \setminus (M_{N,r} \times_{s_1, s_2} \mathcal{F}l_G) \rightarrow K_s \setminus_{s_1, s_2} \mathcal{F}l_G$ is defined in [10]. Thus by definition of $gRes$ and J_p^* and the commutativity of the diagram above, we get

$$J_p^* \overleftarrow{H}_G(\mathcal{T}, \mathcal{K}) \xrightarrow{\sim} \overleftarrow{H}_L(gRes(\mathcal{T}), J_p^* \mathcal{K})[?],$$

To determine the shift, one may consider the following special case, where \mathcal{K} is the constant perverse sheaf I_0 on M and \mathcal{T} equals L_{r^λ} , where λ is a dominant cocharacter of G . For large enough N, r , we have the

following diagram

$$(5.8) \quad \begin{array}{ccc} M_{N,r} & \xleftarrow{\alpha_M} & M_{0,r} \tilde{\times} \mathcal{F} l_P^{\lambda} \\ \uparrow i_{N,r} & & \uparrow \\ V_{N,r} & \xleftarrow{\alpha_V} & V_{0,r} \tilde{\times} \mathcal{F} l_P^{\lambda} \end{array}$$

Remind that $\mathcal{F} l_P^{\lambda}$ is the $U_B(\mathcal{O})$ -orbit through $t^{\lambda} I_P$ in $\mathcal{F} l_P$. The scheme $M_{0,r} \tilde{\times} \mathcal{F} l_P^{\lambda}$ is the scheme classifying pairs (gI_P, m) , where gI_P is in $\mathcal{F} l_P$ and m is in $gM/t^r M$. Similarly the scheme $V_{0,r} \tilde{\times} \mathcal{F} l_P^{\lambda}$ is the scheme classifying pairs (gI_P, v) , where gI_P is in $\mathcal{F} l_P^{\lambda}$ and v is in $gV/t^r V$. For large enough r , we have $gV \cap t^r M = t^r V$. So the right vertical arrow in Diagram (5.8) is a closed immersion. Denote by IC the IC-sheaf of $M_{0,r} \tilde{\times} \mathcal{F} l_P^{\lambda}$. We have canonically

$$\overleftarrow{H}_G(L_{r^{\lambda}}, I_0) \xrightarrow{\sim} \alpha_{M!}(\text{IC}),$$

and additionally

$$\dim(M_{0,r} \tilde{\times} \mathcal{F} l_P^{\lambda}) = \langle \lambda, 2\check{\rho}_G \rangle + r \dim M_0 - \langle \lambda, \check{\mu} \rangle,$$

Hence we have

$$(5.9) \quad \begin{aligned} J_P^* \overleftarrow{H}_G(L_{r^{\lambda}}, I_0) &\xrightarrow{\sim} i_{N,r}^* \alpha_{M!}(\text{IC})[-ra] \\ &\xrightarrow{\sim} \alpha_{V!} \overline{\mathbb{Q}}_{\ell}[\langle \lambda, 2\check{\rho}_G \rangle + r \dim V_0 - \langle \lambda, \check{\mu} \rangle]. \end{aligned}$$

The map α_V factors through

$$(5.10) \quad V_{0,r} \tilde{\times} \mathcal{F} l_P^{\lambda} \longrightarrow V_{0,r} \tilde{\times} \mathcal{F} l_L^{\lambda} \xrightarrow{\alpha_L} V_{N,r},$$

where the first map is a trivial affine fibration with an affine fibre of dimension $\langle \lambda, 2(\check{\rho}_G - \check{\rho}_L) \rangle$. This gives us

$$\dim(V_{0,r} \tilde{\times} \mathcal{F} l_L^{\lambda}) = \langle \lambda, 2\check{\rho}_L \rangle + r \dim V_0 - \langle \lambda, \check{\nu} \rangle.$$

By definition we have

$$(5.11) \quad \overleftarrow{H}_L(L_{r^{\lambda}}, I_0) \xrightarrow{\sim} \alpha_{L!}(\text{IC}).$$

This gives us the desired shift as follows

$$(5.12) \quad \begin{aligned} J_P^* \overleftarrow{H}_G(L_{r^{\lambda}}, I_0) &\xrightarrow{\sim} \alpha_{V!} \overline{\mathbb{Q}}_{\ell}[\langle \lambda, 2\check{\rho}_G \rangle + r \dim V_0 - \langle \lambda, \check{\mu} \rangle] \\ &\xrightarrow{\sim} \alpha_{L!} \overline{\mathbb{Q}}_{\ell}[\langle \lambda, \check{\nu} - \check{\mu} - 2(\check{\rho}_G - \check{\rho}_L) \rangle] \\ &\xrightarrow{\sim} \overleftarrow{H}_L(L_{r^{\lambda}}, I_0)[\langle \lambda, \check{\nu} - \check{\mu} - 2(\check{\rho}_G - \check{\rho}_L) \rangle] \\ &\xrightarrow{\sim} \overleftarrow{H}_L(\mathfrak{gRes}(L_{r^{\lambda}}), I_0)[\langle \lambda, \check{\nu} - \check{\mu} \rangle], \end{aligned}$$

where the first isomorphism is due to (5.9), the second is due to (5.10), the third is due to (5.11) and the last one is due to Lemma 5.5. This shift is compatible with [21, Lemma 5]. \square

