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Fundamentalists vs. chartists:
learning and predictor choice dynamics

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Abstract

In a simple, forward looking univariate model of price determination we investigate the evolution of expectations dynamics in presence of two types of agents: fundamentalists and chartists. In particular, we combine evolutionary selection among heterogeneous classes of models through predictor choice dynamics based on a logit model, with adaptive learning in the form of parameters updating within each class of rules. We find that, for different parameterizations, it can happen that fundamentalists drive chartists completely out of the market or vice-versa, and also that heterogeneous equilibria in which fundamentalists and chartists coexist are possible. Interestingly, though, only

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equilibria in which fundamentalists outperform chartists turn out to be adaptively learnable by agents.

Key words: heterogeneity, expectations, predictor choice, learning.

JEL classification: C15, C62, D83, D84, E37.

1 Introduction

Expectations have long been recognized to play a key role in macroeconomic and finance models. After the rise of rational expectations (RE) in the 70s, in more recent years economists have acknowledged the importance that bounded rationality can have on (macro)economic dynamics and the role it should play in modelling real life situations (Sargent, 1993). In particular, the RE assumption, at least in its most stringent definition, does not allow for heterogeneity of expectations, a feature that seems so commonly observed in real economic situations when agents have to make forecasts.

For some years now, a growing literature has been developing on adaptive learning in macroeconomics (Evans and Honkapohja, 2001): the main idea is to replace agents' rational expectations with learning schemes such as recursive least squares, that are meant to represent the way in which agents revise their expectations over time. An important question then is whether these learning processes converge in the limit to an equilibrium.

Another important line of research has considered the possibility of introducing heterogeneity in the expectations formation processes, allowing agents to choose among different predictors. In the literature, heterogeneous expectations with endogenous predictor choice have been analyzed in various

settings. In the context of a Cobweb model, Brock and Hommes (1997) allow agents to have either rational or naive expectations, while Lasselle et al. (2005) have agents split between rational expectations and adaptive expectations. Brock and Hommes (1998) consider a model of asset pricing where agents can choose from a finite set of predictors of future prices, while in De Grauwe, Dewachter and Embrechts (1993), the authors consider models of exchange rates with two classes of agents, fundamentalists and chartists; Chiarella (1992), Lux (1995) and Sethi (1996) analyze instead stock market models with fundamentalists and chartists. None of these works, though, couple the endogenous determination of the relative fraction of agents using each model with learning dynamics about parameter values for each model.

Guse (2005) and Berardi (2007) analyze heterogeneous adaptive learning, but a limitation of those works is that the proportion of agents using each model is exogenously determined. Guse (2010) tries to amend this problem in the same framework as Guse (2005) by introducing a game theoretic analysis of interactions between two types of agents endowed with different perceived laws of motion. Diks and Dindo (2008) consider an asset pricing model where bounded rational agents choose among different degrees of information while also learning the growth rate of the dividend generating process. Finally, Branch and Evans (2006) consider a cobweb model with heterogeneous expectations, where agents can use different (but all misspecified) models, and show that in the limit intrinsic heterogeneity is possible.

The aim of this paper is to analyze a simple model of price determination where two types of agents, fundamentalists and chartists, interact through

adaptive learning and predictor choice dynamics. It is meant to give a simplified representation of how an asset price (or any other price that displays positive feedback from expectations of future prices on current values) is determined on the market. The main question we try to answer is whether it is possible to find equilibria in which fundamentalists and chartists coexist in the economy, or if instead fundamentalists (chartists) will always run chartists (fundamentalists) out of the market in the long run. Moreover, we want to understand whether these equilibria, if they exist, can be obtained as limiting outcomes of adaptive learning dynamics. In this way, we combine two types of learning, within a model and across models.

We follow the adaptive learning literature in letting agents learn the values of the parameters in their forecasting models through econometrics techniques such as recursive least squares (RLS), and we introduce heterogeneity in the forecasting models (perceived law of motion - PLM) available to them. In particular, we let agents choose between two possible models, one that corresponds to the fundamental or minimum state variables (MSV) solution for the system,¹ and the other that is a simple AR(1) model. Agents who choose the first model are called fundamentalists, as they correctly recognize the exogenous driving force for the economy and use it in their forecasts; agents who instead choose the AR(1) model are called chartists, as they only base their predictions on past data, trying to extract (and exploit) the autocorrelation structure present in the data. We assume that chartists use an AR(1) model because it seems to us the simplest model that an agent

¹The MSV solution is the one that includes the minimum number of variables possible, and is based only on fundamentals. See McCallum (1983) and McCallum (1999) for an introduction to the concept.

trying to extract patterns from data might use, and one that does not require any knowledge about exogenous variables. The important question our work tries to answer is whether heterogeneity of expectations can survive in the economy even when a model consistent with the fundamental solution is available to agents: in other words, while in the misspecification equilibrium (ME) of Branch and Evans (2006) all agents used a misspecified model, we are seeking an heterogeneous equilibrium where some agents use a model that correctly includes all the relevant state variables for the economy.

In order to impose some rigor on what would otherwise be an arbitrary imposition of heterogeneity, we let agents choose endogenously among the two different predictors, i.e., we let the fraction of fundamentalists vs. chartists be determined endogenously in the economy. We model this choice between predictors using a logit model, following the seminal work by Brock and Hommes (1997). This modeling strategy captures a sort of bounded rationality, in the sense that the best performing forecasting model (in any measure we care to use) is not chosen by all agents all the time.

In this way, we combine adaptive learning, in the form of parameter updating within a class of forecasting models, with evolutionary selection among heterogeneous classes of forecasting models. Combined learning dynamics of similar type have been analyzed by Branch and Evans (2006) and Hommes (2009), both in the context of a cobweb model.

Besides these two papers, our work is perhaps mostly related to Guse (2010), though it differs from it in several dimensions. One key difference is that while in Guse (2010) there is no correlation in the exogenous driving force for the system and all persistence in the economy comes from agents

using an AR(1) forecasting model, in our work instead the exogenous component is an AR(1) process and this can lead to interesting dynamics as a result of the interactions between the exogenously and the endogenously generated correlations. In fact, Guse (2010)'s setting can be considered as a special case of ours, one that emerges when we set the AR(1) parameter for the exogenous driving force equal to zero. Moreover, and crucially, in Guse (2010) one PLM nests the other and in fact one group of agents is simply learning a constant, which means that equilibrium values for the parameters in the two models do not depend on the endogenous covariance structures generated in the system. In terms of results, one striking difference is that we find regions of the parameter space where the relative fraction of agents cycles permanently and therefore heterogeneity is sustained over time, for finite values of the intensity of choice parameter. For an infinite value of the intensity of choice parameters, we are able to show that equilibria in which only fundamentalists survive, only chartists survive or where both groups coexist are possible depending on the parameter values of the model. A key result of the paper, though, is that only when the forecasting model consistent with the MSV solution prevails, agents are able to adaptively learn the relevant equilibrium in the economy.

The structure of the paper is as follows: Section 2 introduces the model under homogeneous expectations, solves it under RE and analyses learnability (E-stability) of the equilibrium when only one type of agents is present; Section 3 introduces heterogeneity, solves the model and analyses its E-stability properties with a fixed proportion of agents; Section 4 allows the proportion of each type of agents to vary over time by introducing endoge-

nous predictor choice dynamics, and analyses the E-stability condition of the endogenous equilibrium that emerges; Section 5 concludes.

2 The model

The model representing the system is a simple univariate forward looking model²

$$y_t = AE_t y_{t+1} + Bw_t \quad (1)$$

where y is the price level and w the exogenous driving force represented by a stationary AR(1) process:

$$w_t = \rho w_{t-1} + \varepsilon_t \quad (2)$$

with $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$ and $0 \leq \rho < 1$.

The model can be thought of as a model of financial markets, where net demand (and therefore current price) of the asset depends positively on its future expected price and on a fundamental driving process (here w_t) that could for example represent firm's profits. A is strictly positive, meaning that there is a positive feedback from expectations of future values on current values, as it is common in financial markets. The semi-reduced form (1) could also represent a model of exchange rates, with the exogenous process representing, for example, interest rates or inflation differentials.

There are two types of traders, fundamentalists and chartists. Fun-

²A similar model has been analysed in Evans and Honkapohja (2009), p. 198. A more general model, including lagged endogenous variables, has been considered in Berardi (2007).

damentalists use a model that it is consistent with the fundamental MSV solution that would prevail if all agents were rational and in which the endogenous variable depends only on the exogenous driving process. Chartists use instead an AR(1) model in the attempt to discover statistical regularities in the time-series for prices.

