Sample Path Large Deviations for Squares of Stationary Gaussian Processes

Marguerite Zani

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Abstract

In this paper, we show large deviations for random step functions of type

$$Z_n(t) = \frac{1}{n} \sum_{k=1}^{[nt]} X_k^2 \,,$$

where $\{X_k\}_k$ is a stationary Gaussian process. We deal with the associated random measures $\nu_n = \frac{1}{n} \sum_{k=1}^{n} X_k^2 \delta_{k/n}$. The proofs require a Szegö theorem for generalized Toeplitz matrices, which is presented in the Appendix and is analogous to a result of Kac, Murdoch and Szegö [10]. We also study the polygonal line built on $Z_n(t)$ and show moderate deviations for both random families.

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1 Introduction

The aim of this paper is to provide a large deviations principle (LDP) for random functions of type

$$Z_n(t) = \frac{1}{n} \sum_{k=1}^{[nt]} X_k^2 \,, \tag{1}$$

and the associated polygonal line

$$\tilde{Z}_n(t) = Z_n(t) + \Big(t - \frac{[nt]}{n}\Big) X_{[nt]+1}^2 \,, \tag{2}$$

where $\{X_n\}_n$ is a stationary Gaussian process having spectral density f defined on the torus $\mathbb{T} =]-\pi, \pi]$. We assume f is continuous positive on $\mathbb{T}$.

Large deviations for random measures date back to Sanov [19] who showed a LDP for the family of empirical measures

$$\frac{1}{n} \sum_{i=1}^{n} \delta_{X_i} \,, \tag{3}$$

*Laboratoire d'Analyse et de Mathématiques Appliquées, CNRS UMR 8050, Université Paris Est Créteil, 61 av du Gal de Gaulle, 94010, Créteil, France. e-mail: zani@u-pec.fr

where X_i are i.i.d. random variables.
Then, the first results on large deviations for random paths were given by Borovkov [2] and Varadhan [20]. In [2], Borovkov provides a LDP for the random polygonal line joining the points $(\frac{k}{n}, \frac{S_k}{x})$ where $S_k = \sum_{i=1}^k X_i$ and $x = x(n)$ is in the range

$$\limsup_{n\to\infty} \frac{x}{n} < \infty\,, \quad \lim_{n\to\infty} \frac{x}{\sqrt{n\ln n}} = \infty \tag{4}$$

He also showed large deviations for the paths $\eta(nt)/x$ where $0 \leq t \leq 1$ and η is a separable process with independent increments. The large deviations are given in the spaces $\mathcal{C}([0,1])$ (the set of continuous functions on $[0,1]$) or $\mathcal{D}([0,1])$ (the set of cadlag functions on $[0,1]$) endowed with the uniform metric. Meanwhile, Varadhan [20] proved functional large deviations in $\mathcal{D}([0,1])$ for the random step functions

$$S_n(t) = \frac{1}{n}\sum_{i=1}^{[nt]} X_i \tag{5}$$

where $t \in [0,T]$ and $[nt]$ denotes the integer part of nt. Later on, Mogulskii ([13]) improved these results: he proved large deviations for the polygonal line $(\frac{k}{n}, \frac{S_k}{x})$ in the range

$$\limsup_{n\to\infty} \frac{x}{n} < \infty\,, \quad \lim_{n\to\infty} \frac{x}{\sqrt{n}} = \infty \tag{6}$$

in the space $\mathcal{D}([0,1])$ endowed with the Skorokhod metric. For more general results on large deviations for processes with independent increments, see also Lynch and Sethuraman [11], de Acosta [3] and Mogulskii [14].

The results of [2, 20, 13] concerning step functions and continuous random polygonal lines built on sums of i.i.d. random variables can be found in the books of Dupuis and Ellis [6] and Dembo and Zeitouni [5].

In our paper, to derive the large deviations, we consider the distribution derivative of $t \to Z_n(t)$ and $t \to \tilde{Z}_n(t)$. Therefore we deal with the random measures ν_n and $\tilde{\nu}_n$ given by

$$\langle \nu_n, h\rangle = \frac{1}{n}\sum_{k=1}^n X_k^2 h(\frac{k}{n}) \tag{7}$$

and

$$\langle \tilde{\nu}_n, h\rangle = \sum_{k=1}^n X_k^2 \int_{(k-1)/n}^{k/n} h(s)ds\,, \tag{8}$$

for h in $\mathcal{C}([0,1])$. Let $\mathcal{M}([0,1])$ be the set of positive bouded measures on $[0,1]$ endowed with the weak topology. Therefore ν_n and $\tilde{\nu}_n$ are a.s. in $\mathcal{M}([0,1])$.

