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Derivation of hierarchies of reduced MHD models in Tokamak geometry

Bruno Després* and Rémy Sart†

March 4, 2013

Abstract: We study the mathematical structure of reduced MHD models. We show how to modify the entropy moment methods to obtain a hierarchy of models with a correct energy balance. Our procedure is very adapted to the complicated geometry of the torus. We obtain a comparison principle between all these models. A new correction term due to the geometry is derived which is important for the stability of the model.

Reduced magnetohydrodynamics (MHD) systems provide a simplified reduced modeling of the dynamics of magnetic flows in some specific geometrical situations. These models are called *reduced* because they derive from a simplification (i.e. a reduction) of the system of full MHD (2). This is justified to filter non essential magnetosonic waves [24] for flows close to incompressibility. Two major examples are astrophysics [8, 12], and the modeling of Tokamaks in axisymmetric geometries for which we refer to the seminal works of Strauss [39, 40, 38, 41, 20, 42], see also [2, 18] and references therein. Despite the interest for applications of this family of models, our understanding is that reduced MHD models have not received enough mathematical attention. Our concern in this work is precisely the mathematical structure of such models, having in mind one key domain of application which is the numerical modeling of MHD stability in Tokamaks for which we refer to the recent simulations evoked in [10, 11, 15, 21]. See [3] for a more general mathematical and numerical introduction of the topic of MHD stability by different means. Truly agreeing with Krüger and al [25], we consider that energy conservation and closure of the set of equations is important. But contrary to what is usually done in plasma physics where energy balance is checked a posteriori up to some lower order terms [25] and at the price of a very complicated structure of the equations [6], we consider that energy conservation or energy balance must be satisfied a priori. The main reason is that mathematical stability of a set of complicated PDEs and its numerical stability are known to be quite sensitive to such a principle, which is a first step in the direction of proving the well-posedness: at the numerical level, numerical stability is required to have a control of the stability of the code, and can be an help to design optimal solvers [7]. This is why we consider that having a derivation of reduced MHD models with an exact balance of energy is a fundamental issue. In this direction we will develop a systematic procedure which generates a hierarchy of reduced MHD models with a correct balance of energy and is a way to analyze the complex algebraic structure of such models.

This procedure uses an original extension of the entropy closure method [9, 4, 5] A systematic application of the entropy closure method to model reduction is to be found in [22]. A more geometrical treatment is in [13], and application to the modeling of uncertainties is in [33, 34]. We will show that the entropy closure method can be used in toroidal geometries, which is one of the main theoretical contribution of this work¹. What we believe an important output of this approach is a new correction term Q in for equation of the magnetic potential in axisymmetric domains. We will indicate why standard reduced MHD models in Tokamak geometry (axisymmetric geometry) which do not have this Q may exhibit some linear ill-posedness. The correction term is defined through the solution of a simple linear well-posed variational formulation: this is an interesting property in view of future numerical developments.

To illustrate now more precisely the kind of problems addressed in this work, we first recall the simplest model in the family of reduced MHD models. For simplicity we will consider only simple boundary conditions on the

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¹As an historical comment, we point out that our approach makes compatible two parts of the seminal contributions of Grad which seemed to have independent life in the scientific literature: the first axis is equations of equilibrium states in magnetic plasmas [20], the famous Grad-Shafranov equation; the second axis is closure methods for the derivation of physically sound non stationary models to compute the dynamics of flows around a given equilibrium [19]. To our knowledge this is the first time that an attempt is made to unify these two fields.

boundary of a fixed domain: see [3] for free boundary equations in Tokamaks. Our simplest model problem is the standard incompressible isothermal reduced resistive MHD model in a two-dimensional domain $(x, z) \in \mathcal{D}$

$$\begin{cases} \partial_t \psi = [\psi, \phi] + \eta \Delta_{\perp} \psi, \\ \partial_t \omega = [\omega, \phi] + [\psi, \Delta_{\perp} \psi] + \nu \Delta_{\perp} \omega, \\ \Delta_{\perp} \phi = \omega. \end{cases} \quad (1)$$

Here $\psi(x, z)$ is the magnetic potential, $\phi(x, z)$ is the velocity potential and $\omega(x, z)$ is the vorticity. The tri-dimensional magnetic field and velocity field are defined by

$$\mathbf{B} = \nabla \psi \wedge \nabla y \text{ and } \mathbf{u} = \nabla \phi \wedge \nabla y, \quad \nabla y = (0, 1, 0).$$

By construction \mathbf{B} and \mathbf{u} do not depend on the second variable y . The Poisson bracket is $[a, b] = \partial_x a \partial_z b - \partial_z a \partial_x b$. The perpendicular Laplacian is $\Delta_{\perp} = \partial_{xx} + \partial_{zz}$. The resistivity coefficient is $\eta > 0$, while the viscosity coefficient is $\nu > 0$. In practice, that is for the kind of dilute plasmas encountered in Tokamaks, the resistivity η and the viscosity ν have very small values (in relative units): see for example the simulations reported in [10]. This is why the stability or well-posedness of the reduced model is fundamental in this regime of vanishing diffusivity. As is well known, the model (1) can be understood as an approximation of the full tri-dimensional system of viscous resistive MHD in a cylinder invariant in the y direction $\mathcal{C} = \{(x, y, z); (x, z) \in \mathcal{D}, y \in \mathbb{R}\}$

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \\ \partial_t \mathbf{B} = \nabla \wedge (\mathbf{u} \wedge \mathbf{B}) - \eta \nabla \wedge (\nabla \wedge \mathbf{B}), \\ \partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p + \nabla \cdot \left(\frac{|\mathbf{B}|^2}{2} \mathbf{I} - \mathbf{B} \otimes \mathbf{B} \right) = \nu \Delta \mathbf{u}, \quad \mathbf{J} = \nabla \wedge \mathbf{B}, \\ \partial_t (\rho e) + \nabla \cdot (\rho \mathbf{u} e + p \mathbf{u}) + \nabla \cdot ((\mathbf{u} \wedge \mathbf{B}) \cdot \mathbf{B}) = \nu \nabla \cdot (\mathbf{u} \nabla \mathbf{u}) + \eta \nabla \cdot (\mathbf{B} \wedge \mathbf{J}). \end{cases} \quad (2)$$

In the equation (2), ρ is the density, \mathbf{u} is the velocity, \mathbf{B} is the magnetic field, \mathbf{J} is the current and $e = \frac{p}{(\gamma-1)\rho} + \frac{1}{2} |\mathbf{u}|^2 + \frac{1}{2\rho} |\mathbf{B}|^2$ the density of total energy. Comparing (1) and (2), the reduction of the complexity of the model is in the dimension of the problem (two-dimensional instead of three-dimensional), in the number of variables and in the nature of these variables. The two-dimensional model (1) is used for example in [28, 29] for the numerical simulation of magnetic stability in Tokamaks. A state of the art reference is [23].

A important property of (1) is that there is a physically correct balance equation for the total energy

$$\frac{d}{dt} \int \frac{|\nabla \psi|^2 + |\nabla \phi|^2}{2} = -\eta \int (\Delta_{\perp} \psi)^2 - \nu \int (\Delta_{\perp} \phi)^2. \quad (3)$$

Such identity exists of course already for (2), even in a different form, since the total energy is a variable of the system. Such balance law may appear as very fortunate because the usual way of deriving (1) from (2) by means of asymptotic expansion and model reduction does not preserve a priori such structure. In some sense it is only by chance that (1) is endowed with an energy law (3) which is on the one hand physically correct and on the other hand very useful to prove well-posedness.

The generalization of (1) in axisymmetric geometry is appealing for the modeling of MHD flows in Tokamaks. In this direction one usually starts from representations like

$$\mathbf{B} = F \nabla \theta + \nabla \psi \wedge \nabla \theta \text{ and } \mathbf{u} = \lambda \nabla \theta + \nabla \phi \wedge \nabla \theta. \quad (4)$$

Here θ is the angular (toroidal) variable of the axisymmetric set of variables (r, θ, z) deduced from the Cartesian set of variable (x, y, z) : the correspondence is $r = \sqrt{x^2 + z^2}$, $x = r \cos \theta$ and $z = r \sin \theta$. Clearly $\nabla \theta$ plays the role of ∇y before. In Tokamaks F is a forcing term that represents the exterior coils: it is given. On the other hand λ is the coefficient of parallel velocity and is an unknown of an extension of (1). In some cases all variables are independent of the toroidal variable θ . In more comprehensive models all variables display a dependency with respect to θ . Considering the intrinsic complexity of the expansion (4) it is not evident a priori that all kind of reduced MHD models obtained from a reduction of (2) with the representation (4) are endowed with a energy identity. In some cases it has been checked [31] that the energy identity is lost. Additional remarks are as follows.

- A constant forcing term $F = F_0$ help to design a model with a energy identity, see [17, 25, 24]. Notice moreover that for a constant F_0 the associated magnetic force is zero since $\nabla \wedge \nabla \theta = 0$: that is the forcing term is, physically, force-free.

- A non constant forcing term has never been proved to generate a reduced model with energy balance: see all references therein this work. A priori, non constant F yields a non force-free magnetic force $\nabla \wedge (F\nabla\theta) \wedge (F\nabla\theta) \neq 0$. This is why a energy balance should be looked for in this case, and not a energy conservation. The theory developed in this work will provide a systematic tool to analyze this situation: see the abstract balance equation (19).
- In physical situations we are interested in, the flow is nearly incompressible. This is compatible, in some sense, with the perpendicular part of the velocity (4) which is divergence free. But in this case incompressibility is reached only if the density is constant in space due to

$$\partial_t \rho + \mathbf{u} \cdot \nabla \rho = 0.$$

It corresponds to the basic reduced model (1).

- The Ansatz in toroidal geometry mostly used in the physical literature [6, 17, 24, 25, 10, 11, 21, 25, 35, 37, 40, 42] is based on $\mathbf{u} = \mathbf{u}_{\parallel} + \mathbf{u}_{\perp}$ with

$$\mathbf{u}_{\perp} = r^2 \nabla \phi \wedge \nabla \theta. \quad (5)$$

That is the perpendicular velocity is not divergence free a priori, even if it is nearly divergence free for small curvatures for which the variation of r is negligible in first approximation. The idea behind this choice is some compatibility with the Ohm's law $\mathbf{E} + \mathbf{u} \wedge \mathbf{B} = 0$ which is a key ingredient to eliminate the electric field in the set of full Maxwell's equation: with this equivalence $\partial_t \mathbf{B} + \nabla \wedge \mathbf{E} = 0$ is indeed equivalent to $\partial_t \mathbf{B} = \nabla \wedge \mathbf{u} \wedge \mathbf{B}$. Let us now consider a magnetic field under the form $\mathbf{B} = F_0 \nabla \theta$ and let assume a perpendicular velocity (5) with a potential invariant in the toroidal direction: $\partial_{\theta} \phi = 0$. With these hypotheses, one has after simplifications ($\nabla \theta = \frac{1}{r} \mathbf{e}_{\theta}$)

$$\mathbf{E} = -r^2 (\nabla \phi \wedge \nabla \theta) \wedge (F_0 \nabla \theta) = F_0 \nabla \phi.$$

That is the electric field is potential. A restriction is $\partial_{\theta} \phi = 0$. This idea is developed in [17, 25, 24]) and references therein.

To our knowledge a representation like (5) is not an obstruction to obtain an energy identity for reduced models where the density variable is still an unknown of the model. But if one desires to define a reduced model with vanishing parallel velocity \mathbf{u}_{\parallel} and without tracking of the density, then the use of such \mathbf{u}_{\perp} is an obstruction to obtain energy balance. This is clear if one tries to redo the algebra sketched in [25, 24].

- In this work we prefer to develop closure relations in incompressible or nearly incompressible regimes starting directly from the full MHD system (2). In this direction it has been observed in [15] that incompressibility (that is $\partial_t \rho = 0$) with a non trivial and non constant in space density can be obtained with

$$\mathbf{u}_{\perp} = \frac{1}{\rho_0} \nabla \phi \wedge \nabla \theta, \quad \rho(t=0) = \rho_0. \quad (6)$$

We will develop such a model in our work. It must be noticed that the usual Ansatz (5) is formerly recovered as $\rho_0 = r^{-2}$. This is why there is no contradiction between (5) and (6). We will use this similarity in (59) to write the formulas adapted to the Ansatz used in the physical literature.

- The material presented in this work can be generalized without difficulty for a different modeling of the parallel velocity. One can use

$$\mathbf{u}_{\parallel} = \lambda \mathbf{B}_0$$

where \mathbf{B}_0 is a given a priori and frozen magnetic field which therefore replaces $\nabla \theta$. For example one can take the magnetic field at initial time.

The purpose of this work is precisely to examine all these issues at the light of a procedure, called hyperbolic compatibility in the following, which is an extension of the method of moments [9, 4, 13]. We will show that a systematic use of this method provides a hierarchy of reduced MHD systems, and that all of them are compatible with the conservation of total energy in a sense that will be detailed below. This method of construction is fundamentally different from the usual one which uses asymptotic expansions with respect to small parameters to show that the magnetic and velocity fields admit potential representation. The global structure of our models

is very close (equal in many cases) to the structure of the models used in the plasma physics community for practical simulations.

An important theoretical result is that the dynamics any model written within this framework gives a lower estimate of the dynamics of the full initial model. This is precized in theorem 7. This result can be considered as a fundamental justification of the use of reduced models.

We will also obtain new and original reduced MHD models with a correction term which is not present in standard models. We will write explicitly the equation of the correction term. Two cases of interest for the modeling of Tokamaks will finally be detailed. The first example deals with non constant forcing term F : we will show the compatibility with the Grad-Shafranov equation which described physically correct equilibrium initial data. The second example concerns the dependency with respect to the toroidal variable: we will show that the conservation of energy can be used to prove the existence of a weak solution. We will end with some remarks about the new correction term constructed in our work, which show that it may have strong impact on the well-posedness of reduced MHD models when taking into account the full toroidal dependency.

1 General methods to reduce the dimension

We start from general considerations about hyperbolic systems of conservation laws [26], and more precisely about model reduction of hyperbolic systems of conservation laws.

1.1 Moment models

Following [9, 4] we consider a hyperbolic system of conservation laws in dimension one

$$\partial_t U + \partial_x f(U) = 0, \quad x \in \mathbb{R}, \quad (7)$$

where the unknown is $U(x, t) \in \mathbb{R}^n$ and the flux is a regular function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$. We assume this system is endowed with an entropy-entropy flux pair $U \mapsto (S(U), F(U)) \in \mathbb{R}^2$ with the compatibility properties

$$\nabla S \nabla f = \nabla F \text{ and } d^2 S > 0 \quad (8)$$

where $\nabla S, \nabla F \in \mathbb{R}^n$, $\nabla f \in \mathbb{R}^{n \times n}$ and $d^2 S = d^2 S^t : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ is the Hessian matrix of S . For convenience we define the entropy or adjoint variable

$$V = \nabla S \in \mathbb{R}^n. \quad (9)$$

Since S is strictly convex, the transformation $U \mapsto V$ is a diffeomorphism from \mathbb{R}^n into (a subset of) \mathbb{R}^n . A simple type of reduced models are the ones such that some moments are prescribed

$$(U, Z_i) = a_i \in \mathbb{R}, \quad 1 \leq i \leq p \leq n, \quad (10)$$

where the vectors $Z_i \in \mathbb{R}^n$ are given, so independent of the time and space variables. The moments (10) are in some sense the new degrees of freedom that one tries to incorporate in (7). Since these constraints have no reason to be invariants of (7) one has to model them, in a stable manner if possible. In this work we rely on a procedure exists that guarantees some transmission of well-posedness from one model to the other. It is based on the minimization of the entropy under constraints (10).

