

Sample path large deviations for squares of stationary Gaussian processes

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Abstract

In this paper, we show large deviations for random step functions of type

$$Z_n(t) = \frac{1}{n} \sum_{k=1}^{[nt]} X_k^2,$$

where $\{X_k\}_k$ is a stationary Gaussian process. We deal with the associated random measures $\nu_n = \frac{1}{n} \sum_{k=1}^n X_k^2 \delta_{k/n}$. The proofs require a Szegö theorem for generalized Toeplitz matrices, which is presented in the Appendix and is analogous to a result of Kac, Murdoch and Szegö [10]. We also study the polygonal line built on $Z_n(t)$ and show moderate deviations for both random families.

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1 Introduction

The aim of this paper is to provide a large deviations principle (LDP) for random functions of type

$$Z_n(t) = \frac{1}{n} \sum_{k=1}^{[nt]} X_k^2, \tag{1}$$

and the associated polygonal line

$$\tilde{Z}_n(t) = Z_n(t) + \left(t - \frac{[nt]}{n}\right) X_{[nt]+1}^2, \tag{2}$$

where $\{X_n\}_n$ is a stationary Gaussian process having spectral density f defined on the torus $\mathbb{T} =]-\pi, \pi]$. We assume f is continuous positive on \mathbb{T} .

Large deviations for random measures date back to Sanov [19] who showed a LDP for the family of empirical measures

$$\frac{1}{n} \sum_{i=1}^n \delta_{X_i}, \tag{3}$$

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where X_i are i.i.d. random variables.

Then, the first results on large deviations for random paths were given by Borovkov [2] and Varadhan [20]. In [2], Borovkov provides a LDP for the random polygonal line joining the points $(\frac{k}{n}, \frac{S_k}{x})$ where $S_k = \sum_{i=1}^k X_i$ and $x = x(n)$ is in the range

$$\limsup_{n \rightarrow \infty} \frac{x}{n} < \infty, \quad \lim_{n \rightarrow \infty} \frac{x}{\sqrt{n \ln n}} = \infty \quad (4)$$

He also showed large deviations for the paths $\eta(nt)/x$ where $0 \leq t \leq 1$ and η is a separable process with independent increments. The large deviations are given in the spaces $\mathcal{C}([0, 1])$ (the set of continuous functions on $[0, 1]$) or $\mathcal{D}([0, 1])$ (the set of cadlag functions on $[0, 1]$) endowed with the uniform metric. Meanwhile, Varadhan [20] proved functional large deviations in $\mathcal{D}([0, 1])$ for the random step functions

$$S_n(t) = \frac{1}{n} \sum_{i=1}^{[nt]} X_i \quad (5)$$

where $t \in [0, T]$ and $[nt]$ denotes the integer part of nt . Later on, Mogulskii ([13]) improved these results: he proved large deviations for the polygonal line $(\frac{k}{n}, \frac{S_k}{x})$ in the range

$$\limsup_{n \rightarrow \infty} \frac{x}{n} < \infty, \quad \lim_{n \rightarrow \infty} \frac{x}{\sqrt{n}} = \infty \quad (6)$$

in the space $\mathcal{D}([0, 1])$ endowed with the Skorokhod metric. For more general results on large deviations for processes with independent increments, see also Lynch and Sethuraman [11], de Acosta [3] and Mogulskii [14].

The results of [2, 20, 13] concerning step functions and continuous random polygonal lines built on sums of i.i.d. random variables can be found in the books of Dupuis and Ellis [6] and Dembo and Zeitouni [5].

In our paper, to derive the large deviations, we consider the distribution derivative of $t \rightarrow Z_n(t)$ and $t \rightarrow \tilde{Z}_n(t)$. Therefore we deal with the random measures ν_n and $\tilde{\nu}_n$ given by

$$\langle \nu_n, h \rangle = \frac{1}{n} \sum_{k=1}^n X_k^2 h\left(\frac{k}{n}\right) \quad (7)$$

and

$$\langle \tilde{\nu}_n, h \rangle = \sum_{k=1}^n X_k^2 \int_{(k-1)/n}^{k/n} h(s) ds, \quad (8)$$

for h in $\mathcal{C}([0, 1])$. Let $\mathcal{M}([0, 1])$ be the set of positive bounded measures on $[0, 1]$ endowed with the weak topology. Therefore ν_n and $\tilde{\nu}_n$ are a.s. in $\mathcal{M}([0, 1])$.