2.1 Determinacy

Defining the expectational error in (1) $\eta_t = y_{t+1} - E_t y_{t+1}$, we can write the general solution for the model as

$$y_t = \frac{1}{A}y_{t-1} - \frac{B}{A}w_{t-1} + \eta_t. \quad (3)$$

If $A > 1$, this solution is stationary for any stationary i.i.d. stochastic process η_t and the equilibrium is indeterminate.³ If instead $A < 1$, this equilibrium is in general non stationary. To find the unique stationary equilibrium in this case we define the error term η_t as

$$\eta_t = \frac{B}{1 - \rho A} \varepsilon_t, \quad (4)$$

which allows us to rewrite (3) in the form

$$y_t = \frac{1}{A}y_{t-1} + \frac{B}{1 - \rho A}w_t - \frac{B}{A(1 - \rho A)}w_{t-1}$$

³For an analysis of determinacy and E-stability of a similar model, see Evans and Honkapohja (2001), p. 220.

and then, collecting $(1 - \frac{1}{A}L)$ from both sides (with L being the lag operator), we arrive at the locally unique MSV REE

$$y_t = \frac{B}{1 - \rho A} w_t. \quad (5)$$

2.2 E-stability

An interesting question is whether these equilibria could be learned by agents through some form of recursive algorithm that allows them to extract from past data all the information available and update their beliefs accordingly. The concept that we use to analyze the learnability of an equilibrium is that of E-stability,⁴ which establishes a link between adaptive learning in real time through recursive least squares (or similar techniques) and a set of differential equations that describe the limiting behavior of such a learning process. The E-stability principle states that, under certain conditions, the limiting point of the real time learning dynamics and the set of differential equations is the same.

Suppose all agents use a PLM consistent with the fundamental solution (5):

$$PLM : y_t = k + aw_t \Rightarrow E_t y_{t+1} = k + a\rho w_t. \quad (6)$$

As shown by Evans and Honkapohja (2001), p. 198-200, the equilibrium in this case is E-stable if and only if $A < 1$.

Suppose instead that agents use a PLM that is consistent with the equi-

⁴See Evans and Honkapohja (2001) for an exhaustive introduction to the concept and techniques.

librium representation (3):

$$y_t = k + by_{t-1} + cw_{t-1} \Rightarrow E_t y_{t+1} = k(1+b) + b^2 y_{t-1} + c(\rho+b)w_{t-1} + c\varepsilon_t. \quad (7)$$

In this case, as shown by Honkapohja and Mitra (2004), the equilibrium is never E-stable.⁵

2.3 An alternative PLM for chartists

Besides the two PLMs discussed in the previous section, a third case is worth consideration, which will represent the basis of much of our analysis for the heterogeneous setting in the following sections. This is the case where agents use an AR(1) model that, contrary to (7), is not well specified, since it misses out completely the role of the fundamental in driving the economy. This forecasting model is meant to represent the behavior of chartists, agents that have no knowledge whatsoever of the economy fundamentals and only try to infer the behavior of the endogenous variable statistically through its past values. Specifically, the PLM for these agents can be obtained by imposing $c = 0$ in (7), and the E-stability analysis for an economy populated only by such agents will be presented in Section 3.1.1, as a special case of the heterogeneous setting.

⁵Note that the ARMA(1,1) equilibrium could instead be learnable if agents were to use a PLM consistent with its common factor representation, as shown by Evans and McGough (2005).

3 Heterogeneous expectations equilibrium

In the heterogeneous setting, we consider a unitary mass of agents split between α fundamentalists (denoted by superscript f) who correctly recognize the exogenous driving process and use information about it to form their expectations

$$PLM^f : y_t = k^f + aw_t \Rightarrow E_t^f y_{t+1} = k^f + a\rho w_t; \quad (8)$$

and $(1 - \alpha)$ chartists (denoted by superscript c), who use past observations to predict future ones, using a simple AR(1) model

$$PLM^c : y_t = k^c + by_{t-1} \Rightarrow E_t^c y_{t+1} = k^c(1 + b) + b^2 y_{t-1}. \quad (9)$$

Aggregate expectations are then computed taking the convex combination of the expectations of the two groups, weighted by the relative size in the economy:

$$E_t y_{t+1} = \alpha E_t^f y_{t+1} + (1 - \alpha) E_t^c y_{t+1}. \quad (10)$$

Parameters k^f , k^c , a and b in the two PLMs are chosen by agents as optimal linear projections, so as to minimize the mean square errors (MSEs)

incurred in making forecasts:

$$\begin{bmatrix} \bar{k}^f \\ \bar{a} \end{bmatrix} : E \left[\begin{bmatrix} 1 \\ w_t \end{bmatrix} (y_t - (k^f + aw_t)) \right] = 0 \quad (11)$$

$$\begin{bmatrix} \bar{k}^c \\ \bar{b} \end{bmatrix} : E \left[\begin{bmatrix} 1 \\ y_{t-1} \end{bmatrix} (y_t - (k^c + by_{t-1})) \right] = 0. \quad (12)$$

Conditions (11)-(12) are standard moment conditions, where expectations are taken over the invariant distribution of the stochastic variables for given belief parameters k^f , k^c , a and b . Equilibrium values \bar{k}^f , \bar{k}^c , \bar{a} and \bar{b} depend on parameters A , B , ρ and α . While the first three are structural parameters, α should be regarded as endogenous, coming from the choice of agents about their forecasting model. As a first step, we start here by treating α as a structural and exogenous parameter and compute the heterogeneous expectations equilibrium (HEE)⁶ that emerges for different values of it. In a later section, we will then relax this assumption and model α as the outcome of endogenous predictor choice dynamics.

Definition 1 *A heterogeneous expectations equilibrium (HEE) is a stationary stochastic process for y_t which solves the expectational difference equation (1), where expectations are formed according to (8)-(10) and parameters k^f, k^c, a, b solve conditions (11)-(12).*

Note that this equilibrium is equivalent to the ME of Branch and Evans

⁶For a discussion of the concept of HEE see Branch and McGough (2004). We will follow closely Berardi (2007) in the definition of the concept used here. From a learning perspective, in a HEE parameter estimates have converged to the optimal value, the one that minimizes the MSE, for all agents involved.

(2006) with fixed proportions, except that we do not force all agents to use a model that is underparameterized with respect to the fundamental solution: in fact, fundamentalists are using here a model that is consistent with the MSV equilibrium for the economy, and it turns out to be misspecified only because of the presence of chartists in the economy. This difference is particularly important if we consider the equilibrium as a limiting point of evolutionary dynamics, as we will do in the following section. While in the setting of Branch and Evans (2006) the economy could never converge towards the MSV REE, in our setting this could happen if chartists were to be driven out of the economy by fundamentalists: when this does not happen, the economy converges instead towards an equilibrium where heterogeneity in expectations persists.

Substituting agents' expectations from (8) and (9) into (1) we can find the ALM for y_t :

$$y_t = A\alpha k^f + A(1 - \alpha)(1 + b)k^c + (A\alpha a\rho + B)w_t + A(1 - \alpha)b^2y_{t-1}. \quad (13)$$

Note that, while PLM^f would be correctly specified in the fundamental MSV REE, when there are also chartists in the economy the model used by fundamentalists is misspecified (underparameterized), as it misses the endogenous lag component that is introduced into the economy by chartists.

Knowing the law of motion for the economy, we can now solve (11) and

(12)⁷ for the equilibrium values of k^f , k^c , a and b :

$$\bar{k}^f = \bar{k}^c = 0 \quad (14)$$

$$\bar{a} = \frac{B}{1 - A\alpha\rho - A(1 - \alpha)\rho\bar{b}^2} \quad (15)$$

and

$$\bar{b} : b^3 A\rho(1 - \alpha) - b^2 A(1 - \alpha) + b - \rho = 0. \quad (16)$$

Note that the only equilibrium value for k^f and k^c is zero: this reflects the fact that there is no constant term in the structural model (1). As for the other parameters, \bar{a} depends on \bar{b} and the solution for the latter comes from a 3th-degree polynomial. There are of course 3 solutions, of which at least one real. Using Descartes' Sign Rule we can see that, for A and ρ positive, there are a maximum of 3 positive real roots and a maximum of 0 negative real roots, i.e., all the real roots will be positive.⁸ Moreover, using Cardano's methods, it is possible to show (see Appendix A) that for $\rho > \frac{1}{3}$ there is one unique real solution to (16). If instead $\rho \leq \frac{1}{3}$, there are regions of the parameter space (depending on A) where there are multiple real solutions.

