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Dominances on fuzzy variables based on credibility measure

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Abstract

This paper studies three notions of fuzzy dominance based on credibility measure, namely, the fuzzy mean-risk dominance, the first credibilistic dominance and the second credibilistic dominance.
More precisely, we introduce and examine some properties of the Fuzzy Lower Partial Moments (FLPM) of a fuzzy variable and, we deduce the Fuzzy Kappa index (FK) of a fuzzy variable, that is, a risk-adjusted performance measure of an asset or a portfolio with fuzzy return. Based on the aforementioned notion, we introduce the fuzzy mean-risk dominance of two fuzzy variables and we characterize it in three specific and interesting cases. We recall the first credibilistic dominance and the second credibilistic dominance for fuzzy variables

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introduced earlier by Peng et al. [20]. We characterize the first credibilistic dominance and determine some of its properties. We introduce and characterize the notion of crossing points of distributions of two fuzzy numbers and use them to characterize the second credibilistic dominance for fuzzy numbers. We justify that the first credibilistic dominance is stronger than the fuzzy mean-risk dominance and the second credibilistic dominance, and neither of these two later implies the other.

Key Words: Credibility measure; Fuzzy variable; Fuzzy Lower partial moment; Fuzzy mean-Risk dominance; First credibilistic dominance; Second credibilistic dominance; crossing points.

1 Introduction

In traditional decision theory and decision analysis, there are some situations where one lottery (such as probability distribution over possible outcomes) can be ranked as superior to another lottery. In such situations, following [7] and [2], we can use the term stochastic dominance as a form of stochastic ordering (see [23], section 1.E). It is based on preferences regarding outcomes. A preference might be a simple ranking of outcomes from the best to the least favored, or it might also employ a value measure (e.g., a number associated with each outcome that allows comparison of one outcome with another, such as two instances of winning a euro vs. one instance of winning two euros). In many theoretical or as well practical situations, one has a desirability to make a prediction about a decision-maker preference between given pairs of risky variables (random variables or fuzzy variables) without having knowledge of the decision maker utility function. In the literature, two main approaches have been proposed in the literature to compare two or multiple risky variables.

For the first approach, from seminal works of Markowitz [17] and Tobin [25], many scholars suggested to use the characteristics of such variables, (that are in general k-moments of a variable and in particular mean, variance, semi-variance, skewness, kurtosis, semi-kurtosis) to determine optimal portfolio when asset return is a (random or fuzzy) variable. One limitation of their analysis is due to the utilization of only the (first) moments (mean and variance) of the distribution, since the expected utility is in general, a function of all moments of the (probability or credibility) distribution. In

this paper, we do not consider this approach.

The second approach we consider, first works are based on dominance of random variables. Therefore, we have three main alternatives to [17] and [25]. Firstly, the possibility to focus on downside stochastic dominance namely mean-downside risk dominance, that is, a stochastic dominance of two variables under certain threshold. One example is the so-called lower partial moment (LPM) introduced by Bawa [2]. There are several applications of the mean-LPM for portfolio selection (see [3] and [11]). Notice that the mean-risk dominance is generally used in finance and economics for asset allocation (see [3], [9] and [18]). The role of the lower partial moment in economics can be seen in [24].
Secondly, in the case where the utility function of a decision-maker is unknown, a general stochastic dominance of any order that is based in comparison of two or multiple random variables can be possible and valuable for any utility function in a certain class which exhibits non-increasing marginal utility everywhere. Every expected utility maximizer with an increasing utility function will prefer lottery A over lottery B if A first-order stochastically dominates B. Thirdly, the other commonly used type of stochastic dominance is second-order stochastic dominance. Roughly speaking, for two lotteries A and B, lottery A has second-order stochastic dominance over lottery B if the former is more predictable (i.e. involves less risk) and has at least as high a mean. All risk-averse expected-utility maximizers (that is, those with increasing and concave utility functions) prefer a second-order stochastically dominant gamble to a dominated gamble. Since portfolio analysis typically assumes that all investors are risk averse, no investor would choose a portfolio that is second-order stochastically dominated by some other portfolio.