Analogous random measures have been investigated before by Dembo and Zeitouni [4], and Gamboa and Gassiat [7]. Previous works on LDP for this kind of random functions can be found in Gamboa, Rouault and Zani [8] and Perrin and Zani [16] for stationary Gaussian processes, and in Najim [15] and Maïda, Najim and Péché [12] for i.i.d. sequences. We provide here a functional LDP for $\{\nu_n\}$ and $\{\tilde{\nu}_n\}$, and derive the associated LDP for $\{Z_n\}$ and $\{\tilde{Z}_n\}$. We also prove moderate deviations. The central limit theorem is known. Although part of this work was already presented in [21] the present work provide a full version with proofs and some extensions.

The remaining of the paper is organized as follows. We present in Section 2 the large and moderate deviations results. Section 3 is devoted to the proofs of Theorems. Deriving the LD result, we needed a Szegö type theorem for generalized Toeplitz matrices. This precise result is unknown to our knowledge and despite a very similar result has been shown in Kac Murdoch and Szego (see [10] and [9]), for seek of completenes we prove it in the Appendix. The remaining of the Appendix gather the proofs of technical lemmas.

2 Large and moderate deviations

For any h in $\mathcal{C}([0,1])$, define

$$\Lambda(h) = \begin{cases} -\dfrac{1}{4\pi}\displaystyle\int_{[0,1]}\int_{\mathbb{T}} \log(1-2h(t)f(\theta))\,d\theta\,dt & \text{if } \forall (t,\theta)\in[0,1]\times\mathbb{T},\ h(t)f(\theta)<1/2 \\ +\infty \quad \text{otherwise} \end{cases}$$

Let Λ^* be the Legendre dual of Λ. From Rockafellar [18], we can detail this dual function as following:

Proposition 2.1 *Let ν be the measure in $\mathcal{M}([0,1])$ defined for any h in $\mathcal{C}([0,1])$ by*

$$\langle \nu, h\rangle = \frac{1}{2\pi}\int_{\mathbb{T}} f(\theta)d\theta \int_{[0,1]} h(x)dx\,.$$

Let $\mu\in\mathcal{M}([0,1])$ having the following Lebesgue decomposition with respect to ν: $\mu = l\nu + \mu^{\perp}$ where $l\in\mathcal{C}([0,1])$ and $\mu^{\perp}$ is the singular part. Then

$$\Lambda^*(\mu) = \int_{[0,1]} u^*(l(t))\,\nu(dt) + \int_{[0,1]} \frac{\mu^{\perp}(dt)}{2M}\,,$$

where

$$u(x) = -\frac{1}{4\pi}\int_{\mathbb{T}} \log(1-2xf(\theta))\,d\theta\,,$$

and

$$M = \text{esssup} f\,.$$

The function u is $\mathcal{C}^2$ on $(-\infty, 1/2M)$, and

$$u'(x) = \frac{1}{2\pi}\int_{\mathbb{T}} \frac{f(\theta)}{1-2xf(\theta)}\,d\theta$$

$$u''(x) = \frac{1}{\pi}\int_{\mathbb{T}} \frac{f(\theta)^2}{(1-2xf(\theta))^2}\,d\theta > 0$$

Hence u' is strictly increasing, and $\lim_{x\to-\infty} u'(x) = 0$. On the other hand, we denote $u'(1/2M) := \lim_{x\to+\infty} u'(x) \leq +\infty$ (e.g. if $f \in \mathcal{C}^2$, $u'(1/2M) = +\infty$). The recession function (see Theorem 8.5 of [18]) is $r(u^*; y) = y/2M$.

2.1 Large Deviations

We can now state the LDP result:

Theorem 2.2 *The families $\{\nu_n\}_{n\in\mathbb{N}}$ and $\{\tilde{\nu}_n\}_{n\in\mathbb{N}}$ satisfy a LDP in $\mathcal{M}([0,1])$ with speed n and rate function Λ^*.*