Let $\delta_U : \mathbb{G}_m \times M_0 \rightarrow M_0$ be a linear action, whose fixed points set is V_0 . Assume δ_U contracts M_0 onto V_0 . Let r be a integer, denote by $\nu : \mathbb{G}_m \rightarrow L$ the cocharacter of the center of L acting on V by $x \rightarrow x^r$. Now consider the case where δ_U is the map sending x in \mathbb{G}_m to $\nu(x)x^{-r}$. For any x in \mathbb{G}_m and m in $M_{N,r}$ consider the action of \mathbb{G}_m on $M_{N,r}$ defined by $(x, m) = xm$. Let K be a \mathbb{G}_m -equivariant perverse sheaf in $P_{I_G}(M(F))$

with respect to this action of \mathbb{G}_m on $M_{N,r}$. Then for any w in \widetilde{W}_G , both K and $\widetilde{H}_G(L_w, K)$ are \mathbb{G}_m -equivariant with respect to the δ_U -action on $M(F)$. We get a new version of Proposition 5.7 as follows:

Corollary 5.13. *Let K be a \mathbb{G}_m -equivariant perverse sheaf in $P_{I_G}(M(F))$ for the δ_U -action on $M_{N,r}$, for N, r large enough. Assume that \mathbf{k} admits a \mathbf{k}' -structure for some finite subfield \mathbf{k}' of \mathbf{k} , and as such is pure of weight zero. Then $J_p^*(K)$ is pure of weight zero.*

This is an analogue of [21, Corollary 3] in the Iwahori case affirming that the geometric Jacquet functors preserve the pure perverse sheaves of weight zero.

6. APPENDIX A

The construction of Hecke functors has been done in [10, §3]. We will recall here its big lines for sake of completeness. Let G be a split connected reductive group over \mathbf{k} . Let T_G be the maximal standard torus and B_G be the standard Borel subgroup containing T_G in G . Denote by I_G the corresponding Iwahori subgroup. Let M_0 be a faithful finite-dimensional representation of G and let $M = M_0 \otimes_{\mathbf{k}} \mathcal{O}$. The definitions of the derived category $D_{I_G}(M(F))$ of ℓ -adic sheaves on $M(F)$ and the category $P_{I_G}(M(F))$ of ℓ -adic perverse sheaves on $M(F)$ are given in [10, §3]. For any two integers $N, r \geq 0$ with $N + r > 0$, set $M_{N,r} = t^{-N}M/t^rM$. The subgroup $G(\mathcal{O})$ acts on $M_{N,r}$ via its finite dimensional quotient $G(\mathcal{O}/t^{N+r}\mathcal{O})$. Denote by I_s the kernel of the map $G(\mathcal{O}) \rightarrow G(\mathcal{O}/t^s\mathcal{O})$. The Iwahori subgroup I_G acts on $M_{N,r}$ via its finite-dimensional quotient I_G/I_{N+r} . For $s > 0$ denote by K_s the quotient I_G/I_s . Let $s_1, s_2 \geq 0$ and set

$$(6.1) \quad {}_{s_1, s_2}G(F) = \{g \in G(F) \mid t^{s_1}M \subset gM \subset t^{-s_2}M\}.$$

Then ${}_{s_1, s_2}G(F) \subset G(F)$ is closed and stable under the left and right multiplication by $G(\mathcal{O})$. Further, ${}_{s_1, s_2}\mathcal{F}l_G = {}_{s_1, s_2}G(F)/I_G$ is closed in $\mathcal{F}l_G$. For $s'_1 \geq s_1$ and $s'_2 \geq s_2$, we have the closed embeddings ${}_{s_1, s_2}\mathcal{F}l_G \hookrightarrow {}_{s'_1, s'_2}\mathcal{F}l_G$ and the union of ${}_{s_1, s_2}\mathcal{F}l_G$ is the affine flag variety $\mathcal{F}l_G$. The map sending g to g^{-1} yields an isomorphism between ${}_{s_1, s_2}G(F)$ and ${}_{s_2, s_1}G(F)$. Denote by $\check{\mu}$ in X^+ the character by which G acts on $\det(M_0)$. The connected components of the affine Grassmannian Gr_G are indexed by the algebraic fundamental group $\pi_1(G)$ of G . For θ a cocharacter in $\pi_1(G)$, choose λ in X^+ whose image in $\pi_1(G)$ equals θ . Denote by Gr_G^θ the connected component of Gr_G containing Gr_G^λ . The affine flag manifold $\mathcal{F}l_G$ is a fibration over Gr_G with the typical fibre G/B . Hence the connected components of the affine flag variety $\mathcal{F}l_G$ are also indexed by $\pi_1(G)$. For θ in $\pi_1(G)$, denote by $\mathcal{F}l_G^\theta$ the preimage of Gr_G^θ in $\mathcal{F}l_G$. Set ${}_{s_1, s_2}\mathcal{F}l_G^\theta = \mathcal{F}l_G^\theta \cap {}_{s_1, s_2}\mathcal{F}l_G$. According to [10, Lemma 4.2], There exists an inverse image functor