Let us consider the Lagrangian

$$L(U, \lambda) = S(U) - \sum_{i=1}^p \lambda_i ((U, Z_i) - a_i).$$

The optimality conditions write

$$V - \sum_{i=1}^p \lambda_i Z_i = 0. \quad (11)$$

Considering (10) or (11) and assuming the Z_i are linearly independent, the unknown U lives in a variety of dimension p and co-dimension $n - p$. The p equations that describe the dynamics of this reduced model are

$$\partial_t a_i + \partial_x (f(U), Z_i) = 0, \quad 1 \leq i \leq p. \quad (12)$$

1.2 Another formulation

The previous reduced set of conservation laws (10)-(12) admits a more general formulation that has been developed in a completely different context [13, 33, 34]. It is based on the observation that the variety in which the solution lives is more easily described in the adjoint variable. In particular (11) may be rewritten as geometrical constraint

$$V \in K = \text{Span}_{1 \leq i \leq p} \{Z_i\}. \quad (13)$$

Here K is a d -dimensional vectorial subspace of \mathbb{R}^n . It gives hints that, using the entropy variables, model reduction admits a geometrical formulation that we detail now.

The general problem is now to model the constraints $V \in K$ where K is a given closed variety in \mathbb{R}^n . Considering (13), we will assume that K is an affine hyperplane

$$\exists V_0 \in \mathbb{R}^n \text{ such that } K = V_0 + \text{Span}_{1 \leq i \leq p} \{Z_i\}. \quad (14)$$

The cone of admissible direction is a vectorial subspace independent of V

$$dK = \text{Span}_{1 \leq i \leq p} \{Z_i\}.$$

The formulation of the reduced system is

$$\begin{cases} \partial_t(U, Z_i) + \partial_x(f(U), Z_i) = 0, & 1 \leq i \leq p, \\ V \in K. \end{cases} \quad (15)$$

There are many ways to prove that this reduced system is hyperbolic. The simplest one consists in remarking that $V - V_0 \in dK$. Therefore smooth solutions of (15) satisfy

$$(V - V_0, \partial_t U) + (V - V_0, \partial_x f(U)) = 0.$$

Making use that V is the entropy variable (8), it can be rearranged as

$$\partial_t \widehat{S}(U) + \partial_x \widehat{F}(U) = 0, \quad \widehat{S}(U) = S(U) - (V_0, U), \quad \widehat{F}(U) = F(U) - (V_0, f(U)). \quad (16)$$

Notice that $U \mapsto \widehat{S}(U)$ is a strictly convex functional with respect to U since $d^2 \widehat{S} = d^2 S > 0$. Up to non essential verifications left to the reader, it shows that $(\widehat{S}(U), \widehat{F}(U))$ is an entropy-entropy flux pair for the reduced system (15). A well known consequence is that (10)-(12) or (11)-(12) is hyperbolic by construction, see [4, 9, 13].

1.3 Hyperbolic compatibility

At inspection of the type of constraints (4) that we desire to model and analyze, it is nevertheless necessary to extend the previous formulation to functions.

We now consider the following problem, where $\Omega \subset \mathbb{R}^n$ is a given open domain and the space variable is denoted as $\mathbf{x} \in \Omega$. For the problems we have in mind, the domain may be a cylinder $\Omega = \mathcal{C}$ or a torus $\Omega = \mathcal{T}$. The flux is $f(U)$ with $f : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times d}$. The starting point is the system of conservation laws in dimension d

$$\partial_t U + \nabla \cdot f(U) = 0.$$

We will denote for convenience

$$X = \mathcal{C}^1(\Omega).$$

Remark 1. *The \mathcal{C}^1 regularity is enough for the weak formulation (18) to make sense. It is not a key ingredient and will be modified in the sequel. For example the strong formulation (60) needs more regularity to make sense. Convenient functional spaces in the context of the Lions-Temam theory will be used in our last section.*

Let us consider any affine subspace of X^n defined by

$$\mathcal{K} = V_0 + d\mathcal{K} \subset X^n \quad (17)$$

where $V_0 \in X^n$ is now a function and $d\mathcal{K}$ is a closed vectorial subspace of X^n . The dimension of $d\mathcal{K}$ is a priori infinite.

Definition 2 (Hyperbolic compatibility). *Neglecting at this stage the boundary conditions, any model that can be written under the form*

$$\begin{cases} \int_{\Omega} [\partial_t U + \nabla \cdot f(U), Z] dv = 0, & \forall Z \in d\mathcal{K}, \\ V \in \mathcal{K}, \end{cases} \quad (18)$$

will be said to be hyperbolic compatible.

Remark 3. *Boundary conditions, though simple ones, will be reintroduced along this work.*

Additional constraints must often be considered to obtain a physically sound set of equations. Such additional constraints can be characterized by the definition of open possibly unbounded domain $\mathcal{P} \subset X^n$. Typically any natural physical constraint such as $\rho > 0$ or $T > 0$ is incorporated in the definition of \mathcal{P} . In this case a more precise formulation could be $V \in \mathcal{K} \cap \mathcal{P}$ instead of $V \in \mathcal{K}$. However such considerations add nothing to the main point discussed in this work, at the price of heavier notations. So for the sake of simplicity, we will write the constraint $V \in \mathcal{K}$, having in mind that physically sound unknowns lie in a subset of \mathcal{K} . It is also interesting to recall the distinction that Dirac [16] made between **primary constraints** and **secondary constraints**. Primary constraints are naturally respected by the initial model so do not need to be modeled and can be analyzed by almost immediate considerations. The issue is about **secondary constraints**. For example the constraint $\rho > 0$ is a primary constraint. On the other hand $\mathbf{B} = \nabla\psi \wedge \nabla y$ or $\mathbf{B} = F\nabla\theta + \nabla\phi \wedge \nabla\theta$ are secondary constraints. We may make use of this distinction in the following to indicate which constraints are the important ones.

Let us now define the shifted entropy

$$\widehat{S}_0(U, \mathbf{x}) = S(U) - (V_0(\mathbf{x}), U)$$

which is the generalization of (16).

Proposition 4 (A formal entropy or energy identity). *A model with hyperbolic compatibility satisfies the identity*

$$\frac{d}{dt} \int_{\Omega} \widehat{S}_0(U, \mathbf{x}) dv = \int_{\Omega} (\nabla V_0(\mathbf{x}) : f(U)) dv + b.c. \quad (19)$$

where b.c. represents integrals on the boundary $\partial\Omega$ and $:$ is the contraction of tensors.

Proof. By definition \mathcal{K} is affine and $V - V_0 \in d\mathcal{K}$. So (18) yields

$$\int_{\Omega} [(\partial_t U, V - V_0) + (\nabla f(U), V - V_0)] dv = 0.$$

It yields

$$\int_{\Omega} \partial_t \widehat{S}_0(U, \mathbf{x}) dv = - \int_{\Omega} \nabla F(U) dv + \int_{\Omega} (\partial_x f(U), V_0) dv.$$

Integration by parts yields

$$\int_{\Omega} \partial_t \widehat{S}_0(U, \mathbf{x}) dv = - \int_{\partial\Omega} (F(U), \mathbf{n}) d\sigma + \int_{\partial\Omega} (f(U) V_0, \mathbf{n}) d\sigma - \int_{\Omega} (\nabla V_0(x) : f(U)) dv.$$

The proof is ended. □

Depending on the regularity of V_0 and the coercivity properties of S , the identity (19) may give some informations about the stability of the model.

Another result in the same direction concerns the stability of the linearized equations. Let us start with a reference rest state U'_0 , that is $V'_0 = \nabla_{U'_0} S \in \mathcal{K}$, and

$$\nabla \cdot f(U'_0) = 0 \quad (20)$$

Assuming that boundary conditions do not yield additional perturbation the function $U(x, t) = U'_0(x)$ is a solution of (16). Notice that constant (in space) states are rest states of course, but constant states might not belong to \mathcal{K} . This is in practice the case for equilibrium solutions in Tokamaks since physical rest states satisfy the Grad-Shafranov equation (64). This will be developed in the second part of this work.

So we add a perturbation to the initial condition $U_\varepsilon(0) = \widehat{U}_\varepsilon$, that is

$$\widehat{U}_\varepsilon = U'_0 + \varepsilon U_1 + o(\varepsilon). \quad (21)$$

Let us now consider the solution with initial condition $U_\varepsilon(0)$. A natural question is to determine the evolution of the perturbation, at least in the linear regime.

By subtraction one gets

$$\int_{\Omega} \left[\left(\partial_t \frac{U_\varepsilon - U'_0}{\varepsilon}, Z \right) + \left(\nabla \frac{f(U_\varepsilon) - f(U'_0)}{\varepsilon}, Z \right) \right] dv = 0, \quad \forall Z \in d\mathcal{K}. \quad (22)$$

The linearized adjoint unknown

$$V_\varepsilon = V'_0 + \varepsilon V_1 + o(\varepsilon). \quad (23)$$

A Taylor expansion yields

$$\frac{U_\varepsilon - U'_0}{\varepsilon} = (\nabla_{V'_0} U) V_1 + o(1),$$

and

$$\frac{f(U_\varepsilon) - f(U'_0)}{\varepsilon} = (\nabla_{V'_0} f(U)) V_1 + o(1).$$

Take care that $\nabla_{V'_0} f(U) \in \mathbb{R}^{d \times n \times n}$ is a tensor. One has the classical relations

$$\nabla_V U = d_V^2 S^* = \left(d_{U(V)}^2 S \right)^{-1} \quad \text{and} \quad \nabla_V f(U) = d_V^2 F^*$$

where the Legendre (resp. polar) transform of S (resp. F) is

$$S^*(V) = (V, U(V)) - S(U(V)) \quad (\text{resp. } F^*(V) = (V, f(U(V))) - F(U(V))).$$

Therefore the formal limit of (22) yields the linearized formulation

$$\int_{\Omega} \left[\partial_t \left(d_{V'_0}^2 S^* V_1 \right) + \nabla \cdot \left(d_{V'_0}^2 F^* V_1 \right) \right] dv = 0, \quad \forall Z \in d\mathcal{K}. \quad (24)$$

Note that both $d_{V'_0}^2 S^*$ and $d_{V'_0}^2 F^*$ are a priori dependent of the space variable \mathbf{x} since $V'_0 \in \mathcal{K}$ also depends on the space variable. Since $V_\varepsilon, V'_0 \in \mathcal{K}$, one can pass to the limit in $V_1 = \lim_\varepsilon \frac{V_\varepsilon - V'_0}{\varepsilon}$ to obtain

$$V_1 \in d\mathcal{K}. \quad (25)$$

Proposition 5 (Linearized stability). *Solutions of the linearized formulation (24-25) satisfy*

$$\frac{d}{dt} \int_{\Omega} \left(V_1, d_{V'_0}^2 S^* V_1 \right) dv = \int_{\Omega} \left(\left(\nabla \cdot d_{V'_0}^2 F^* \right) V_1, V_1 \right) dv + b.c. \quad (26)$$

Proof. The proof proceeds by taking $Z = V_1$ in (24) and integration in space using the identity

$$\left(\nabla \cdot \left(d_{V'_0}^2 F^* V_1 \right), V_1 \right) = \nabla \cdot \frac{\left(\left(d_{V'_0}^2 F^* V_1 \right), V_1 \right)}{2} - \left(\left(\nabla \cdot d_{V'_0}^2 F^* \right) V_1, V_1 \right).$$

□

Let us assume, for simplicity, that there exists $\alpha > 0$ such that $d_{V'_0}^2 S^* \geq \alpha \mathbf{I} > 0$. It yields α -convexity: in this case it is possible in theory to use (26) with a Gronwall lemma to obtain a control of $V_1(t)$ in some adapted L^2 based norm. This is why the identity is fundamentally a stability result. It establishes, for hyperbolic compatible models, the linearized stability of rest states.

1.4 An abstract comparison principle

It is well known that the method of moment is endowed with a comparison principle [9, 4, 5, 13] for the eigenvalues of the Jacobian matrix. It is therefore natural to question about a generalization of this principle to our problem. We found the following generalization which can be interpreted as a evaluation of the growth rate of modes in the linear regime.

It is convenient for mathematical development to insure a minimal amount of compactness since we deal with functions. This is why we add dissipation to the initial model and consider

$$\begin{cases} \int_{\Omega} [\partial_t U + \nabla \cdot f(U) - \nu \Delta(U - U'_0), Z] dv = 0, & \forall Z \in d\mathcal{K}, \\ V \in \mathcal{K}. \end{cases}$$

Notice that U'_0 is incorporated in the dissipation term to respect the fact that U'_0 is a rest state. In the context of the modeling of Tokamaks, the so-called bootstrap current [28, 29, 15] is exactly of this nature. The dissipation right hand side can be replaced by a more general dissipative operator $\nabla \cdot (D(U)\nabla(U - U'_0))$ where $D(U) = D(U)^t$ is a given positive dissipation tensor compatible with the entropy principle [9, 4, 5]. For the sake of simplicity we consider here only the simplified dissipative term $\Delta(U - U'_0)$ with a viscous coefficient $\nu > 0$. From now on we only comment the consequences of the introduction of this very specific dissipation term.

The linearized equations (21-25) write now

$$\int_{\Omega} \left[\partial_t \left(d_{V'_0}^2 S^* V_1 \right) + \nabla \cdot \left(d_{V'_0}^2 F^* V_1 \right) - \nu \nabla \cdot \left(d_{V'_0}^2 S^* \nabla V_1 \right), Z \right] dv = 0, \quad \forall Z \in d\mathcal{K},$$

still with $V_1 \in d\mathcal{K}$. Taking $Z = V_1$ and assuming that all boundary terms vanish, one obtains by integration by parts the generalization of (26)

$$\frac{d}{dt} \int_{\Omega} \left(V_1, d_{V'_0}^2 S^* V_1 \right) dv = \int_{\Omega} \left(\left(\nabla \cdot d_{V'_0}^2 F^* \right) V_1, V_1 \right) dv - 2\nu \int_{\Omega} \left(d_{V'_0}^2 S^* \nabla V_1 : \nabla V_1 \right) dv. \quad (27)$$

Let us define the closed in $H_0^1(\Omega)^n$ subspace

$$Y(d\mathcal{K}) = \overline{d\mathcal{K} \cap H_0^1(\Omega)^n}$$

and the real number $\lambda(V'_0, d\mathcal{K}) \in \mathbb{R}$

$$\lambda(V'_0, d\mathcal{K}) = \sup_{V_1 \in Y(\mathcal{K})} \frac{\int_{\Omega} \left(\left(\nabla \cdot d_{V'_0}^2 F^* \right) V_1, V_1 \right) dv - 2\nu \int_{\Omega} \left(d_{V'_0}^2 S^* \nabla V_1 : \nabla V_1 \right) dv}{\int_{\Omega} \left(V_1, d_{V'_0}^2 S^* V_1 \right) dv}. \quad (28)$$

We make usual assumptions on the boundedness [1] of the symmetric matrices $d_{V'_0}^2 S^*$ and $\nabla \cdot d_{V'_0}^2 F^*$

$$0 < \alpha_- \leq d_{V'_0}^2 S^*, \quad \text{and} \quad \left\| d_{V'_0}^2 S^* \right\|_{L^\infty(\Omega)^{n \times n}} + \left\| \nabla \cdot d_{V'_0}^2 F^* \right\|_{L^\infty(\Omega)^{n \times n}} < \infty. \quad (29)$$

Proposition 6. *Under the previous assumptions, $\lambda(V'_0, d\mathcal{K})$ is well defined by (28).*

Proof. Let us define $A = \nabla \cdot d_{V'_0}^2 F^*$ and $B = d_{V'_0}^2 S^*$. Let us take a sufficiently large number $l > 0$ and set $C = lA - B$ which is non negative: $C \geq 0$. Then

$$\lambda(V'_0, d\mathcal{K}) - l = \sup_{W \in Y(\mathcal{K})} \frac{-(CW, W) - (B\nabla W : \nabla W)}{(BW, W)}.$$

So it is equivalent to say that

$$(\lambda(V'_0, d\mathcal{K}) - l)^{-1} = \sup_{W \in Y(\mathcal{K})} \frac{(BW, W)}{(CW, W) + (B\nabla W : \nabla W)}.$$

The denominator is an equivalent norm in $H_0^1(\Omega)^n$. So compactness properties of $Y(\mathcal{K})$ implies that the sup is also a max [1]. \square

Set $a(t) = \left(\int_{\Omega} \left(V_1, d_{V_0'}^2 S^* V_1 \right) dv \right) (t)$. The relation (27) yields

$$a'(t) \leq \lambda(V_0', d\mathcal{K}) a(t) \Rightarrow a(t)^{\frac{1}{2}} \leq e^{\frac{1}{2}\lambda(V_0', d\mathcal{K})t} a(0)^{\frac{1}{2}}.$$

This is why $\frac{1}{2}\lambda(V_0', d\mathcal{K})$ is an upper bound of the growth rate of modes in the linear regime. If by chance the rest state is such that $\lambda(V_0', d\mathcal{K}) = 0$, then this rest state is endowed with very strong stability properties since the exponential growth is actually constant in time. An even better situation is of course if $\lambda(V_0', d\mathcal{K}) < 0$. The next result states that $\lambda(V_0', d\mathcal{K})$ is a lower bound of the same quantity for the initial model.