We can carry the previous LDP to the random functions Z_n and $\tilde{Z}_n$. Following Lynch and Sethuraman [11] and de Acosta [3], we introduce some notations. Let $D([0,1],\mathbb{R})$ be the space of cadlag real functions on $[0,1]$, and $bv([0,1],\mathbb{R}) \subset D([0,1],\mathbb{R})$ the space of bounded variation functions. We can identify $bv([0,1],\mathbb{R})$ with $\mathcal{M}([0,1])$: to h in $bv([0,1],\mathbb{R})$ corresponds μ_h in $\mathcal{M}([0,1])$ characterized by $\mu_h([0,t]) = h(t)$. Up to this identification, the topological dual of $bv([0,1],\mathbb{R})$ is the set $\mathcal{C}([0,1])$. We endow $bv([0,1],\mathbb{R})$ with the w^*–topology written σ, i.e. the topology induced by $\mathcal{C}([0,1])$ on $\mathcal{M}([0,1])$. Now, let us define the rate function associated to Z_n and $\tilde{Z}_n$: let h be in $bv([0,1],\mathbb{R})$ and μ_h the associated measure in $\mathcal{M}([0,1])$; let $\mu_h = (\mu_h)_a + (\mu_h)_s$ be the Lebesgue decomposition of μ_h in absolutely continuous and singular terms with respect to the Lebesgue measure on $[0,1]$; let $h_a(t) = (\mu_h)_a([0,t])$ and $h_s(t) = (\mu_h)_s([0,t])$. Set

$$\Phi(h) = \int_{[0,1]} u^*(h'_a)(t)\,\nu(dt) + r h_s(1)\,,$$

where u^* and r are defined in Proposition 2.1.

Theorem 2.3 *The families of random functions $\{Z_n\}$ and $\{\tilde{Z}_n\}$ satisfy a LDP on the space $(bv([0,1],\mathbb{R}),\sigma)$, with speed n and rate function Φ.*

2.2 Moderate deviations

We can state also in this case a moderate deviation principle. We detail it for ν_n, it is the same for $\tilde{\nu}_n$. Let $\{a_n\}$ be a sequence of positive real numbers such that $a_n \to 0$ and $na_n \to +\infty$ when $n \to +\infty$. Set

$$Y_n = \sqrt{na_n}(\nu_n - E(\nu_n))\,.$$

We have the following moderate deviations principle

Theorem 2.4 *$\{Y_n\}$ satisfy a LDP with speed a_n^{-1} and good rate function defined, for all $\mu \in \mathcal{M}([0,1])$ by*

$$I(\mu) = \begin{cases} \dfrac{\pi}{2\bar{f^2}} \displaystyle\int_{[0,1]} l(x)^2\,dx & \text{if} \quad \mu(dx) = l(x)\,dx \\ +\infty \quad \text{otherwise}\,, \end{cases}$$

where

$$\bar{f^2} = \frac{1}{2\pi}\int_{\mathbb{T}} f^2\,.$$

2.3 Generalizations

The previous results can be generalized to some other random functions.

2.3.1 Weighted random variables

Assume g is a continuous function on $[0,1]$ and define

$$W_n = \frac{1}{n}\sum_{k=1}^{[nt]} g\left(\frac{k}{n}\right) X_k^2\,, \tag{9}$$

For any h in $\mathcal{C}([0,1])$, define

$$\Lambda(h) = \begin{cases} -\dfrac{1}{4\pi}\displaystyle\int_{[0,1]}\int_{\mathbb{T}} \log(1-2h(t)g(t)f(\theta))\,d\theta\,dt & \text{if } \forall (t,\theta)\in[0,1]\times\mathbb{T},\ h(t)g(t)f(\theta)<1/2 \\ +\infty \quad \text{otherwise} \end{cases}$$

The previous large deviations results apply with rate function Λ^*.

2.3.2 Quadratic forms built on the stationary process

We define

$$m = \text{essinf} f$$

and assume $m > 0$. Let F be a continuous positive function on $[m, M]$. Let O be an orthonormal matrix such that $O^* T_n(f) O$ is the diagonal matrix whose i–th diagonal element is $\mu_{i,n}$ the i–th eigenvalue of $T_n(f)$. Define

$$F(T_n(f)) = O D_f O^*$$

where D_f is the diagonal matrix whose i-th element is $F(\mu_{i,n})$. Define the following quadratic form

$$W_n = \frac{1}{n} X^* F(T_n(f)) X = \frac{1}{n} Y^* Y\,,$$

where $Y = (Y_1, \cdots Y_n)$ is the vector defined by

$$Y = F(T_n(f))^{1/2} X\,.$$

In this case, W_n satisfies a LDP and moderate deviations theorem with rate function Λ^* where for any h in $\mathcal{C}([0,1])$

$$\Lambda(h) = \begin{cases} -\dfrac{1}{4\pi}\displaystyle\int_{[0,1]}\int_{\mathbb{T}} \log[1-2h(t)f(\theta)F[f(\theta)]]\,d\theta\,dt & \text{if } \forall (t,\theta)\in[0,1]\times\mathbb{T},\ h(t)f(\theta)<1/2 \\ +\infty \quad \text{otherwise} \end{cases}$$

.