$$act_q^* : D_{I_G}(M(F)) \times D_{I_G}(\mathcal{F}l_G) \longrightarrow D_{I_G}(M(F) \times \mathcal{F}l_G)$$

which preserves perversity and is compatible with the Verdier duality in the following way: for any \mathcal{K} in $D_{I_G}(M(F))$ and \mathcal{F} in $D_{I_G}(\mathcal{F}l_G)$ we have

$$\mathbb{D}(act_q^*(\mathcal{K}, \mathcal{F})) \xrightarrow{\sim} act_q^*(\mathbb{D}(\mathcal{K}), \mathbb{D}(\mathcal{F})).$$

Given $N, r, s_1, s_2 \geq 0$ with $r \geq s_1$ and $s \geq \max\{N + r, s_1 + s_2 + 1\}$, one can define the following commutative diagram

$$\begin{array}{ccccc}
 M_{N,r} \times_{s_1, s_2} G(F) & \xrightarrow{\text{act}} & M_{N+s_1, r-s_1} & & \\
 \downarrow q_G & & \downarrow q_M & & \\
 M_{N,r} & \xleftarrow{pr_1} & M_{N,r} \times_{s_1, s_2} \mathcal{F}l_G & \xrightarrow{\text{act}_q} & K_s \backslash M_{N+s_1, r-s_1} \\
 \downarrow & & \downarrow & \nearrow \text{act}_{q,s} & \\
 K_s \backslash M_{N,r} & \xleftarrow{pr} & K_s \backslash (M_{N,r} \times_{s_1, s_2} \mathcal{F}l_G) & \xrightarrow{pr_2} & K_s \backslash ({}_{s_1, s_2} \mathcal{F}l_G)
 \end{array}$$

The action map act sends the couple (v, g) to $g^{-1}v$. The maps pr_1 , pr_2 and pr are projections. The map q_G sends the couple (v, g) to (v, gI_G) . All the vertical arrows are stack quotients for the action of the corresponding group. The group K_s acts diagonally on $M_{N,r} \times_{s_1, s_2} \mathcal{F}l_G$ and the map act_q is equivariant with respect to this action. This functor sends $(\mathcal{K}, \mathcal{T})$ to

$$K \boxtimes \mathcal{T} := (act_{q,s}^* \mathcal{K}) \otimes pr_2^* \mathcal{T}[\dim(K_s) - c + s_1 \dim M_0]$$

where c equals $\langle \theta, \check{\mu} \rangle$ over ${}_{s_1, s_2} \mathcal{F}l_G^\theta$.

For any N, r, s_1, s_2 greater than zero satisfying the condition $s \geq \max\{N + r, s_1 + s_2 + 1\}$, consider the projection

$$pr : K_s \backslash (M_{N,r} \times_{s_1, s_2} \mathcal{F}l_G) \longrightarrow K_s \backslash M_{N,r}$$

For any \mathcal{K} in $D_{I_G}(M(F))$ and \mathcal{T} in $D_{I_G}(\mathcal{F}l_G)$, the Hecke functor

$$\overleftarrow{H}_G(\cdot, \cdot) : D_{I_G}(\mathcal{F}l_G) \times D_{I_G \times I_H}(\Pi(F)) \rightarrow D_{I_G \times I_H}(\Pi(F))$$

is defined by

$$\overleftarrow{H}_G(\mathcal{T}, \mathcal{K}) = pr_1((K \boxtimes \mathcal{T}))$$

Moreover, this functor is compatible with the convolution product on $D_{I_G}(\mathcal{F}l_G)$. Namely, given $\mathcal{T}_1, \mathcal{T}_2$ in $D_{I_G}(\mathcal{F}l_G)$ and \mathcal{K} in $D_{I_G}(M(F))$, one has naturally

$$\overleftarrow{H}_G(\mathcal{T}_1, \overleftarrow{H}_G(\mathcal{T}_2, \mathcal{K})) \xrightarrow{\sim} \overleftarrow{H}_G(\mathcal{T}_1 \star \mathcal{T}_2, \mathcal{K}).$$

