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# Sample path large deviations for squares of stationary Gaussian processes

Marguerite Zani \*

## Abstract

In this paper, we show large deviations for random step functions of type

$$Z_n(t) = \frac{1}{n} \sum_{k=1}^{[nt]} X_k^2,$$

where  $\{X_k\}_k$  is a stationary Gaussian process. We deal with the associated random measures  $\nu_n = \frac{1}{n} \sum_{k=1}^n X_k^2 \delta_{k/n}$ . The proofs require a Szegö theorem for generalized Toeplitz matrices, which is presented in the Appendix and is analogous to a result of Kac, Murdoch and Szegö [10]. We also study the polygonal line built on  $Z_n(t)$  and show moderate deviations for both random families.

*AMS classification:* primary: 60G15, 60F10, 47B35 secondary: 60G10, 60G17.

*Keywords:* Gaussian processes, Large deviations, Szegö theorem, Toeplitz matrices.

## 1 Introduction

The aim of this paper is to provide a large deviations principle (LDP) for random functions of type

$$Z_n(t) = \frac{1}{n} \sum_{k=1}^{[nt]} X_k^2, \quad (1)$$

and the associated polygonal line

$$\tilde{Z}_n(t) = Z_n(t) + \left(t - \frac{[nt]}{n}\right) X_{[nt]+1}^2, \quad (2)$$

where  $\{X_n\}_n$  is a stationary Gaussian process having spectral density  $f$  defined on the torus  $\mathbb{T} = ]-\pi, \pi]$ . We assume  $f$  is continuous positive on  $\mathbb{T}$ .

Large deviations for random measures date back to Sanov [19] who showed a LDP for the family of empirical measures

$$\frac{1}{n} \sum_{i=1}^n \delta_{X_i}, \quad (3)$$

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where  $X_i$  are i.i.d. random variables.

Then, the first results on large deviations for random paths were given by Borovkov [2] and Varadhan [20]. In [2], Borovkov provides a LDP for the random polygonal line joining the points  $(\frac{k}{n}, \frac{S_k}{x})$  where  $S_k = \sum_{i=1}^k X_i$  and  $x = x(n)$  is in the range

$$\limsup_{n \rightarrow \infty} \frac{x}{n} < \infty, \quad \lim_{n \rightarrow \infty} \frac{x}{\sqrt{n \ln n}} = \infty \quad (4)$$

He also showed large deviations for the paths  $\eta(nt)/x$  where  $0 \leq t \leq 1$  and  $\eta$  is a separable process with independent increments. The large deviations are given in the spaces  $\mathcal{C}([0, 1])$  ( the set of continuous functions on  $[0, 1]$ ) or  $\mathcal{D}([0, 1])$  ( the set of cadlag functions on  $[0, 1]$ ) endowed with the uniform metric. Meanwhile, Varadhan [20] proved functional large deviations in  $\mathcal{D}([0, 1])$  for the random step functions

$$S_n(t) = \frac{1}{n} \sum_{i=1}^{[nt]} X_i \quad (5)$$

where  $t \in [0, T]$  and  $[nt]$  denotes the integer part of  $nt$ . Later on, Mogulskii ([13]) improved these results: he proved large deviations for the polygonal line  $(\frac{k}{n}, \frac{S_k}{x})$  in the range

$$\limsup_{n \rightarrow \infty} \frac{x}{n} < \infty, \quad \lim_{n \rightarrow \infty} \frac{x}{\sqrt{n}} = \infty \quad (6)$$

in the space  $\mathcal{D}([0, 1])$  endowed with the Skorokhod metric. For more general results on large deviations for processes with independent increments, see also Lynch and Sethuraman [11], de Acosta [3] and Mogulskii [14].

The results of [2, 20, 13] concerning step functions and continuous random polygonal lines built on sums of i.i.d. random variables can be found in the books of Dupuis and Ellis [6] and Dembo and Zeitouni [5].

In our paper, to derive the large deviations, we consider the distribution derivative of  $t \rightarrow Z_n(t)$  and  $t \rightarrow \tilde{Z}_n(t)$ . Therefore we deal with the random measures  $\nu_n$  and  $\tilde{\nu}_n$  given by

$$\langle \nu_n, h \rangle = \frac{1}{n} \sum_{k=1}^n X_k^2 h\left(\frac{k}{n}\right) \quad (7)$$

and

$$\langle \tilde{\nu}_n, h \rangle = \sum_{k=1}^n X_k^2 \int_{(k-1)/n}^{k/n} h(s) ds, \quad (8)$$

for  $h$  in  $\mathcal{C}([0, 1])$ . Let  $\mathcal{M}([0, 1])$  be the set of positive bounded measures on  $[0, 1]$  endowed with the weak topology. Therefore  $\nu_n$  and  $\tilde{\nu}_n$  are a.s. in  $\mathcal{M}([0, 1])$ .