However, in order to complement this approach, we are sometimes faced with the situation where the values of random variables are not completely known. There are many other information and knowledge that cannot generally be described well by random variables because of the lack of historical data. For instance, investors in energy sector would like to estimate the coal reserves in some area, but even so after exploration analysis drawn by appraisers will always be "about billions of tons". The precede estimation "billions of tons" is a value expert's estimation rather than observations because the coal reserve has on exact true value that we don't know but estimate. In such a situation, because statistics and probability theory are unsuitable and Possibility theory proposed by Zadeh [26] is not self-dual, Liu [16] suggested to deal with it by introducing a *self-dual* credibility measure that is the av-

erage of possibility measure and necessity measure. The credibility measure theory provides an alternative framework for modeling economics and financial problems. Recently, Peng et al. [13, 20] also provided an extension of stochastic dominance to fuzzy variables based on credibility distributions. Apart from economics and Finance, it is of interest to define such extension to solve the problem of extending the natural ordering of the real line to fuzzy intervals has produced a large and scattered literature.
In this paper, we use the credibility theory to study and characterize some dominance relations between fuzzy variables.

In this paper, we consider a risk factor as a fuzzy variable based on credibility theory of Liu [16].

In this paper, after the introduction of the so-called Fuzzy lower partial moment (FLPM) based on credibility measure, we propose some of its properties. We also deduced the n-th order fuzzy kappa index (FK) as a the quotient between risk premium and the n-th squared root of the FLPM of a fuzzy variable. The precede, is a risk-adjusted performance measure of a given fuzzy variable. The fuzzy kappa index can be a helpful tool to classified assets or portfolios with different risk factors. We complete our credibility fuzzy dominance with a general analysis of the mean-risk dominance based on FLPM, the first and second order dominance of fuzzy variables. Our analysis complements the one of Peng et al. [13] and [20] in different ways. Because in some situations, it is not absolutely possible to compare two or multiple fuzzy variables, following Osuna [19] and in order to characterize the second order fuzzy dominance, we also introduce and characterize the crossings points between two fuzzy variables distributions.

After the introduction in Section 1, the paper is planned as follows: Section 2 reviews notions on credibility measure of [16], introduces FLPM, gives some of its examples for illustration, justifies that it becomes some well-known notions in some specific cases. We end this section by determining some interesting properties of the FLPM. Section 3 analyzes three dominance relations on fuzzy variables based on credibility theory. More precisely, we introduce and characterize the Mean-risk dominance relation based on FLPM on the set of fuzzy variables. We recall the first credibilistic dominance relation and the second credibilistic dominance relation on fuzzy variables introduced recently by Li et Peng [13]. We characterize and determine some of the properties of the first credibilistic dominance relation for fuzzy numbers. We introduce fuzzy counterparts of the two notions of Interval of Coincidence and Crossing Points of two random variables introduced by Osuna [19]. We use these two notions to characterize the second credibilistic dominance relation and, we characterize crossing points for fuzzy numbers. We establish

the relationship between the three relations.

2 PRELIMINARIES

2.1 Review on credibility measure

Let ξ be a fuzzy variable with membership function μ where for any $x \in \mathbb{R}$, $\mu(x)$ represents the possibility that ξ takes value x. ξ is normal if $\exists x_0/\mu(x_0) = 1$. ξ is non-negative fuzzy variable if $\forall x < 0, \mu(x) = 0$. A support of ξ is the crisp subset of $\mathbb{R}$ defined by $Supp(\xi) = \{x \in \mathbb{R}, \mu(x) > 0\}$ and its core is also a crisp subset of $\mathbb{R}$ defined by $Cor(\xi) = \{x \in \mathbb{R}, \mu(x) = 1\}$.

Throughout this paper, we assume that all fuzzy variables are normal, that is, $\exists x_0 \in \mathbb{R}, \mu(x_0) = 1$.

In Section 3.2, we will use a fuzzy number ξ which is a fuzzy variable satisfying: $\exists a, b, c, d \in \mathbb{R}$ with $a \leq b \leq c \leq d$ and (i) μ is upper semi-continuous, (ii) $\forall r \notin [a, d], \mu(r) = 0$, (iii) μ is increasing on $[a, b]$, (iv) $\forall r \in [b, c], \mu(r) = 1$ and (v) μ is decreasing on $[c, d]$.
We denote a fuzzy number by $\xi = (a, b, c, d)$ with $a \leq b \leq c \leq d$ and its support is the bounded interval $Supp(\xi) = Supp((a, b, c, d)) = [a, d]$.