Theorem 7. *Assume $d\mathcal{K}_1 \subset d\mathcal{K}_2 \subset H_0^1(\Omega)^n$. Then*

$$\lambda(V_0', d\mathcal{K}_1) \leq \lambda(V_0', d\mathcal{K}_2) \leq \lambda(V_0', H_0^1(\Omega)^n).$$

Proof. This is immediate from the definition (28). □

An interpretation of this abstract comparison principle is the following. Let us assume that one is interested by rest states V_0' which are also stable with respect to small perturbation. In such a situation, it is worthwhile to seek for states V_0' such that $\lambda(V_0', H_0^1(\Omega)^n)$ is the smallest as possible, which in turns implies boundedness and control of $\lambda(V_0', d\mathcal{K})$ for any \mathcal{K} . In our mind such an interpretation could be indicative of the stability of the state V_0' . With this respect instability of one single reduced model may be a good indication of the instability of the initial model before reduction.

2 Application to reduced MHD modeling

Our purpose is to apply the previous material to define a hierarchy of reduced MHD models with the hyperbolic compatibility. Any model obtained within this formalism will be called an hyperbolic compatible model.

2.1 The adjoint variable

The first step is to identify the formulation of full ideal MHD such that \mathbf{B} and \mathbf{u} are components of the adjoint variable, so that relations like (4) can be used to define \mathcal{K} . We use the fact that no shocks exist in Tokamaks. Therefore one can use either the energy formulation (2) rewritten for convenience in its non viscous and non resistive version

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \\ \partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p + \nabla \cdot \left(\frac{|\mathbf{B}|^2}{2} \mathbf{I} - \mathbf{B} \otimes \mathbf{B} \right) = 0, \\ \partial_t \mathbf{B} - \nabla \wedge (\mathbf{u} \wedge \mathbf{B}) = 0, \\ \partial_t (\rho e) + \nabla \cdot (\rho \mathbf{u} e + p \mathbf{u}) + \nabla \cdot ((\mathbf{u} \wedge \mathbf{B}) \cdot \mathbf{B}) = 0, \end{cases} \quad (30)$$

or its absolutely equivalent isentropic formulation

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \\ \partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p + \nabla \cdot \left(\frac{|\mathbf{B}|^2}{2} \mathbf{I} - \mathbf{B} \otimes \mathbf{B} \right) = 0, \\ \partial_t \mathbf{B} - \nabla \wedge (\mathbf{u} \wedge \mathbf{B}) = 0, \\ \partial_t (\rho s) + \nabla \cdot (\rho \mathbf{u} s) = 0. \end{cases} \quad (31)$$

The equivalence for smooth flows between (30) and (31) is described in many textbooks. It uses the free-divergence $\nabla \cdot \mathbf{B} = 0$ which is true for all times since it is true at initial time.

The exact form of the density or entropy equation is not important. For example the entropy equation can be replaced by the pressure equation

$$\partial_t p + \mathbf{u} \cdot \nabla p + \rho c^2 \nabla \cdot \mathbf{u} = 0, \quad c^2 = \frac{\gamma p}{\rho},$$

without changing our results. For the simplicity of mathematical notations, we prefer to use the entropy equation (31).

Next step is to determine the entropy variable. It is a consequence of the fundamental principle of thermodynamics

$$T ds = d\varepsilon + p d\tau.$$

Defining $e = \varepsilon + \frac{1}{2} |\mathbf{u}|^2 + \frac{1}{2} \tau |\mathbf{B}|^2$, it implies

$$Td s = de - \mathbf{u} \cdot d\mathbf{u} - \mathbf{B} \cdot d(\tau \mathbf{B}) + (p + |\mathbf{B}|^2) d\tau.$$

and then

$$Td(\rho s) = d(\rho e) - \mathbf{u} \cdot d(\rho \mathbf{u}) - \mathbf{B} \cdot d\mathbf{B} + \mu d\rho$$

where $\mu = Ts - e + |\mathbf{u}|^2 + \tau |\mathbf{B}|^2 - p\tau$ is the Gibbs thermodynamical potential. This differential relation can be summarized as follows.

Proposition 8. *The convex entropy of system (30) is $S = -\rho s$ with*

$$U = (\rho, \rho \mathbf{u}, \mathbf{B}, \rho e) \text{ and adjoint variable } V = -\frac{1}{T} (\mu, -\mathbf{u}, -\mathbf{B}, 1).$$

The convex entropy of system (31) is $S = \rho e$ with

$$U = (\rho, \rho \mathbf{u}, \mathbf{B}, \rho s) \text{ and adjoint variable } V = (-\mu, \mathbf{u}, \mathbf{B}, T).$$

By comparison with (4) it is immediate to realize that (4) can be interpreted as a constraint for the isentropic system (31) and not for (30) because of the $\frac{1}{T}$ term which spoils the structure. That is why we will now consider only (31) in the following.

2.2 Derivation of the fundamental model (1) in a cylinder

We now desire to explain how to recover, in planar geometry, the reduced model (1) through the definition of an appropriate set \mathcal{K} . The domain is a cylinder $\Omega = \mathcal{C} = \{(x, y, z); (x, z) \in \mathcal{D}, y \in \mathbb{R}\}$ where \mathcal{D} is a smooth bounded domain $\mathcal{D} \subset \mathbb{R}^2$. The integration measure is $dv = dx dy dz$.

Proposition 9. *Let $F_0 \in \mathbb{R}$ be a constant and*

$$Y = \{(a, \phi, \psi, b) \in X^4; \partial_y \phi = \partial_y \psi = 0\} \subset X^4.$$

Consider the reduced model in the transverse 2D domain \mathcal{D} , deduced from the set $\mathcal{K} \subset X^8$

$$\mathcal{K} = (0, 0, F_0 \nabla y, 0) + \text{Span}_{(a, \phi, \psi, b) \in Y} \{a, \nabla \phi \wedge \nabla y, \nabla \psi \wedge \nabla y, b\}. \quad (32)$$

Assume the initial data is $\rho \equiv 1$ and $T \equiv 1$. Assume boundary data for the magnetic potential and for the velocity potential: $\psi = \phi = \partial_n \phi = 0$ on $\partial\Omega$.

The corresponding reduced model is the hyperbolic part of the basic reduced MHD model (1).

Remark 10. *The boundary condition is chosen for its simplicity. It insures that $\mathbf{u} = 0$ on the wall, which is related to the fact that the total mass is preserved in the domain. Moreover it is essential below to show that the solution of a certain Poisson problem with Dirichlet boundary condition vanishes.*

Proof. One has

$$d\mathcal{K} = \text{Span}_Y \{a, \nabla \phi \wedge \nabla y, \nabla \psi \wedge \nabla y, b\} \subset X^8.$$

We split the proof in three parts. The first part is the thermodynamic part which is easy since $\rho = 1$ and $T = 1$ are primary constraints. The second part is the velocity part which is a secondary constraint: the manipulations are somewhat identical to the ones performed in the usual physical derivation. The third part is for the magnetic field, it is also a secondary constraint: the result is the usual one even if the method is slightly different for reasons explained below. For convenience we use $X_0 = \mathcal{D}_0^\infty(\Omega) \subset X$ the set of smooth function with compact support in Ω .

Thermodynamic part: Consider the test vector $Z = (a, 0, 0, 0) \in d\mathcal{K}$ for all $a \in X_0$. One deduces from (18) that

$$\int_{\mathcal{C}} (\partial_t \rho + \nabla \cdot (\rho \mathbf{u})) a \, dv = 0, \quad \forall a \in X_0.$$

It yields: $\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0$ for all $\mathbf{x} \in \Omega$. The same algebra holds for the last equation, that is $\partial_t(\rho s) + \nabla \cdot (\rho \mathbf{u} s) = 0$. By construction $\mathbf{u} = \nabla \wedge (\phi \nabla y)$ is divergence free: $\nabla \cdot \mathbf{u} = 0$. Therefore ρ and T are uniformly equal to 1.

Velocity part: The velocity equation can be written as

$$\int_{\mathcal{C}} \left(\partial_t(\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p + \nabla \cdot \left(\frac{|\mathbf{B}|^2}{2} \mathbf{I} - \mathbf{B} \otimes \mathbf{B} \right) \right) \cdot \nabla \wedge (\tilde{\phi} \nabla y) dv = 0$$

for all test functions $\tilde{\phi} \in X_0$ with $\partial_y \tilde{\psi} = 0$. An integration by parts yields

$$\int_{\mathcal{C}} \tilde{\phi} \nabla y \cdot \nabla \wedge \left(\partial_t(\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) - \nabla \cdot (\mathbf{B} \otimes \mathbf{B}) + \nabla \left(p + \frac{|\mathbf{B}|^2}{2} \right) \right) dv = 0.$$

Note that

$$\hat{\psi} = \nabla y \cdot \nabla \wedge \left(\partial_t(\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) - \nabla \cdot (\mathbf{B} \otimes \mathbf{B}) + \nabla \left(p + \frac{|\mathbf{B}|^2}{2} \right) \right)$$

is such that $\hat{\psi} \in X$ and $\partial_y \hat{\psi} = 0$. Since $\tilde{\phi}$ is arbitrary in X_0 (with $\partial_y \tilde{\phi}$), evident simplifications and $\nabla \wedge \nabla \left(p + \frac{|\mathbf{B}|^2}{2} \right) = 0$ simplify into

$$\nabla y \cdot \left(\partial_t(\nabla \wedge \mathbf{u}) + \nabla \wedge \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) - \nabla \wedge \nabla \cdot (\mathbf{B} \otimes \mathbf{B}) \right) = 0, \quad \mathbf{x} \in \Omega. \quad (33)$$

Since by hypothesis $\mathbf{u} = \nabla \wedge (\phi \nabla y)$, then $\mathbf{u} = (-\partial_z \phi, 0, \partial_x \phi)$ and $\nabla \wedge \mathbf{u} = -\Delta_{\perp} \phi \mathbf{e}_y = -\Delta_{\perp} \phi \nabla y$. So $\nabla y \cdot \partial_t(\nabla \wedge \mathbf{u}) = -\partial_t \Delta_{\perp} \phi$. We compute $\nabla \wedge \nabla \cdot (\mathbf{u} \otimes \mathbf{u})$ as follows

$$\mathbf{u} \otimes \mathbf{u} = \begin{pmatrix} \partial_z \phi^2 & 0 & -\partial_x \phi \partial_z \phi \\ 0 & 0 & 0 \\ -\partial_x \phi \partial_z \phi & 0 & \partial_x \phi^2 \end{pmatrix},$$

$$\nabla \cdot (\mathbf{u} \otimes \mathbf{u}) = \begin{pmatrix} \partial_x (\partial_z \phi^2) - \partial_z (\partial_x \phi \partial_z \phi) \\ 0 \\ -\partial_x (\partial_x \phi \partial_z \phi) + \partial_z (\partial_x \phi^2) \end{pmatrix} = \begin{pmatrix} \partial_z \phi \partial_{xz} \phi - \partial_x \phi \partial_{zz} \phi \\ 0 \\ \partial_x \phi \partial_{xz} \phi - \partial_z \phi \partial_{xx} \phi \end{pmatrix},$$

and

$$\begin{aligned} \nabla y \cdot \nabla \wedge \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) &= \\ &= \partial_z (\partial_z \phi \partial_{xz} \phi - \partial_x \phi \partial_{zz} \phi) - \partial_x (\partial_x \phi \partial_{xz} \phi - \partial_z \phi \partial_{xx} \phi) \\ &= -\partial_x \phi \partial_z \Delta_{\perp} \phi + \partial_z \phi \partial_x \Delta_{\perp} \phi = -[\phi, \Delta_{\perp} \phi]. \end{aligned}$$

Similarly, since $\mathbf{B} = \nabla \wedge (\psi \nabla y)$: $\nabla y \cdot \nabla \wedge \nabla \cdot (\mathbf{B} \otimes \mathbf{B}) = -[\psi, \Delta_{\perp} \psi]$.

Therefore (33) yields

$$\partial_t \omega = [\omega, \phi] + [\psi, \Delta_{\perp} \psi], \quad (x, z) \in \mathcal{D}, \quad (34)$$

where $\omega = \Delta_{\perp} \phi$ is the vorticity.