Proposition 2 For $\rho > \frac{1}{3}$, there is one unique real solution \bar{b} to (16), with $\bar{b} \geq 0$.

⁷The main difficulty in solving for the optimal values for a and b is that variances and covariances involving the endogenous variable depend, in equilibrium, on the value of the parameters that are to be found. Variances and covariances needed in order to find the solutions are available in the Appendix B.

⁸This result relies on the restrictions imposed above, namely $A > 0$, $\rho \geq 0$. We are not considering here models with negative feedback from expectations or with negative autocorrelation in the exogenous driving process.

Proposition 3 For $\rho \leq \frac{1}{3}$, $\exists A_1, A_2 > 0$, $A_1 < A_2$, such that:

- a. For $A < A_1$ or $A > A_2$, (16) admits only one (positive) real root;
- b. For $A_1 < A < A_2$, (16) admits three distinct (positive) real roots;
- c. For $A = A_1$ and $A = A_2$, (16) admits three (positive) real root, two of which are equal.

This shows that if there is enough persistence in the exogenous driving process (i.e., if $\rho > \frac{1}{3}$), only one real solution exists for any values of the remaining parameters.⁹ Note also that, in order to have stationarity in (13), we have to impose the restriction $0 \leq b < \sqrt{\frac{1}{A(1-\alpha)}}$. When more than one real solution exists, our numerical investigations suggest that only one equilibrium implies stationarity in the HEE. In the following part of the paper, therefore, we will restrict our analysis to the regions where a unique solution exists.¹⁰

From (13) and (15) we can also see that as $\alpha \rightarrow 1$, we obtain again the homogeneous MSV RE solution (5).¹¹

Note that for $\rho = 0$, there are two solutions: $\{\bar{a} = B, \bar{b} = 0\}$ and $\{\bar{a} = B, \bar{b} = 1/A(1-\alpha)\}$. If we substitute the ensuing expectations into the structural model, in the first case (with $\bar{b} = 0$) we obtain the MSV RE solution $y_t = Bw_t$ (and fundamentalists turn out to be correct in their

⁹We focus only on real solutions here as complex solutions make no sense for a model of price determination.

¹⁰We have investigated numerically the values of A_1 and A_2 for various combinations of $\{\rho, \alpha\}$, and we find that in the cases when we will set $\rho = .3$ in the following part of the paper, choosing $A = 1.5$ ensures that there will be only one real solution.

¹¹Note that for $\alpha = 1$, the model implies $b = \rho$: as there are only fundamentalists in the economy, the autocorrelation in the price level is the same as that in the exogenous driving force. Of course, there are no chartists in the economy actually using the equilibrium value for b .

beliefs), while for $\bar{b} = 1/A(1 - \alpha)$ we have $y_t = 1/A(1 - \alpha)y_{t-1} + Bw_t$, which is a linear combination of the two models used by the two groups of agents. Ex post, thus, both fundamentalists and chartists are making mistakes, the first because they ignore the effect on prices introduced by the second.

3.1 E-stability of HEE

We are interested now in determining whether the heterogeneous expectations equilibrium defined above could be the outcome of adaptive learning dynamics. In other words, suppose agents did not know the equilibrium values for k^f , k^c , a and b : could they learn these values through adaptive learning schemes such as recursive least squares? To answer this question, we will make use once again of the concept of E-stability and consider an equilibrium learnable if and only if it is E-stable.¹²

We start from the stochastic recursive algorithm (SRA) that represents learning in real time for the two groups of agents

$$\begin{pmatrix} k_{t+1}^f \\ a_{t+1} \end{pmatrix} = \begin{pmatrix} k_t^f \\ a_t \end{pmatrix} + t^{-1}R_t^{-1} \begin{bmatrix} 1 \\ w_t \end{bmatrix} \left(y_t - (k_t^f + a_t w_t) \right) \quad (17)$$

$$R_t = R_{t-1} + t^{-1} \left(\begin{bmatrix} 1 \\ w_t \end{bmatrix} \begin{bmatrix} 1 \\ w_t \end{bmatrix}' - R_{t-1} \right)$$

$$\begin{pmatrix} k_{t+1}^c \\ b_{t+1} \end{pmatrix} = \begin{pmatrix} k_t^c \\ b_t \end{pmatrix} + t^{-1}S_t^{-1} \begin{bmatrix} 1 \\ y_{t-1} \end{bmatrix} (y_t - (k_t^c + b_t y_{t-1})) \quad (18)$$

$$S_t = S_{t-1} + t^{-1} \left(\begin{bmatrix} 1 \\ y_{t-1} \end{bmatrix} \begin{bmatrix} 1 \\ y_{t-1} \end{bmatrix}' - S_{t-1} \right).$$

The E-stability principle (Evans and Honkapohja (2001)) ensures that possi-

¹²Convergence of least squares learning under heterogeneous expectations has been studied by Evans and Honkapohja (1996).

ble limiting points of this stochastic recursive algorithm are also fixed points of an associated system of ODEs that can be obtained through stochastic approximation.

Starting from the learning algorithm used by the fundamentalists, represented by eq. (17), the associated ODEs are defined as

$$\begin{pmatrix} \dot{k}^f \\ \dot{a} \end{pmatrix} = \lim_{t \rightarrow \infty} EQ(t; k^f, a, 1, w_t) \quad (19)$$

where E denotes expectations of $Q(\cdot)$ taken over the invariant distribution of w , for fixed k^f and a . $Q(\cdot)$ comes from the SRA and it is defined as

$$Q(t; k^f, a, 1, w_t) = R^{-1} \begin{bmatrix} 1 \\ w_t \end{bmatrix} \left(y_t - (k^f + aw_t) \right), \quad (20)$$

where R is the asymptotic variance covariance matrix of the regressors. Analogously, for chartists, starting from (18), the relevant ODEs are defined as

$$\begin{pmatrix} \dot{k}^c \\ \dot{b} \end{pmatrix} = \lim_{t \rightarrow \infty} EZ(t; k^c, b, 1, y_{t-1}) \quad (21)$$

where E denotes expectations of $Z(\cdot)$ taken over the invariant distribution of y , for fixed k^c and b . $Z(\cdot)$ comes from the SRA and it is defined as

$$Z(t; k^c, b, 1, y_{t-1}) = S^{-1} \begin{bmatrix} 1 \\ y_{t-1} \end{bmatrix} \left(y_t - (k^c + by_{t-1}) \right), \quad (22)$$

where S is the asymptotic variance covariance matrix of the regressors. From (20) and (22) we can see that the orthogonality conditions (11)-(12) hold at the resting points for this ODEs.

Computing the limits in (19) and (21), we find that the associated ODEs for the two groups of agents are:

$$\dot{k}^f = A\alpha k^f + A(1-\alpha)(1+b)k^c - k^f \quad (23)$$

$$\dot{k}^c = A\alpha k^f + A(1-\alpha)(1+b)k^c - k^c \quad (24)$$

$$\dot{a} = (A\alpha\rho a + B - a)\sigma_w^2 + A(1-\alpha)b^2\sigma_{wy_{-1}}^2 \quad (25)$$

$$\dot{b} = \Gamma(\sigma_y^2)^{-1}\sigma_{w,y_{-1}}^2 + \Omega - b \quad (26)$$

where Γ , Ω are as defined in Appendix B.

Note that (25)-(26) are independent from (23)-(24): using results from Appendix B regarding the endogenous variance covariance structures, we can rewrite the last two ODEs as

$$\dot{a} = \frac{A\alpha\rho a + B}{1 - \rho A(1-\alpha)b^2} - a \quad (27)$$

$$\dot{b} = \frac{\left(1 - [A(1-\alpha)b^2]^2\right)\rho}{(1 + \rho A(1-\alpha)b^2)} + A(1-\alpha)b^2 - b. \quad (28)$$

By evaluating the Jacobian J of the system (see Appendix C) at the point $(\bar{k}^f, \bar{k}^c, \bar{a}, \bar{b})$ that solves (23)-(26), we can then assess the E-stability of the equilibrium: this requires the above system of ODEs to be locally asymptotically stable, i.e., all eigenvalues of J must have negative real part. Note that, when evaluated at the equilibrium point $(\bar{k}^f, \bar{k}^c, \bar{a}, \bar{b})$, with $\bar{k}^f = \bar{k}^c = 0$, matrix J is block diagonal.