In the particular case where μ is a straight line on $[a, b]$ and $[c, d]$, then $\xi = (a, b, c, d)$ is the usual and well-known trapezoidal fuzzy number. If $b = c$, then $\xi = (a, b, d)$ is a triangular fuzzy number.

Note that for ξ taking values in B, Zadeh [26] has defined the possibility measure of B by

$$Pos(\{\xi \in B\}) = \sup_{x \in B} \mu(x)$$

and the necessity measure of ξ by

$$Nec(\{\xi \in B\}) = 1 - \sup_{x \in B^c} \mu(x).$$

But neither of these measures are self-dual. Therefore, Liu and Liu [16] introduced the credibility measure as the average of possibility measure and necessity measure as follows: for any set B,

$$Cr(\{\xi \in B\}) = \frac{1}{2}\left(\sup_{x \in B} \mu(x) \ - \ \sup_{x \in B^c} \mu(x) + 1\right). \tag{1}$$

It is easy to show that credibility measure is self-dual. That is,

$$Cr(\{\xi \in B\}) + Cr(\{\xi \in B^c\}) = 1.$$

Liu and Liu [16] introduced the expected value of ξ defined as follows

$$E[\xi] = e = \int_0^{+\infty} Cr\{\xi \geq r\}\, dr \; - \int_{-\infty}^{0} Cr\{\xi \leq r\}\, dr \tag{2}$$

provided that at least one of the above integrals is finite. Note that, expected value is one of the most important concept for a fuzzy variable which gives the center of its distribution.
Liu [14] also defined the credibility distribution $\Phi : \mathbb{R} \to [0, 1]$ of a fuzzy variable ξ as follows:

$$\forall t \in \mathbb{R}, \Phi(t) = Cr\{\xi \leq t\} = \frac{1}{2}[1 + \sup_{x \in]-\infty;t]} \mu(x) - \sup_{x \in]t;+\infty[} \mu(x)]. \tag{3}$$

When Φ is absolutely continuous, we have the credibility density function $\phi : \mathbb{R} \to [0, \infty[$ such that

$$\forall t \in \mathbb{R}, \Phi(t) = \int_{-\infty}^{t} \phi(u) du. \tag{4}$$

Obviously, we have $\int_{-\infty}^{+\infty} \phi(u) du = 1$.

A distribution function Φ of ξ is a non-degenerate distribution function if $\forall t \in \mathbb{R}$, $t \neq E[\xi] \Rightarrow \Phi(t) \neq 0$.

The distribution function Φ of a fuzzy number $\xi = (a, b, c, d)$ is defined by:

$$\forall r \in \mathbb{R}, \Phi(r) = \begin{cases} 0 \text{ if } r < a \\ \frac{1}{2}\mu(r) \text{ if } a \leq r < b \\ \frac{1}{2} \text{ if } b \leq r < c \\ 1 - \frac{1}{2}\mu(r) \text{ if } c \leq r < d \\ 1 \text{ if } d \leq r \end{cases}. \tag{5}$$

It is an increasing function, that means, $\forall x \in [a, b], \forall y \in [b, c], \forall z \in [c, d]$, $\Phi(x) \leq \Phi(y) \leq \Phi(z)$.

In the next Paragraph, we will introduce the lower partial moment of a fuzzy variable, namely, the Fuzzy Lower Partial Moment (FLPM) of ξ and examine some of its properties.

2.2 Lower Partial Moments of fuzzy variables

In the following, we define the FLPM of a fuzzy variable and deduce some of its properties that seem similar to some well-known properties of lower partial moment (LPM) of a random variable under a probability measure. For illustration, we particularly focus on the determination of the FLPM for trapezoidal and triangular fuzzy variables.