Magnetic part: The magnetic equation yields

$$\int_{\mathcal{C}} (\partial_t \mathbf{B} - \nabla \wedge (\mathbf{u} \wedge \mathbf{B})) \cdot \nabla \wedge (\tilde{\psi} \nabla y) dv = 0$$

for all test functions $\tilde{\psi} \in X_0$ such that $\partial_y \tilde{\psi} = 0$. It is possible to integrate the equation by part directly. However it is preferable to notice first that $\partial_t \mathbf{B} = \partial_t (\nabla \psi \wedge \nabla y) = \nabla \wedge (\partial_t \psi \nabla y)$. We obtain the weak formulation

$$\int_{\mathcal{C}} \nabla \wedge (\partial_t \psi \nabla y - \mathbf{u} \wedge \mathbf{B}) \cdot \nabla \wedge (\tilde{\psi} \nabla y) dv = 0$$

Next we integrate by part

$$\int_{\mathcal{C}} (\partial_t \psi \nabla y - \mathbf{u} \wedge \mathbf{B}) \cdot \nabla \wedge \nabla \wedge (\tilde{\psi} \nabla y) dv = 0.$$

Since $\nabla \cdot \tilde{\psi} \nabla y = 0$ by hypothesis, then $\nabla \wedge \nabla \wedge (\tilde{\psi} \nabla y) = -\Delta \tilde{\psi} \nabla y = -\Delta_{\perp} \tilde{\psi} \nabla y$. Therefore after simplifications

$$\int_{\mathcal{C}} (\partial_t \psi - (\mathbf{u} \wedge \mathbf{B}) \cdot \nabla y) \Delta_{\perp} \tilde{\psi} = 0, \quad \forall \tilde{\psi} \in X_0, \quad \partial_y \tilde{\psi} = 0.$$

By construction the function $\partial_t \psi - (\mathbf{u} \wedge \mathbf{B}) \cdot \nabla y$ is independent of the variable y . So we obtain that $\partial_t \psi - (\mathbf{u} \wedge \mathbf{B}) \cdot \nabla y$ is in the kernel of $\Delta_{\perp}^* = \Delta_{\perp}$. That is

$$\partial_t \psi - (\mathbf{u} \wedge \mathbf{B}) \cdot \nabla y = Q, \quad \Delta_{\perp} Q = 0, \quad (x, z) \in \mathcal{D}. \quad (35)$$

The hypotheses of the theorem assume natural boundary conditions such as $\psi = 0$ and $\mathbf{u} = 0$ on the boundary of the domain. It yields $Q = 0$ on the boundary \mathcal{D} . In this case $Q = 0$. We obtain the simplified equation

$$\partial_t \psi - (\mathbf{u} \wedge \mathbf{B}) \cdot \nabla y = 0.$$

Since $\mathbf{u} = (-\partial_z \phi, 0, \partial_x \phi)$ and $\mathbf{B} = (-\partial_z \psi, F_0, \partial_x \psi)$, evident calculations yield $(\mathbf{u} \wedge \mathbf{B}) \cdot \nabla y = [\psi, \phi]$. We finally obtain

$$\partial_t \psi = [\psi, \phi], \quad (x, z) \in \mathcal{D}. \quad (36)$$

The proof is ended. \square

Remark 11. *The identity (19) can be written, retaining only the variables which makes sense for (32) or equivalently for (1): that is $V_0 = 0$ and $\widehat{S}_0 = \frac{1}{2}|\mathbf{B}|^2 + \frac{1}{2}|\mathbf{u}|^2$ and $\rho \equiv 1$. At the boundary, it is a mixed of homogeneous Dirichlet and Neumann conditions. Therefore we obtain the identity*

$$\int_{\mathcal{D}} \frac{|\nabla \psi|^2 + |\nabla \phi|^2}{2} dx dz = 0.$$

This is by construction exactly (3), discarding the viscous the resistive parts which do not enter in (32).

Therefore the method of hyperbolic compatibility provides effectively a systematic way to design reduced models with the preservation of the total energy. Since our method of construction is general, we are now in position to describe a hierarchy of reduced MHD models in axisymmetric configuration, in the context of the modeling of Tokamaks.

2.3 Models with toroidal invariance

The domain is a torus $\Omega = \mathcal{T} = \mathcal{D} \times [0, 2\pi]$ where \mathcal{D} is a smooth bounded domain $\mathcal{D} \subset \mathbb{R}^2$. The measure of integration in \mathcal{T} is naturally $dv = dx dy dz = 2\pi r dr d\theta dz$. The reduced models written in \mathcal{D} will use the reduced measure of integration $dr dz$.

The local direct orthogonal basis is $(\mathbf{e}_r, \mathbf{e}_{\theta}, \mathbf{e}_z)$ with

$$\mathbf{e}_r = \nabla r, \quad \frac{1}{r} \mathbf{e}_{\theta} = \nabla \theta, \quad \mathbf{e}_z = \nabla z.$$

It turns into $\nabla \wedge \mathbf{e}_r = \nabla \wedge \mathbf{e}_z = \nabla \wedge (\frac{1}{r} \mathbf{e}_{\theta}) = 0$. We also notice that $\Delta \theta = 0$ since θ is an harmonic function. The space of functions independent of the toroidal variable is

$$X_{\perp} = \{h \in X, \partial_{\theta} h = 0\} \subset X.$$

We also define $Y = X \times X_{\perp}^2 \times X$.

Our method of presentation is to introduce various effects through convex sets with increasing complexity. We hope that it helps to distinguish between the various phenomenons, still having the possibility to identify the related algebra.

2.3.1 A first poloidal model

Here we assume $\rho \equiv 1$, $s \equiv 1$ and consider

$$\mathcal{K}_1 = \text{Span}_{(a,\phi,\psi,b) \in Y} (a, \nabla\phi \wedge \nabla\theta, \nabla\psi \wedge \nabla\theta, b) \subset X^8. \quad (37)$$

Since the velocity is divergence free, it is sufficient to assume that $\rho = s = 1$ at initial time. Therefore the temperature is constant also. This the reason why density and temperature do not show up in the next model.

Proposition 12. *Assume Dirichlet-Neumann boundary data for the magnetic and velocity potentials: $\psi = \phi = \partial_n \phi = 0$ on $\partial\Omega$. The incompressible and isothermal reduced model in \mathcal{D} deduced from \mathcal{K}_1 writes in strong form*

$$\begin{cases} \partial_t \psi = \frac{1}{r} [\psi, \phi], \\ \partial_t \omega = r \left[\frac{1}{r^2} \omega, \phi \right] - r \left[\frac{1}{r^2} \Delta^* \psi, \psi \right], \end{cases} \quad (38)$$

where the vorticity is $\omega = \Delta^* \phi$.

This model is endowed with the formal energy identity

$$\frac{d}{dt} \int_{\mathcal{D}} \frac{|\nabla\psi|^2 + |\nabla\phi|^2}{2r} dr dz = 0. \quad (39)$$

Remark 13. *This model is used in [10] for numerical purposes. It is constructed in [15] by different means. See [14] for numerical discretization.*

Proof. Starting with (18) related to the MHD model (31), with test functions $Z = (0, \nabla\tilde{\phi} \wedge \nabla\theta, \nabla\tilde{\psi} \wedge \nabla\theta, 0) \in d\mathcal{K}$, one writes

$$\begin{cases} \int_{\mathcal{T}} \left[\partial_t \mathbf{u} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) + \nabla p + \nabla \cdot \left(\frac{|\mathbf{B}|^2}{2} \mathbf{I} - \mathbf{B} \otimes \mathbf{B} \right) \right] \cdot \nabla\tilde{\phi} \wedge \nabla\theta \, dv = 0, \\ \int_{\mathcal{T}} \left[\partial_t \mathbf{B} - \nabla \wedge (\mathbf{u} \wedge \mathbf{B}) \right] \cdot \nabla\tilde{\psi} \wedge \nabla\theta \, dv = 0, \\ \mathbf{u} = \nabla\phi \wedge \nabla\theta, \quad \mathbf{B} = \nabla\psi \wedge \nabla\theta. \end{cases}$$

The manipulations of such weak formulations need some formulas of differential calculus in the torus which are recalled now. These formulas will also be used in the next models.

To simplify the notations, we will use a subset of test functions, namely $\tilde{\psi}, \tilde{\phi} \in X_0 \cap X_{\perp}$, so the test functions vanish (and also all their derivatives) at the boundary.

Velocity part: Recalling that $\nabla\tilde{\phi} \wedge \nabla\theta = \nabla \wedge (\tilde{\phi} \nabla\theta)$, and integrating by parts, we get

$$\int_{\mathcal{T}} \nabla \wedge \left[\partial_t \mathbf{u} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) + \nabla p + \nabla \cdot \left(\frac{|\mathbf{B}|^2}{2} \mathbf{I} - \mathbf{B} \otimes \mathbf{B} \right) \right] \cdot \nabla\theta \, \tilde{\phi} \, dv = 0, \quad \forall \tilde{\phi} \in X_0 \cap X_{\perp}.$$

One notices that $\nabla \wedge \nabla \left(p + \frac{|\mathbf{B}|^2}{2} \right) = 0$. Since all functions under the integral are independent of the variable θ , and since this integral equation is satisfied for all arbitrary test function $\tilde{\phi}$, we deduce

$$\nabla \wedge [\partial_t \mathbf{u} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) - \nabla \cdot (\mathbf{B} \otimes \mathbf{B})] \cdot e_{\theta} = 0, \quad \mathbf{x} \in \mathcal{T}. \quad (40)$$

Since this function does not depend on the toroidal variable, the same equality can be written equivalently in the reduced domain $(r, z) \in \mathcal{D}$.

We write $\mathbf{u} = \nabla\phi \wedge \nabla\theta = \left(-\frac{1}{r} \partial_z \phi, 0, \frac{1}{r} \partial_r \phi \right)$. We note $\alpha = -\frac{1}{r} \partial_z \phi$ and $\beta = \frac{1}{r} \partial_r \phi$ so that $\mathbf{u} = (\alpha, 0, \beta)$. Then, we calculate

$$(\nabla \wedge \mathbf{u}) \cdot e_{\theta} = \partial_z \left(-\frac{1}{r} \partial_z \phi \right) - \partial_r \left(\frac{1}{r} \partial_r \phi \right) = -\frac{1}{r} \partial_z^2 \phi - \frac{1}{r} \partial_r^2 \phi + \frac{1}{r^2} \partial_r \phi = -\frac{1}{r} \Delta_{\perp} \phi + \frac{1}{r^2} \partial_r \phi.$$

Since the Grad-Shafranov operator is defined by $\Delta^* \phi = \Delta_\perp \phi - \frac{1}{r^2} \partial_r \phi$, one obtains

$$\nabla \wedge (\partial_t \mathbf{u}) \cdot e_\theta = -\frac{1}{r} \partial_t \Delta^* \phi \quad (41)$$

Next, remarking that $\nabla \cdot (\mathbf{u} \otimes \mathbf{u}) = \mathbf{u} \cdot \nabla \mathbf{u}$ (since $\nabla \cdot \mathbf{u} = 0$),

$$\nabla \cdot (\mathbf{u} \otimes \mathbf{u}) = \begin{pmatrix} \alpha \partial_r \alpha + \beta \partial_z \alpha \\ 0 \\ \alpha \partial_r \beta + \beta \partial_z \beta \end{pmatrix}$$

and

$$\begin{aligned} \nabla \wedge \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) \cdot e_\theta &= \partial_z (\alpha \partial_r \alpha + \beta \partial_z \alpha) - \partial_r (\alpha \partial_r \beta + \beta \partial_z \beta) \\ &= \alpha \partial_r (\partial_z \alpha - \partial_r \beta) + \beta \partial_z (\partial_z \alpha - \partial_r \beta) + (\partial_z \alpha - \partial_r \beta) (\partial_r \alpha + \partial_z \beta) \end{aligned}$$

Since $\partial_z \alpha - \partial_r \beta = \partial_z (-\frac{1}{r} \partial_z \phi) - \partial_r (\frac{1}{r} \partial_r \phi) = -\frac{1}{r} \partial_z^2 \phi - \frac{1}{r} \partial_r^2 \phi + \frac{1}{r^2} \partial_r \phi = -\frac{1}{r} \Delta^* \phi$ and $\partial_r \alpha + \partial_z \beta = \partial_r (-\frac{1}{r} \partial_z \phi) + \partial_z (\frac{1}{r} \partial_r \phi) = -\frac{1}{r} \partial_r \partial_z \phi + \frac{1}{r^2} \partial_z \phi + \frac{1}{r} \partial_z \partial_r \phi = \frac{1}{r^2} \partial_z \phi$, we finally obtain

$$\begin{aligned} \nabla \wedge (\mathbf{u} \otimes \mathbf{u}) \cdot e_\theta &= -\frac{1}{r} \partial_z \phi \partial_r \left(-\frac{1}{r} \Delta^* \phi \right) + \frac{1}{r} \partial_r \phi \partial_z \left(-\frac{1}{r} \Delta^* \phi \right) - \frac{1}{r^3} \Delta^* \phi \partial_z \phi \\ &= \partial_z \phi \partial_r \left(\frac{1}{r^2} \Delta^* \phi \right) - \partial_r \phi \partial_z \left(\frac{1}{r^2} \Delta^* \phi \right) \\ &= \left[\frac{1}{r^2} \Delta^* \phi, \phi \right] \end{aligned} \quad (42)$$

Similarly, we deal with the magnetic term and get

$$\nabla \wedge (\mathbf{B} \otimes \mathbf{B}) \cdot e_\theta = \left[\frac{1}{r^2} \Delta^* \psi, \psi \right] \quad (43)$$

and finally write the final form of the equation (40) including the relations (41), (42) and (43):

$$-\frac{1}{r} \partial_t \Delta^* \phi + \left[\frac{1}{r^2} \Delta^* \phi, \phi \right] - \left[\frac{1}{r^2} \Delta^* \psi, \psi \right] = 0, \quad (r, z) \in \mathcal{D}.$$

Magnetic part: Using the rotational expression $\mathbf{B} = \nabla \wedge (\psi \nabla \theta)$, we get

$$\int_{\mathcal{T}} \nabla \wedge \left[\partial_t (\psi \nabla \theta) - \mathbf{u} \wedge \mathbf{B} \right] \cdot \nabla \wedge (\tilde{\psi} \nabla \theta) dv = 0, \quad \forall \tilde{\psi} \in X_0 \cap X_\perp.$$

Integrating by parts, it yields

$$\int_{\mathcal{T}} \left[\partial_t (\psi \nabla \theta) - \mathbf{u} \wedge \mathbf{B} \right] \cdot \nabla \theta \Delta \tilde{\psi} dv = 0 \quad (44)$$

where we used the fact that $\nabla \cdot (\tilde{\psi} \nabla \theta) = 0$ to simplify $\nabla \wedge (\nabla \wedge (\tilde{\psi} \nabla \theta)) = -\Delta \tilde{\psi} \nabla \theta$.

From the relations $\mathbf{u} = \nabla \phi \wedge \nabla \theta = (-\frac{1}{r} \partial_z \phi, 0, \frac{1}{r} \partial_r \phi)$ and $\mathbf{B} = \nabla \psi \wedge \nabla \theta = (-\frac{1}{r} \partial_z \psi, 0, \frac{1}{r} \partial_r \psi)$, we deduce

$$(\mathbf{u} \wedge \mathbf{B}) \cdot e_\theta = -\frac{1}{r^2} \partial_r \phi \partial_z \psi + \frac{1}{r^2} \partial_r \psi \partial_z \phi = \frac{1}{r^2} [\psi, \phi] \quad (45)$$

and then, the magnetic equation (44) with (45) writes

$$\int_{\mathcal{T}} \left(\partial_t \psi - \frac{1}{r} [\psi, \phi] \right) \Delta \tilde{\psi} dv = 0.$$

We observe that $\partial_t \psi - \frac{1}{r} [\psi, \phi] \in X_\perp$ and that $\tilde{\psi} \in X_\perp \cap X_0$. So we finally get

$$\partial_t \psi - \frac{1}{r} [\psi, \phi] = Q, \quad \Delta Q = 0.$$

Homogeneous boundary conditions for ψ and ϕ yield that $\partial_t \psi = [\psi, \phi] = 0$ on the boundary: we conclude that $Q = 0$ and

$$\partial_t \psi - \frac{1}{r} [\psi, \phi] = 0, \quad (x, z) \in \mathcal{D}.$$

Energy check: At the abstract level, it can be checked either starting from (19). To verify that the boundary terms are zero, one must use the boundary conditions. This is performed by multiplying the first equation of (38) by $\frac{1}{r}\Delta^*\psi$, the second equation of (38) by $\frac{1}{r}\phi$, and by a series of integration by parts which use all boundary conditions.

The proof is ended. \square

2.3.2 A second poloidal model

Let $\rho_0 > 0$ be the initial density which is constant in the toroidal direction but not necessarily constant in the poloidal plane, that is $\rho_0 = \rho_0(r, z)$. We will see that the next reduced model preserves the initial density $\rho = \rho_0$: the model is incompressible (in time). On the other hand the entropy is assumed uniformly constant in space ($s = s_0 = 1$ for example).