Analogous random measures have been investigated before by Dembo and Zeitouni [4], and Gamboa and Gassiat [7]. Previous works on LDP for this kind of random functions can be found in Gamboa, Rouault and Zani [8] and Perrin and Zani [16] for stationary Gaussian processes, and in Najim [15] and Maïda, Najim and P  ch   [12] for

i.i.d. sequences. We provide here a functional LDP for  $\{\nu_n\}$  and  $\{\tilde{\nu}_n\}$ , and derive the associated LDP for  $\{Z_n\}$  and  $\{\tilde{Z}_n\}$ . We also prove moderate deviations. The central limit theorem is known. Although part of this work was already presented in [21] the present work provide a full version with proofs and some extensions.

The remaining of the paper is organized as follows. We present in Section 2 the large and moderate deviations results. Section 3 is devoted to the proofs of Theorems. Deriving the LD result, we needed a Szegő type theorem for generalized Toeplitz matrices. This precise result is unknown to our knowledge and despite a very similar result has been shown in Kac Murdoch and Szego (see [10] and [9]), for seek of completeness we prove it in the Appendix. The remaining of the Appendix gather the proofs of technical lemmas.

## 2 Large and moderate deviations

For any  $h$  in  $\mathcal{C}([0, 1])$ , define

$$\Lambda(h) = \begin{cases} -\frac{1}{4\pi} \int_{[0,1]} \int_{\mathbb{T}} \log(1 - 2h(t)f(\theta)) d\theta dt & \text{if } \forall(t, \theta) \in [0, 1] \times \mathbb{T}, h(t)f(\theta) < 1/2 \\ +\infty & \text{otherwise} \end{cases}$$

Let  $\Lambda^*$  be the Legendre dual of  $\Lambda$ . From Rockafellar [18], we can detail this dual function as following:

**Proposition 2.1** *Let  $\nu$  be the measure in  $\mathcal{M}([0, 1])$  defined for any  $h$  in  $\mathcal{C}([0, 1])$  by*

$$\langle \nu, h \rangle = \frac{1}{2\pi} \int_{\mathbb{T}} f(\theta) d\theta \int_{[0,1]} h(x) dx.$$

*Let  $\mu \in \mathcal{M}([0, 1])$  having the following Lebesgue decomposition with respect to  $\nu$ :  $\mu = l\nu + \mu^\perp$  where  $l \in \mathcal{C}([0, 1])$  and  $\mu^\perp$  is the singular part. Then*

$$\Lambda^*(\mu) = \int_{[0,1]} u^*(l(t)) \nu(dt) + \int_{[0,1]} \frac{\mu^\perp(dt)}{2M},$$

where

$$u(x) = -\frac{1}{4\pi} \int_{\mathbb{T}} \log(1 - 2xf(\theta)) d\theta,$$

and

$$M = \text{esssup} f.$$

The function  $u$  is  $\mathcal{C}^2$  on  $(-\infty, 1/2M)$ , and

$$u'(x) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{f(\theta)}{1 - 2xf(\theta)} d\theta$$

$$u''(x) = \frac{1}{\pi} \int_{\mathbb{T}} \frac{f(\theta)^2}{(1 - 2xf(\theta))^2} d\theta > 0$$

Hence  $u'$  is strictly increasing, and  $\lim_{x \rightarrow -\infty} u'(x) = 0$ . On the other hand, we denote  $u'(1/2M) := \lim_{x \rightarrow +\infty} u'(x) \leq +\infty$  (e.g. if  $f \in \mathcal{C}^2$ ,  $u'(1/2M) = +\infty$ ). The recession function ( see Theorem 8.5 of [18]) is  $r(u^*; y) = y/2M$ .

## 2.1 Large Deviations

We can now state the LDP result:

**Theorem 2.2** *The families  $\{\nu_n\}_{n \in \mathbb{N}}$  and  $\{\tilde{\nu}_n\}_{n \in \mathbb{N}}$  satisfy a LDP in  $\mathcal{M}([0, 1])$  with speed  $n$  and rate function  $\Lambda^*$ .*