Analogous random measures have been investigated before by Dembo and Zeitouni [4], and Gamboa and Gassiat [7]. Previous works on LDP for this kind of random functions can be found in Gamboa, Rouault and Zani [8] and Perrin and Zani [16] for stationary Gaussian processes, and in Najim [15] and Maïda, Najim and P      [12] for

i.i.d. sequences. We provide here a functional LDP for $\{\nu_n\}$ and $\{\tilde{\nu}_n\}$, and derive the associated LDP for $\{Z_n\}$ and $\{\tilde{Z}_n\}$. We also prove moderate deviations. The central limit theorem is known. Although part of this work was already presented in [21] the present work provide a full version with proofs and some extensions.

The remaining of the paper is organized as follows. We present in Section 2 the large and moderate deviations results. Section 3 is devoted to the proofs of Theorems. Deriving the LD result, we needed a Szegő type theorem for generalized Toeplitz matrices. This precise result is unknown to our knowledge and despite a very similar result has been shown in Kac Murdoch and Szego (see [10] and [9]), for seek of completeness we prove it in the Appendix. The remaining of the Appendix gather the proofs of technical lemmas.

2 Large and moderate deviations

For any h in $\mathcal{C}([0, 1])$, define

$$\Lambda(h) = \begin{cases} -\frac{1}{4\pi} \int_{[0,1]} \int_{\mathbb{T}} \log(1 - 2h(t)f(\theta)) d\theta dt & \text{if } \forall(t, \theta) \in [0, 1] \times \mathbb{T}, h(t)f(\theta) < 1/2 \\ +\infty & \text{otherwise} \end{cases}$$

Let Λ^* be the Legendre dual of Λ . From Rockafellar [18], we can detail this dual function as following:

Proposition 2.1 *Let ν be the measure in $\mathcal{M}([0, 1])$ defined for any h in $\mathcal{C}([0, 1])$ by*

$$\langle \nu, h \rangle = \frac{1}{2\pi} \int_{\mathbb{T}} f(\theta) d\theta \int_{[0,1]} h(x) dx.$$

Let $\mu \in \mathcal{M}([0, 1])$ having the following Lebesgue decomposition with respect to ν : $\mu = l\nu + \mu^\perp$ where $l \in \mathcal{C}([0, 1])$ and μ^\perp is the singular part. Then

$$\Lambda^*(\mu) = \int_{[0,1]} u^*(l(t)) \nu(dt) + \int_{[0,1]} \frac{\mu^\perp(dt)}{2M},$$

where

$$u(x) = -\frac{1}{4\pi} \int_{\mathbb{T}} \log(1 - 2xf(\theta)) d\theta,$$

and

$$M = \text{esssup} f.$$

The function u is \mathcal{C}^2 on $(-\infty, 1/2M)$, and

$$u'(x) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{f(\theta)}{1 - 2xf(\theta)} d\theta$$

$$u''(x) = \frac{1}{\pi} \int_{\mathbb{T}} \frac{f(\theta)^2}{(1 - 2xf(\theta))^2} d\theta > 0$$

Hence u' is strictly increasing, and $\lim_{x \rightarrow -\infty} u'(x) = 0$. On the other hand, we denote $u'(1/2M) := \lim_{x \rightarrow +\infty} u'(x) \leq +\infty$ (e.g. if $f \in \mathcal{C}^2$, $u'(1/2M) = +\infty$). The recession function (see Theorem 8.5 of [18]) is $r(u^*; y) = y/2M$.

2.1 Large Deviations

We can now state the LDP result:

Theorem 2.2 *The families $\{\nu_n\}_{n \in \mathbb{N}}$ and $\{\tilde{\nu}_n\}_{n \in \mathbb{N}}$ satisfy a LDP in $\mathcal{M}([0, 1])$ with speed n and rate function Λ^* .*