The set of constraints that we consider is

$$\mathcal{K}_2 = \text{Span}_{(a, \phi, \psi, b) \in X \times X^2_{\perp} \times X} \left(a, \frac{1}{\rho_0} \nabla \phi \wedge \nabla \theta, \nabla \psi \wedge \nabla \theta, b \right) \subset X^8. \quad (46)$$

Proposition 14. *Assume Dirichlet-Neumann boundary data for the magnetic and velocity potentials: $\psi = \phi = \partial_n \phi = 0$ on $\partial\Omega$. The isentropic reduced model deduced from \mathcal{K}_2 is incompressible $\rho = \rho_0$ and writes in strong form*

$$\begin{cases} \partial_t \psi = \frac{1}{\rho r} [\psi, \phi], \\ \partial_t \omega = \rho r \left[\frac{1}{(\rho r)^2} \omega, \phi \right] - \rho r \left[\frac{1}{\rho r^2} \Delta^* \psi, \psi \right] \end{cases} \quad (47)$$

where the vorticity is $\omega = \Delta_\rho \phi$ and the operator Δ_ρ is defined by $\Delta_\rho g = \Delta^* g - \frac{1}{\rho} \nabla \rho \cdot \nabla g = \rho r \nabla \cdot \left(\frac{1}{\rho r} \nabla g \right)$. This model is endowed with the formal energy identity

$$\frac{d}{dt} \int_{\mathcal{D}} \frac{|\nabla \psi|^2}{2r} + \frac{|\nabla \phi|^2}{2\rho r} dr dz = 0. \quad (48)$$

Remark 15. *This model has been derived in [15] by a completely different method, very similar to the usual one in plasma physics literature. Here we use the variational setting.*

Proof. Starting with (18) related to the MHD model (31), with test functions $Z = (\tilde{a}, \frac{1}{\rho_0} \nabla \tilde{\phi} \wedge \nabla \theta, \nabla \tilde{\psi} \wedge \nabla \theta, 0) \in d\mathcal{K}$, one writes

$$\begin{cases} \int_{\mathcal{T}} (\partial_t \rho + \nabla \cdot (\rho \mathbf{u})) \tilde{a} dv = 0, \\ \int_{\mathcal{T}} [\partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p - (\nabla \wedge \mathbf{B}) \wedge \mathbf{B}] \cdot \frac{1}{\rho_0} \nabla \tilde{\phi} \wedge \nabla \theta dv = 0, \\ \int_{\mathcal{T}} [\partial_t \mathbf{B} - \nabla \wedge (\mathbf{u} \wedge \mathbf{B})] \cdot \nabla \tilde{\psi} \wedge \nabla \theta dv = 0, \\ \mathbf{u} = \frac{1}{\rho_0} \nabla \phi \wedge \nabla \theta, \quad \mathbf{B} = \nabla \psi \wedge \nabla \theta. \end{cases}$$

Thermodynamic part: Since this equality is satisfied for all $\tilde{a} \in \mathbb{R}$, the mass conservation is satisfied in the whole domain. Moreover :

$$\mathbf{u} = \frac{1}{\rho_0} \nabla \phi \wedge \nabla \theta \implies \rho_0 \mathbf{u} = \nabla \wedge (\phi \nabla \theta) \implies \nabla \cdot (\rho_0 \mathbf{u}) = \nabla \cdot \nabla \wedge (\phi \nabla \theta) = 0 \implies \partial_t \rho_0 = 0$$

Then, the density ρ is conserved for all time : $\forall t > 0, \rho(t) = \rho_0$. The constraint on the entropy is trivial.

Velocity part: Recalling that $\nabla \tilde{\phi} \wedge \nabla \theta = \nabla \wedge (\tilde{\phi} \nabla \theta)$, and integrating by parts, we get

$$\int_{\mathcal{T}} \nabla \wedge \left[\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{\nabla p}{\rho} - \frac{(\nabla \wedge \mathbf{B}) \wedge \mathbf{B}}{\rho} \right] \cdot \nabla \theta \tilde{\phi} dv = 0$$

Due to the isentropic condition the pressure can be expressed as a function of the density only, that is $p = p(\rho)$: so the quantity $\frac{\nabla p}{\rho}$ is a gradient and its rotational vanishes. The preceding integral equation is satisfied for all arbitrary test function $\tilde{\phi} \in X_{\perp} \cap X_0$. Moreover the other term also belongs to X_{\perp} . So

$$\nabla \wedge \left[\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \frac{(\nabla \wedge \mathbf{B}) \wedge \mathbf{B}}{\rho} \right] \cdot e_{\theta} = 0, \quad \mathbf{x} \in \mathcal{T}. \quad (49)$$

First of all, we write $\mathbf{u} = \frac{1}{\rho} \nabla \phi \wedge \nabla \theta = \left(-\frac{1}{\rho r} \partial_z \phi, 0, \frac{1}{\rho r} \partial_r \phi \right)$, and note $\alpha_{\rho} = -\frac{1}{\rho r} \partial_z \phi$ and $\beta_{\rho} = \frac{1}{\rho r} \partial_r \phi$ so that $\mathbf{u} = (\alpha_{\rho}, 0, \beta_{\rho})$. Then, we calculate

$$\begin{aligned} (\nabla \wedge \mathbf{u}) \cdot e_{\theta} &= \partial_z \left(-\frac{1}{\rho r} \partial_z \phi \right) - \partial_r \left(\frac{1}{\rho r} \partial_r \phi \right) \\ &= -\frac{1}{\rho r} \partial_z^2 \phi - \frac{1}{\rho r} \partial_r^2 \phi + \frac{1}{\rho r^2} \partial_r \phi + \frac{1}{\rho^2 r} (\partial_r \rho \partial_r \phi + \partial_z \rho \partial_z \phi) \\ &= -\frac{1}{\rho r} \left(\Delta_{\perp} \phi - \frac{1}{r} \partial_r \phi - \frac{1}{\rho} (\nabla \rho \cdot \nabla \phi) \right) \\ &= -\frac{1}{\rho r} \left(\Delta^* \phi - \frac{1}{\rho} (\nabla \rho \cdot \nabla \phi) \right) \end{aligned}$$

Then, noting $\Delta_{\rho} \phi = \Delta^* \phi - \frac{1}{\rho} \nabla \rho \cdot \nabla \phi$, we get :

$$\nabla \wedge (\partial_t \mathbf{u}) \cdot e_{\theta} = -\frac{1}{\rho r} \partial_t \Delta_{\rho} \phi \quad (50)$$

Next,

$$\mathbf{u} \cdot \nabla \mathbf{u} = \begin{pmatrix} \alpha_{\rho} \partial_r \alpha_{\rho} + \beta_{\rho} \partial_z \alpha_{\rho} \\ 0 \\ \alpha_{\rho} \partial_r \beta_{\rho} + \beta_{\rho} \partial_z \beta_{\rho} \end{pmatrix}$$

and

$$\begin{aligned} \nabla \wedge (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot e_{\theta} &= \partial_z (\alpha_{\rho} \partial_r \alpha_{\rho} + \beta_{\rho} \partial_z \alpha_{\rho}) - \partial_r (\alpha_{\rho} \partial_r \beta_{\rho} + \beta_{\rho} \partial_z \beta_{\rho}) \\ &= \alpha_{\rho} \partial_r (\partial_z \alpha_{\rho} - \partial_r \beta_{\rho}) + \beta_{\rho} \partial_z (\partial_z \alpha_{\rho} - \partial_r \beta_{\rho}) + (\partial_z \alpha_{\rho} - \partial_r \beta_{\rho}) (\partial_r \alpha_{\rho} + \partial_z \beta_{\rho}) \end{aligned}$$

Since $\partial_z \alpha_{\rho} - \partial_r \beta_{\rho} = \partial_z \left(-\frac{1}{\rho r} \partial_z \phi \right) - \partial_r \left(\frac{1}{\rho r} \partial_r \phi \right) = -\frac{1}{\rho r} \Delta_{\rho} \phi$ and

$$\begin{aligned} \partial_r \alpha_{\rho} + \partial_z \beta_{\rho} &= \partial_r \left(-\frac{1}{\rho r} \partial_z \phi \right) + \partial_z \left(\frac{1}{\rho r} \partial_r \phi \right) \\ &= -\frac{1}{\rho r} \partial_r \partial_z \phi + \frac{1}{\rho r^2} \partial_z \phi + \frac{1}{\rho^2 r} \partial_r \rho \partial_z \phi + \frac{1}{\rho r} \partial_z \partial_r \phi - \frac{1}{\rho^2 r} \partial_z \rho \partial_r \phi \\ &= -\partial_r \left(\frac{1}{\rho r} \right) \partial_z \phi + \partial_z \left(\frac{1}{\rho r} \right) \partial_r \phi \\ &= -\left[\frac{1}{\rho r}, \phi \right], \end{aligned}$$

we finally obtain

$$\begin{aligned} \nabla \wedge (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot e_{\theta} &= -\frac{1}{\rho r} \partial_z \phi \partial_r \left(-\frac{1}{\rho r} \Delta_{\rho} \phi \right) + \frac{1}{\rho r} \partial_r \phi \partial_z \left(-\frac{1}{\rho r} \Delta_{\rho} \phi \right) + \frac{1}{\rho r} \Delta^* \phi \left[\frac{1}{\rho r}, \phi \right] \\ &= \partial_z \phi \partial_r \left(\frac{1}{(\rho r)^2} \Delta_{\rho} \phi \right) - \partial_r \phi \partial_z \left(\frac{1}{(\rho r)^2} \Delta_{\rho} \phi \right) \\ &= \left[\frac{1}{(\rho r)^2} \Delta_{\rho} \phi, \phi \right] \end{aligned} \quad (51)$$

Next, we deal with the magnetic term. From $\mathbf{B} = \nabla\psi \wedge \nabla\theta = \left(-\frac{1}{r}\partial_z\psi, 0, \frac{1}{r}\partial_r\psi\right)$ and $\nabla \wedge \mathbf{B} = \left(0, \partial_z\left(-\frac{1}{r}\partial_z\psi\right) - \partial_r\left(\frac{1}{r}\partial_r\psi\right), 0\right) = \left(0, -\frac{\Delta^*\psi}{r}, 0\right)$, one obtains $\frac{(\nabla \wedge \mathbf{B}) \wedge \mathbf{B}}{\rho} = \left(-\frac{1}{\rho r}\partial_r\psi\frac{\Delta^*\psi}{r}, 0, -\frac{1}{\rho r}\partial_z\psi\frac{\Delta^*\psi}{r}\right)$ and

$$\begin{aligned} \nabla \wedge \left(\frac{(\nabla \wedge \mathbf{B}) \wedge \mathbf{B}}{\rho} \right) \cdot e_\theta &= \partial_z \left(-\frac{1}{\rho r} \partial_r \psi \frac{\Delta^* \psi}{r} \right) + \partial_r \left(\frac{1}{\rho r} \partial_z \psi \frac{\Delta^* \psi}{r} \right) \\ &= -\partial_r \psi \partial_z \left(\frac{1}{\rho r^2} \Delta^* \psi \right) + \partial_z \psi \partial_r \left(\frac{1}{\rho r^2} \Delta^* \psi \right) \\ &= \left[\frac{1}{\rho r^2} \Delta^* \psi, \psi \right] \end{aligned} \quad (52)$$

Introducing (50), (51) and (52) in the equation (49), it comes

$$-\frac{1}{\rho r} \partial_t \Delta_\rho \phi + \left[\frac{1}{\rho r^2} \Delta_\rho \phi, \phi \right] - \left[\frac{1}{\rho r^2} \Delta^* \psi, \psi \right] = 0, \quad (x, z) \in \mathcal{D}.$$

Magnetic part: Using the rotational expression $\mathbf{B} = \nabla \wedge (\psi \nabla \theta)$, we get

$$\int_{\mathcal{T}} \nabla \wedge \left[\partial_t (\psi \nabla \theta) - \mathbf{u} \wedge \mathbf{B} \right] \cdot \nabla \wedge (\tilde{\psi} \nabla \theta) \, dv = 0$$

and, integrating by parts and by $\nabla \theta = \frac{e_\theta}{r}$, it yields for $\tilde{\psi} \in X_\perp \cap X_0$

$$\int_{\mathcal{T}} \left[\partial_t \left(\frac{\psi e_\theta}{r} \right) - \mathbf{u} \wedge \mathbf{B} \right] \cdot e_\theta \Delta \tilde{\psi} \, dv = 0.$$

From the relations $\mathbf{u} = \frac{1}{\rho} \nabla \phi \wedge \nabla \theta = \left(-\frac{1}{\rho r} \partial_z \phi, 0, \frac{1}{\rho r} \partial_r \phi\right)$ and $\mathbf{B} = \nabla \psi \wedge \nabla \theta = \left(-\frac{1}{r} \partial_z \psi, 0, \frac{1}{r} \partial_r \psi\right)$ we deduce

$$(\mathbf{u} \wedge \mathbf{B}) \cdot e_\theta = -\frac{1}{\rho r^2} \partial_r \phi \partial_z \psi + \frac{1}{\rho r^2} \partial_r \psi \partial_z \phi = \frac{1}{\rho r^2} [\psi, \phi]$$

and then, the magnetic equation writes

$$\int_{\mathcal{T}} \left(\partial_t \psi - \frac{1}{\rho r} [\psi, \phi] \right) \Delta \tilde{\psi} \, dv = 0 \quad (53)$$

As before, this weak equation being true for all $\tilde{\psi} \in X_\perp \cap X_0$, it means that

$$\partial_t \psi - \frac{1}{\rho r} [\psi, \phi] = Q, \quad \Delta Q = 0.$$

By hypothesis, homogeneous boundary conditions are required $\psi = \phi = 0$ on *partial* Ω : we conclude that

$$\partial_t \psi - \frac{1}{\rho r} [\psi, \phi] = 0, \quad (x, z) \in \mathcal{D}.$$

Energy check: it can be verified by two methods, as in the proof of the proposition. Except that the second equation must be multiplied by $\frac{1}{\rho r} \phi$, which is algebraically compatible with $\omega = \rho r \nabla \cdot \left(\frac{1}{\rho r} \nabla \varphi \right)$. Detailed verifications are in [15].

The proof is ended. □

2.3.3 A third poloidal model

As before the initial density $\rho_0(r, z)$ is constant in the toroidal direction but not necessarily constant in the poloidal plane. We introduce a non constant "forcing term" $F = F(r, z)$ in the set of constraints, that is

$$\mathcal{K}_3 = V_0 + \text{Span}_{(a, \phi, \psi, b) \in X \times X_\perp^2 \times X} \left(a, \frac{1}{\rho_0} \nabla \phi \wedge \nabla \theta, \nabla \psi \wedge \nabla \theta, b \right) \quad (54)$$

with $V_0 = (0, 0, F \nabla \theta, 0)$.

With this choice the representation of the magnetic field is exactly $\mathbf{B} = F \nabla \theta + \nabla \psi \wedge \nabla \theta$. A non zero F is the minimum to represents correctly the helical structure of the magnetic field in Tokamaks [18].

Proposition 16. Assume Dirichlet-Neumann boundary data for the magnetic and velocity potentials: $\psi = \phi = \partial_n \phi = 0$ on $\partial\Omega$. The reduced model of (31) related to the adjoint variable set \mathcal{K}_3 is incompressible $\rho = \rho_0$ and writes

$$\begin{cases} \partial_t \psi = \frac{1}{\rho r} [\psi, \phi], \\ \partial_t \omega = \rho r \left[\frac{1}{(\rho r)^2} \omega, \phi \right] - \rho r \left[\frac{F}{\rho r^2}, F \right] - \rho r \left[\frac{1}{\rho r^2} \Delta^* \psi, \psi \right] \end{cases} \quad (55)$$

where the vorticity is $\omega = \Delta_\rho \phi$.