We can carry the previous LDP to the random functions  $Z_n$  and  $\tilde{Z}_n$ . Following Lynch and Sethuraman [11] and de Acosta [3], we introduce some notations. Let  $D([0, 1], \mathbb{R})$  be the space of cadlag real functions on  $[0, 1]$ , and  $bv([0, 1], \mathbb{R}) \subset D([0, 1], \mathbb{R})$  the space of bounded variation functions. We can identify  $bv([0, 1], \mathbb{R})$  with  $\mathcal{M}([0, 1])$ : to  $h$  in  $bv([0, 1], \mathbb{R})$  corresponds  $\mu_h$  in  $\mathcal{M}([0, 1])$  characterized by  $\mu_h([0, t]) = h(t)$ . Up to this identification, the topological dual of  $bv([0, 1], \mathbb{R})$  is the set  $\mathcal{C}([0, 1])$ . We endow  $bv([0, 1], \mathbb{R})$  with the  $w^*$ -topology written  $\sigma$ , i.e. the topology induced by  $\mathcal{C}([0, 1])$  on  $\mathcal{M}([0, 1])$ . Now, let us define the rate function associated to  $Z_n$  and  $\tilde{Z}_n$ : let  $h$  be in  $bv([0, 1], \mathbb{R})$  and  $\mu_h$  the associated measure in  $\mathcal{M}([0, 1])$ ; let  $\mu_h = (\mu_h)_a + (\mu_h)_s$  be the Lebesgue decomposition of  $\mu_h$  in absolutely continuous and singular terms with respect to the Lebesgue measure on  $[0, 1]$ ; let  $h_a(t) = (\mu_h)_a([0, t])$  and  $h_s(t) = (\mu_h)_s([0, t])$ . Set

$$\Phi(h) = \int_{[0,1]} u^*(h'_a)(t) \nu(dt) + rh_s(1),$$

where  $u^*$  and  $r$  are defined in Proposition 2.1.

**Theorem 2.3** *The families of random functions  $\{Z_n\}$  and  $\{\tilde{Z}_n\}$  satisfy a LDP on the space  $(bv([0, 1], \mathbb{R}), \sigma)$ , with speed  $n$  and rate function  $\Phi$ .*

## 2.2 Moderate deviations

We can state also in this case a moderate deviation principle. We detail it for  $\nu_n$ , it is the same for  $\tilde{\nu}_n$ . Let  $\{a_n\}$  be a sequence of positive real numbers such that  $a_n \rightarrow 0$  and  $na_n \rightarrow +\infty$  when  $n \rightarrow +\infty$ . Set

$$Y_n = \sqrt{na_n}(\nu_n - E(\nu_n)).$$

We have the following moderate deviations principle

**Theorem 2.4**  *$\{Y_n\}$  satisfy a LDP with speed  $a_n^{-1}$  and good rate function defined, for all  $\mu \in \mathcal{M}([0, 1])$  by*

$$I(\mu) = \begin{cases} \frac{\pi}{2\bar{f}^2} \int_{[0,1]} l(x)^2 dx & \text{if } \mu(dx) = l(x) dx \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$\bar{f}^2 = \frac{1}{2\pi} \int_{\mathbb{T}} f^2.$$

## 2.3 Generalizations

The previous results can be generalized to some other random functions.

### 2.3.1 Weighted random variables

Assume  $g$  is a continuous function on  $[0, 1]$  and define

$$W_n = \frac{1}{n} \sum_{k=1}^{\lfloor nt \rfloor} g\left(\frac{k}{n}\right) X_k^2, \quad (9)$$

For any  $h$  in  $\mathcal{C}([0, 1])$ , define

$$\Lambda(h) = \begin{cases} -\frac{1}{4\pi} \int_{[0,1]} \int_{\mathbb{T}} \log(1 - 2h(t)g(t)f(\theta)) d\theta dt & \text{if } \forall(t, \theta) \in [0, 1] \times \mathbb{T}, h(t)g(t)f(\theta) < 1/2 \\ +\infty & \text{otherwise} \end{cases}$$

The previous large deviations results apply with rate function  $\Lambda^*$ .

### 2.3.2 Quadratic forms built on the stationary process

We define

$$m = \text{essinf} f$$

and assume  $m > 0$ . Let  $F$  be a continuous positive function on  $[m, M]$ . Let  $O$  be an orthonormal matrix such that  $O^*T_n(f)O$  is the diagonal matrix whose  $i$ -th diagonal element is  $\mu_{i,n}$  the  $i$ -th eigenvalue of  $T_n(f)$ . Define

$$F(T_n(f)) = OD_fO^*$$

where  $D_f$  is the diagonal matrix whose  $i$ -th element is  $F(\mu_{i,n})$ . Define the following quadratic form

$$W_n = \frac{1}{n} X^* F(T_n(f)) X = \frac{1}{n} Y^* Y,$$

where  $Y = (Y_1, \dots, Y_n)$  is the vector defined by

$$Y = F(T_n(f))^{1/2} X.$$

In this case,  $W_n$  satisfies a LDP and moderate deviations theorem with rate function  $\Lambda^*$  where for any  $h$  in  $\mathcal{C}([0, 1])$

$$\Lambda(h) = \begin{cases} -\frac{1}{4\pi} \int_{[0,1]} \int_{\mathbb{T}} \log[1 - 2h(t)f(\theta)F[f(\theta)]] d\theta dt & \text{if } \forall(t, \theta) \in [0, 1] \times \mathbb{T}, h(t)f(\theta) < 1/2 \\ +\infty & \text{otherwise} \end{cases}$$

### 3 Proof of the large and moderate deviations

We first give some asymptotic properties for the families  $\{\nu_n\}_n$  and  $\{\tilde{\nu}_n\}_n$ .