We can carry the previous LDP to the random functions Z_n and \tilde{Z}_n . Following Lynch and Sethuraman [11] and de Acosta [3], we introduce some notations. Let $D([0, 1], \mathbb{R})$ be the space of cadlag real functions on $[0, 1]$, and $bv([0, 1], \mathbb{R}) \subset D([0, 1], \mathbb{R})$ the space of bounded variation functions. We can identify $bv([0, 1], \mathbb{R})$ with $\mathcal{M}([0, 1])$: to h in $bv([0, 1], \mathbb{R})$ corresponds μ_h in $\mathcal{M}([0, 1])$ characterized by $\mu_h([0, t]) = h(t)$. Up to this identification, the topological dual of $bv([0, 1], \mathbb{R})$ is the set $\mathcal{C}([0, 1])$. We endow $bv([0, 1], \mathbb{R})$ with the w^* -topology written σ , i.e. the topology induced by $\mathcal{C}([0, 1])$ on $\mathcal{M}([0, 1])$. Now, let us define the rate function associated to Z_n and \tilde{Z}_n : let h be in $bv([0, 1], \mathbb{R})$ and μ_h the associated measure in $\mathcal{M}([0, 1])$; let $\mu_h = (\mu_h)_a + (\mu_h)_s$ be the Lebesgue decomposition of μ_h in absolutely continuous and singular terms with respect to the Lebesgue measure on $[0, 1]$; let $h_a(t) = (\mu_h)_a([0, t])$ and $h_s(t) = (\mu_h)_s([0, t])$. Set

$$\Phi(h) = \int_{[0,1]} u^*(h'_a)(t) \nu(dt) + r h_s(1),$$

where u^* and r are defined in Proposition 2.1.

Theorem 2.3 *The families of random functions $\{Z_n\}$ and $\{\tilde{Z}_n\}$ satisfy a LDP on the space $(bv([0, 1], \mathbb{R}), \sigma)$, with speed n and rate function Φ .*

2.2 Moderate deviations

We can state also in this case a moderate deviation principle. We detail it for ν_n , it is the same for $\tilde{\nu}_n$. Let $\{a_n\}$ be a sequence of positive real numbers such that $a_n \rightarrow 0$ and $na_n \rightarrow +\infty$ when $n \rightarrow +\infty$. Set

$$Y_n = \sqrt{na_n}(\nu_n - E(\nu_n)).$$

We have the following moderate deviations principle

Theorem 2.4 *$\{Y_n\}$ satisfy a LDP with speed a_n^{-1} and good rate function defined, for all $\mu \in \mathcal{M}([0, 1])$ by*

$$I(\mu) = \begin{cases} \frac{\pi}{2\bar{f}^2} \int_{[0,1]} l(x)^2 dx & \text{if } \mu(dx) = l(x) dx \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$\bar{f}^2 = \frac{1}{2\pi} \int_{\mathbb{T}} f^2.$$

2.3 Generalizations

The previous results can be generalized to some other random functions.

2.3.1 Weighted random variables

Assume g is a continuous function on $[0, 1]$ and define

$$W_n = \frac{1}{n} \sum_{k=1}^{\lfloor nt \rfloor} g\left(\frac{k}{n}\right) X_k^2, \quad (9)$$

For any h in $\mathcal{C}([0, 1])$, define

$$\Lambda(h) = \begin{cases} -\frac{1}{4\pi} \int_{[0,1]} \int_{\mathbb{T}} \log(1 - 2h(t)g(t)f(\theta)) d\theta dt & \text{if } \forall (t, \theta) \in [0, 1] \times \mathbb{T}, h(t)g(t)f(\theta) < 1/2 \\ +\infty & \text{otherwise} \end{cases}$$

The previous large deviations results apply with rate function Λ^* .

2.3.2 Quadratic forms built on the stationary process

We define

$$m = \text{essinf} f$$

and assume $m > 0$. Let F be a continuous positive function on $[m, M]$. Let O be an orthonormal matrix such that $O^* T_n(f) O$ is the diagonal matrix whose i -th diagonal element is $\mu_{i,n}$ the i -th eigenvalue of $T_n(f)$. Define

$$F(T_n(f)) = O D_f O^*$$

where D_f is the diagonal matrix whose i -th element is $F(\mu_{i,n})$. Define the following quadratic form

$$W_n = \frac{1}{n} X^* F(T_n(f)) X = \frac{1}{n} Y^* Y,$$

where $Y = (Y_1, \dots, Y_n)$ is the vector defined by

$$Y = F(T_n(f))^{1/2} X.$$

In this case, W_n satisfies a LDP and moderate deviations theorem with rate function Λ^* where for any h in $\mathcal{C}([0, 1])$

$$\Lambda(h) = \begin{cases} -\frac{1}{4\pi} \int_{[0,1]} \int_{\mathbb{T}} \log[1 - 2h(t)f(\theta)F[f(\theta)]] d\theta dt & \text{if } \forall (t, \theta) \in [0, 1] \times \mathbb{T}, h(t)f(\theta) < 1/2 \\ +\infty & \text{otherwise} \end{cases}$$

3 Proof of the large and moderate deviations

We first give some asymptotic properties for the families $\{\nu_n\}_n$ and $\{\tilde{\nu}_n\}_n$.