3 Proof of the large and moderate deviations

We first give some asymptotic properties for the families $\{\nu_n\}_n$ and $\{\tilde{\nu}_n\}_n$.

3.1 Weak convergence of ν_n and $\{\tilde{\nu}_n\}_n$

Lemma 3.1 *Let h be in $\mathcal{C}([0,1])$.*

$$\langle \nu_n, h\rangle \to \langle \nu, h\rangle \quad \textit{in probability} \quad \textit{as } n \to +\infty \tag{10}$$

and

$$\langle \tilde{\nu}_n, h\rangle \to \langle \nu, h\rangle \quad \textit{in probability} \quad \textit{as } n \to +\infty$$

where

$$\langle \nu, h\rangle = \bar{f} \int_{[0,1]} h(x)\, dx\,,$$

and

$$\bar{f} = \frac{1}{2\pi} \int_{\mathbb{T}} f(\theta)\, d\theta\,.$$

Proof :

Let h be in $\mathcal{C}([0,1])$, and consider

$$\langle \nu_n, h\rangle = \frac{1}{n} \sum_{k=1}^{n} X_k^2 h(\frac{k}{n})\,.$$

Set X the Gaussian vector $(X_1, X_2, \cdots, X_n)$ and Δ_h the diagonal matrix

$$\begin{pmatrix} h(\frac{1}{n}) & 0 & 0 & 0 \\ 0 & h(\frac{2}{n}) & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & h(1) \end{pmatrix}$$

Therefore we can write

$$\langle \nu_n, h\rangle = \frac{1}{n} X^* \,\Delta_h\, X\,,$$

where X^* denote the transpose of X. By an orthonormal change of basis,

$$\langle \nu_n, h\rangle = \frac{1}{n} U_n^*\, T_n(f)^{1/2} \Delta_h T_n(f)^{1/2}\, U_n\,,$$

where U_n is a standard normal vector and $T_n(f)$ the order-n Toeplitz matrix associated to f. Therefore

$$\langle \nu_n, h\rangle = \frac{1}{n} \sum_{k=1}^{n} \lambda_{k,n} Z_{k,n} \tag{11}$$

where $\{Z_{k,n}\}$ are independent $\chi^2(1)$-distributed random variables, and $\{\lambda_{k,n}\}$ are the eigenvalues of $T_n(f)^{1/2}\Delta_h T_n(f)^{1/2}$.
We can write as well

$$\langle \tilde{\nu}_n, h\rangle = \frac{1}{n}\sum_{k=1}^{n} \tilde{\lambda}_{k,n} Z_{k,n} \tag{12}$$

where $\{Z_{k,n}\}$ are independent $\chi^2(1)$-distributed random variables, and $\{\tilde{\lambda}_{k,n}\}$ are the eigenvalues of $T_n(f)^{1/2}A_h T_n(f)^{1/2}$, and the matrix A_h is diagonal with k-th diagonal term

$$(A_h)_{k,k} = \int_{(k-1)/n}^{k/n} h(s)\,ds\,.$$

We have the two following results on the distributions $\{\lambda_{k,n}\}$ and $\{\tilde{\lambda}_{k,n}\}$, which proofs are postponed to the Appendix.

Lemma 3.2 *The sequences $\{\lambda_{k,n}\}$ and $\{\tilde{\lambda}_{k,n}\}$ are bounded as follows:*

$$\forall n \in \mathbb{N}\,,\ \forall\, 1 \le k \le n\,, \qquad \begin{aligned} |\lambda_{k,n}| &\le \|h\|_\infty \|f\|_\infty \\ |\tilde{\lambda}_{k,n}| &\le \|h\|_\infty \|f\|_\infty \end{aligned}$$

Lemma 3.3 *For any p in $\mathbb{N}$, $p \ge 1$,*

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} \lambda_{k,n}^p = \frac{1}{2\pi}\int_{[0,1]}\int_{\mathbb{T}} (h(t)f(\theta))^p\, dt d\theta\,.$$

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} (\tilde{\lambda}_{k,n})^p = \frac{1}{2\pi}\int_{[0,1]}\int_{\mathbb{T}} (h(t)f(\theta))^p\, dt d\theta\,.$$

With the above lemma,

$$\lim_{n\to+\infty} E(\langle \nu_n, h\rangle) = \langle \nu, h\rangle\,.$$

Moreover,

$$\lim_{n\to+\infty} n\mathrm{Var}\langle \nu_n, h\rangle = \frac{2}{n}\sum_{k=1}^{n}\lambda_{k,n}^2 = \frac{1}{\pi}\int_{[0,1]}\int_{\mathbb{T}} (h(t)f(\theta))^2 dt d\theta\,.$$

We do as well for $\tilde{\nu}_n$, and it ends the proof of lemma 3.1.