#### 3.1 Weak convergence of $\nu_n$ and $\{\tilde{\nu}_n\}_n$

**Lemma 3.1** *Let  $h$  be in  $\mathcal{C}([0, 1])$ .*

$$\langle \nu_n, h \rangle \rightarrow \langle \nu, h \rangle \quad \text{in probability as } n \rightarrow +\infty \quad (10)$$

and

$$\langle \tilde{\nu}_n, h \rangle \rightarrow \langle \nu, h \rangle \quad \text{in probability as } n \rightarrow +\infty$$

where

$$\langle \nu, h \rangle = \bar{f} \int_{[0,1]} h(x) dx,$$

and

$$\bar{f} = \frac{1}{2\pi} \int_{\mathbb{T}} f(\theta) d\theta.$$

*Proof :*

Let  $h$  be in  $\mathcal{C}([0, 1])$ , and consider

$$\langle \nu_n, h \rangle = \frac{1}{n} \sum_{k=1}^n X_k^2 h\left(\frac{k}{n}\right).$$

Set  $X$  the Gaussian vector  $(X_1, X_2, \dots, X_n)$  and  $\Delta_h$  the diagonal matrix

$$\begin{pmatrix} h\left(\frac{1}{n}\right) & 0 & 0 & 0 \\ 0 & h\left(\frac{2}{n}\right) & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & h(1) \end{pmatrix}$$

Therefore we can write

$$\langle \nu_n, h \rangle = \frac{1}{n} X^* \Delta_h X,$$

where  $X^*$  denote the transpose of  $X$ . By an orthonormal change of basis,

$$\langle \nu_n, h \rangle = \frac{1}{n} U_n^* T_n(f)^{1/2} \Delta_h T_n(f)^{1/2} U_n,$$

where  $U_n$  is a standard normal vector and  $T_n(f)$  the order- $n$  Toeplitz matrix associated to  $f$ . Therefore

$$\langle \nu_n, h \rangle = \frac{1}{n} \sum_{k=1}^n \lambda_{k,n} Z_{k,n} \quad (11)$$

where  $\{Z_{k,n}\}$  are independent  $\chi^2(1)$ -distributed random variables, and  $\{\lambda_{k,n}\}$  are the eigenvalues of  $T_n(f)^{1/2}\Delta_h T_n(f)^{1/2}$ .

We can write as well

$$\langle \tilde{\nu}_n, h \rangle = \frac{1}{n} \sum_{k=1}^n \tilde{\lambda}_{k,n} Z_{k,n} \quad (12)$$

where  $\{Z_{k,n}\}$  are independent  $\chi^2(1)$ -distributed random variables, and  $\{\tilde{\lambda}_{k,n}\}$  are the eigenvalues of  $T_n(f)^{1/2}A_h T_n(f)^{1/2}$ , and the matrix  $A_h$  is diagonal with  $k$ -th diagonal term

$$(A_h)_{k,k} = \int_{(k-1)/n}^{k/n} h(s) ds.$$

We have the two following results on the distributions  $\{\lambda_{k,n}\}$  and  $\{\tilde{\lambda}_{k,n}\}$ , which proofs are postponed to the Appendix.

**Lemma 3.2** *The sequences  $\{\lambda_{k,n}\}$  and  $\{\tilde{\lambda}_{k,n}\}$  are bounded as follows:*

$$\forall n \in \mathbb{N}, \forall 1 \leq k \leq n, \quad \begin{aligned} |\lambda_{k,n}| &\leq \|h\|_\infty \|f\|_\infty \\ |\tilde{\lambda}_{k,n}| &\leq \|h\|_\infty \|f\|_\infty \end{aligned}$$

**Lemma 3.3** *For any  $p$  in  $\mathbb{N}$ ,  $p \geq 1$ ,*

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=1}^n \lambda_{k,n}^p &= \frac{1}{2\pi} \int_{[0,1]} \int_{\mathbb{T}} (h(t)f(\theta))^p dt d\theta. \\ \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=1}^n (\tilde{\lambda}_{k,n})^p &= \frac{1}{2\pi} \int_{[0,1]} \int_{\mathbb{T}} (h(t)f(\theta))^p dt d\theta. \end{aligned}$$

With the above lemma,

$$\lim_{n \rightarrow +\infty} E(\langle \nu_n, h \rangle) = \langle \nu, h \rangle.$$

Moreover,

$$\lim_{n \rightarrow +\infty} n \text{Var} \langle \nu_n, h \rangle = \frac{2}{n} \sum_{k=1}^n \lambda_{k,n}^2 = \frac{1}{\pi} \int_{[0,1]} \int_{\mathbb{T}} (h(t)f(\theta))^2 dt d\theta.$$

We do as well for  $\tilde{\nu}_n$ , and it ends the proof of lemma 3.1.