Remark 4 *The heterogeneous expectations equilibrium defined in Definition 1 is E-stable when all eigenvalues of the Jacobian matrix for the system*

(23)-(26) (as defined by (36) in Appendix C) have negative real part.

In order to compute numerically these eigenvalues, we consider the parameter space delimited by $.01 \leq A \leq 3$,¹³ $.01 \leq \alpha \leq .99$ and $\rho = \{.3, .6, .9\}$:¹⁴ the contour map in Figure 1 shows the region of E-stability (white) for the (unique) real stationary solution. We can see that as α approaches 1, the E-stability condition (for any value of ρ) becomes that of the homogeneous case in section (2.2) when agents use a PLM consistent with the fundamental solution, i.e., $A < 1$. Note that for high values of A and low values of α , there are combinations of these two parameters that lead to a violation of the restriction $\Omega\rho < 1$ necessary for having a finite covariance structure in the system (see Appendix B): all these combinations, though, reside in a subregion of the E-instability area.

3.1.1 Special case: $\alpha = 0$

By setting $\alpha = 0$, we are able to analyze E-stability for the restricted perceptions equilibrium (RPE) that emerges when the economy is populated only by chartists.¹⁵ As shown in Figure 2, the RPE turns out to be E-stable for any value of ρ considered when $A < .5$, and as A increases up to 1, E-stability requires a lower ρ . For values of $A > 1$, the RPE is never learnable.¹⁶ This result is quite interesting, especially when compared to

¹³This includes a region of determinacy ($A < 1$) and one of indeterminacy ($A > 1$) for the structural model (1).

¹⁴Note that B does not affect the stability properties of (27) and (28).

¹⁵An RPE emerges when the PLM used by agents is underspecified or "restricted" with respect to the ALM for the economy. For a detailed explanation of the concept, see Evans and Honkapohja (2001), p. 320.

¹⁶Also in this case, only one solution to (12) can be real and E-stable in the region where the restriction on $\Omega\rho$ is satisfied.

the result reported in Section 2.2 that the equilibrium emerging when all agents use PLM (7) is never E-stable: it appears that neglecting the exogenous lag component allows the learning process of agents to converge, though of course it converges not to a rational expectations equilibrium but to a restricted perceptions one.

This RPE is an example of a consistent expectations equilibrium (CEE) in the sense of Hommes and Sorger (1998): the true law of motion is a linear stochastic process driven by an exogenous linear random process, while agents believe in a simple AR(1) process driven by i.i.d. noise.

Comparing the E-stability properties of the RPE, the HEE and the MSV REE, we can see how the limiting properties of one equilibrium merge into the other's as α moves from 0 to 1. For $\alpha = 0$ the HEE becomes an RPE and the E-stability properties of the former coincide with that of the latter (as it can be seen by comparing Figure 1 with Figure 2). For $\alpha = 1$ instead, the HEE becomes the MSV REE and it shares its E-stability condition ($A < 1$).

3.1.2 Special case: $\rho = 0$

We can also consider the special case with $\rho = 0$: in this case it is possible to show analytically by computing the Jacobian (36) in Appendix C that the heterogeneous equilibrium that emerges with $b = \frac{1}{A(1-\alpha)}$ is never E-stable. Nonetheless, imposing $\rho = 0$ will allow us to derive some analytical results in a later section, where we will use this case to help understand the more complicated case with $\rho > 0$. Note that for $\rho = 0$, there is no intrinsic autocorrelation in the driving process, and all the autocorrelation in prices comes from chartists' behavior: in this sense, it is a sort of "self-confirming"

autocorrelation; for $\rho > 0$, instead, there is some persistence in the driving process that is passed on to prices, and this can be picked up by chartists. This could explain why in the first case the HEE is not E-stable, while in the second case it can be.

4 Endogenous heterogeneous expectations equilibrium

So far, we have treated α , the relative size of the two groups of agents in the economy, as an exogenous parameter and have derived equilibrium values for parameters $(\bar{k}^f, \bar{k}^c, \bar{a}, \bar{b})$ as function of (A, B, ρ, α) . One objective of this paper, though, is to allow agents to endogenously choose which predictor (i.e., which PLM) to use in their forecasts, i.e., to allow α to be endogenously determined in the economy.

We therefore look now for an endogenous heterogeneous expectations equilibrium, where parameters $(\bar{k}^f, \bar{k}^c, \bar{a}, \bar{b}, \bar{\alpha})$ are functions of the structural parameters (A, B, ρ) . The new equilibrium concept that we define, where heterogeneity is endogenized, is similar in spirit to the misspecifications equilibrium (ME) introduced in Branch and Evans (2006), except that in their case agents choose between models that are misspecified with respect to the fundamental MSV solution. This is an important difference, because it will allow us to assess whether intrinsic heterogeneity can persist even when a model consistent with the fundamental MSV equilibrium is available to agents (at no extra cost).

4.1 A measure of performance for forecasting models

We assume that the predictor choice of agents is based on the relative performance of the two forecasting models as given by the unconditional mean squared error (MSE), which can be derived from a quadratic loss function for errors:

$$MSE^f = E \left[y_t - (k^f + aw_t) \right]^2 \quad (29)$$

$$MSE^c = E [y_t - (k^c + by_{t-1})]^2. \quad (30)$$

The specific expressions for MSE^c and MSE^f are given in Appendix B. Here we compute the $MSEs$ for optimal values of the parameters in each forecasting model, i.e., using $(\bar{k}^f, \bar{k}^c, \bar{a}, \bar{b})$. To have an idea of the relative performance of the two models for different values of α we show in Figure 3 the sign of the difference in the mean squared errors ($MSE^c - MSE^f$) as a contour map in the space (A, α) , where the white color represents regions of the parameter space where $MSE^c - MSE^f > 0$, while the black color shows regions where chartists outperform fundamentalists (i.e., $MSE^c < MSE^f$). We report 3 cases, for $\rho = \{.3, .6, .9\}$: as it can be seen in the figure, for $\rho = .3$ fundamentalists always have smaller forecast errors than chartists, while for $\rho = .6$ and $\rho = .9$ there is a stripe cutting across the graph where chartists are able to outperform fundamentalists ($MSE^c < MSE^f$).¹⁷ These results suggest that there must be enough persistence in the exogenous data generating process for an AR(1) model to be able to replicate successfully the

¹⁷All these regions are in the area where the restriction $\Omega\rho < 1$ is satisfied. This condition is violated only in a subregion of the area below the black stripes.

dynamics of the endogenous variable and allow chartists to outperform fundamentalists. When this is the case, there are combinations of (A, α) that ensure $MSE^c < MSE^f$, and the set of these combinations (the black stripe in Figure 3) enlarges as the degree of persistence in the exogenous shock increases.

4.1.1 Special case: $\rho = 0$

It is useful at this point to consider the case with $\rho = 0$, as in this case we are able to compute analytically the solution and the MSE for the two groups of agents. In this case, while $\bar{k}^f = \bar{k}^c = 0$ and $\bar{a} = B$, there are two solutions for b : $\{\bar{b} = 0\}$ and $\{\bar{b} = 1/A(1 - \alpha)\}$. The solution that gives a non-degenerate model for chartists is the second one,¹⁸ and with this we have

$$\Delta = MSE^c - MSE^f = \frac{[B^2(A(1 - \alpha))^2 - 2B^2]}{(A(1 - \alpha))^2 - 1} \sigma_\varepsilon^2. \quad (31)$$

It is then easy to show that $\Delta = MSE^c - MSE^f < 0$ if and only if $1 < [A(1 - \alpha)]^2 < 2$: since A and α are positive, the condition requires $1 < A(1 - \alpha) < \sqrt{2}$. In this case, chartists outperform fundamentalists.

Proposition 5 *In an economy described by (1) and (2) with $\rho = 0$ and expectations given by (8)-(10), where k^f , k^c , a and b are as defined by (14)-(16), agents using the forecasting model (9) (chartists) have smaller mean squared errors than agents using the forecasting model (8) (fundamentalists)*

¹⁸Note that Figure 3 shows $\text{sign}(\Delta)$ using a solution that corresponds to the first one here, which is the only real solution in that case. The solution used in Figure 3, and throughout the paper, has $b \rightarrow 0$ as $\rho \rightarrow 0$, i.e., the estimated correlation in the endogenous variable goes to zero as the correlation in the exogenous driving process goes to zero.

when $1 < A(1 - \alpha) < \sqrt{2}$ or, equivalently, $1 - \frac{\sqrt{2}}{A} < \alpha < 1 - \frac{1}{A}$.