3.2 Proof of Theorem 2.2:

The proof follows exactly the scheme [8]. We detail here for ν_n, it is similar for $\tilde{\nu}_n$. With the decomposition (11), we get the n.c.g.f. of ν_n: for any $h \in \mathcal{C}([0,1])$,

$$\Lambda_n(h) = \frac{1}{n}\log E(\exp\{n\langle \nu_n, h\rangle\}) = \begin{cases} -\dfrac{1}{2n}\displaystyle\sum_{k=1}^{n}\log(1-2\lambda_{k,n}) & \text{if } \forall k,\ \lambda_{k,n} < 1/2 \\ +\infty & \text{otherwise} \end{cases} \tag{13}$$

From Lemma 3.3, we can determine the limit of Λ_n in two cases:

- if $\forall (t,\theta) \in [0,1]\times\mathbb{T} \quad h(t)f(\theta) < 1/2$, then

$$\lim_{n\to+\infty} \Lambda_n(h) = -\frac{1}{4\pi}\int_{[0,1]}\int_{\mathbb{T}} \log(1-2h(t)f(\theta))\,d\theta\,dt = \Lambda(h)\,.$$

- if $\exists (t,\theta) \in [0,1]\times\mathbb{T}; \quad h(t)f(\theta) > 1/2$, then for n large enough, $\Lambda_n(h) = +\infty$ and

$$\lim_{n\to+\infty} \Lambda_n(h) = +\infty = \Lambda(h)\,.$$

These two cases do not cover the whole set $\mathcal{C}([0,1])$. Nevertheless, this will be sufficient for the LDP, since they contain a dense subset of exposing hyperplanes of Λ^*.

Upper bound
From Theorem 4.5.3 b) of [5], and the following lemma, which proof is postponed to the Appendix, the upper bound holds for compact sets.

Lemma 3.4 *For any $\delta > 0$ and μ in $\mathcal{M}([0,1])$, there exists h_δ in $\mathcal{C}([0,1])$ such that:*

$$\forall (t,\theta),\ h_\delta(t)f(\theta) < 1/2$$

$$\int_{[0,1]} h_\delta(t)\,d\mu(t) - \Lambda(h_\delta) \geq \Lambda^*_\delta(\mu) \tag{14}$$

where

$$\Lambda^*_\delta(\mu) = \min\{\Lambda^*(\mu)-\delta, \frac{1}{\delta}\}\,.$$

Exponential tightness
Remark that for a real number a,

$$\{\sup_{\|h\|_\infty\leq 1} \langle \nu_n, h\rangle \geq a\} \subset \{\nu_n(1)\geq a\}\,.$$

If $M = \mathrm{esssup}_\theta f(\theta)$, for any $y < 1/2M$,

$$\limsup_n \frac{1}{n}\log P(\nu_n(1)\geq a) \leq -ya - \frac{1}{4\pi}\int_{[0,1]}\int_{\mathbb{T}} \log(1-2yf(\theta))\,d\theta\,,$$

and

$$\lim_{a\to+\infty}\limsup_n \frac{1}{n}\log P(\nu_n(1)\geq a) = -\infty\,.$$

Hence the sequence (ν_n) is exponentially tight, and the upper bound holds for any closed set of $\mathcal{M}([0,1])$.

Lower bound
We study the set of exposed points of Λ^* (see [5]). Let

$$\mathcal{H} = \{\mu\in\mathcal{M}([0,1]);\ \mu = l\nu,\ 0 < l < u'(1/2M),\ l \text{ continuous on } [0,1]\}\,.$$

The following two lemmas, which proofs are postponed to the Appendix, show that that $\mathcal{H}$ is a dense subset of the exposed points of Λ^*, which concludes the proof of Theorem 2.2.

Lemma 3.5 *Let $\mu = l\nu$ be in $\mathcal{H}$. There exists h_l in $\mathcal{C}([0,1])$ such that*

$$\begin{aligned} \forall(t,\theta) \in [0,1]\times\mathbb{T} \quad h_l(t)f(\theta) < 1/2 \\ \forall\xi \in \mathcal{M}([0,1]) \quad \Lambda^*(\mu) - \Lambda^*(\xi) < (\mu-\xi)(h_l) \end{aligned} \tag{15}$$

Furthermore, there exists $\gamma > 1$ such that $\Lambda(\gamma l) < +\infty$.