### 3.2 Proof of Theorem 2.2:

The proof follows exactly the scheme [8]. We detail here for  $\nu_n$ , it is similar for  $\tilde{\nu}_n$ . With the decomposition (11), we get the n.c.g.f. of  $\nu_n$ : for any  $h \in \mathcal{C}([0, 1])$ ,

$$\Lambda_n(h) = \frac{1}{n} \log E(\exp\{n \langle \nu_n, h \rangle\}) = \begin{cases} -\frac{1}{2n} \sum_{k=1}^n \log(1 - 2\lambda_{k,n}) & \text{if } \forall k, \lambda_{k,n} < 1/2 \\ +\infty & \text{otherwise} \end{cases} \quad (13)$$

From Lemma 3.3, we can determine the limit of  $\Lambda_n$  in two cases:



- if  $\forall(t, \theta) \in [0, 1] \times \mathbb{T}$   $h(t)f(\theta) < 1/2$ , then

$$\lim_{n \rightarrow +\infty} \Lambda_n(h) = -\frac{1}{4\pi} \int_{[0,1]} \int_{\mathbb{T}} \log(1 - 2h(t)f(\theta)) d\theta dt = \Lambda(h).$$

- if  $\exists(t, \theta) \in [0, 1] \times \mathbb{T}$ ;  $h(t)f(\theta) > 1/2$ , then for  $n$  large enough,  $\Lambda_n(h) = +\infty$  and

$$\lim_{n \rightarrow +\infty} \Lambda_n(h) = +\infty = \Lambda(h).$$

These two cases do not cover the whole set  $\mathcal{C}([0, 1])$ . Nevertheless, this will be sufficient for the LDP, since they contain a dense subset of exposing hyperplanes of  $\Lambda^*$ .

### Upper bound

From Theorem 4.5.3 b) of [5], and the following lemma, which proof is postponed to the Appendix, the upper bound holds for compact sets.

**Lemma 3.4** *For any  $\delta > 0$  and  $\mu$  in  $\mathcal{M}([0, 1])$ , there exists  $h_\delta$  in  $\mathcal{C}([0, 1])$  such that:*

$$\begin{aligned} & \forall(t, \theta), h_\delta(t)f(\theta) < 1/2 \\ & \int_{[0,1]} h_\delta(t) d\mu(t) - \Lambda(h_\delta) \geq \Lambda_\delta^*(\mu) \end{aligned} \quad (14)$$

where

$$\Lambda_\delta^*(\mu) = \min\{\Lambda^*(\mu) - \delta, \frac{1}{\delta}\}.$$

### Exponential tightness

Remark that for a real number  $a$ ,

$$\left\{ \sup_{\|h\|_\infty \leq 1} \langle \nu_n, h \rangle \geq a \right\} \subset \{ \nu_n(1) \geq a \}.$$

If  $M = \text{esssup}_\theta f(\theta)$ , for any  $y < 1/2M$ ,

$$\limsup_n \frac{1}{n} \log P(\nu_n(1) \geq a) \leq -ya - \frac{1}{4\pi} \int_{[0,1]} \int_{\mathbb{T}} \log(1 - 2yf(\theta)) d\theta,$$

and

$$\lim_{a \rightarrow +\infty} \limsup_n \frac{1}{n} \log P(\nu_n(1) \geq a) = -\infty.$$

Hence the sequence  $(\nu_n)$  is exponentially tight, and the upper bound holds for any closed set of  $\mathcal{M}([0, 1])$ .

### Lower bound

We study the set of exposed points of  $\Lambda^*$  (see [5]). Let

$$\mathcal{H} = \{\mu \in \mathcal{M}([0, 1]); \mu = l\nu, 0 < l < u'(1/2M), l \text{ continuous on } [0, 1]\}.$$

The following two lemmas, which proofs are postponed to the Appendix, show that that  $\mathcal{H}$  is a dense subset of the exposed points of  $\Lambda^*$ , which concludes the proof of Theorem 2.2.

**Lemma 3.5** *Let  $\mu = l\nu$  be in  $\mathcal{H}$ . There exists  $h_l$  in  $\mathcal{C}([0, 1])$  such that*

$$\begin{aligned} \forall (t, \theta) \in [0, 1] \times \mathbb{T} \quad h_l(t)f(\theta) &< 1/2 \\ \forall \xi \in \mathcal{M}([0, 1]) \quad \Lambda^*(\mu) - \Lambda^*(\xi) &< (\mu - \xi)(h_l) \end{aligned} \tag{15}$$

*Furthermore, there exists  $\gamma > 1$  such that  $\Lambda(\gamma l) < +\infty$ .*

Hence  $\mu$  is an exposed point of  $\Lambda^*$  with exposing hyperplane  $h_l$ .