Our last condition in Proposition 5 is exactly the same as the result in Guse (2010), Section 3.2: there need to be enough chartists in the economy¹⁹ for an AR(1) model to be able to outperform the MSV forecasting model. Only in this case, in fact, agents expectations generate enough persistence to make their forecasts superior. Note that, when $\rho = 0$, there is no intrinsic correlation in the economy, and all persistence in data comes from the expectations of agents. This is not the case in general with $\rho > 0$, as in this case persistence is built into the economy through the exogenous driving force.

4.2 Endogenous predictor choice dynamics

To analyze the endogenous dynamics of α , we assume that at each time t agents choose their predictor (i.e., their identity as fundamentalists or chartists) according to a discrete logit model based on relative performance:

$$\alpha_{t+1} = \frac{\exp\{-\beta MSE_t^f\}}{\exp\{-\beta MSE_t^f\} + \exp\{-\beta MSE_t^c\}} \quad (32)$$

or equivalently

$$\alpha_{t+1} = \frac{1}{2} \left(\tanh \left[\frac{\beta}{2} (MSE_t^c - MSE_t^f) \right] + 1 \right), \quad (33)$$

¹⁹Note that A needs to be greater than 1 for the condition in Proposition 5 to hold and chartists to be able to outperform fundamentalists. For example, for $A = 1.5$, the condition reduces to $\alpha < .33$; for $A = 2$, the condition requires $\alpha < .5$.

where β is the "intensity of choice" parameter (Brock and Hommes, 1997), a measure of how fast agents switch predictors, and MSE_t^f and MSE_t^c are the unconditional mean square errors (see Appendix B), for fundamentalists and chartists, computed at time t with optimal values for $(\bar{k}^f, \bar{k}^c, \bar{a}, \bar{b})$, for a given value of α .²⁰ In other words, at each point in time agents evaluate the relative performance of the two models, given the current distribution of "types" in the population (i.e., given α_t), and decide whether to switch to the alternative model or not: α gets therefore updated accordingly. In choosing to use the unconditional MSE instead of real time realized errors, we have followed Branch and Evans (2006), who point out that this solution is better in stochastic environments where agents could otherwise be misled by short period anomalies.

Denoting $\Delta_t = MSE_t^c - MSE_t^f$, and noting how this measure depends on the structural parameters (A, B, ρ) and on α_t (see Appendix B), we can write

$$\begin{aligned}\alpha_{t+1} &= f(\Delta_t; \beta) \\ \Delta_t &= g(\alpha_t; A, B, \rho, \sigma_\varepsilon^2)\end{aligned}$$

and therefore, with some economy of notation

$$\alpha_{t+1} = f[g(\alpha_t)]. \quad (34)$$

Note that as α itself is a function of the structural parameters, the long-

²⁰Note that these optimal values $(\bar{k}^f, \bar{k}^c, \bar{a}, \bar{b})$ depend on α_t , and so change over time.

run behavior of the system will ultimately be determined by $(A, B, \rho, \beta, \sigma_\varepsilon^2)$ and by the initial condition α_0 . In other words, a fixed point $\bar{\alpha}$ of equation (34) will be a function of $(A, B, \rho, \beta, \sigma_\varepsilon^2; \alpha_0)$.

Definition 6 *An endogenous heterogeneous expectations equilibrium (EHEE) is defined as an heterogeneous expectations equilibrium (as per Definition 1) with $\bar{\alpha}$ determined as a fixed point of (33).*

Note that for $\beta = \infty$, we have a deterministic choice model where agents are fully rational and all choose the predictor with the smaller MSE, while for $\beta = 0$ agents split equally between the two predictors independently of the relative performance.

4.2.1 The case with $\beta = \infty$

We start by setting $\beta = \infty$ and so find the limiting behavior of the system when all agents chose the best performing model at all times. The dynamics of α in this case depends only on the sign of Δ :

$$\Delta > 0 \implies \alpha = 1$$

$$\Delta < 0 \implies \alpha = 0.$$

To check whether $\alpha = 1$ and/or $\alpha = 0$ are equilibrium points in this system, we need to check the sign of Δ at these two points. We will use the fact that $\bar{k}^f = \bar{k}^c = 0$ for any value of the parameters $(A, B, \rho; \alpha)$.

For $\alpha = 1$, it is easy to show that

$$\begin{aligned} MSE^c &= \left(\frac{B}{1 - A\rho} \right)^2 \sigma_w^2 > 0 \\ MSE^f &= 0 \end{aligned}$$

which implies $\Delta > 0$ and therefore $\alpha = 1$ is an equilibrium point for the dynamics of α .

Proposition 7 *For $\beta = \infty$, the point $\alpha = 1$ is an absorbing state for the dynamics of α_t described in (33).*

For $\alpha = 0$, we can show that

$$\begin{aligned} MSE^c &= \left[\frac{(1 + \bar{b}^2 - 2A\bar{b}^3) [B^2 (1 + A\rho\bar{b}^2)]}{(1 - A\rho\bar{b}^2) (1 - (A\bar{b}^2)^2)} - \frac{2B^2\rho\bar{b}}{(1 - A\rho\bar{b}^2)} \right] \sigma_w^2 \\ MSE^f &= \left[\frac{B^2 (1 + A\rho\bar{b}^2)}{(1 - A\rho\bar{b}^2) (1 - (A\bar{b}^2)^2)} + \bar{a}^2 - \frac{2\bar{a}B}{(1 - A\rho\bar{b}^2)} \right] \sigma_w^2 \end{aligned}$$

and therefore, using the solution expression for \bar{a} from (15):

$$\Delta = \left[\frac{\bar{b}^2 (1 - 2A\bar{b}) [(1 + A\rho\bar{b}^2)]}{(1 - A\rho\bar{b}^2) (1 - (A\bar{b}^2)^2)} - \frac{2\rho\bar{b}}{(1 - A\rho\bar{b}^2)} + \frac{1}{(1 - A\rho\bar{b}^2)^2} \right] B^2 \sigma_w^2. \quad (35)$$

Though the sign of Δ seems ambiguous, we have already seen that it is possible to find analytical results for $\rho = 0$. As shown in Section 4.1.1, $\Delta < 0$ for $1 < A(1 - \alpha) < \sqrt{2}$: for $\alpha = 0$, this condition means that when $1 < A < \sqrt{2}$ we have $MSE^c < MSE^f$ and $\alpha = 0$ is an equilibrium. This implies the following proposition:

Proposition 8 *For $\beta = \infty$, when $\rho = 0$, $1 < A < \sqrt{2}$ and chartists use the solution given by $\bar{b} = 1/A(1 - \alpha)$, the point $\alpha = 0$ is an absorbing state for the dynamics of α_t described in (33).*

The last two propositions say that if all agents choose the best performing model at all times (i.e., $\beta = \infty$), $\alpha = 1$ and $\alpha = 0$ can be equilibrium points in the endogenous dynamics. Proposition 7 says that $\alpha = 1$ is always an absorbing state, for any parameterization of the model, and this rules out cycles between the two fixed points as possible limiting behavior. Proposition 8 reveals that there are parameterizations for which also $\alpha = 0$ is an absorbing state for the dynamics of α : in particular, $\alpha = 0$ is an absorbing state for $\rho = 0$ and $1 < A < \sqrt{2}$.

Note that Proposition 5 also defines the basin of attraction for the equilibrium point $\alpha = 0$ when $\beta = \infty$ and $\rho = 0$: if $1 - \frac{\sqrt{2}}{A} < \alpha_0 < 1 - \frac{1}{A}$, then $\Delta < 0$ and $\alpha \rightarrow 0$. Since, from Proposition (8), $1 < A < \sqrt{2}$, we have that $1 - \frac{\sqrt{2}}{A} < 0$, and the previous condition reduces to $\alpha_0 < 1 - \frac{1}{A}$.