2.2.1 Definitions, specific cases and examples

Definition 1. *Let ξ be a fuzzy variable, $\tau \in \mathbb{R}$ and $\alpha \in \mathbb{R}_+^*$. The Fuzzy Lower Partial Moment (FLPM) of ξ with order α and target value τ is defined as*

$$FLPM_{\alpha,\tau}[\xi] = E[\max(\tau - \xi, 0)^\alpha] \tag{6}$$

where the expectation operator E is defined by (2).

Following the paper of Kaplan and Knowles [11], we introduce the Fuzzy Kappa index which is a fuzzy risk-adjusted performance measure by using the fuzzy lower partial moment (FLPM) as a risk measure for a fuzzy variable.

Definition 2. *For $\tau \in \mathbb{R}$ and $\alpha \in \mathbb{R}_+^*$, the fuzzy Kappa index (FK) of the fuzzy variable ξ is:*

$$FK_{\alpha,\tau}[\xi] = \frac{E[\xi] - \tau}{(FLPM_{\alpha,\tau}[\xi])^{\frac{1}{\alpha}}}. \tag{7}$$

Throughout this paper, we will simply say "The Fuzzy Lower Partial Moment (FLPM) of ξ" instead of "The Fuzzy Lower Partial Moment (FLPM) of ξ with order α and target value τ."

Let us deduce from the previous definition some notions that are fuzzy counterpart of some well-known notion on LPM of a random variable under probability theory. We also provide an alternative definition of the FLPM of ξ based on its credibility density function.

Remark 1. *Let ξ be a fuzzy variable, $\tau \in \mathbb{R}$ and $\alpha \in \mathbb{R}_+^*$.*

1. *If the target value $\tau = E[\xi] = \mu$, then we obtain the following known notions:*
   - *For $\alpha \in 2\mathbb{N} - \{0\}$, $FLPM_{\alpha,\mu}[\xi]$ is the semi-moment of order α of ξ introduced earlier by Sadefo Kamdem et al.[21].*
   - *For $\alpha = 4$, $FLPM_{\alpha,\mu}[\xi]$ becomes the fuzzy semi-kurtosis of ξ introduced by Sadefo Kamdem et al.[21].*

- *For $\alpha = 2$, $FLPM_{\alpha,\mu}[\xi]$ becomes the fuzzy semi-variance of ξ introduced by Huang [10].*

2. *For some specific values of α, we obtain fuzzy counterparts of some well-known notions of the downside risk under probability theory:*

   - *We introduce the so-called credibility of loss equals to the 0th order $FLPM_{0,\tau}$ of ξ.*
   - *In the case where $\alpha = 1$, $FLPM_{1,\tau} = E[\max(\tau - \xi, 0)]$ is called the expected loss of ξ. Here the constant target value τ can be considered as the threshold point separating returns in two parts (downside returns and upside returns depending of the threshold).*
   - *For $\alpha = 2$, $FLPM_{2,\tau} = E[\max(\tau - \xi, 0)^2]$ is the Credibilistic Fuzzy Target Semi-Variance (CFTSV).*
   - *By setting $\alpha = 4$, we obtain $FLPM_{4,\tau}$) which is the Credibilistic Fuzzy Target Semi-Kurtosis (CFTSV).*

3. *The FLPM of ξ can be defined by means of the density function of ξ as follows:*

$$\begin{aligned} FLPM_{\alpha,\tau}[\xi] &= \int_0^{+\infty} Cr\{\max(\tau - \xi, 0)^\alpha \geq r\}\, dr = \alpha \int_{-\infty}^{\tau} (\tau - u)^{\alpha-1}\Phi(u)du \\ &= \int_{-\infty}^{\tau} (\tau - u)^\alpha d\Phi(u) = \int_{-\infty}^{\tau} (\tau - u)^\alpha \phi(u)du. \end{aligned} \tag{8}$$

The previous definition of $FLPM_{\alpha,\tau}$ shows that it is a function of the credibility distribution function and, it is a non-decreasing function of its target τ return. As τ increases, $FLPM_{\alpha,\tau}$ also increases.

We end this Section with the expressions of FLPM of trapezoidal and triangular fuzzy numbers.