An example of computation of Hecke functors. Let $R, r \geq 0$ and $t^r M \subset V \subset t^{-R} M$ be an intermediate lattice stable under I_G . Let $K \in P_{I_G}(M_{R,r})$ be a shifted local system on $V/t^r M \subset t^{-R} M/t^r M$. We are going to explain the above construction explicitly in this case. Let \mathcal{T} be in $D_{I_G}({}_{s_1, s_2} \mathcal{F}l_G)$. Choose $r_1 \geq r + s_1$. If g is a point in ${}_{s_1, s_2} \mathcal{F}l_G$ then $t^{r_1} M \subset gV$. So we can define the scheme $(V/t^r M) \tilde{\times}_{s_1, s_2} \mathcal{F}l_G$ as the scheme classifying pairs (gI_G, m) such that gI_G is an element of ${}_{s_1, s_2} \mathcal{F}l_G$ and m is in $(gV)/(t^{r_1} M)$. For a point (m, g) of this scheme we have $g^{-1}m$ in $V/t^r M$. Assuming $s \geq R + r$ we get the digram

$$M_{R+s_2, r_1} \xleftarrow{p} (V/t^r M) \tilde{\times}_{s_1, s_2} \mathcal{F}l_G \xrightarrow{act_{q,s}} K_s \backslash (V/t^r M),$$

where p is the map sending (gI_G, m) to m . For $gG(O$ in Gr_G^θ , the virtual dimension of V/gV is $\langle \theta, \check{\mu} \rangle$. The space $(V/t^1 M) \times_{s_1, s_2} \mathcal{F}l_G^\theta$ is locally trivial fibration over $s_1, s_2 \mathcal{F}l_G^\theta$ with fibre isomorphic to an affine space of dimension $\dim(V/t^1 M) - \langle \theta, \check{\mu} \rangle$. Since K is a shifted local system, the tensor product $act_{q,s}^* K \otimes pr_2^* \mathcal{T}$ is a shifted perverse sheaf. Let $K \boxtimes \mathcal{T}$ be the perverse sheaf $act_{q,s}^* K \otimes pr_2^* \mathcal{T}[\dim]$. The shift $[\dim]$ in the definition depends on the dimension of the connected component and hence on $\check{\mu}$ as explained above and is such that the sheaf $act_{q,s}^* K \otimes pr_2^* \mathcal{T}[\dim]$ is perverse. Then $\overleftarrow{H}_G(\mathcal{T}, K) = p_!(K \boxtimes \mathcal{T})$.

7. APPENDIX B

The aim of this appendix is to compute the complex $\overleftarrow{H}_H(L_\tau, \mathcal{I}^\mu)$ in the derived category $D_{I_H \times I_G}(\Pi(F))$ for any L_τ in $P_{I_H}(\mathcal{F}l_H)$. We will first consider the case of τ being a simple reflection the finite Weyl group of H then for τ being the unique simple affine reflection in \widetilde{W}_H . The action of length zero elements being evident, this completes the action of simple objects of $P_{I_H}(\mathcal{F}l_H)$ on \mathcal{I}^μ in the derived category $D_{I_H \times I_G}(\Pi(F))$. For any point hI_H in $\overline{\mathcal{F}l_H}$ we write $U'_i = hU_i$ and we fix a complete flag $U'_1 \subset \dots \subset U'_m$ on U'/tU' . Let $\overline{\Pi}_1^\mu \times \overline{\mathcal{F}l_H}$ be the scheme classifying pairs (v, hI_H) , where hI_H is in $\overline{\mathcal{F}l_H}$ and a v is a map from L^* to U'/tU' such that $v(e_i^*) \in U'_{v(i)}$ for all $i = 1, \dots, n$. Let

$$(7.1) \quad \pi : \overline{\Pi}_1^\mu \times \overline{\mathcal{F}l_H} \rightarrow \Pi_{0,1}$$

be the map sending (v, hI_H) to v . By definition we have

$$\overleftarrow{H}_H(L_\tau, \mathcal{I}^\mu) \xrightarrow{\sim} \pi_!(\overline{\mathbb{Q}}_\ell \boxtimes L_\tau).$$

Let $\Pi_1^\mu \times \mathcal{F}l_H$ be the open subscheme in $\overline{\Pi}_1^\mu \times \overline{\mathcal{F}l_H}$ consisting of pairs (v, hI_H) in $\Pi_1^\mu \times \mathcal{F}l_H$ such that $v(e_i^*) \notin U'_{v(i)-1}$ for all $i = 1, \dots, n$. If π^0 is the restriction of π to the open subscheme $\Pi_1^\mu \times \mathcal{F}l_H$, then we have

$$\overleftarrow{H}_H(L_{\tau^!}, \mathcal{I}^\mu) \xrightarrow{\sim} \pi_1^0(\overline{\mathbb{Q}}_\ell \boxtimes L_\tau).$$

For $1 \leq i < m$ we will denote by τ_i the simple reflection $(i, i+1)$ in W_H .