Could there be an interior solution as well, with $\rho = 0$, besides $\alpha = 0$ and $\alpha = 1$? Equation (31) reveals that when $\rho = 0$, while for $A(1 - \alpha) = 1$, $\Delta = -\infty$ and we have a discontinuity, for $A(1 - \alpha) = \sqrt{2}$, $\Delta = 0$: this means that $\alpha = 1 - \frac{\sqrt{2}}{A}$ could be an interior solution for the dynamics of α , one in which chartists and fundamentalists coexist and have the same forecasting error. But we know from Proposition 8 that, in order to have $\alpha = 0$ as a solution, it must be $1 < A < \sqrt{2}$, which means that $\alpha = 1 - \frac{\sqrt{2}}{A} < 0$ and so it is not a feasible solution. Thus, for parameterizations where $\alpha = 0$ and $\alpha = 1$ are equilibria, there can not be an interior solution, and the basins

of attraction for the two equilibria are separated by a discontinuity in Δ . There can be though an interior solution for $\rho = 0$ and $A > \sqrt{2}$ (which rules out $\alpha = 0$ as an equilibrium) and this happens when $\alpha = 1 - \frac{\sqrt{2}}{A}$. In this case $\Delta = 0$ and agents have no incentive to switch model. By computing the Jacobian (36) with $\rho = 0$, though, we can see that this heterogeneous expectations equilibrium is not E-stable.

What about the generic case with $\rho > 0$? From Figure 3, we can then see that also for $\rho > 0$, $\alpha = 0$ can be an absorbing state for the dynamics of α when $\beta = \infty$: this happens when Δ is negative at $\alpha = 0$. Specifically, this is the case (second quadrant) for $\rho = .6$ and A roughly between .75 and 1, and (third quadrant) for $\rho = .9$ and A roughly between .55 and 1. This means that, contrary to the case with $\rho = 0$, when there is enough autocorrelation in the driving force for the economy, the feedback from expectations to outcomes (i.e., parameter A) can be smaller than one and still have chartists outperforming fundamentalists and driving them out of the market.

Results of this section imply that when all agents choose the best performing model at all times, both an economy populated only by fundamentalists (for any parameterization) and an economy populated only by chartists (for some parameterizations) can represent equilibria. Moreover, there is also the possibility of an equilibrium where chartists and fundamentalists coexist, with their relative proportion depending on the structural parameter A . This equilibrium though is never learnable by agents.

4.2.2 Finite β

We analyze now the system for finite values of the intensity of choice parameter β . When $\beta < \infty$, a second dimension of bounded rationality is introduced in the model besides adaptive learning: agents, though choosing the model to use for their forecasts according to the relative performance of competing models, do not choose the best performing model at all times.

In order to analyze the dynamics of α_t , given a certain initial value for α_0 , the key element to define is $\frac{\delta\alpha_{t+1}}{\delta\alpha_t} = \frac{\delta\alpha_{t+1}}{\delta\Delta_t} \frac{\delta\Delta_t}{\delta\alpha_t}$: while the first part is always positive, the sign of the second is ambiguous and not constant over the parameter space. Numerical investigations reveal in fact that Δ is non-monotonic both in the dimension of A and α : by plotting the value of Δ against A and α , we can see that the surface has many local minimum and maximum points. An example for specific parameter ranges is shown in Figure 4.

We take $B = .1$ and $\sigma_c^2 = .1$ as baseline values (note from (35) that these two parameters affect Δ monotonically, and do not affect its sign and derivatives w.r.t. A and α), and investigate the dynamics of α_t for different values of A and ρ . We set $\beta = 10$, and initialize the system at $\alpha_0 = .5$. Numerical simulations suggest that for values of A below 1, fundamentalists always prevail, and α_t converges to a fixed point $\bar{\alpha} \in (.5, 1]$. We then try to start the system with a lower α , setting $\alpha_0 = .1$, in regions where $MSE^c < MSE^f$: even in this case, the dynamics of endogenous predictor choice drive the system towards a resting value for α greater than .5. This means that, even though, when $\beta = \infty$, $\alpha = 0$ can be an absorbing state,

with a relatively small intensity of choice parameter the system will not be driven towards this outcome. This happens because, even when Δ is negative at the initial α_0 , its (absolute) value is small, and tends to move the system towards a region where fundamentalists have smaller forecast errors.

Results change significantly if we introduce instead a strong feedback from expectations to current endogenous variables by setting $A > 1$. We will focus here on three scenarios, respectively defined by $(A, \rho) = (1.5, .3)$, $(1.5, .6)$, $(1.5, .9)$, with $\alpha_0 = .5$.²¹ In Figure 5 we report results for the dynamics of α_t in each of these three different cases. As it can be seen, for low values of ρ , fundamentalists still prevail even when there is high feedback from expectations to current variables, though as ρ increases the advantage of fundamentalists decreases. For high values of ρ , then, chartists are able to outnumber fundamentalists, as shown in the third quadrant. This suggests that there must be a transition between one scenario and the other where interesting dynamics might emerge. We in fact find such a region for values of ρ about .62, where the system shows cycling dynamics, as shown in Figure 5.²² These cycles increase their frequency as ρ increases to about .63, and then disappear as the system quickly enters into a region where the restriction on $\Omega\rho$ is violated. As ρ increases even further, this condition is then satisfied again at about $\rho \simeq .87$, at which point chartists prevail over fundamentalists. The same type of behavior emerges for different combination of the parameter values: in particular, if A is increased, a lower ρ is sufficient

²¹Other combinations of A and ρ can produce the same three scenarios: in general, lower values of A require higher values of ρ in order to produce the same scenario.

²²In all results reported, we have checked that the condition on $\Omega\rho$ is satisfied.

to trigger cycles and then to make chartists outperform fundamentalists.

Comparing Figure 3 and 5, we can see that for $\rho = .9$, α settles down to a value smaller than .5: this is probably due to the fact that we started the system at $\alpha_0 = .5$, in the black area of Figure 3, which means that chartists outperform fundamentalists and so α_t decreases below .5 and settles down there. It appears then that the cycles we observe for $\rho = .62$ are due to the fact that in this case the system starts (with $\alpha_0 = .5$) right on the border between the black and the white regions, and the dynamics induced by (33) make α_t bounce repeatedly from one region to the other.

These simulations show different types of intrinsic heterogeneity that can emerge in this model in different regions of the parameter space: the relative size of the two groups can stabilize on a fixed value, with one group or the other prevailing in size, or it can keep moving in cycles, as the two groups alternate in taking the lead.

A regularity that has emerged from our simulations is that there must be enough persistence in the exogenous driving process for chartists' AR(1) model to be able to compete with the fundamentalists' one. When there is a strong (positive) feedback effect from expectations to current endogenous variables, and a high autocorrelation coefficient in the exogenous driving process, chartists are able to outperform fundamentalists and the economy can settle down into an equilibrium where $\alpha < .5$.

Remark 9 *In an economy described by (1) and (2) with i) expectations given by (8)-(10), where k^f , k^c , a and b are as defined by (14)-(16), and α is determined endogenously by (33); ii) and fixing parameter values $B =$*

.1, $\beta = 10, \sigma_\varepsilon^2 = .1$; there are 3 possible outcomes depending on the values assigned to A, ρ :

a) agents using the forecasting model (8) outperform agents using the forecasting model (9) and α stabilizes on a value $> .5$;

b) agents using the forecasting model (9) outperform agents using the forecasting model (8) and α stabilizes on a value $< .5$;

c) agents using forecasting models (8) and (9) alternate in outperforming each other and α shows cycling dynamics.

In particular, scenario a) is observed for $A = 1.5, \rho < .62$; scenario b) for $A = 1.5, \rho \geq .87$; and scenario c) for $A = 1.5, \rho = .62$.

It is worth commenting on the cycle arising for $\rho = .62$, with α_0 initialized at .5: as it can be seen in Figure 5, α decreases slowly for a while, then drops abruptly before jumping up to 1; at this point we have only fundamentalists in the economy, but this situation doesn't last long, as immediately α starts dropping, the fraction of chartists is restored and the cycle repeats itself. This means that with a relatively low intensity of choice parameter, even $\alpha = 1$ is not an equilibrium value and chartists are not driven out of the market permanently.

Special case: $\rho = 0$ We consider now the case with $\rho = 0$: in this case, as pointed out before, Δ is defined by equation 31. Using this expression we have

$$\alpha_{t+1} = \frac{1}{2} \left(\tanh \left[\frac{\beta}{2} \frac{[B^2 (A(1 - \alpha_t))^2 - 2B^2]}{(A(1 - \alpha_t))^2 - 1} \sigma_\varepsilon^2 \right] + 1 \right).$$

We can thus compute numerically the equilibrium value for α , i.e., a fixed point of the above map, depending on the fundamental parameters A , B , β and σ_ε^2 and on the initial value α_0 . In particular, we fix B , β and σ_ε^2 as before and focus our attention on the parameter A .²³

In Figure 6 we set $\alpha_0 = .5$ and show the resting point for α for different values of A . As expected from Proposition 5, for $1 - \frac{\sqrt{2}}{A} < \alpha_0 < 1 - \frac{1}{A}$ we have that chartists outnumber fundamentalists and α settles down to a value $< .5$: given our choice of the initial value $\alpha_0 = .5$, this requires $2\sqrt{2} > A > 2$.