Corollary 1. *1. The FLPM of the trapezoidal fuzzy variable $\xi = (a, b, c, d)$ is:*

$$FLPM_{\alpha,\tau}[\xi] = \begin{cases} 0 \ \textit{if } \tau < a \\ \frac{(\tau-a)^{\alpha+1}}{2(\alpha+1)(b-a)} \ \textit{if } a \leq \tau < b \\ \frac{[(\tau-a)^{\alpha+1}-(\tau-b)^{\alpha+1}]}{2(\alpha+1)(b-a)} \ \textit{if } b \leq \tau < c \\ \frac{[(\tau-a)^{\alpha+1}-(\tau-b)^{\alpha+1}]}{2(\alpha+1)(b-a)} + \frac{(\tau-c)^{\alpha+1}}{2(\alpha+1)(d-c)} \ \textit{if } c \leq \tau < d \\ \frac{[(\tau-a)^{\alpha+1}-(\tau-b)^{\alpha+1}]}{2(\alpha+1)(b-a)} + \frac{[(\tau-c)^{\alpha+1}-(\tau-d)^{\alpha+1}]}{2(\alpha+1)(d-c)} \ \textit{if } \tau \geq d \end{cases}. \tag{9}$$

2. The FLPM of the triangular fuzzy number (a,b,d) is:

$$FLPM_{\alpha,\tau}[\xi] = \begin{cases} 0 \ \textit{if}\ \tau < a \\ \frac{(\tau-a)^{\alpha+1}}{2(\alpha+1)(b-a)} \ \textit{if}\ a \leq \tau < b \\ \frac{[(\tau-a)^{\alpha+1}-(\tau-b)^{\alpha+1}]}{2(\alpha+1)(b-a)} + \frac{(\tau-b)^{\alpha+1}}{2(\alpha+1)(d-b)} \ \textit{if}\ b \leq \tau < d \\ \frac{[(\tau-a)^{\alpha+1}-(\tau-b)^{\alpha+1}]}{2(\alpha+1)(b-a)} + \frac{[(\tau-b)^{\alpha+1}-(\tau-d)^{\alpha+1}]}{2(\alpha+1)(d-b)} \ \textit{if}\ \tau \geq d \end{cases}. \tag{10}$$

In the appendix, we give some examples of FLPM of trapezoidal fuzzy number and, we introduce and study the Fuzzy Kappa index of such numbers.

In the next Paragraph, we determine some properties of FLPM of a fuzzy variable ξ based on its absolutely continuous credibility distribution function.

2.2.2 Some results on $\mathbf{FLPM}_{\alpha,\tau}$

The following result determines the credibility distribution function of ξ in term of derivatives of its FLPM when $\alpha \in \mathbb{N}^*$. More precisely, it establishes that we can determine the credibility distribution $\Phi(\tau)$ uniquely given $\mathrm{FLPM}_{\alpha,\tau}$ with $\alpha \in \mathbb{N}^*$.

Proposition 1. *The credibility distribution function Φ of a fuzzy variable ξ satisfies the following relation:*

$$\frac{d^\alpha}{d\tau^\alpha} FLPM_{\alpha,\tau} = \alpha!\,\Phi(\tau), \ \textit{that is}, \ \Phi(\tau) = \frac{1}{\alpha!}\frac{d^\alpha}{d\tau^\alpha} FLPM_{\alpha,\tau} \tag{11}$$

Proof: Let Φ be the credibility distribution function of the fuzzy variable ξ. We have:

$$\begin{aligned} \frac{d^\alpha}{d\tau^\alpha}\mathrm{FLPM}_{\alpha,\tau} &= \frac{d^\alpha}{d\tau^\alpha}[\int_{-\infty}^{\tau} (\tau-u)^\alpha d\Phi(u)] \\ &= \int_{-\infty}^{\tau} \frac{d^\alpha}{d\tau^\alpha}[(\tau-u)^\alpha d\Phi(u)]. \end{aligned}$$

It is easy to check that $\forall \alpha \in \mathbb{N}^*$, $\frac{d^\alpha}{d\tau^\alpha}(\tau-u)^\alpha = \alpha!$ and finally, we have:

$$\begin{aligned} \frac{d^\alpha}{d\tau^\alpha}\mathrm{FLPM}_{\alpha,\tau} &= \int_{-\infty}^{\tau} \alpha! d\Phi(u) = \alpha! \int_{-\infty}^{\tau} d\Phi(u) = \alpha![\Phi(\tau) - \lim_{u\to-\infty}\Phi(u)] \\ &= \alpha![\Phi(\tau) - 0] = \alpha!\Phi(\tau). \end{aligned}$$

Hence the result. □

The following result determines necessary and sufficient condition on a FLPM under which the density function ϕ of ξ belongs to exponential family.