3.1 Weak convergence of ν_n and $\{\tilde{\nu}_n\}_n$

Lemma 3.1 *Let h be in $\mathcal{C}([0, 1])$.*

$$\langle \nu_n, h \rangle \rightarrow \langle \nu, h \rangle \quad \text{in probability as } n \rightarrow +\infty \quad (10)$$

and

$$\langle \tilde{\nu}_n, h \rangle \rightarrow \langle \nu, h \rangle \quad \text{in probability as } n \rightarrow +\infty$$

where

$$\langle \nu, h \rangle = \bar{f} \int_{[0,1]} h(x) dx,$$

and

$$\bar{f} = \frac{1}{2\pi} \int_{\mathbb{T}} f(\theta) d\theta.$$

Proof :

Let h be in $\mathcal{C}([0, 1])$, and consider

$$\langle \nu_n, h \rangle = \frac{1}{n} \sum_{k=1}^n X_k^2 h\left(\frac{k}{n}\right).$$

Set X the Gaussian vector (X_1, X_2, \dots, X_n) and Δ_h the diagonal matrix

$$\begin{pmatrix} h(\frac{1}{n}) & 0 & 0 & 0 \\ 0 & h(\frac{2}{n}) & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & h(1) \end{pmatrix}$$

Therefore we can write

$$\langle \nu_n, h \rangle = \frac{1}{n} X^* \Delta_h X,$$

where X^* denote the transpose of X . By an orthonormal change of basis,

$$\langle \nu_n, h \rangle = \frac{1}{n} U_n^* T_n(f)^{1/2} \Delta_h T_n(f)^{1/2} U_n,$$

where U_n is a standard normal vector and $T_n(f)$ the order- n Toeplitz matrix associated to f . Therefore

$$\langle \nu_n, h \rangle = \frac{1}{n} \sum_{k=1}^n \lambda_{k,n} Z_{k,n} \quad (11)$$

where $\{Z_{k,n}\}$ are independent $\chi^2(1)$ -distributed random variables, and $\{\lambda_{k,n}\}$ are the eigenvalues of $T_n(f)^{1/2}\Delta_h T_n(f)^{1/2}$.

We can write as well

$$\langle \tilde{\nu}_n, h \rangle = \frac{1}{n} \sum_{k=1}^n \tilde{\lambda}_{k,n} Z_{k,n} \quad (12)$$

where $\{Z_{k,n}\}$ are independent $\chi^2(1)$ -distributed random variables, and $\{\tilde{\lambda}_{k,n}\}$ are the eigenvalues of $T_n(f)^{1/2}A_h T_n(f)^{1/2}$, and the matrix A_h is diagonal with k -th diagonal term

$$(A_h)_{k,k} = \int_{(k-1)/n}^{k/n} h(s) ds.$$

We have the two following results on the distributions $\{\lambda_{k,n}\}$ and $\{\tilde{\lambda}_{k,n}\}$, which proofs are postponed to the Appendix.

Lemma 3.2 *The sequences $\{\lambda_{k,n}\}$ and $\{\tilde{\lambda}_{k,n}\}$ are bounded as follows:*

$$\begin{aligned} \forall n \in \mathbb{N}, \forall 1 \leq k \leq n, \quad & |\lambda_{k,n}| \leq \|h\|_\infty \|f\|_\infty \\ & |\tilde{\lambda}_{k,n}| \leq \|h\|_\infty \|f\|_\infty \end{aligned}$$

Lemma 3.3 *For any p in \mathbb{N} , $p \geq 1$,*

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=1}^n \lambda_{k,n}^p &= \frac{1}{2\pi} \int_{[0,1]} \int_{\mathbb{T}} (h(t)f(\theta))^p dt d\theta. \\ \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=1}^n (\tilde{\lambda}_{k,n})^p &= \frac{1}{2\pi} \int_{[0,1]} \int_{\mathbb{T}} (h(t)f(\theta))^p dt d\theta. \end{aligned}$$

With the above lemma,

$$\lim_{n \rightarrow +\infty} E(\langle \nu_n, h \rangle) = \langle \nu, h \rangle.$$

Moreover,

$$\lim_{n \rightarrow +\infty} n \text{Var} \langle \nu_n, h \rangle = \frac{2}{n} \sum_{k=1}^n \lambda_{k,n}^2 = \frac{1}{\pi} \int_{[0,1]} \int_{\mathbb{T}} (h(t)f(\theta))^2 dt d\theta.$$

We do as well for $\tilde{\nu}_n$, and it ends the proof of lemma 3.1.