The energy identity is

$$\frac{d}{dt} \int_{\mathcal{D}} \frac{|\nabla \psi|^2}{2r} + \frac{|\nabla \phi|^2}{2\rho r} dr dz = \int_{\mathcal{D}} \left[\frac{F}{\rho r^2}, F \right] \phi dr dz. \quad (56)$$

The new term with respect to (47) is the source term $\left[\frac{F}{\rho r^2}, F \right]$. We will show in proposition 20 that it is essential to obtain the compatibility with the Grad-Shrafranov equation [3, 36] for equilibrium in Tokamaks. If $F = F_0$ is constant in space, the source term vanishes: it also vanishes if ρr^2 is a constant.

Remark 17. The interpretation of this energy balance is as follows. First the physical energy is the sum of the internal energy, the kinetic energy and the magnetic energy $\rho e = \rho \varepsilon + \frac{\rho |\mathbf{u}|^2}{2} + \frac{|\mathbf{B}|^2}{2}$. Plugging the chosen representation \mathcal{K}_3 , we obtain

$$\rho e = \left(\rho \varepsilon + \frac{|F_0 \nabla \theta|^2}{2} \right) + \frac{|\nabla \phi \wedge \mathbf{e}_\theta|^2}{2\rho r^2} + \frac{|\nabla \psi \wedge \mathbf{e}_\theta|^2}{2r^2} = K + \frac{|\nabla \phi|^2}{2\rho r^2} + \frac{|\nabla \psi|^2}{2r^2}$$

since $\psi, \phi \in X_\perp$. The term K being is independent of the time variable and vanishes after derivation with respect to t . The time derivative of the integral (with weight $r dr dz$) of the remaining term is exactly the left hand side of the energy balance (56). Notice that the linear part of (19) vanishes

$$\frac{d}{dt} \int_{\Omega} V_0(\mathbf{x}) \cdot U = \frac{d}{dt} \int_{\Omega} (F \nabla \theta) \cdot ((F \nabla \theta + \nabla \psi \wedge \theta) dx dy dz) = 0.$$

Since the model is hyperbolic compatible, the right hand side of (56) is by construction equal to the right hand side of (19). This non trivial fact deserves a separate verification. Indeed one has the series of equalities (discarding all terms on the boundary)

$$\begin{aligned} \int_{\Omega} \nabla V_0(\mathbf{x}) : f(U) dx dy dz &= \int_{\Omega} V_0(\mathbf{x}) \cdot \nabla \cdot f(U) dx dy dz = \int_{\Omega} F \nabla \theta \cdot \nabla \wedge (\mathbf{u} \wedge \mathbf{B}) dx dy dz \\ &= \int_{\Omega} \nabla \wedge (F \nabla \theta) \cdot \mathbf{u} \wedge \mathbf{B} dx dy dz = \int_{\Omega} (\nabla F \wedge \theta) \cdot \left(\left(\frac{1}{\rho} \nabla \phi \wedge \nabla \theta \right) \wedge (F \nabla \theta + \nabla \psi \wedge \theta) \right) dx dy dz. \end{aligned} \quad (57)$$

Simplifications due to $\nabla \psi \cdot \mathbf{e}_\theta = \nabla \phi \cdot \mathbf{e}_\theta = 0$ yield

$$\int_{\Omega} \nabla V_0(\mathbf{x}) : f(U) dx dy dz = - \int_{\Omega} \frac{F}{\rho r^3} (\nabla F \wedge \mathbf{e}_\theta) \cdot \nabla \phi dx dy dz = -2\pi \int_{\mathcal{D}} \frac{F}{\rho r^2} [\phi, F] dr dz.$$

One more integration by parts of the Poisson bracket shows that

$$\int_{\Omega} \nabla V_0(\mathbf{x}) : f(U) dx dy dz = 2\pi \int_{\mathcal{D}} \left[\frac{F}{\rho r^2}, F \right] \phi dr dz$$

which is exactly the right hand side of (56), up to the factor 2π of course.

A physical interpretation of this source term is in our opinion the following. The forcing term is due to this $F \nabla \theta$. In case $F = F_0$, the forcing term is curl free since $\nabla \wedge (F_0 \nabla \theta)$. Since a curl free magnetic force is force free, does not change the energy balance which is therefore still an energy conservation. The situation is completely different with a non curl free magnetic force, as considered in this model.

Proof. Since this model is very close to the previous one, we concentrate only on the important part of the proof. Starting with (18) related to the MHD model (31), with test functions $Z = (\frac{1}{\rho_0} \nabla \tilde{\phi} \wedge \nabla \theta, \nabla \tilde{\psi} \wedge \nabla \theta, 0) \in d\mathcal{K}$, one writes

$$\begin{cases} \int_{\mathcal{T}} [\partial_t(\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p - (\nabla \wedge \mathbf{B}) \wedge \mathbf{B}] \cdot \frac{1}{\rho_0} \nabla \tilde{\phi} \wedge \nabla \theta = 0 \\ \int_{\mathcal{T}} [\partial_t \mathbf{B} - \nabla \wedge (\mathbf{u} \wedge \mathbf{B})] \cdot \nabla \tilde{\psi} \wedge \nabla \theta = 0 \\ \mathbf{u} = \frac{1}{\rho_0} \nabla \phi \wedge \nabla \theta, \quad \mathbf{B} = F \nabla \theta + \nabla \psi \wedge \nabla \theta \end{cases}$$

The hydrodynamic (ρ, s) part can be treated as for the second poloidal model.

Velocity part: The only contribution of the additional term $F \nabla \theta$ in \mathbf{B} appears in the Lorentz term of the velocity equation:

$$\int_{\mathcal{T}} (\nabla \wedge \mathbf{B}) \wedge \mathbf{B} \cdot \frac{1}{\rho} \nabla \tilde{\phi} \wedge \nabla \theta = \int_{\mathcal{T}} \nabla \wedge \left(\frac{(\nabla \wedge \mathbf{B}) \wedge \mathbf{B}}{\rho} \right) \cdot \Delta \tilde{\phi} \nabla \theta$$

The new expression of \mathbf{B} gives

$$\mathbf{B} = F \nabla \theta + \nabla \psi \wedge \nabla \theta = \begin{pmatrix} -\frac{1}{r} \partial_z \psi \\ \frac{F}{r} \\ \frac{1}{r} \partial_r \psi \end{pmatrix} \quad \text{and} \quad \nabla \wedge \mathbf{B} = \begin{pmatrix} -\frac{1}{r} \partial_z F \\ \partial_z \left(-\frac{1}{r} \partial_z \psi \right) - \partial_r \left(\frac{1}{r} \partial_r \psi \right) \\ \frac{1}{r} \partial_r F \end{pmatrix} = \begin{pmatrix} -\frac{1}{r} \partial_z F \\ -\frac{\Delta^* \psi}{r} \\ \frac{1}{r} \partial_r F \end{pmatrix}$$

and then

$$\frac{(\nabla \wedge \mathbf{B}) \wedge \mathbf{B}}{\rho} = \begin{pmatrix} -\frac{F}{\rho r^2} \partial_r F - \frac{1}{\rho r} \partial_r \psi \frac{\Delta^* \psi}{r} \\ -\frac{1}{\rho r^2} \partial_r \psi \partial_z F + \frac{1}{\rho r^2} \partial_z \psi \partial_r F \\ -\frac{F}{\rho r^2} \partial_z F - \frac{1}{\rho r} \partial_z \psi \frac{\Delta^* \psi}{r} \end{pmatrix}$$

and

$$\begin{aligned} \nabla \wedge \left(\frac{(\nabla \wedge \mathbf{B}) \wedge \mathbf{B}}{\rho} \right) \cdot e_\theta &= -\partial_z \left(\frac{F}{\rho r^2} \partial_r F + \frac{1}{\rho r} \partial_r \psi \frac{\Delta^* \psi}{r} \right) + \partial_r \left(\frac{F}{\rho r^2} \partial_z F + \frac{1}{\rho r} \partial_z \psi \frac{\Delta^* \psi}{r} \right) \\ &= \left[\frac{F}{\rho r^2}, F \right] + \left[\frac{1}{\rho r^2} \Delta^* \psi, \psi \right]. \end{aligned} \quad (58)$$

This is the reason of the quadratic source term in the vorticity equation.

Magnetic part: The additional part $F \nabla \theta$ in \mathbf{B} has no contribution to the magnetic equation. Indeed,

$$\int_{\mathcal{T}} \left[\partial_t F \nabla \theta - \nabla \wedge (\mathbf{u} \wedge F \nabla \theta) \right] \cdot \nabla \tilde{\psi} \wedge \nabla \theta = \int_{\mathcal{T}} \partial_t F \nabla \theta \cdot (\nabla \tilde{\psi} \wedge \nabla \theta) - (\mathbf{u} \wedge F \nabla \theta) \cdot \Delta \tilde{\psi} \nabla \theta = 0$$

since $\partial_t F = 0$ and $(\mathbf{u} \wedge \nabla \theta) \cdot \nabla \theta = 0$.

Energy balance: it can be checked as for the previous models. See also remark 17.

The proof is ended. □

2.4 Toroidal dependency

We now consider a more complete case with parallel velocity, under the form

$$\mathcal{K}_4 = V_0 + \text{Span}_{(a,b) \in \mathbb{R}^2, (\phi, \psi) \in \mathcal{X}^2} (a, \lambda \nabla \theta + G \nabla \phi \wedge \nabla \theta, \nabla \psi \wedge \nabla \theta, b) \quad (59)$$

with a forcing term

$$V_0 = (0, 0, F \nabla \theta, 0) \quad \text{and} \quad F = F(r, z).$$

The most important feature of this model is that all functions $(\psi, \phi$ and $\lambda)$ have dependency with respect to the toroidal variable θ . As before the magnetic forcing F does not depend on the toroidal variable: $\nabla F \perp \nabla \theta$. The parallel velocity is $\mathbf{u}_\parallel = \lambda(t, r, \theta, z) \nabla \theta$. The perpendicular velocity is $\mathbf{u}_\perp = G \nabla \phi \wedge \nabla \theta$ where G is an arbitrary coefficient independent of the toroidal variable $\nabla G \perp \nabla \theta$ which can be taken as $G = 1$, $G = \rho_0^{-1}$ or $G = r^2$ as in the previous models.

Proposition 18. Assume an homogeneous boundary conditions $\psi = \phi = 0$.
The reduced model based on \mathcal{K}_4 can be written under the strong form

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, & \mathbf{x} \in \mathcal{T}, \\ \nabla \theta \cdot (\partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p - (\nabla \wedge \mathbf{B}) \wedge \mathbf{B}) = 0, & \mathbf{x} \in \mathcal{T}, \\ \nabla \theta \cdot [\nabla \wedge (\partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) - (\nabla \wedge \mathbf{B}) \wedge \mathbf{B})] = 0, & \mathbf{x} \in \mathcal{T}, \\ \partial_t \psi = \frac{G}{r} [\psi, \phi] + \frac{G}{r^2} \partial_\theta (F\phi) - \frac{1}{r^2} \partial_\theta (\lambda \psi) + Q, & \mathbf{x} \in \mathcal{T}, \\ \partial_t (\rho s) + \nabla \cdot (\rho \mathbf{u} s) = 0, & \mathbf{x} \in \mathcal{T}, \end{cases} \quad (60)$$

where the source term is solution of a series of poloidal variational formulations

$$\forall \theta, \quad a(Q(\theta), \tilde{\psi}) = (g_\theta, \tilde{\psi}), \quad \forall \tilde{\psi} \in V_\perp$$

where $V_\perp \subset H^1(\mathcal{D})$ encounters of the boundary condition: typically $V_\perp = H_0^1(\mathcal{D})$. Let $\tilde{\psi}$ denote any test function in V_\perp . The bilinear form is

$$a(Q, \tilde{\psi}) = \int_{\mathcal{D}} \frac{1}{r} \left(\partial_r Q \partial_r \tilde{\psi} + \partial_z Q \partial_z \tilde{\psi} \right).$$

The right hand side is given by

$$(g_\theta, \tilde{\psi}) = \int \frac{1}{r^3} \left(\partial_\theta (-\phi \partial_z (FG) + \psi \partial_z \lambda) \partial_z \tilde{\psi} + \partial_\theta (-\phi \partial_r (FG) + \psi \partial_r \lambda) \partial_r \tilde{\psi} \right) + 2 \int \frac{1}{r^4} \partial_\theta (FG\phi - \lambda \psi) \partial_r \tilde{\psi}.$$

The strong form of the equations correspond to what is usually encountered in plasma physics articles [6, 10, 11, 21, 25, 35, 37, 40, 42]. A difference is in the source term Q which is needed to guarantee the conservation of total energy. The equation of Q is more conveniently written in weak or variational form. The strong form is

$$\Delta^* Q = r \partial_z \left(\frac{1}{r^3} \partial_\theta (-\phi \partial_z (FG) + \psi \partial_z \lambda) \right) + r \partial_r \left(\frac{1}{r^3} \partial_\theta (-\phi \partial_r (FG) + \psi \partial_r \lambda) + \frac{2}{r^4} \partial_\theta (FG\phi - \lambda \psi) \right).$$

Remark 19. The energy balance associated to this system writes

$$\frac{d}{dt} \int_{\Omega} \rho e dx dy dz = \int_{\Omega} \left[\frac{G}{r^2} F, F \right] \phi dx dy dz - \int_{\Omega} \left[\frac{1}{r^3} \lambda, F \right] \psi dx dy dz + b.c. \quad (61)$$

The verification is the same as in remark 17, inserting in equation (57) the Ansatz $\mathbf{u} = \lambda \nabla \theta + G \nabla \phi \wedge \nabla \theta$. Taking $G = r^2$ and $\lambda = 0$ in (61) yields an exact energy conservation (up to boundary terms): with this choice the magnetic field does not have long range influence in the interior of the system which can therefore be considered as isolated from the energy point of view. Taking $G = \frac{1}{\rho}$, one partially recovers 17 which is a priori non zero. A possible physical interpretation is the following: if one desires to stabilize the system with a density profile $\rho_0 = G \neq r^2$ or a non zero parallel velocity, then one has to expect some long range energy exchange between the medium and the exterior coils of the Tokamak.

Proof. Using the same method as before, the thermodynamic equations of the reduced model are

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0,$$

and

$$\partial_t (\rho s) + \nabla \cdot (\rho \mathbf{u} s) = 0.$$

The velocity part of the trial space $d\mathcal{K}$ is generated by $\tilde{\nabla} \theta$ and $G \nabla \wedge (\tilde{\phi} \nabla \theta)$ where the test functions are $\tilde{\lambda}$ and $\tilde{\phi}$. Testing against all possible test functions $\tilde{\lambda}$ one obtains first

$$\nabla \theta \cdot (\partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p - (\nabla \wedge \mathbf{B}) \wedge \mathbf{B}) = 0.$$

Testing against all possible test functions $\tilde{\phi}$ one obtains after integration by parts

$$\nabla \theta \cdot (G \nabla \wedge [\partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p - (\nabla \wedge \mathbf{B}) \wedge \mathbf{B}]) = 0.$$

We notice that it can be simplified under the form

$$\nabla \theta \cdot (G \nabla \wedge [\partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} - \mathbf{B} \otimes \mathbf{B})]) = 0.$$

The equation is independent of the pressure p .