**Lemma 3.6** *Let  $\mu$  be in  $\mathcal{M}([0, 1])$  such that  $\Lambda^*(\mu) < +\infty$ . There exists a sequence  $(\mu_n) \in \mathcal{H}$  such that  $\mu_n \Rightarrow \mu$  and  $\lim_{n \rightarrow +\infty} \Lambda^*(\mu_n) = \Lambda^*(\mu)$ .*

### 3.3 Proof of Theorem 2.4:

The n.c.g.f. of  $Y_n$  is given for any  $h$  in  $\mathcal{C}[m, M]$  by

$$\begin{aligned} \Lambda_n(h) &= a_n \log E(\exp \left\{ \sqrt{\frac{n}{a_n}} (\langle \nu_n, h \rangle - E(\langle \nu_n, h \rangle)) \right\}) \\ &= -\frac{a_n}{2} \sum_{k=1}^n \log \left( 1 - \frac{2}{\sqrt{na_n}} \lambda_{k,n} \right) + \frac{2}{\sqrt{na_n}} \lambda_{k,n} \end{aligned}$$

We recall that  $\{\lambda_{k,n}\}$  are the eigenvalues of the matrix  $T_n(f)^{1/2} \Delta_h T_n(f)^{1/2}$ . We can assert

$$\Lambda_n(h) = \frac{1}{n} \sum_{k=1}^n \lambda_{k,n}^2 + O\left(\frac{1}{n\sqrt{na_n}} \sum_{k=1}^n |\lambda_{k,n}|^3\right).$$

From the convergence (10), Therefore

$$\lim_{n \rightarrow +\infty} \Lambda_n(h) = \Lambda = \bar{f}^2 \int_{[0,1]} h(x)^2 dx \tag{16}$$

This function is defined on all  $\mathcal{C}[0, 1]$ , then the rate function is the Legendre dual of  $\Lambda$  which is, from Rockafellar [18],

$$I(\mu) = \frac{\pi}{2\bar{f}^2} \int_{[0,1]} l(x)^2 dx,$$

where  $d_\mu(t) = l(x) dx$ .

## 4 Appendix

### 4.1 A Szegő Theorem for generalized Toeplitz matrices

In this paragraph we show a result on the distribution of eigenvalues of some kind of generalized Toeplitz matrices.

Suppose  $g$  is a real function defined on  $[0, 1] \times \mathbb{T}$  such that for any  $x \in [0, 1]$ ,  $g(x, \cdot) \in L^1(\mathbb{T})$ . Define

$$\hat{g}_k(x) = \frac{1}{2\pi} \int_{\mathbb{T}} g(x, \theta) e^{-ik\theta} d\theta,$$

and

$$T_n^{\text{gen}}(g)_{k,l} = \hat{g}_{l-k} \left( \frac{k}{n} \right). \quad (17)$$

Denote by

$$\|\hat{g}_k\|_{\infty} = \sup_{x \in [0,1]} |\hat{g}_k(x)|.$$

**Theorem 4.1** *Under assumption*

$$M := \sum_k \|\hat{g}_k\|_{\infty} < \infty, \quad (18)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \text{tr}(T_n^{\text{gen}}(g))^p = \frac{1}{2\pi} \int_0^1 \int_{\mathbb{T}} g(x, \theta)^p d\theta dx. \quad (19)$$

*Proof:* This proof is analogous to the one of [10]. Let  $\varepsilon > 0$  be fixed and  $m \in \mathbb{N}$  chosen such that:

$$\sum_{|k| > m} \|\hat{g}_k\|_{\infty} < \varepsilon$$

Consider the trigonometric polynomial of degree  $m$ :

$$g^m(x, \theta) = \sum_{k=-m}^m \hat{g}_k(x) e^{ik\theta} \quad (20)$$

Let  $T_n^{\text{gen}}(g^m)$  be the generalized Topelitz matrix associated to  $g^m$  as in (17). Therefore

$$T_n^{\text{gen}}(g) = T_n^{\text{gen}}(g^m) + R$$

and the sum of the moduli of the elements of any row of  $R$  is less than  $\varepsilon$ . Hence the same is true for the eigenvalues of  $R$  i.e. for the eigenvalues of  $T_n^{\text{gen}}(g) - T_n^{\text{gen}}(g^m)$ . From the Weyl-Courant Lemma, we can therefore bound

$$|\lambda_{k,n} - \lambda_{k,n}^m| \leq \varepsilon,$$

where  $\{\lambda_{k,n}\}_k$  and  $\{\lambda_{k,n}^m\}_k$  are the eigenvalues of  $T_n^{\text{gen}}(g)$  and  $T_n^{\text{gen}}(g^m)$  respectively non-decreasingly ordered. From assumption (18),