Hence μ is an exposed point of Λ^* with exposing hyperplane h_l.

Lemma 3.6 *Let μ be in $\mathcal{M}([0,1])$ such that $\Lambda^*(\mu) < +\infty$. There exists a sequence $(\mu_n) \in \mathcal{H}$ such that $\mu_n \Rightarrow \mu$ and $\lim_{n\to+\infty}\Lambda^*(\mu_n) = \Lambda^*(\mu)$.*

3.3 Proof of Theorem 2.4:

The n.c.g.f. of Y_n is given for any h in $\mathcal{C}[m,M]$ by

$$\begin{aligned} \Lambda_n(h) &= a_n \log E(\exp\left\{\sqrt{\frac{n}{a_n}}(\langle\nu_n,h\rangle - E(\langle\nu_n,h\rangle))\right\}) \\ &= -\frac{a_n}{2}\sum_{k=1}^{n}\log\left(1-\frac{2}{\sqrt{na_n}}\lambda_{k,n}\right) + \frac{2}{\sqrt{na_n}}\lambda_{k,n} \end{aligned}$$

We recall that $\{\lambda_{k,n}\}$ are the eigenvalues of the matrix $T_n(f)^{1/2}\Delta_h T_n(f)^{1/2}$. We can assert

$$\Lambda_n(h) = \frac{1}{n}\sum_{k=1}^{n}\lambda_{k,n}^2 + O\left(\frac{1}{n\sqrt{na_n}}\sum_{k=1}^{n}|\lambda_{k,n}|^3\right).$$

From the convergence (10), Therefore

$$\lim_{n\to+\infty}\Lambda_n(h) = \Lambda = \bar{f}^2\int_{[0,1]} h(x)^2\,dx \tag{16}$$

This function is defined on all $\mathcal{C}[0,1]$, then the rate function is the Legendre dual of Λ which is, from Rockafellar [18],

$$I(\mu) = \frac{\pi}{2\bar{f}^2}\int_{[0,1]} l(x)^2\,dx,$$

where $d_\mu(t) = l(x)\,dx$.

4 Appendix

4.1 A Szegö Theorem for generalized Toeplitz matrices

In this paragraph we show a result on the distribution of eigenvalues of some kind of generalized Toeplitz matrices.

Suppose g is a real function defined on $[0,1] \times \mathbb{T}$ such that for any $x \in [0,1]$, $g(x,\cdot) \in L^1(\mathbb{T})$. Define

$$\hat{g}_k(x) = \frac{1}{2\pi} \int_{\mathbb{T}} g(x,\theta) e^{-ik\theta} d\theta \,,$$

and

$$T_n^{\text{gen}}(g)_{k,l} = \hat{g}_{l-k}\left(\frac{k}{n}\right) . \tag{17}$$

Denote by

$$\|\hat{g}_k\|_\infty = \sup_{x\in[0,1]} |\hat{g}_k(x)| \,.$$

Theorem 4.1 *Under assumption*

$$M := \sum_k \|\hat{g}_k\|_\infty < \infty \,, \tag{18}$$

$$\lim_{n\to\infty} \frac{1}{n} tr\,(T_n^{gen}(g))^p = \frac{1}{2\pi} \int_0^1 \int_{\mathbb{T}} g(x,\theta)^p d\theta dx \,. \tag{19}$$

Proof: This proof is analogous to the one of [10]. Let $\varepsilon > 0$ be fixed and $m \in \mathbb{N}$ chosen such that:

$$\sum_{|k|>m} \|\hat{g}_k\|_\infty < \varepsilon$$

Consider the trigonometric polynom of degree m:

$$g^m(x,\theta) = \sum_{k=-m}^{m} \hat{g}_k(x) e^{ik\theta} \tag{20}$$

Let $T_n^{\text{gen}}(g^m)$ be the generalized Topelitz matrix associated to g^m as in (17). Therefore

$$T_n^{\text{gen}}(g) = T_n^{\text{gen}}(g^m) + R$$

and the sum of the moduli of the elements of any row of R is less than ε. Hence the same is true for the eigenvalues of R i.e. for the eigenvalues of $T_n^{\text{gen}}(g) - T_n^{\text{gen}}(g^m)$. From the Weyl-Courant Lemma, we can therefore bound

$$|\lambda_{k,n} - \lambda_{k,n}^m| \leq \varepsilon \,,$$

where $\{\lambda_{k,n}\}_k$ and $\{\lambda_{k,n}^m\}_k$ are the eigenvalues of $T_n^{\text{gen}}(g)$ and $T_n^{\text{gen}}(g^m)$ respectively non-decreasingly ordered. From assumption (18),