The magnetic equation needs more attention. The weak form writes

$$\int_{\mathcal{T}} (\partial_t \mathbf{B} - \nabla \wedge (\mathbf{u} \wedge \mathbf{B})) \cdot \nabla \wedge (\tilde{\psi} \nabla \theta) = 0.$$

Since by hypothesis $\partial_t \mathbf{B} = \nabla \wedge (\psi \nabla \theta)$, one can substitute and integrate by parts to obtain

$$\int (\partial_t \psi \nabla \theta - \mathbf{u} \wedge \mathbf{B}) \cdot \nabla \wedge \nabla \wedge (\tilde{\psi} \nabla \theta) \, dx dy dz = 0. \quad (62)$$

Let us denote $\tilde{\mathbf{B}} = \nabla \wedge (\tilde{\psi} \nabla \theta)$ and $\tilde{\mathbf{J}} = \nabla \wedge \tilde{\mathbf{B}}$. Careful calculations show that since $\tilde{\mathbf{B}} = -\frac{\partial_z \tilde{\psi}}{r} \mathbf{e}_r + \frac{\partial_r \tilde{\psi}}{r} \mathbf{e}_z$ and $\nabla \wedge \mathbf{e}_r = \nabla \wedge \mathbf{e}_z = 0$, then

$$\tilde{\mathbf{J}} = -\frac{1}{r} \Delta^* \tilde{\psi} \mathbf{e}_\theta + \frac{1}{r^2} \left(\partial_{r\theta}^2 \tilde{\psi} \mathbf{e}_r + \partial_{z\theta}^2 \tilde{\psi} \mathbf{e}_z \right).$$

One also has

$$\begin{aligned} \mathbf{u} \wedge \mathbf{B} &= (\lambda \nabla \theta + G \nabla \phi \wedge \nabla \theta) \wedge (F \nabla \theta + \nabla \psi \wedge \nabla \theta) \\ &= \frac{G}{r^2} [\psi, \phi] \mathbf{e}_\theta - \frac{FG}{r^2} (\partial_z \phi \mathbf{e}_z + \partial_r \phi \mathbf{e}_r) + \frac{\lambda}{r^2} (\partial_z \psi \mathbf{e}_z + \partial_r \psi \mathbf{e}_r). \end{aligned}$$

The measure of integration is $dx \, dy \, dz = 2\pi r \, dr \, d\theta \, dz$. Substitutions in (62) yield

$$\begin{aligned} &\int \left(\left(-\frac{1}{r} \partial_t \psi + \frac{G}{r^2} [\psi, \phi] \right) \Delta^* \tilde{\psi} \right. \\ &\left. + \frac{1}{r^3} (FG \partial_z \phi - \lambda \partial_z \psi) \partial_{z\theta}^2 \tilde{\psi} + \frac{1}{r^4} (FG \partial_r \phi - \lambda \partial_r \psi) \partial_{r\theta}^2 \tilde{\psi} \right) dr \, d\theta \, dz = 0. \end{aligned}$$

It is convenient to define what we call a correction term Q as

$$Q = \partial_t \psi - \frac{G}{r} [\psi, \phi] - \frac{1}{r^2} \partial_\theta (FG \phi - \lambda \psi)$$

where we keep FG inside the ∂_θ to shorten the notations. We get

$$\begin{aligned} &\int \left(-\frac{1}{r} Q - \frac{1}{r^3} \partial_\theta (FG \phi - \lambda \psi) \right) \Delta^* \tilde{\psi} \\ &+ \int \left(\frac{1}{r^3} (FG \partial_z \phi - \lambda \partial_z \psi) \partial_{z\theta}^2 \tilde{\psi} + \frac{1}{r^3} (FG \partial_r \phi - \lambda \partial_r \psi) \partial_{r\theta}^2 \tilde{\psi} \right) dr \, d\theta \, dz = 0. \end{aligned}$$

Integration by parts with respect to the Grad-Shafranov operator and to the toroidal variable θ yields

$$\begin{aligned} &\int \frac{1}{r} \left(\partial_r Q \partial_r \tilde{\psi} + \partial_z Q \partial_z \tilde{\psi} \right) dr \, d\theta \, dz = \int \frac{1}{r^3} \partial_\theta (FG \phi - \lambda \psi) \Delta^* \tilde{\psi} \\ &+ \int \left(\frac{1}{r^3} \partial_\theta (FG \partial_z \phi - \lambda \partial_z \psi) \partial_z \tilde{\psi} + \frac{1}{r^3} \partial_\theta (FG \partial_r \phi - \lambda \partial_r \psi) \partial_r \tilde{\psi} \right) \\ &= \int \frac{1}{r^3} \partial_\theta (FG \phi - \lambda \psi) \Delta^* \tilde{\psi} \\ &+ \int \left(\frac{1}{r^3} \partial_{z\theta}^2 (FG \phi - \lambda \psi) \partial_z \tilde{\psi} + \frac{1}{r^3} \partial_{r\theta}^2 (FG \phi - \lambda \psi) \partial_r \tilde{\psi} \right). \\ &+ \int \left(\frac{1}{r^3} \partial_\theta (-\phi \partial_z (FG) + \psi \partial_z \lambda) \partial_z \tilde{\psi} + \frac{1}{r^3} \partial_\theta (-\phi \partial_r (FG) + \psi \partial_r \lambda) \partial_r \tilde{\psi} \right). \end{aligned}$$

An integration by parts yields

$$\int \frac{1}{r^3} \partial_\theta (FG \phi - \lambda \psi) \Delta^* \tilde{\psi} = - \int \frac{1}{r} \nabla_{r,z} \left(\frac{1}{r^2} \partial_\theta (FG \phi - \lambda \psi) \right) \cdot \nabla_{r,z} \tilde{\psi}$$

$$= - \int \frac{1}{r^3} \nabla_{r,z} \partial_\theta (FG\phi - \lambda\psi) \cdot \nabla_{r,z} \tilde{\psi} + 2 \int \frac{1}{r^4} \partial_\theta (FG\phi - \lambda\psi) \partial_r \tilde{\psi}.$$

We obtain

$$\begin{aligned} & \int \frac{1}{r} \left(\partial_r Q \partial_r \tilde{\psi} + \partial_z Q \partial_z \tilde{\psi} \right) dr d\theta dz \\ &= \int \left(\frac{1}{r^3} \partial_\theta (-\phi \partial_z (FG) + \psi \partial_z \lambda) \partial_z \tilde{\psi} + \frac{1}{r^3} \partial_\theta (-\phi \partial_r (FG) + \psi \partial_r \lambda) \partial_r \tilde{\psi} \right) \\ & \quad + 2 \int \frac{1}{r^4} \partial_\theta (FG\phi - \lambda\psi) \partial_r \tilde{\psi}. \end{aligned}$$

Under this form, the correction term Q is a the solution of a series of Grad-Shafranov like weak formulations for all θ . It ends the proof. \square

3 Existence of a weak solution

In this section we study more precisely the existence of a solution by means of analogy with the Teman-Lions theory. This makes necessary to incorporate some diffusive operators in the previous models. In the following, we have retained diffusive operators with two properties: they are the closest to what is in the physical literature; it is possible to prove existence of a weak solution using the regularity/compactness of the additional viscous part. Indeed a key tool to prove the existence of weak solutions is the energy identity which provides a control of the regularity of the solution. In "real life" physical situations and numerical simulations we are aware of, such diffusive operators are always present, even if with quite small viscous or resistive constants.

For these two reasons we incorporate hereafter viscous and resistive operators to the reduced models derived before, and concentrate on the mathematical properties of a weak solution.

3.1 Poloidal model with non constant forcing term

The temperature and pressure do not show up in the final model. On the other hand we keep a forcing term $F = F_0$ independent of the space and time variables. We also disregard the parallel velocity $\lambda \equiv 0$. However we keep the dependence of other physical variables with respect to the toroidal variable θ . By comparison with (59) one has $G = \rho^{-1}$. For the sake of mathematical simplicity we consider isotropic viscosity and resistivity. Consider

$$\begin{cases} \partial_t \psi = \frac{1}{\rho r} [\psi, \phi] + \eta \Delta^* \psi, \\ \partial_t \omega = \rho r \left[\frac{1}{(\rho r)^2} \omega, \phi \right] - \rho r \left[\frac{F}{\rho r^2}, F \right] - \rho r \left[\frac{1}{\rho r^2} \Delta^* \psi, \psi \right] + \nu \rho r \Delta_\rho \omega, \\ \omega = \Delta_\rho \phi. \end{cases} \quad (63)$$

We consider the Grad-Shafranov equation [20] for stationary states

$$\Delta^* \bar{\psi} = -r^2 \frac{d\bar{p}}{d\bar{\psi}} - \frac{1}{2} \frac{d\bar{F}^2}{d\bar{\psi}} \quad (64)$$

where $\bar{\psi} \mapsto \bar{p}(\bar{\psi})$ and $\bar{\psi} \mapsto \bar{F}(\bar{\psi})$ are two given smooth functions. It is well known that in axisymmetric geometry the equilibrium of force between the pressure and the magnetic force holds at $t = 0$ if and only if

$$F = \bar{F}(\bar{\psi}), \quad p = \bar{p}(\bar{\psi}), \quad \text{and} \quad \psi = \bar{\psi}. \quad (65)$$

The pressure can also be calculated in function of the density in an isentropic model. Therefore one also has

$$\rho = \bar{\rho}(\bar{\psi}). \quad (66)$$

It is necessary to add a source term in the magnetic equation to respect this rest state. In the context of Tokamak's modeling, this is the role of the bootstrap current for which we refer to [28, 29, 15] and references therein. The system is modified

$$\begin{cases} \partial_t \psi = \frac{1}{\rho r} [\psi, \phi] + \eta \Delta^* \psi - \eta \Delta^* \bar{\psi}, \\ \partial_t \omega = \rho r \left[\frac{1}{(\rho r)^2} \omega, \phi \right] - \rho r \left[\frac{F}{\rho r^2}, F \right] - \rho r \left[\frac{1}{\rho r^2} \Delta^* \psi, \psi \right] + \nu \rho r \Delta_\rho \omega, \\ \omega = \Delta_\rho \phi. \end{cases} \quad (67)$$

Proposition 20. *Solutions of the Grad-Shafranov equation (65-66) are also stationary solutions of (67).*

Proof. One has

$$\left[\frac{\bar{F}}{\rho r^2}, \bar{F} \right] = \bar{F} \bar{F}'(\bar{\psi}) \left[\frac{1}{\rho r^2}, \bar{\psi} \right].$$

Moreover

$$\begin{aligned} \left[\frac{1}{\rho r^2} \Delta^* \bar{\psi}, \bar{\psi} \right] &= - \left[\frac{1}{\bar{\rho}} \frac{d\bar{p}}{d\bar{\psi}}, \bar{\psi} \right] - \frac{1}{2} \left[\frac{1}{\bar{\rho} r^2} \frac{d\bar{F}^2}{d\bar{\psi}}, \bar{\psi} \right] \\ &= 0 - \bar{F} \bar{F}'(\bar{\psi}) \left[\frac{1}{\rho r^2}, \bar{\psi} \right]. \end{aligned}$$

Therefore the sum of these two terms vanishes, $\left[\frac{F}{\rho r^2}, F \right] - \left[\frac{1}{\rho r^2} \Delta^* \psi, \psi \right] = 0$, which proves the result. \square

The previous property is an important enhancement of our initial model [15]. Another property is the well-posedness, at least in the weak sense.

Theorem 21. *There exists a weak solution to the problem (63) equipped with mixed Dirichlet-Neumann boundary condition $\psi = \phi = \frac{\partial \phi}{\partial n} = 0$.*

The proof is based on the result proved in [15]. Indeed the source term $-\left[\frac{F}{\rho r^2}, F \right]$ does not modify the structure of the proof, neither the result.

3.2 Model with toroidal dependency and constant forcing term

The set of equations that we consider in the torus $\Omega = \{(x, z) \in \mathcal{D} \text{ and } \theta \in [0, 2\pi]\}$ writes in strong form

$$\begin{cases} \partial_t \psi = \frac{1}{r} [\psi, \phi] + \eta_{\perp} \Delta^* \psi + \eta_{\parallel} \partial_{\theta}^2 \psi + \frac{1}{r^2} F_0 \partial_{\theta} \phi + Q, \\ \partial_t \omega = r \left[\frac{1}{r^2} \omega, \phi \right] + r \left[\psi, \frac{1}{r^2} \Delta^* \psi \right] + \nu_{\perp} \Delta^* \omega + \nu_{\parallel} \partial_{\theta}^2 \omega + \frac{1}{r^2} F_0 \Delta^{***} \partial_{\theta} \psi, \\ \omega = \Delta^* \phi, \end{cases} \quad (68)$$

and the source term Q is given by the weak form

$$\int_{\mathcal{D}} \frac{1}{r} \left(\partial_r Q \partial_r \tilde{\psi} + \partial_z Q \partial_z \tilde{\psi} \right) = 2F_0 \int_{\mathcal{D}} \frac{1}{r^4} \partial_{\theta} \phi \partial_r \tilde{\psi}, \quad \forall \tilde{\psi} \in H_0^1(\mathcal{D}). \quad (69)$$

By comparison with (59) we have taken $G = 1$ but our results are the same with the two other possibilities, namely $G = \rho^{-1}$ and $G = r^2$. We implicitly consider boundary conditions of Dirichlet and Neumann type

$$\psi = \phi = \frac{\partial \phi}{\partial n} = 0 \text{ on } \partial \mathcal{D}. \quad (70)$$

For simplicity the source also satisfies a Dirichlet boundary condition $Q = 0$ on $\partial \mathcal{D}$.

Proposition 22. *One has the formal energy identity*

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \frac{1}{r} \left(|\nabla_{r,z} \psi|^2 + |\nabla_{r,z} \phi|^2 \right) + \eta_{\perp} \int_{\Omega} \frac{|\Delta^* \psi|^2}{r} + \nu_{\perp} \int_{\Omega} \frac{|\Delta^* \phi|^2}{r} + \eta_{\parallel} \int_{\Omega} \frac{|\partial_{\theta} \nabla_{r,z} \psi|^2}{r} + \nu_{\parallel} \int_{\Omega} \frac{|\partial_{\theta} \nabla_{r,z} \phi|^2}{r} = 0 \quad (71)$$

Proof. Summing the integral form of equations (68) with test functions $-\frac{\Delta^* \psi}{r}$ for the first equation and $-\frac{\phi}{r}$ for the second one, we get the classical part of energy identity for the standard incompressible reduced resistive MHD model, already detailed in [15] for general conditions of density, and some additional terms related to the parallel viscosities η_{\parallel} , ν_{\parallel} , the forcing term F_0 and the source term Q . We will only deal with the extra terms here.

It is trivial for the parallel diffusion terms through integrations by parts:

$$\int \eta_{\parallel} \partial_{\theta}^2 \psi \frac{\Delta^* \psi}{r} + \int \nu_{\parallel} \partial_{\theta}^2 \omega \frac{\phi}{r} = \eta_{\parallel} \int \frac{|\partial_{\theta} \nabla_{r,z} \psi|^2}{r} + \nu_{\parallel} \int \frac{|\partial_{\theta} \nabla_{r,z} \phi|^2}{r},$$

and the forcing and source terms may be treated together as follows:

$$\begin{aligned} \int \left(\frac{1}{r^2} F_0 \partial_\theta \phi + Q \right) \frac{\Delta^* \psi}{r} + \int \frac{1}{r^2} F_0 \Delta^{***} \partial_\theta \psi \frac{\phi}{r} &= \int \frac{1}{r^3} F_0 \partial_\theta \phi (\Delta^* \psi - \Delta^{***} \psi) - \frac{\nabla_{r,z} Q \cdot \nabla_{r,z} \psi}{r} \\ &= \int \frac{2}{r^4} F_0 \partial_\theta \phi \partial_r \psi - \frac{1}{r} (\partial_r Q \partial_r \psi + \partial_z Q \partial_z \psi) \end{aligned}$$

since $\Delta^* \psi - \Delta^{***} \psi = \left(\Delta \psi - \frac{\partial_r \psi}{r} \right) - \left(\Delta \psi + 3 \frac{\partial_r \psi}{r} \right) = 2 \frac{\partial_r \psi}{r}$. This contribution vanishes thanks to (69). \square

Theorem 23. *There exists a weak solution to the problem (68–70).*

Proof. We will here rely on the construction of approximate solutions $(\psi_{\Delta t}, \phi_{\Delta t})$ detailed in [15] for standard 2-D incompressible reduced resistive MHD model and focus on the stability of solutions. In this construction, $\Delta t > 0$ is a time step which is used to define a specific and well posed form of the equations, with a splitting in time of the different operators. It is sufficient in the following to consider that $\Delta t > 0$ is an abstract regularization parameter and, as usual in the Teman-Lions theory for such systems, the fundamental issue is to show the continuity of non linear operators with respect to the norms which are controlled by the energy identity.