$$|\lambda_{k,n}| \leq M, \quad |\lambda_{k,n}^m| \leq M.$$

Hence for any positive integer  $s$

$$|(\lambda_{k,n})^s - (\lambda_{k,n}^m)^s| \leq \varepsilon s M^{s-1}.$$

We can bound similarly  $|g(x, \theta)^s - g^m(x, \theta)^s|$  and therefore to show (19) it is enough to consider the polynomial  $g^m$ . We derive

$$\lim_{n \rightarrow \infty} \frac{1}{n} \text{tr} (T_n^{\text{gen}}(g^m))^p = \sum_{D_p} \sum_{j=1}^m \hat{g}_{l_1} \left( \frac{j+l_1}{n} \right) \hat{g}_{l_2} \left( \frac{j+l_1+l_2}{n} \right) \cdots \hat{g}_{l_p} \left( \frac{j}{n} \right),$$

where  $D_p = \{(l_1, \dots, l_p) \in \mathbb{Z}^p; \sum l_i = 0\}$  and the second sum in the RHS above is on  $j$  such that  $j + \sum_1^k l_i$  – for  $k$  from 1 to  $p$  – is in the range  $1, \dots, n$ , i.e.  $sp \leq j \leq n - sp$ . Therefore we have to suppress at most  $2sp + 1$  terms. From classical results on Riemann sums,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{D_p} \sum_{j=1}^m \hat{g}_{l_1} \left( \frac{j+l_1}{n} \right) \hat{g}_{l_2} \left( \frac{j+l_1+l_2}{n} \right) \cdots \hat{g}_{l_p} \left( \frac{j}{n} \right) \\ = \sum_{D_p} \int_0^1 \hat{g}_{l_1}(x) \hat{g}_{l_2}(x) \cdots \hat{g}_{l_p}(x) dx \\ = \sum_{(l_1, \dots, l_p) \in \mathbb{Z}^p} \frac{1}{2\pi} \int_{\mathbb{T}} e^{i(l_1+l_2+\dots+l_p)\theta} d\theta \int_0^1 g_{l_1}(x) \hat{g}_{l_2}(x) \cdots \hat{g}_{l_p}(x) dx \\ = \frac{1}{2\pi} \int_0^1 \int_{\mathbb{T}} g(x, \theta)^p d\theta dx. \end{aligned}$$

## 4.2 Proof of Proposition 2.1

This lemma is a consequence of Theorem 5 of Rockafellar [18]. For the sake of clarity, we recall the framework of that paper. Let  $h$  be in  $\mathcal{C}([m, M])$ , and

$$\Lambda(h) = \int_{[m, M]} u(t, h(t)) d\nu(t),$$

where  $u(t, x)$  defined on  $[m, M] \times \mathbb{R} \rightarrow \mathbb{R}$  is a function convex in  $x$ , and  $\nu$  a non-negative,  $\sigma$ -finite measure. For any  $\mu$  in  $\mathcal{M}([m, M])$  having, with respect to  $\nu$  the Lebesgue decomposition  $\mu = l\nu + \mu^\perp$ , where  $l \in \mathcal{C}([m, M])$ , and  $\mu^\perp$  is the singular part, then

$$\Lambda^*(\mu) = \int_{[m, M]} u^*(t, l(t)) d\nu(t) + \int_{[m, M]} r(u^*(t, \cdot); d\mu^\perp/d\eta(t)) d\eta(t) \quad (21)$$

where  $\eta$  is any nonnegative measure of  $\mathcal{M}([m, M])$  with respect to which  $\mu^\perp$  is absolutely continuous, and  $u^*(t, \cdot)$  is the dual function of  $u(t, \cdot)$ :

$$\forall t, \quad u^*(t, y) = \sup_{x \in \mathbb{R}} \{xy - u(t, x)\}.$$

Applying the result of (21) to  $u(t, x) = -(1/t) \log(1 - 2tx)$ , we have the formula of Proposition 2.1

### 4.3 Proof of Lemma 3.2

From Proposition V 1.8 and Theorem X 1.1 of Bhatia [1], since  $T_n(f)$  is an hermitian positive matrix,

$$\|T_n(f)^{1/2}\Delta_h T_n(f)^{1/2}\| \leq \|T_n(f)\| \|\Delta_h\| \quad (22)$$

From Grenander and Szegö ([9] p.64)

$$\|T_n(f)\| \leq \|f\|_\infty.$$

In addition,

$$\|\Delta_h\| \leq \sup_k \sum_s |(\Delta_h)_{ks}| \leq \|h\|_\infty \quad (23)$$

Getting together inequalities (22) and (23), we get the result.

### 4.4 Proof of Lemma 3.3

This lemma is a direct consequence of Theorem 4.1 above, for both random measures.