Proposition 7.2. *Let i be an integer such that $1 \leq i < m$*

- (1) *If $i \notin I_\mu$ then the complex $\overleftarrow{H}_H(L_{\tau_i}, \mathcal{I}^\mu)$ is canonically isomorphic to $\mathcal{I}^\mu \otimes \text{R}\Gamma(\mathbb{P}^1, \overline{\mathbb{Q}}_\ell)[1]$.*
- (2) *If $i \in I_\mu$ then*

$$\overleftarrow{H}_H(L_{\tau_i}, \mathcal{I}^\mu) \xrightarrow{\sim} \begin{cases} \mathcal{I}^{\tau_i \circ v} \oplus \mathcal{I}^{\tau_{i-1} \circ v} & \text{if } i > 1 \text{ and } i-1 \notin I_\mu \\ \mathcal{I}^{\tau_i \circ v} \oplus \text{IC}(Y'') & \text{otherwise,} \end{cases}$$

where Y'' is a specific locally closed subscheme of $\Pi_{0,1}$ (whose construction will be given in the proof below).

Proof. The scheme $\overline{\mathcal{F}l_H}$ is the projective space of lines in U_{i+1}/U_{i-1} .

- (1) Consider the projection $p_H : \mathcal{F}l_H \rightarrow Gr_H$. Then $p_{H!}(L_{\tau_i})$ is canonically isomorphic to $\text{R}\Gamma(\mathbb{P}^1, \overline{\mathbb{Q}}_\ell)[1]$. This implies that

$$\overleftarrow{H}_H(L_{\tau_i}, \mathcal{I}^\mu) \xrightarrow{\sim} \mathcal{I}^\mu \otimes \text{R}\Gamma(\mathbb{P}^1, \overline{\mathbb{Q}}_\ell)[1].$$

- (2) Let us describe the image of the map π in (7.1). It is contained in the closed subscheme Y' of $\Pi_{0,1}$ given by the following two conditions:
 - a) For $j \neq v^{-1}(i)$, $v(e_j^*) \in U_{v(j)}$.
 - b) For $j = v^{-1}(i)$, $v(e_j^*) \in U_{i+1}$.

Let Y'' be the closed subscheme in Y' consisting of elements v such that $v(e_j^*)$ belongs to U_{i-1} if $j = v^{-1}(i)$. Over a point of Y'' the fibre of the map (7.1) is \mathbb{P}^1 whence over a point of $Y' - Y''$ the fibre of the map (7.1) is a point. Thus

$$\overleftarrow{H}_H(L_\tau, \mathcal{I}^\mu) \xrightarrow{\sim} \mathrm{IC}(Y') \oplus \mathrm{IC}(Y'').$$

Now we need to identify Y' and Y'' .

If $i+1 \notin I_\mu$, let $I_{\mu'}$ be the subset of $\{1, \dots, m\}$ obtained from I_μ by throwing i away and adding $i+1$. In this case Y' is isomorphic to $\overline{\Pi}_{0,1}^{\mu'}$ so $\mathrm{IC}(Y')$ is canonically isomorphic to $\mathcal{I}^{\mu'}$. Now let $v' : \{1, \dots, n\} \rightarrow \{1, \dots, m\}$ be strictly decreasing with image $I_{\mu'}$. Considering v' as an element of $S_{n,m}$, we get that $\Pi_{0,1}^{\nu'} = \Pi_{0,1}^{\mu'}$.

If $i+1 \in I_\mu$, let $v' : \{1, \dots, n\} \rightarrow \{1, \dots, m\}$ be the map v composed with the permutation τ_i . The image of v' is the subset I_μ (but v' is not strictly decreasing). Viewing v' as an element of $S_{n,m}$ enables us to identify Y' with the closure of $I_H \times I_G$ -orbit $\Pi_{0,1}^{\nu'}$.

Thus, in both cases $\nu' = \tau_i \circ v$ is the composition $\{1, \dots, n\} \xrightarrow{v} \{1, \dots, m\} \xrightarrow{\tau_i} \{1, \dots, m\}$ and $\mathrm{IC}(Y')$ is isomorphic to $\mathcal{I}^{\tau_i \circ v}$.