3.2 Proof of Theorem 2.2:

The proof follows exactly the scheme [8]. We detail here for ν_n , it is similar for $\tilde{\nu}_n$. With the decomposition (11), we get the n.c.g.f. of ν_n : for any $h \in \mathcal{C}([0,1])$,

$$\Lambda_n(h) = \frac{1}{n} \log E(\exp\{n \langle \nu_n, h \rangle\}) = \begin{cases} -\frac{1}{2n} \sum_{k=1}^n \log(1 - 2\lambda_{k,n}) & \text{if } \forall k, \lambda_{k,n} < 1/2 \\ +\infty & \text{otherwise} \end{cases} \quad (13)$$

From Lemma 3.3, we can determine the limit of Λ_n in two cases:

- if $\forall (t, \theta) \in [0, 1] \times \mathbb{T}$ $h(t)f(\theta) < 1/2$, then

$$\lim_{n \rightarrow +\infty} \Lambda_n(h) = -\frac{1}{4\pi} \int_{[0,1]} \int_{\mathbb{T}} \log(1 - 2h(t)f(\theta)) d\theta dt = \Lambda(h).$$

- if $\exists (t, \theta) \in [0, 1] \times \mathbb{T}$; $h(t)f(\theta) > 1/2$, then for n large enough, $\Lambda_n(h) = +\infty$ and

$$\lim_{n \rightarrow +\infty} \Lambda_n(h) = +\infty = \Lambda(h).$$

These two cases do not cover the whole set $\mathcal{C}([0, 1])$. Nevertheless, this will be sufficient for the LDP, since they contain a dense subset of exposing hyperplanes of Λ^* .

Upper bound

From Theorem 4.5.3 b) of [5], and the following lemma, which proof is postponed to the Appendix, the upper bound holds for compact sets.

Lemma 3.4 *For any $\delta > 0$ and μ in $\mathcal{M}([0, 1])$, there exists h_δ in $\mathcal{C}([0, 1])$ such that:*

$$\begin{aligned} & \forall (t, \theta), \quad h_\delta(t)f(\theta) < 1/2 \\ & \int_{[0,1]} h_\delta(t) d\mu(t) - \Lambda(h_\delta) \geq \Lambda_\delta^*(\mu) \end{aligned} \tag{14}$$

where

$$\Lambda_\delta^*(\mu) = \min\{\Lambda^*(\mu) - \delta, \frac{1}{\delta}\}.$$

Exponential tightness

Remark that for a real number a ,

$$\left\{ \sup_{\|h\|_\infty \leq 1} \langle \nu_n, h \rangle \geq a \right\} \subset \{\nu_n(1) \geq a\}.$$

If $M = \text{esssup}_\theta f(\theta)$, for any $y < 1/2M$,

$$\limsup_n \frac{1}{n} \log P(\nu_n(1) \geq a) \leq -ya - \frac{1}{4\pi} \int_{[0,1]} \int_{\mathbb{T}} \log(1 - 2yf(\theta)) d\theta,$$

and

$$\lim_{a \rightarrow +\infty} \limsup_n \frac{1}{n} \log P(\nu_n(1) \geq a) = -\infty.$$

Hence the sequence (ν_n) is exponentially tight, and the upper bound holds for any closed set of $\mathcal{M}([0, 1])$.

Lower bound

We study the set of exposed points of Λ^* (see [5]). Let

$$\mathcal{H} = \{\mu \in \mathcal{M}([0, 1]); \mu = l\nu, \quad 0 < l < u'(1/2M), \quad l \text{ continuous on } [0, 1]\}.$$

The following two lemmas, which proofs are postponed to the Appendix, show that \mathcal{H} is a dense subset of the exposed points of Λ^* , which concludes the proof of Theorem 2.2.

Lemma 3.5 *Let $\mu = l\nu$ be in \mathcal{H} . There exists h_l in $\mathcal{C}([0, 1])$ such that*

$$\begin{aligned} \forall (t, \theta) \in [0, 1] \times \mathbb{T} \quad h_l(t)f(\theta) &< 1/2 \\ \forall \xi \in \mathcal{M}([0, 1]) \quad \Lambda^*(\mu) - \Lambda^*(\xi) &< (\mu - \xi)(h_l) \end{aligned} \quad (15)$$

Furthermore, there exists $\gamma > 1$ such that $\Lambda(\gamma l) < +\infty$.