Note that there is a discontinuity at $(A(1 - \alpha))^2 = 1$ (i.e., $\alpha = 1 - \frac{1}{A}$) where Δ goes to infinity before changing sign. As the system approaches the discontinuity, cycles of period 2 emerge in the dynamics of α : the relative size of agents jumps above and below .5 repeatedly as $(A(1 - \alpha))^2$ jumps on either side of 1. Instead, for $(A(1 - \alpha))^2 = 2$, (i.e., $\alpha = 1 - \frac{\sqrt{2}}{A}$) Δ changes sign going through zero, and the value of α simply goes from being $< .5$ to being $> .5$ (or vice-versa). Note that if $A = 2\sqrt{2}$, $\alpha = 1 - \frac{\sqrt{2}}{A} = \frac{1}{2}$, and this is a resting point for the system.

4.3 E-stability of the EHEE

Suppose now that agents didn't know the optimal values for parameters in the two forecasting models and needed to learn them in order to decide which model to use: which EHEE found above would survive under adaptive learning dynamics?

²³Simulations show that results do not change qualitatively for β ranging between 1 and 100: relatively high values of β only tend to extremize the equilibrium values for α .

We have already seen for the case with $\beta = \infty$ that the heterogeneous expectations equilibrium that emerges there with $\alpha = 1 - \frac{\sqrt{2}}{A}$ is not learnable. What about equilibria that emerge with finite β ? It is easy to see, by comparing Figure 1 and Figure 3, that all regions where $MSE^c < MSE^f$, i.e., all regions where chartists outperform fundamentalists, are regions of E-instability for the HEE. This means that equilibria where chartists prevail can not be learned by agents. In other words, the system can never converge, under joint endogenous predictor choice and learning dynamics, to an EHEE where chartists prevail over fundamentalists (i.e., $\bar{\alpha} < .5$), as these outcomes are not E-stable.

Remark 10 *Numerical investigations show that in regions where $MSE^c < MSE^f$, the E-stability condition of Remark (4) is never satisfied.*

5 Conclusions

In this paper we have analyzed a simple univariate forward-looking model populated by agents that are heterogeneous in terms of their forecasting models. Following a growing literature in economics and finance, we have allowed for two types of agents, fundamentalists and chartists, whose relative group size is determined endogenously on the basis of the predictive power of the two models. While fundamentalists form their expectations using a model that correctly recognizes the role of an exogenous driving process, chartists employ a model that only uses past endogenous values to predict future ones.

Parameters in each model are chosen by agents so as to minimize the

expected mean squared errors of the ensuing predictions, and if these values are not known, they are learned using econometrics techniques such as RLS. A main finding of this paper is that fundamentalists, though using a model that is consistent with the fundamental REE, do not necessarily outperform chartists, and are not always able to drive them out of the market permanently. Situations in which chartists prevail, and others where the two groups alternate in taking the lead, also emerged in some regions of the parameter space. This means that intrinsic heterogeneity can persist even when a model consistent with the fundamental MSV equilibrium is available to agents for selection, and without the need to introduce ad-hoc costs of information gathering or processing.

Interestingly, though, we found that only equilibria where fundamentalists prevail can be adaptively learned by agents. Equilibria where chartists perform better than fundamentalists, or where cycles emerge in the dynamics of the endogenous size of the two groups, turn out not to be learnable. These results show that, in the simple setting we have used, agents are able to learn simultaneously in two dimensions, "within" a model and "across" models, only for equilibria where the majority of agents are using a forecasting model consistent with the fundamental solution of the economy.

More in general, results of this paper add to the existing literature and indicate that intrinsic heterogeneity is a feature that is very resilient to the specification of forecasting models available to agents and that it does not require ad-hoc costs to persist.

6 Appendix

6.1 Appendix A

Following Cardano's method (see Nickalls, 1993), it is possible to determine the number of real roots to (16) by studying the Cardano's equation $G^2 + 4H^3$, where

$$\begin{aligned} G &= -A^2(1-\alpha)^2\rho^3 + \frac{A^2(1-\alpha)^2\rho}{3} + 2\left(\frac{-A(1-\alpha)}{3}\right)^3 \\ H &= \frac{A\rho(1-\alpha)}{3} - \left(\frac{A(1-\alpha)}{3}\right)^2. \end{aligned}$$

It is possible to distinguish three cases:

1. $G^2 + 4H^3 > 0 \implies$ there exists only one real root to (16);
2. $G^2 + 4H^3 = 0 \implies$ there exists 3 real roots to (16), two of which are equal;
3. $G^2 + 4H^3 < 0 \implies$ there exists 3 distinct real roots to (16).

Note that $G^2 + 4H^3 = 0$ for $A = 0$, $\rho = 0$ or $\alpha = 1$: in any of these three cases, there are 3 real roots to (16), two of which are equal.

For A , $\rho \neq 0$, $\alpha \neq 1$, let's consider Cardano's equation as a function of A . It is then possible to show that there are only two non-zero roots to the equation $G^2 + 4H^3 = 0$, and these roots are real only for $\rho \leq 1/3$. Let's call these roots A_1 and A_2 , with $A_1 \leq A_2$.²⁴ In this case, since at $A = 0$ the first and second derivative of $G^2 + 4H^3$ w.r.t. A are equal to zero and the third derivative is positive, it follows that for $0 < A < A_1$, $G^2 + 4H^3 > 0$,

²⁴It is possible to show that $A_1, A_2 > 0$ if and only if $\rho < \sqrt{\frac{1}{3} + \sqrt{\frac{4}{3^3}}}$. So for any $\rho < 1/3$ the two roots are positive.

for $A_1 < A < A_2$, $G^2 + 4H^3 < 0$ and for $A > A_2$, $G^2 + 4H^3 > 0$.

For $\rho > 1/3$ the two (non-zero) roots of $G^2 + 4H^3 = 0$ are complex. By noting that, at $A = 0$, the first and second derivative of $G^2 + 4H^3$ w.r.t. A are equal to zero, and the third derivative is positive, it follows that for any $A > 0$, $G^2 + 4H^3 > 0$, and therefore there is only one real root to (16).

6.2 Appendix B

Denoting

$$\Gamma(a) = A\alpha a\rho + B$$

$$\Omega(b) = A(1 - \alpha)b^2$$

we can write recursively

$$\begin{aligned}\sigma_w^2 &= Ew_t^2 = \frac{1}{1 - \rho^2} \sigma_\varepsilon^2 \\ \sigma_{w,w_{-i}}^2 &= Ew_t w_{t-i} = \rho^i \sigma_w^2 \\ \sigma_{w,y}^2 &= Ew_t y_t = \frac{\Gamma}{1 - \Omega\rho} \sigma_w^2 \\ \sigma_{w,y_{-1}}^2 &= Ew_t y_{t-1} = \Gamma\rho \sigma_w^2 \sum_{i=0}^{\infty} (\Omega\rho)^i = \frac{\Gamma\rho}{1 - \Omega\rho} \sigma_w^2 \\ \sigma_y^2 &= Ey_t^2 = \left(\frac{\Gamma^2(1 + \Omega\rho)}{(1 - \Omega\rho)(1 - \Omega^2)} \right) \sigma_w^2 + \frac{(A\alpha k^f + A(1 - \alpha)(1 + b)k^c)^2}{1 - \Omega^2} \\ \sigma_{y,y_{-1}}^2 &= Ey_t y_{t-1} = \frac{(A\alpha k^f + A(1 - \alpha)(1 + b)k^c)^2}{1 - \Omega^2} + \frac{\Gamma^2(\Omega + \rho)}{(1 - \Omega\rho)(1 - \Omega^2)} \sigma_w^2.\end{aligned}$$

The expression for $\sigma_{w,y_{-1}}^2$ of course holds only for $\Omega\rho < 1$, which is a restriction we impose on the model in order to have finite covariance structures in

our economy.