If $i > 1$ and $i-1 \notin I_\mu$ then $\tau_{i-1} \circ v : \{1, \dots, n\} \rightarrow \{1, \dots, m\}$ is strictly decreasing and Y'' is isomorphic to the closure of $\Pi_{0,1}^{\tau_{i-1} \circ v}$. So we get $\mathrm{IC}(Y'') = \mathcal{I}^{\tau_{i-1} \circ v}$. This proves the assertion. \square

Proposition 7.3. *Let i be an integer such that $1 \leq i < m$*

(1) *If neither i nor $i+1$ is in I_μ then*

$$\overleftarrow{H}_H(L_{\tau_i}, \mathcal{I}^{\mu^!}) \xrightarrow{\sim} \mathcal{I}^{\mu^!} \otimes \mathrm{R}\Gamma(\mathbb{P}^1, \overline{\mathbb{Q}}_\ell)[1]$$

(2) *If i is not in I_μ and $i+1$ is in I_μ then the composition $\tau_i \circ v$ is again strictly decreasing and there is a distinguished triangle*

$$\mathcal{I}^{\mu^!}[-1] \longrightarrow \overleftarrow{H}_H(L_{\tau_i}, \mathcal{I}^{\mu^!}) \longrightarrow \mathcal{I}^{(\tau_i \circ v)^!} \xrightarrow{+1}$$

(3) *If i is in I_μ and $i+1$ is not an element of I_μ then there is a distinguished triangle*

$$\mathcal{I}^{(\tau_i \circ v)^!} \longrightarrow \overleftarrow{H}_H(L_{\tau_i}, \mathcal{I}^{\mu^!}) \longrightarrow \mathcal{I}^{\nu^!}[1] \xrightarrow{+1}.$$

(4) *If both i and $i+1$ are in I_μ then there is a distinguished triangle*

$$\mathcal{I}^{(\tau_i \circ v)^!} \longrightarrow \overleftarrow{H}_H(L_{\tau_i}, \mathcal{I}^{\mu^!}) \longrightarrow \mathcal{I}^{\mu^!}[1] \xrightarrow{+1}.$$

Proof. (1) This is straightforward as in Proposition 7.2.

(2) Let Y' be the locally closed subscheme of $\Pi_{0,1}$ given by the conditions :

- a) For $1 \leq j \leq n$, $v(e_j^*) \in U_{v(j)}$.
- b) For $j \neq v^{-1}(i+1)$, $v(e_j^*) \notin U_{v(j)-1}$.
- c) For $j = v^{-1}(i+1)$, $v(e_j^*) \notin U_{v(j)-2}$.

The scheme Y' is the union of two $I_H \times I_G$ -orbits corresponding to ν and $\tau_i \circ \nu$. Moreover, we have

$$\dim(\Pi_{0,1}^\nu) = 1 + \dim(\Pi_{0,1}^{\tau_i \circ \nu}).$$

Hence

$$\overleftarrow{H}_H(L_{\tau_i}, \mathcal{I}^{\mu^!}) \xrightarrow{\sim} \mathrm{IC}(Y')[-1]$$

and the assertion follows.

(3) Let Y' be the scheme classifying elements ν in $\Pi_{0,1}$ such that :

- a) For $j \neq \nu^{-1}(i)$, $\nu(e_j^*) \in U_{\nu(j)}$ and $\nu(e_j^*) \notin U_{\nu(j)-1}$.
- b) For $j = \nu^{-1}(i)$, $\nu(e_j^*) \in U_{i+1}$ and $\nu(e_j^*) \notin U_{i-1}$.

Then Y' is the union of two orbits $\Pi_{0,1}^\nu$ and $\Pi_{0,1}^{\tau_i \circ \nu}$. Thus $\overleftarrow{H}_H(L_{\tau_i}, \mathcal{I}^{\mu^!})$ is the extension by zero of $\mathrm{IC}(Y')$ from Y' to $\Pi_{0,1}$. Hence we have a distinguished triangle

$$\mathcal{I}^{(\tau_i \circ \nu)^!} \longrightarrow \mathrm{IC}(Y') \longrightarrow \mathcal{I}^{\nu^!}[1] \xrightarrow{+1} .$$

(4) Let Y' be a locally closed subscheme $\Pi_{0,1}$ be the scheme given by the conditions:

- a) For $\nu(j) \neq i, i+1$, $\nu(e_j^*) \in U_{\nu(j)} - U_{\nu(j)-1}$.
- b) For $j = \nu^{-1}(i)$, $\nu(e_j^*)$ and $\nu(e_{j-1}^*)$ belong to U_{i+1} , and their classes modulo U_{i-1} form a basis of U_{i+1}/U_{i-1} .