$$|\lambda_{k,n}| \leq M \,, \quad |\lambda_{k,n}^m| \leq M \,.$$

Hence for any positive integer s

$$|(\lambda_{k,n})^s - (\lambda_{k,n}^m)^s| \leq \varepsilon s M^{s-1} \,.$$

We can bound similarly $|g(x,\theta)^s - g^m(x,\theta)^s|$ and therefore to show (19) it is enough to consider the polynomial g^m. We derive

$$\lim_{n\to\infty} \frac{1}{n} \mathrm{tr}\left(T_n^{\mathrm{gen}}(g^m)\right)^p = \sum_{D_p} \sum_{j=1}^{m} \hat{g}_{l_1}\left(\frac{j+l_1}{n}\right) \hat{g}_{l_2}\left(\frac{j+l_1+l_2}{n}\right) \cdots \hat{g}_{l_p}\left(\frac{j}{n}\right),$$

where $D_p = \{(l_1, \cdots l_p) \in \mathbb{Z}^p; \sum l_i = 0\}$ and the second sum in the RHS above is on j such that $j + \sum_1^k l_i$ – for k from 1 to p – is in the range $1, \ldots, n$, i.e. $sp \le j \le n - sp$. Therefore we have to suppress at most $2sp+1$ terms. From classical results on Riemann sums,

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \sum_{D_p} \sum_{j=1}^{m} \hat{g}_{l_1}\left(\frac{j+l_1}{n}\right) \hat{g}_{l_2}\left(\frac{j+l_1+l_2}{n}\right) \cdots \hat{g}_{l_p}\left(\frac{j}{n}\right) \\
= \sum_{D_p} \int_0^1 \hat{g}_{l_1}(x)\hat{g}_{l_2}(x)\cdots\hat{g}_{l_p}(x)dx \\
= \sum_{(l_1,\cdots l_p)\in\mathbb{Z}^p} \frac{1}{2\pi}\int_{\mathbb{T}} e^{i(l_1+l_2+\cdots l_p)}d\theta \int_0^1 g_{l_1}(x)\hat{g}_{l_2}(x)\cdots\hat{g}_{l_p}(x)dx \\
= \frac{1}{2\pi}\int_0^1\int_{\mathbb{T}} g(x,\theta)^p d\theta dx\,.
\end{aligned}$$

4.2 Proof of Proposition 2.1

This lemma is a consequence of Theorem 5 of Rockafellar [18]. For the sake of clarity, we recall the framework of that paper. Let h be in $\mathcal{C}([m,M])$, and

$$\Lambda(h) = \int_{[m,M]} u(t, h(t))\, d\nu(t)\,,$$

where $u(t,x)$ defined on $[m,M]\times\mathbb{R}\to\mathbb{R}$ is a function convex in x, and ν a non–negative, σ–finite measure. For any μ in $\mathcal{M}([m,M])$ having, with respect to ν the Lebesgue decomposition $\mu = l\nu + \mu^\perp$, where $l\in\mathcal{C}([m,M])$, and $\mu^\perp$ is the singular part, then

$$\Lambda^*(\mu) = \int_{[m,M]} u^*(t, l(t))\, d\nu(t) + \int_{[m,M]} r(u^*(t,\cdot); d\mu^\perp/d\eta(t))\, d\eta(t) \tag{21}$$

where η is any nonnegative measure of $\mathcal{M}([m,M])$ with respect to which $\mu^\perp$ is absolutely continuous, and $u^*(t,\cdot)$ is the dual function of $u(t,\cdot)$:

$$\forall t\,,\quad u^*(t,y) = \sup_{x\in\mathbb{R}}\{xy - u(t,x)\}\,.$$

Applying the result of (21) to $u(t,x) = -(1/t)\log(1-2tx)$, we have the formula of Proposition 2.1

4.3 Proof of Lemma 3.2

From Proposition V 1.8 and Theorem X 1.1 of Bhatia [1], since $T_n(f)$ is an hermitian positive matrix,

$$\|T_n(f)^{1/2}\Delta_h T_n(f)^{1/2}\| \leq \|T_n(f)\| \, \|\Delta_h\| \tag{22}$$

From Grenander and Szegö ([9] p.64)

$$\|T_n(f)\| \leq \|f\|_\infty \, .$$

In addition,

$$\|\Delta_h\| \leq \sup_k \sum_s |(\Delta_h)_{ks}| \leq \|h\|_\infty \tag{23}$$

Getting together inequalities (22) and (23), we get the result.