Proposition 24. $\nabla_{r,z} \psi_{\Delta t}$ and $\nabla_{r,z} \phi_{\Delta t}$ are uniformly bounded in $L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ and by interpolation in $L^{\frac{2}{1-\xi}}(0, T; L^{\frac{6}{2\xi+1}}(\Omega))$, for all $\xi \in]0, 1[$.

Proof. Integrating the energy identity (71) on $[0, T]$, one obtains the a priori uniform bounds:

$$\nabla_{r,z} \psi_{\Delta t} \in L^\infty(0, T; L^2(\Omega)), \quad \nabla_{r,z} \phi_{\Delta t} \in L^\infty(0, T; L^2(\Omega)) \quad (72)$$

$$\Delta^* \psi_{\Delta t} \in L^2(0, T; L^2(\Omega)), \quad \Delta^* \phi_{\Delta t} \in L^2(0, T; L^2(\Omega)) \quad (73)$$

$$\partial_\theta \nabla_{r,z} \psi_{\Delta t} \in L^2(0, T; L^2(\Omega)), \quad \partial_\theta \nabla_{r,z} \phi_{\Delta t} \in L^2(0, T; L^2(\Omega)) \quad (74)$$

Putting together (72), (73) and (74) we conclude that $\nabla_{r,z} \psi_{\Delta t}$ and $\nabla_{r,z} \phi_{\Delta t}$ are bounded in $L^\infty(0, T; L^2(\Omega))$ and $L^2(0, T; H^1(\Omega))$. Next, since $\Omega \subset \mathbb{R}^3$, one has the embedding $H^1(\Omega) \subset L^6(\Omega)$ and then we conclude with the following classical interpolation result

$$\forall \xi \in (0, 1), \quad L^{p_1}(L^{q_1}(\Omega)) \cap L^{p_2}(L^{q_2}(\Omega)) \subset L^p(L^q(\Omega)), \quad \text{with } \frac{1}{p} = \frac{\xi}{p_1} + \frac{1-\xi}{p_2} \quad \text{and} \quad \frac{1}{q} = \frac{\xi}{q_1} + \frac{1-\xi}{q_2}$$

applied for $p_1 = +\infty, q_1 = 2, p_2 = 2, q_2 = 6$. \square

Proposition 25. $\nabla_{r,z} \psi_{\Delta t}$ and $\nabla_{r,z} \phi_{\Delta t}$ are compactly embedded in $L^2(0, T; L^2(\Omega))$.

Proof. This result will come from the compact injection of $H = \{v \in L^2(0, T; H^1(\Omega)); \partial_t v \in L^p(0, T; B)\}$ into $L^2(0, T; L^2(\Omega))$ for all $1 < p < \infty$ and any reflexive Banach space B such that the embedding $L^2(\Omega) \subset B$ is continuous. So, let us show that $\nabla_{r,z} \psi_{\Delta t}$ and $\nabla_{r,z} \phi_{\Delta t}$ belong to H for appropriate p and B .

First, by Lemma 24, $\nabla_{r,z} \psi_{\Delta t}$ and $\nabla_{r,z} \phi_{\Delta t}$ are uniformly bounded in $L^2(0, T; H^1(\Omega))$.

Next, we get bounds on the time derivatives $\partial_t \nabla_{r,z} \psi_{\Delta t}$ and $\partial_t \nabla_{r,z} \phi_{\Delta t}$ through the following equations:

$$\begin{cases} \partial_t \psi_{\Delta t} = \overbrace{\frac{1}{r} [\psi_{\Delta t}, \phi_{\Delta t}]}^{S_1} + \overbrace{\eta_\perp \Delta^* \psi_{\Delta t}}^{S_2} + \overbrace{\eta_\parallel \partial_\theta^2 \psi_{\Delta t}}^{S_3} + \overbrace{\frac{1}{r^2} F_0 \partial_\theta \phi_{\Delta t} + Q}_{S_4} \\ \partial_t \omega_{\Delta t} = r \underbrace{\left[\frac{1}{r^2} \omega_{\Delta t}, \phi_{\Delta t} \right]}_{T_1} + r \underbrace{\left[\psi_{\Delta t}, \frac{1}{r^2} \Delta^* \psi_{\Delta t} \right]}_{T_2} + \underbrace{\nu_\perp \Delta^* \omega_{\Delta t}}_{T_3} + \underbrace{\nu_\parallel \partial_\theta^2 \omega_{\Delta t}}_{T_4} + \underbrace{\frac{1}{r^2} F_0 \Delta^{***} \partial_\theta \psi_{\Delta t}}_{T_5} \end{cases}$$

Notice that r, F_0 and diffusion coefficients are bounded from above and below and thus play no role here. We also remark that the Poisson brackets can be written in the following different forms:

$$[a, b] = \nabla_{r,z} a \cdot \nabla_{r,z}^\perp b = \text{div}_{r,z} (a \nabla_{r,z}^\perp b), \quad \text{where } \nabla_{r,z}^\perp b = (-\partial_z b, 0, \partial_r b).$$

By Lemma 24, $\nabla_{r,z}\psi_{\Delta t} \in L^\infty(0, T; L^2(\Omega))$ and $\nabla_{r,z}^\perp\phi_{\Delta t} \in L^2(0, T; L^6(\Omega))$, and

$$S_1 = \frac{1}{r}\nabla_{r,z}\psi_{\Delta t} \cdot \nabla_{r,z}^\perp\phi_{\Delta t} \text{ belong to } L^2(0, T; L^s(\Omega)), \text{ for all } 1 < s \leq \frac{3}{2}.$$

By Lemma 24 and (73), we get $\omega_{\Delta t} \cdot \nabla^\perp\phi_{\Delta t} \in L^2(L^2(\Omega)) \times L^{\frac{2}{1-\xi}}(0, T; L^{\frac{6}{2\xi+1}}(\Omega))$ for all $\xi \in]0, 1[$. This leads to $\omega_{\Delta t} \cdot \nabla^\perp\phi_{\Delta t} \in L^{\frac{2}{2-\xi}}(0, T; L^{\frac{3}{\xi+2}}(\Omega))$ for all $\xi \in]0, 1[$ and then

$$T_1 = r \operatorname{div}_{r,z} \left(\frac{1}{r^2} \omega_{\Delta t} \cdot \nabla^\perp\phi_{\Delta t} \right) \in L^{\frac{2}{2-\xi}}(0, T; W^{-1, \frac{3}{\xi+2}}(\Omega)), \text{ for all } \xi \in]0, 1[.$$

Analogously, by Lemma 24 and (73),

$$T_2 = r \operatorname{div}_{r,z} \left(\frac{1}{r^2} \Delta^* \psi_{\Delta t} \cdot \nabla^\perp\psi_{\Delta t} \right) \in L^{\frac{2}{2-\xi}}(0, T; W^{-1, \frac{3}{\xi+2}}(\Omega)), \text{ for all } \xi \in]0, 1[.$$

All the remaining linear terms belong to $L^2(0, T; H^{-2}(\Omega))$ and this is enough to conclude that

$$\partial_t \nabla_{r,z}\psi_{\Delta t} \text{ and } \partial_t \nabla_{r,z}\phi_{\Delta t} \text{ belong to } L^p(0, T; B), \text{ with } B = H^{-3}(\Omega) \text{ and } p = \frac{2}{2-\xi} \text{ for any } \xi \in]0, 1[.$$

□

The compactness properties given in Lemma 25 together with the bounds in Lemma 24, insure the continuity of the non linear terms $[\psi_{\Delta t}, \phi_{\Delta t}]$, $[\frac{1}{r^2}\omega_{\Delta t}, \phi_{\Delta t}]$ and $[\psi_{\Delta t}, \frac{1}{r^2}\Delta^*\psi_{\Delta t}]$.

□

4 Remarks on the role of the correction term

It is instructing to analyze the role of the correction term Q (68) with respect to the well-posedness of the final reduced models. Our claim is that Q is related to the spectral stability of the partial differential operator with respect to the angular variable θ . More precisely the model might be linearly ill posed in the limit of vanishing resistivity and viscosity.

To understand this fact we simplify (68-69). Retaining only the derivatives with respect to θ , and taking into account that $\Delta^* = \partial_{rr} - \frac{1}{r}\partial_r$ and $\Delta^* = \partial_{rr} - \frac{3}{r}\partial_r$, one gets the set of linear equations

$$\begin{cases} \partial_t \psi = \frac{F_0}{r^2} \partial_\theta \phi + Q, \\ \partial_t \omega = \frac{F_0}{r^2} (\partial_{rr} - \frac{3}{r}\partial_r) \partial_\theta \psi, \\ \omega = (\partial_{rr} - \frac{1}{r}\partial_r) \phi, \\ \int \frac{1}{r} \partial_r Q \partial_r \tilde{\psi} = 2F_0 \int \frac{1}{r^3} \partial_\theta \phi \partial_r \tilde{\psi}, \quad \forall \tilde{\psi} \in Z. \end{cases} \quad (75)$$

It is, of course, assumed that Z is a convenient space of trial function such that at least $H_0^1(I) \subset Z$. In the following we systematically disregard boundary conditions since we are interested in the behavior of the correction term Q inside the domain of study.

We therefore consider for simplicity that the last weak formulation of (75) is replaced by

$$\partial_r Q = \frac{2F_0}{r^3} \partial_\theta \phi. \quad (76)$$

Next we replace any function with its Fourier transform in the direction θ . One obtains the system ($n \in \mathbb{Z}$)

$$\begin{cases} \partial_t \hat{\psi}_n = \mathbf{i}n \frac{F_0}{r^2} \hat{\phi}_n + \mathbf{i}n F_0 \hat{q}_n, \\ \partial_t \hat{\omega}_n = \mathbf{i}n \frac{F_0}{r^2} (\partial_{rr} - \frac{3}{r}\partial_r) \hat{\psi}_n, \\ \hat{\omega}_n = (\partial_{rr} - \frac{1}{r}\partial_r) \hat{\phi}_n, \\ \partial_r \hat{q}_n = \frac{2}{r^3} \hat{\phi}_n. \end{cases} \quad (77)$$

The spectral stability amounts to consider a dependence $\widehat{f}(t, r) = e^{\lambda t} \widetilde{f}(r)$. One obtains the system

$$\begin{cases} \lambda \widetilde{\psi}_n = \mathbf{i}n \frac{F_0}{r^2} \widetilde{\phi}_n + \mathbf{i}n F_0 \widetilde{q}_n, \\ \lambda \widetilde{\omega}_n = \mathbf{i}n \frac{F_0}{r^2} \left(\partial_{rr} - \frac{3}{r} \partial_r \right) \widetilde{\psi}_n, \\ \widetilde{\omega}_n = \left(\partial_{rr} - \frac{1}{r} \partial_r \right) \widetilde{\phi}_n = r \partial_r \left(\frac{1}{r} \partial_r \widetilde{\phi}_n \right), \\ \partial_r \widetilde{q}_n = \frac{2}{r^3} \widetilde{\phi}_n. \end{cases} \quad (78)$$

The analogous of the energy identity is as follows. First

$$\begin{aligned} \lambda \int \frac{|\widetilde{\psi}'_n|^2 + |\widetilde{\phi}'_n|^2}{r} dr &= -\lambda \int \frac{1}{r} \widetilde{\psi}_n \left(\partial_{rr} - \frac{1}{r} \partial_r \right) \overline{\widetilde{\psi}_n} + \frac{1}{r} \omega_n \overline{\widetilde{\phi}_n} \\ &= -\mathbf{i}n F_0 \int \frac{1}{r} \left(\frac{1}{r^2} \widetilde{\phi}_n + \widetilde{q}_n \right) \left(\partial_{rr} \overline{\widetilde{\psi}_n} - \frac{1}{r} \partial_r \overline{\widetilde{\psi}_n} \right) + \frac{1}{r} \left(\frac{1}{r^2} \left(\partial_{rr} - \frac{3}{r} \partial_r \right) \widetilde{\psi}_n \right) \overline{\widetilde{\phi}_n} \\ &= -\mathbf{i}n F_0 \underbrace{\int \frac{1}{r} \widetilde{q}_n \left(\partial_{rr} \overline{\widetilde{\psi}_n} - \frac{1}{r} \partial_r \overline{\widetilde{\psi}_n} \right) - \frac{1}{r^3} \widetilde{\phi}_n \frac{2}{r} \partial_r \overline{\widetilde{\psi}_n}}_{=A} \\ &\quad - \mathbf{i}n F_0 \underbrace{\int \frac{1}{r^3} \left(\widetilde{\phi}_n \left(\partial_{rr} \overline{\widetilde{\psi}_n} - \frac{1}{r} \partial_r \overline{\widetilde{\psi}_n} \right) + \left(\partial_{rr} \widetilde{\psi}_n - \frac{1}{r} \partial_r \widetilde{\psi}_n \right) \overline{\widetilde{\phi}_n} \right)}_{=B}. \end{aligned}$$

The second term is real $B \in \mathbb{R}$. An integration by parts (still forgetting about the boundary terms) yields

$$A = \int \frac{1}{r} \partial_r \widetilde{q}_n \partial_r \overline{\widetilde{\psi}_n} - \frac{2}{r^4} \widetilde{\phi}_n \partial_r \overline{\widetilde{\psi}_n} = 0$$

due to the definition of the correction term. It yields

$$\lambda = -\mathbf{i}n F_0 B \left(\int \frac{|\widetilde{\psi}'_n|^2 + |\widetilde{\phi}'_n|^2}{r} dr \right)^{-1} \in \mathbf{i}\mathbb{R}$$

which in turns corresponds to the linear of spectral stability of (77) or (78). Once again this analysis is qualitative and valid away from the boundary.

If one forgets the correction, the algebra is almost the same except that now

$$A = - \int \frac{1}{r} \frac{2}{r^4} \widetilde{\phi}_n \partial_r \overline{\widetilde{\psi}_n}.$$

The first equation of (78) is also modified $\lambda \widetilde{\psi}_n = \mathbf{i}n \frac{F_0}{r^2} \widetilde{\phi}_n$. Now A has no reason a priori to vanish and also no reason to be purely imaginary. If A admits a real part, the model might be linear unstable. More studies are needed to confirm, or not, this hypothesis. In any case, we conclude that the correction term, which is related to the metric, that is to the curvature of the domain (a torus for Tokamaks), guarantees the well-posedness.

It must be noted that the correction term is not present in all models in Tokamak geometry used for numerical simulation that we know about in the literature, see for example [21, 25, 27, 32, 35, 37].

Another point that will deserve attention in a near future is the role of the correction term in the conditioning (good or bad) of the matrices that one has to solve or invert in reduced MHD codes [7].

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