Hence μ is an exposed point of Λ^* with exposing hyperplane h_l .

Lemma 3.6 *Let μ be in $\mathcal{M}([0, 1])$ such that $\Lambda^*(\mu) < +\infty$. There exists a sequence $(\mu_n) \in \mathcal{H}$ such that $\mu_n \Rightarrow \mu$ and $\lim_{n \rightarrow +\infty} \Lambda^*(\mu_n) = \Lambda^*(\mu)$.*

3.3 Proof of Theorem 2.4:

The n.c.g.f. of Y_n is given for any h in $\mathcal{C}[m, M]$ by

$$\begin{aligned} \Lambda_n(h) &= a_n \log E(\exp \left\{ \sqrt{\frac{n}{a_n}} (\langle \nu_n, h \rangle - E(\langle \nu_n, h \rangle)) \right\}) \\ &= -\frac{a_n}{2} \sum_{k=1}^n \log \left(1 - \frac{2}{\sqrt{na_n}} \lambda_{k,n} \right) + \frac{2}{\sqrt{na_n}} \lambda_{k,n} \end{aligned}$$

We recall that $\{\lambda_{k,n}\}$ are the eigenvalues of the matrix $T_n(f)^{1/2} \Delta_h T_n(f)^{1/2}$. We can assert

$$\Lambda_n(h) = \frac{1}{n} \sum_{k=1}^n \lambda_{k,n}^2 + O \left(\frac{1}{n\sqrt{na_n}} \sum_{k=1}^n |\lambda_{k,n}|^3 \right).$$

From the convergence (10), Therefore

$$\lim_{n \rightarrow +\infty} \Lambda_n(h) = \Lambda = \bar{f}^2 \int_{[0,1]} h(x)^2 dx \quad (16)$$

This function is defined on all $\mathcal{C}[0, 1]$, then the rate function is the Legendre dual of Λ which is, from Rockafellar [18],

$$I(\mu) = \frac{\pi}{2\bar{f}^2} \int_{[0,1]} l(x)^2 dx,$$

where $d_\mu(t) = l(x) dx$.

4 Appendix

4.1 A Szegő Theorem for generalized Toeplitz matrices

In this paragraph we show a result on the distribution of eigenvalues of some kind of generalized Toeplitz matrices.

Suppose g is a real function defined on $[0, 1] \times \mathbb{T}$ such that for any $x \in [0, 1]$, $g(x, \cdot) \in L^1(\mathbb{T})$. Define

$$\hat{g}_k(x) = \frac{1}{2\pi} \int_{\mathbb{T}} g(x, \theta) e^{-ik\theta} d\theta,$$

and

$$T_n^{\text{gen}}(g)_{k,l} = \hat{g}_{l-k} \left(\frac{k}{n} \right). \quad (17)$$

Denote by

$$\|\hat{g}_k\|_{\infty} = \sup_{x \in [0,1]} |\hat{g}_k(x)|.$$

Theorem 4.1 *Under assumption*

$$M := \sum_k \|\hat{g}_k\|_{\infty} < \infty, \quad (18)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \text{tr}(T_n^{\text{gen}}(g))^p = \frac{1}{2\pi} \int_0^1 \int_{\mathbb{T}} g(x, \theta)^p d\theta dx. \quad (19)$$

Proof: This proof is analogous to the one of [10]. Let $\varepsilon > 0$ be fixed and $m \in \mathbb{N}$ chosen such that:

$$\sum_{|k| > m} \|\hat{g}_k\|_{\infty} < \varepsilon$$

Consider the trigonometric polynom of degree m :

$$g^m(x, \theta) = \sum_{k=-m}^m \hat{g}_k(x) e^{ik\theta} \quad (20)$$

Let $T_n^{\text{gen}}(g^m)$ be the generalized Topelitz matrix associated to g^m as in (17). Therefore

$$T_n^{\text{gen}}(g) = T_n^{\text{gen}}(g^m) + R$$

and the sum of the moduli of the elements of any row of R is less than ε . Hence the same is true for the eigenvalues of R i.e. for the eigenvalues of $T_n^{\text{gen}}(g) - T_n^{\text{gen}}(g^m)$. From the Weyl-Courant Lemma, we can therefore bound

$$|\lambda_{k,n} - \lambda_{k,n}^m| \leq \varepsilon,$$

where $\{\lambda_{k,n}\}_k$ and $\{\lambda_{k,n}^m\}_k$ are the eigenvalues of $T_n^{\text{gen}}(g)$ and $T_n^{\text{gen}}(g^m)$ respectively non-decreasingly ordered. From assumption (18),