4.4 Proof of Lemma 3.3

This lemma is a direct consequence of Theorem 4.1 above, for both random measures.

4.5 Proof of Lemma 3.4

From the definition of Λ^*, for any $\delta > 0$, there exists h_δ in $\mathcal{C}([0,1])$ such that inequality (14) holds. In case we only have

$$\forall (t,\theta) \in [0,1] \times \mathbb{T} \quad h_\delta(t) f(\theta) \leq \frac{1}{2} \, ,$$

we choose h_ε with $\varepsilon > 0$ such that

$$\int_{[0,1]} h_\varepsilon(t) \, d\mu(t) - \Lambda(h_\varepsilon) \geq \Lambda^*_\delta(\mu) - \varepsilon \, .$$

(this is possible from the continuity of Λ in a neighborhood of h_δ)
Then (14) holds with another δ. From assumption on f, $f > 0$, then $h_\varepsilon f < 1/2$.

4.6 Proof of Lemma 3.5

For all $0 < y < 1/u'(1/2M)$, there exists a unique x_y in $(-\infty, 1/2M)$ such that $y = u'(x_y)$. For such a pair (y, x_y),

$$u^*(y) = y x_y - u(x_y) \, .$$

Since u' is strictly increasing and continuous, u^* is strictly convex on $0 < y < u'(1/2M)$. For such an y and $z > 0\,,\; z \neq y$,

$$u^*(y) - u^*(z) < (y - z)x_y \tag{24}$$

(then y is an exposed point of u^* with exposing hyperplane x_y) If $\mu = l\nu$ and $\xi = \tilde{l}\nu + \xi^{\perp}$. We apply inequality (24) with $y = l(t)$ and $z = \tilde{l}(t)$, and then we integrate over $[0,1]$ against ν. We obtain the inequality (15) with $h_l(t) = x_{l(t)}$.

4.7 Proof of Lemma 3.6

Following the sketch of proof of [8], we proceed in 4 steps. Assume $u'(1/2M) = +\infty$.

Step 1: Let $\mu = l\nu + \mu^{\perp}$ be in $\mathcal{M}([0,1])$ such that $\Lambda^*(\mu) < \infty$ with l continuous and $l \in (0, u'(\frac{1}{2M})$, and such that $\mu^{\perp}$ is in $L^1([0,1])$. Since ν has full support on $[0,1]$, there exists a sequence of continuous positive functions on $[0,1]$ such that $h_n d\nu \Rightarrow \mu^{\perp}$. From the lower semi-continuity of Λ^*,

$$\liminf_{n \to +\infty} \Lambda^*((l + h_n)\nu) \geq \Lambda^*(\mu)\,.$$

Since u^* is a convex function, from Rockafellar (see [17]), for any $y > 0$ and $z \geq 0$,

$$u^*(y + z) \leq u^*(y) + \frac{z}{2M}\,.$$

Therefore

$$\Lambda^*((l + \tilde{l})\nu) \leq \Lambda^*(l\nu) + \frac{1}{2M}\int \tilde{l}(t) d\nu(t) \tag{25}$$

From inequality above,

$$\Lambda^*((l + h_n)\nu) \leq \Lambda^*(l\nu) + \frac{1}{2M}\int_{0,1]} h_n\, d\nu$$

And then

$$\liminf_{n \to +\infty} \Lambda^*((l + h_n)\nu) \leq \Lambda^*(\mu)\,.$$

We now show that the Lemma is true if $\mu = l\,\nu$ with l ν-a.s. in $(0, u'(\frac{1}{2M})$ and integrable.

Step 2

We prove the result for $\mu = l\,\nu$ assuming that l is in $(0, u'(\frac{1}{2M})$ integrable and that for some $\epsilon > 0$, $l > \epsilon$ ν-a.s. There exists a sequence (l_n) of continuous positive functions such that l_n converges both in $L^1(\nu)$ norm and ν-a.s. to l and $l_n > \epsilon/2$. Since on $(\epsilon/2, u'(\frac{1}{2M})$ the function u^* is Lipschitzian, the lemma holds.

Step 3

Define $l_\epsilon := l1\!\!1_{l>\epsilon} + \epsilon 1\!\!1_{l\leq\epsilon}$. Apply second step and inequality (25) noticing that l_ϵ converges in $L^1(\nu)$ to l and that $l_\epsilon \geq l$.

Step 4

For $\mu = l\nu + \eta$, combine first and third step.

If $u'(1/2M) < +\infty$, we have to modify the second and third step, introducing an additional truncation at level $u'(1/2M) - \varepsilon$.

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