Then $\overleftarrow{H}_H(L_{\tau_i}, \mathcal{I}^{\mu^!})$ is isomorphic to $\mathrm{IC}(Y')$ extended by zero to $\Pi_{0,1}$. The scheme Y' is the union of two orbits $\Pi_{0,1}^\nu$ and $\Pi_{0,1}^{\tau_i \circ \nu}$, and we have a distinguished triangle

$$\mathcal{I}^{(\tau_i \circ \nu)^!} \longrightarrow \mathrm{IC}(Y') \longrightarrow \mathcal{I}^{\nu^!}[1] \xrightarrow{+1} .$$

□

Proposition 7.4. *Let w be the affine simple reflection, i.e. $w = t^\lambda \tau$ where $\lambda = (-1, 0, \dots, 0, 1)$ and $\tau = (1, m)$ is the longest element of W_H .*

(1) *If nor 1 neither m lies in I_μ then*

$$\overleftarrow{H}_H(L_w, \mathcal{I}^{\mu^!}) \xrightarrow{\sim} \mathcal{I}^{\mu^!} \otimes R\Gamma(\mathbb{P}^1, \overline{\mathbb{Q}}_\ell)[1].$$

(2) *If 1 is not in I_μ and m lies in I_μ , let $\lambda' = (-1, 0, \dots, 0)$, and $w' = (\lambda', \tau \circ \nu)$ be an element of $X_G \times S_{n,m}$, then there is a distinguished triangle*

$$\mathcal{I}^{w'^!} \longrightarrow \overleftarrow{H}_H(L_w, \mathcal{I}^{\mu^!}) \longrightarrow \mathcal{I}^{\mu^!}[1] \xrightarrow{+1} .$$

(3) *If 1 is in I_μ , and m is not in I_μ , let $\lambda' = (0, \dots, 0, 1)$, and $w' = (\lambda', \tau \circ \nu)$ be an element of $X_G \times S_{n,m}$, then there is a distinguished triangle*

$$\mathcal{I}^{\mu^!}[-1] \longrightarrow \overleftarrow{H}_H(L_w, \mathcal{I}^{\mu^!}) \longrightarrow \mathcal{I}^{\mu^!}[1] \xrightarrow{+1} .$$

(4) *If 1 and m are both in I_μ , let $\lambda' = (-1, 0, \dots, 0, 1)$ and $w' = (\lambda', \tau \circ \nu)$ then there is a distinguished triangle*

$$\mathcal{I}^{w'^!} \longrightarrow \overleftarrow{H}_H(L_w, \mathcal{I}^{\mu^!}) \longrightarrow \mathcal{I}^{\mu^!}[1] \xrightarrow{+1} .$$

Proof. Denote by $U_{-1} \subset U_0 \subset U_1 \subset \dots \subset U_m = U \subset U_{m+1}$ the standard flag of lattices in $U(F)$. Assume $m > 1$. A point hI_H in $\overline{\mathcal{F}l}_H^w$ is given by a line U'_0/U_{-1} in U_1/U_{-1} . We set $U'_m = t^{-1}U'_0$. Let $\Pi^\mu \overline{\mathcal{F}l}_H^w$ be the scheme classifying pairs (v, hI_H) , where hI_H is in $\overline{\mathcal{F}l}_H^w$ and v is a map from L^* to U_{m+1}/U_{-1} verifying:

- a) For $v(j) \neq m$, $v(e_j^*) \in U_{v(j)}$.
- b) For $v(j) \neq 1$, $v(e_j^*) \in U_{v(j)} - U_{v(j)-1}$.
- c) For $m \in I_\mu$, $v(e_1^*) \in U'_m - U'_{m-1}$.
(The condition $m \in I_\mu$ is equivalent to $v(1) = m$).
- d) If $1 \in I_\mu$ then $v(e_n^*) \in U_1 - U'_0$.
(The condition $1 \in I_\mu$ is equivalent to $v(n) = 1$).

Now let

$$(7.5) \quad \pi : \Pi^\mu \overline{\mathcal{F}l}_H^w \longrightarrow \text{Hom}_{\mathcal{O}}(L^*, U_{m+1}/U_{-1})$$

be the projection sending a couple (v, hI_H) to v . The scheme $\Pi^\mu \overline{\mathcal{F}l}_H^w$ is smooth. Write IC for the intersection cohomology sheaf of $\Pi^\mu \overline{\mathcal{F}l}_H^w$. The sheaf IC is nothing but the constant sheaf shifted to be perverse. Then

$$\overleftarrow{H}_H(L_w, \mathcal{I}^{\mu^1}) \xrightarrow{\sim} \pi_!(\text{IC}).$$

We can now prove the assertions.

- (1) This is straightforward as in Proposition 7.2.
- (2) The space $\text{Hom}_{\mathcal{O}}(L^*, U_{m+1}/U_{-1})$ is an \mathcal{O} -module on which t acts trivially, hence it is a vector space. By definition $\overleftarrow{H}_H(L_w, \mathcal{I}^{\mu^1})$ may be considered as a complex on $\text{Hom}_{\mathcal{O}}(L^*, U_{m+1}/U_{-1})$. Let Y' be the subscheme of $\text{Hom}_{\mathcal{O}}(L^*, U_{m+1}/U_{-1})$ given by the conditions:
 - a) For $j > 1$, $v(e_j^*) \in U_{v(j)} - U_{v(j)-1}$.
 - b) The vector $v(e_1^*)$ does not vanish in U_{m+1}/U_{m-1} .