Moreover:

$$MSE^f = E(y_t - (k^f + aw_t))^2 = (k^f)^2 + \sigma_y^2 + a^2\sigma_w^2 - 2a\sigma_{w,y}$$

$$MSE^c = E(y_t - (k^c + by_{t-1}))^2 = (k^c)^2 + \sigma_y^2 + b^2\sigma_{y,y-1} - 2b\sigma_{y,y-1}$$

6.3 Appendix C

The Jacobian for analyzing E-stability of equilibria is

$$J = \begin{bmatrix} \frac{\delta \dot{k}^f}{\delta k^f} & \frac{\delta \dot{k}^f}{\delta k^c} & \frac{\delta \dot{k}^f}{\delta a} & \frac{\delta \dot{k}^f}{\delta b} \\ \frac{\delta \dot{k}^c}{\delta k^f} & \frac{\delta \dot{k}^c}{\delta k^c} & \frac{\delta \dot{k}^c}{\delta a} & \frac{\delta \dot{k}^c}{\delta b} \\ \frac{\delta \dot{a}}{\delta k^f} & \frac{\delta \dot{a}}{\delta k^c} & \frac{\delta \dot{a}}{\delta a} & \frac{\delta \dot{a}}{\delta b} \\ \frac{\delta \dot{b}}{\delta k^c} & \frac{\delta \dot{b}}{\delta k^c} & \frac{\delta \dot{b}}{\delta a} & \frac{\delta \dot{b}}{\delta b} \end{bmatrix} \quad (36)$$

with

$$\begin{aligned} \frac{\delta \dot{k}^f}{\delta k^f} &= A\alpha - 1; \quad \frac{\delta \dot{k}^f}{\delta k^c} = A(1 - \alpha)(1 + b); \quad \frac{\delta \dot{k}^f}{\delta a} = 0 \\ \frac{\delta \dot{k}^f}{\delta b} &= A(1 - \alpha)k^c; \quad \frac{\delta \dot{k}^c}{\delta k^f} = A\alpha; \quad \frac{\delta \dot{k}^c}{\delta k^c} = A(1 - \alpha)(1 + b) - 1 \\ \frac{\delta \dot{k}^c}{\delta a} &= 0; \quad \frac{\delta \dot{k}^c}{\delta b} = A(1 - \alpha)k^c; \quad \frac{\delta \dot{a}}{\delta k^f} = 0 \\ \frac{\delta \dot{a}}{\delta k^c} &= 0; \quad \frac{\delta \dot{a}}{\delta a} = \frac{A\alpha\rho}{1 - A(1 - \alpha)\rho b^2} - 1; \quad \frac{\delta \dot{a}}{\delta b} = \frac{2A(1 - \alpha)\rho b(A\alpha\rho a + B)}{(1 - A(1 - \alpha)\rho b^2)^2} \\ \frac{\delta \dot{b}}{\delta k^c} &= 0; \quad \frac{\delta \dot{b}}{\delta k^c} = 0; \quad \frac{\delta \dot{b}}{\delta a} = 0 \\ \frac{\delta \dot{b}}{\delta b} &= 2bA(1 - \alpha) + \frac{-4\rho[A(1 - \alpha)]^2 b^3 + 6\rho^2[A(1 - \alpha)]^3 b^5 - 2\rho^2 A(1 - \alpha)b}{(1 + \rho A(1 - \alpha)b^2)^2} - 1 \end{aligned}$$

7 Figures

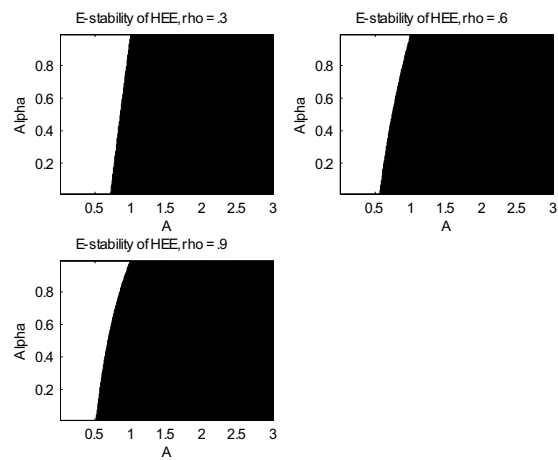


Figure 1: Contour map for the E-stability region (E-stable = white).

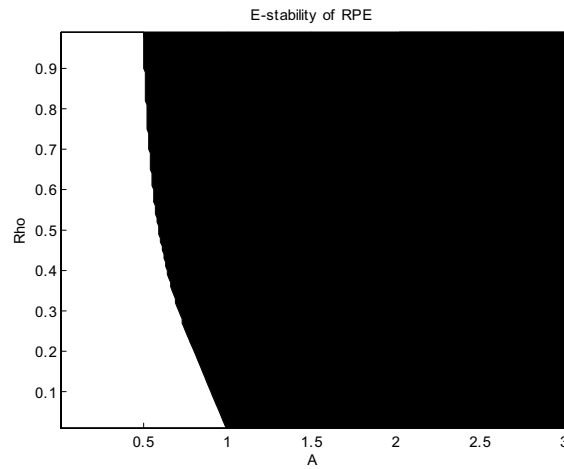


Figure 2: Contour map for the E-stability region for the RPE emerging when $\alpha = 0$.

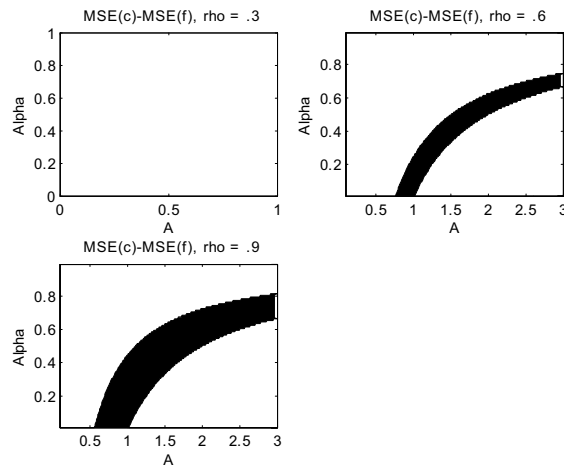
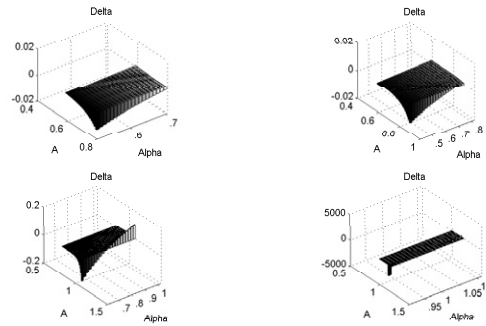
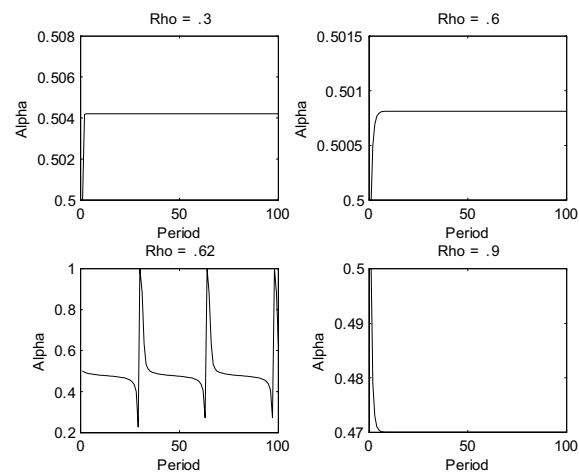


Figure 3: Relative performance of fundamentalists and chartists: black regions indicate chartists outperforming fundamentalists.

Figure 4: Value of Δ .Figure 5: Endogenous dynamics for α .

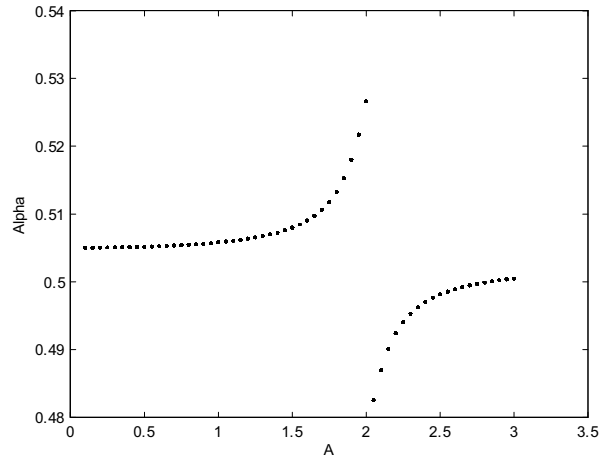


Figure 6: Endogenous limiting value for α with $\rho = 0$.

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