$$|\lambda_{k,n}| \leq M, \quad |\lambda_{k,n}^m| \leq M.$$

Hence for any positive integer s

$$|(\lambda_{k,n})^s - (\lambda_{k,n}^m)^s| \leq \varepsilon s M^{s-1}.$$

We can bound similarly $|g(x, \theta)^s - g^m(x, \theta)^s|$ and therefore to show (19) it is enough to consider the polynomial g^m . We derive

$$\lim_{n \rightarrow \infty} \frac{1}{n} \text{tr} (T_n^{\text{gen}}(g^m))^p = \sum_{D_p} \sum_{j=1}^m \hat{g}_{l_1} \left(\frac{j+l_1}{n} \right) \hat{g}_{l_2} \left(\frac{j+l_1+l_2}{n} \right) \cdots \hat{g}_{l_p} \left(\frac{j}{n} \right),$$

where $D_p = \{(l_1, \dots, l_p) \in \mathbb{Z}^p; \sum l_i = 0\}$ and the second sum in the RHS above is on j such that $j + \sum_1^k l_i$ – for k from 1 to p – is in the range $1, \dots, n$, i.e. $sp \leq j \leq n - sp$. Therefore we have to suppress at most $2sp + 1$ terms. From classical results on Riemann sums,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{D_p} \sum_{j=1}^m \hat{g}_{l_1} \left(\frac{j+l_1}{n} \right) \hat{g}_{l_2} \left(\frac{j+l_1+l_2}{n} \right) \cdots \hat{g}_{l_p} \left(\frac{j}{n} \right) \\ = \sum_{D_p} \int_0^1 \hat{g}_{l_1}(x) \hat{g}_{l_2}(x) \cdots \hat{g}_{l_p}(x) dx \\ = \sum_{(l_1, \dots, l_p) \in \mathbb{Z}^p} \frac{1}{2\pi} \int_{\mathbb{T}} e^{i(l_1+l_2+\dots+l_p)\theta} d\theta \int_0^1 g_{l_1}(x) \hat{g}_{l_2}(x) \cdots \hat{g}_{l_p}(x) dx \\ = \frac{1}{2\pi} \int_0^1 \int_{\mathbb{T}} g(x, \theta)^p d\theta dx. \end{aligned}$$

4.2 Proof of Proposition 2.1

This lemma is a consequence of Theorem 5 of Rockafellar [18]. For the sake of clarity, we recall the framework of that paper. Let h be in $\mathcal{C}([m, M])$, and

$$\Lambda(h) = \int_{[m, M]} u(t, h(t)) d\nu(t),$$

where $u(t, x)$ defined on $[m, M] \times \mathbb{R} \rightarrow \mathbb{R}$ is a function convex in x , and ν a non-negative, σ -finite measure. For any μ in $\mathcal{M}([m, M])$ having, with respect to ν the Lebesgue decomposition $\mu = l\nu + \mu^\perp$, where $l \in \mathcal{C}([m, M])$, and μ^\perp is the singular part, then

$$\Lambda^*(\mu) = \int_{[m, M]} u^*(t, l(t)) d\nu(t) + \int_{[m, M]} r(u^*(t, \cdot); d\mu^\perp/d\eta(t)) d\eta(t) \quad (21)$$

where η is any nonnegative measure of $\mathcal{M}([m, M])$ with respect to which μ^\perp is absolutely continuous, and $u^*(t, \cdot)$ is the dual function of $u(t, \cdot)$:

$$\forall t, \quad u^*(t, y) = \sup_{x \in \mathbb{R}} \{xy - u(t, x)\}.$$

Applying the result of (21) to $u(t, x) = -(1/t) \log(1 - 2tx)$, we have the formula of Proposition 2.1

4.3 Proof of Lemma 3.2

From Proposition V 1.8 and Theorem X 1.1 of Bhatia [1], since $T_n(f)$ is an hermitian positive matrix,

$$\|T_n(f)^{1/2}\Delta_h T_n(f)^{1/2}\| \leq \|T_n(f)\| \|\Delta_h\| \quad (22)$$

From Grenander and Szegö ([9] p.64)

$$\|T_n(f)\| \leq \|f\|_\infty.$$

In addition,

$$\|\Delta_h\| \leq \sup_k \sum_s |(\Delta_h)_{ks}| \leq \|h\|_\infty \quad (23)$$

Getting together inequalities (22) and (23), we get the result.