Then $\overleftarrow{H}_H(L_w, \mathcal{I}^{\mu^1})$ is isomorphic to $\text{IC}(Y)'$ extended by zero to $\text{Hom}_{\mathcal{O}}(L^*, U_{m+1}/U_{-1})$. The subscheme Y' is the union of two $I_H \times I_G$ -orbits, the closed orbit corresponds to $\Pi_{N,r}^\mu$ and the open orbit passes through maps v given by

$$v(e_n^*) = u_{v(n)}, \dots, v(e_2^*) = u_{v(2)}, v(e_1^*) = t^{-1}u_1.$$

The map v can be written as follows:

- a) For $j \in I_\mu - \{m\}$, $v(u_j^*) = e_{\mu(j)}$.
- b) For all other $k \neq 1$, $v(u_k) = 0$; and $v(u_1^*) = t^{-1}e_1$.

So this open orbit corresponds to the element $w' = (\lambda', \tau \circ v)$ in $X_G \times S_{n,m}$, where $\lambda' = (-1, 0, \dots, 0)$. This leads to the desired distinguished triangle.

- (3) In this case $\overleftarrow{H}_H(L_w, \mathcal{I}^{\mu^1})$ is naturally a complex over $\text{Hom}_{\mathcal{O}}(L^*, U_{m-1}/U_{-1})$. The space $\text{Hom}_{\mathcal{O}}(L^*, U_{m-1}/U_{-1})$ is an \mathcal{O} -module on which t acts trivially, hence is a vector space. Denote by Y' the subscheme of $\text{Hom}_{\mathcal{O}}(L^*, U_{m-1}/U_{-1})$ given by the conditions:
 - a) $v(e_n^*) \in U_1 - U_{-1}$.
 - b) $v(e_j^*) \in U_{v(j)} - U_{v(j)-1}$ for $1 \leq j < n$.

Then the fibres of the map (7.5) identify with \mathbb{A}^1 . So $\overleftarrow{H}_H(L_w, \mathcal{I}^{\mu^1})$ is the sheaf $\mathrm{IC}(Y')[-1]$ extended by zero to $\mathrm{Hom}_O(L^*, U_{m-1}/U_{-1})$. The scheme Y' is the union of two $I_H \times I_G$ -orbits, the open corresponding to ν and the closed one passing through ν given by

$$\nu(e_n^*) = tu_m, \nu(e_{n-1}^*) = u_{\nu(n-1)}, \dots, \nu(e_1^*) = u_{\nu(1)}.$$

Let $w' = (\lambda', \tau \circ \nu)$ be in $X_G \times S_{n,m}$ with λ' being equal to $(0, \dots, 0, 1)$. Then we have a distinguished triangle

$$\mathcal{I}^{\mu}[-1] \longrightarrow \overleftarrow{H}_H(L_w, \mathcal{I}^{\mu^1}) \longrightarrow \mathcal{I}^{w'} \xrightarrow{+1}.$$

(4) In this case we have $\nu(1) = m$ and $\nu(n) = 1$. Let Y' be the subscheme of $\mathrm{Hom}_O(L^*, U_{m+1}/U_{-1})$ classifying maps ν satisfying the conditions:

- a) For all $1 < j < n$, $\nu(e_j^*) \in U_{\nu(j)} - U_{\nu(j)-1}$.
- b) $\nu(e_n^*) \in U_1 - U_{-1}$.
- c) $\nu(e_1^*) \in U_{m+1} - U_{m-1}$.
- d) $\{\nu(e_n^*), \nu(e_1^*)\}$ are linearly independent in U_1/U_{-1} .

Then the map (7.5) is an isomorphism onto Y' . Hence $\overleftarrow{H}_H(L_w, \mathcal{I}^{\mu^1})$ identifies with $\mathrm{IC}(Y')$ extended by zero from Y' to $\mathrm{Hom}_O(L^*, U_{m+1}/U_{-1})$. The scheme Y' contains the closed subscheme which is the $I_H \times I_G$ -orbit corresponding to ν . The complement of the latter scheme in Y' is the $I_H \times I_G$ -orbit passing through ν given by

$$\nu(e_1^*) = t^{-1}u_1, \nu(e_2) = u_{\nu(2)}, \dots, \nu(e_{n-1}^*) = u_{\nu(n-1)}, \nu(e_n^*) = tu_m.$$

Let $w' = (\lambda, \tau_i \circ \nu)$ where $\lambda = (-1, 0, \dots, 0, 1)$. Then there is a distinguished triangle

$$\mathcal{I}^{w'} \longrightarrow \mathrm{IC}(Y') \longrightarrow \mathcal{I}^{\mu^1}[1] \xrightarrow{+1}.$$

□

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