### 4.5 Proof of Lemma 3.4

From the definition of  $\Lambda^*$ , for any  $\delta > 0$ , there exists  $h_\delta$  in  $\mathcal{C}([0, 1])$  such that inequality (14) holds. In case we only have

$$\forall (t, \theta) \in [0, 1] \times \mathbb{T} \quad h_\delta(t)f(\theta) \leq \frac{1}{2},$$

we choose  $h_\varepsilon$  with  $\varepsilon > 0$  such that

$$\int_{[0,1]} h_\varepsilon(t) d\mu(t) - \Lambda(h_\varepsilon) \geq \Lambda_\delta^*(\mu) - \varepsilon.$$

(this is possible from the continuity of  $\Lambda$  in a neighborhood of  $h_\delta$ )

Then (14) holds with another  $\delta$ . From assumption on  $f$ ,  $f > 0$ , then  $h_\varepsilon f < 1/2$ .

### 4.6 Proof of Lemma 3.5

For all  $0 < y < 1/u'(1/2M)$ , there exists a unique  $x_y$  in  $(-\infty, 1/2M)$  such that  $y = u'(x_y)$ . For such a pair  $(y, x_y)$ ,

$$u^*(y) = yx_y - u(x_y).$$

Since  $u'$  is strictly increasing and continuous,  $u^*$  is strictly convex on  $0 < y < u'(1/2M)$ . For such an  $y$  and  $z > 0$ ,  $z \neq y$ ,

$$u^*(y) - u^*(z) < (y - z)x_y \quad (24)$$

(then  $y$  is an exposed point of  $u^*$  with exposing hyperplane  $x_y$ ) If  $\mu = l\nu$  and  $\xi = \tilde{l}\nu + \xi^\perp$ . We apply inequality (24) with  $y = l(t)$  and  $z = \tilde{l}(t)$ , and then we integrate over  $[0, 1]$  against  $\nu$ . We obtain the inequality (15) with  $h_l(t) = x_{l(t)}$ .

#### 4.7 Proof of Lemma 3.6

Following the sketch of proof of [8], we proceed in 4 steps. Assume  $u'(1/2M) = +\infty$ .

Step 1: Let  $\mu = l\nu + \mu^\perp$  be in  $\mathcal{M}([0, 1])$  such that  $\Lambda^*(\mu) < \infty$  with  $l$  continuous and  $l \in (0, u'(\frac{1}{2M}))$ , and such that  $\mu^\perp$  is in  $L^1([0, 1])$ . Since  $\nu$  has full support on  $[0, 1]$ , there exists a sequence of continuous positive functions on  $[0, 1]$  such that  $h_n d\nu \Rightarrow \mu^\perp$ . From the lower semi-continuity of  $\Lambda^*$ ,

$$\liminf_{n \rightarrow +\infty} \Lambda^*((l + h_n)\nu) \geq \Lambda^*(\mu).$$

Since  $u^*$  is a convex function, from Rockafellar (see [17]), for any  $y > 0$  and  $z \geq 0$ ,

$$u^*(y + z) \leq u^*(y) + \frac{z}{2M}.$$

Therefore

$$\Lambda^*((l + \tilde{l})\nu) \leq \Lambda^*(l\nu) + \frac{1}{2M} \int \tilde{l}(t) d\nu(t) \quad (25)$$

From inequality above,

$$\Lambda^*((l + h_n)\nu) \leq \Lambda^*(l\nu) + \frac{1}{2M} \int_{[0,1]} h_n d\nu$$

And then

$$\liminf_{n \rightarrow +\infty} \Lambda^*((l + h_n)\nu) \leq \Lambda^*(\mu).$$

We now show that the Lemma is true if  $\mu = l\nu$  with  $l$   $\nu$ -a.s. in  $(0, u'(\frac{1}{2M}))$  and integrable.

##### Step 2

We prove the result for  $\mu = l\nu$  assuming that  $l$  is in  $(0, u'(\frac{1}{2M}))$  integrable and that for some  $\epsilon > 0$ ,  $l > \epsilon$   $\nu$ -a.s. There exists a sequence  $(l_n)$  of continuous positive functions such that  $l_n$  converges both in  $L^1(\nu)$  norm and  $\nu$ -a.s. to  $l$  and  $l_n > \epsilon/2$ . Since on  $(\epsilon/2, u'(\frac{1}{2M}))$  the function  $u^*$  is Lipschitzian, the lemma holds.

##### Step 3

Define  $l_\epsilon := l\mathbb{1}_{l>\epsilon} + \epsilon\mathbb{1}_{l\leq\epsilon}$ . Apply second step and inequality (25) noticing that  $l_\epsilon$  converges in  $L^1(\nu)$  to  $l$  and that  $l_\epsilon \geq l$ .

#### Step 4

For  $\mu = l\nu + \eta$ , combine first and third step.

If  $u'(1/2M) < +\infty$ , we have to modify the second and third step, introducing an additional truncation at level  $u'(1/2M) - \epsilon$ .

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