4.4 Proof of Lemma 3.3

This lemma is a direct consequence of Theorem 4.1 above, for both random measures.

4.5 Proof of Lemma 3.4

From the definition of Λ^* , for any $\delta > 0$, there exists h_δ in $\mathcal{C}([0, 1])$ such that inequality (14) holds. In case we only have

$$\forall (t, \theta) \in [0, 1] \times \mathbb{T} \quad h_\delta(t)f(\theta) \leq \frac{1}{2},$$

we choose h_ε with $\varepsilon > 0$ such that

$$\int_{[0,1]} h_\varepsilon(t) d\mu(t) - \Lambda(h_\varepsilon) \geq \Lambda_\delta^*(\mu) - \varepsilon.$$

(this is possible from the continuity of Λ in a neighborhood of h_δ)

Then (14) holds with another δ . From assumption on f , $f > 0$, then $h_\varepsilon f < 1/2$.

4.6 Proof of Lemma 3.5

For all $0 < y < 1/u'(1/2M)$, there exists a unique x_y in $(-\infty, 1/2M)$ such that $y = u'(x_y)$. For such a pair (y, x_y) ,

$$u^*(y) = yx_y - u(x_y).$$

Since u' is strictly increasing and continuous, u^* is strictly convex on $0 < y < u'(1/2M)$. For such an y and $z > 0$, $z \neq y$,

$$u^*(y) - u^*(z) < (y - z)x_y \quad (24)$$

(then y is an exposed point of u^* with exposing hyperplane x_y) If $\mu = l\nu$ and $\xi = \tilde{l}\nu + \xi^\perp$. We apply inequality (24) with $y = l(t)$ and $z = \tilde{l}(t)$, and then we integrate over $[0, 1]$ against ν . We obtain the inequality (15) with $h_l(t) = x_{l(t)}$.

4.7 Proof of Lemma 3.6

Following the sketch of proof of [8], we proceed in 4 steps. Assume $u'(1/2M) = +\infty$.

Step 1: Let $\mu = l\nu + \mu^\perp$ be in $\mathcal{M}([0, 1])$ such that $\Lambda^*(\mu) < \infty$ with l continuous and $l \in (0, u'(\frac{1}{2M}))$, and such that μ^\perp is in $L^1([0, 1])$. Since ν has full support on $[0, 1]$, there exists a sequence of continuous positive functions on $[0, 1]$ such that $h_n d\nu \Rightarrow \mu^\perp$. From the lower semi-continuity of Λ^* ,

$$\liminf_{n \rightarrow +\infty} \Lambda^*((l + h_n)\nu) \geq \Lambda^*(\mu).$$

Since u^* is a convex function, from Rockafellar (see [17]), for any $y > 0$ and $z \geq 0$,

$$u^*(y + z) \leq u^*(y) + \frac{z}{2M}.$$

Therefore

$$\Lambda^*((l + \tilde{l})\nu) \leq \Lambda^*(l\nu) + \frac{1}{2M} \int \tilde{l}(t) d\nu(t) \quad (25)$$

From inequality above,

$$\Lambda^*((l + h_n)\nu) \leq \Lambda^*(l\nu) + \frac{1}{2M} \int_{[0,1]} h_n d\nu$$

And then

$$\liminf_{n \rightarrow +\infty} \Lambda^*((l + h_n)\nu) \leq \Lambda^*(\mu).$$

We now show that the Lemma is true if $\mu = l\nu$ with l ν -a.s. in $(0, u'(\frac{1}{2M}))$ and integrable.

Step 2

We prove the result for $\mu = l\nu$ assuming that l is in $(0, u'(\frac{1}{2M}))$ integrable and that for some $\epsilon > 0$, $l > \epsilon$ ν -a.s. There exists a sequence (l_n) of continuous positive functions such that l_n converges both in $L^1(\nu)$ norm and ν -a.s. to l and $l_n > \epsilon/2$. Since on $(\epsilon/2, u'(\frac{1}{2M}))$ the function u^* is Lipschitzian, the lemma holds.

Step 3

Define $l_\epsilon := l\mathbb{1}_{l>\epsilon} + \epsilon\mathbb{1}_{l\leq\epsilon}$. Apply second step and inequality (25) noticing that l_ϵ converges in $L^1(\nu)$ to l and that $l_\epsilon \geq l$.

Step 4

For $\mu = l\nu + \eta$, combine first and third step.

If $u'(1/2M) < +\infty$, we have to modify the second and third step, introducing an additional truncation at level $u'(1/2M) - \epsilon$.

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