Proposition 2. *Let ϕ be the credibility density function of a non-negative fuzzy variable ξ.*
ϕ belongs to exponential family, that means,

$$\phi(x) = e^{\gamma x + K(x) + D(\gamma)}, \qquad x \in (0, \infty), \quad \gamma > 0, \tag{12}$$

where $K(\cdot)$ and $D(\cdot)$ are arbitrary functions, if and only if, its $FLPM_{\alpha,\tau}$ satisfy a recurrence relationship

$$FLPM_{\alpha+1,\tau} = (\tau + D^{'}(\gamma))FLPM_{\alpha,\tau} - \frac{d}{d\gamma}FLPM_{\alpha,\tau}, \tag{13}$$

where $D^{'}(\gamma)$ is the derivative of $D(\gamma)$ with respect to γ.

Proof: ($\Rightarrow$) Assume that the credibility density function ϕ is defined by: $\phi(x) = e^{\gamma x + K(x) + D(\gamma)}$ where $x \in (0, \infty)$, $\gamma > 0$, and, K and D are arbitrary functions. Let us prove that $\mathrm{FLPM}_{\alpha+1,\tau} = (\tau + D^{'}(\gamma))\mathrm{FLPM}_{\alpha,\tau} - \frac{d}{d\gamma}\mathrm{FLPM}_{\alpha,\tau}$. We have:

$$\frac{d}{d\gamma}\mathrm{FLPM}_{\alpha,\tau} = \frac{d}{d\gamma}[\int_0^\tau (\tau - u)^\alpha e^{\gamma u + K(u) + D(\gamma)} du] = \int_0^\tau (\tau - u)^\alpha \frac{d}{d\gamma} e^{\gamma u + K(u) + D(\gamma)} du$$

i.e

$$\frac{d}{d\gamma}\mathrm{FLPM}_{\alpha,\tau} = \int_0^\tau (\tau - u)^\alpha (u + D'(\gamma)) e^{\alpha u + K(u) + D(\alpha)} du$$

i.e

$$\frac{d}{d\gamma}\mathrm{FLPM}_{\alpha,\tau} = \int_0^\tau u(\tau - u)^\alpha e^{\alpha u + K(u) + D(\gamma)} du + \int_0^\tau D'(\gamma)(\tau - u)^\alpha e^{\gamma u + K(u) + D(\gamma)} du$$

i.e

$$\frac{d}{d\gamma}\mathrm{FLPM}_{\alpha,\tau} = \int_0^\tau (u - \tau + \tau)(\tau - u)^\alpha e^{\gamma u + K(u) + D(\gamma)} du + D'(\gamma)\mathrm{FLPM}_{\alpha,\tau}$$

i.e

$$\frac{d}{d\gamma}\mathrm{FLPM}_{\alpha,\tau} = -\int_0^\tau (\tau - u)^{\alpha+1} e^{\gamma u + K(u) + D(\gamma)} du + \tau \int_0^\tau (\tau - u)^\alpha e^{\gamma u + K(u) + D(\gamma)} du + D'(\gamma)\mathrm{FLPM}_{\alpha,\tau}$$

i.e

$$\frac{d}{d\gamma}\text{FLPM}_{\alpha,\tau} = -\text{FLPM}_{\alpha+1,\tau} + (\tau + D'(\gamma))\text{FLPM}_{\alpha,\tau}.$$

Hence the result.

($\Leftarrow$) Now we prove the sufficient condition.
We have $\text{FLPM}_{\alpha+1,\tau} = (\tau + D^{'}(\gamma))\text{FLPM}_{\alpha,\tau} - \frac{d}{d\gamma}\text{FLPM}_{\alpha,\tau}$ and $\frac{\frac{d}{d\gamma}\text{FLPM}_{\alpha,\tau}}{\text{FLPM}_{\alpha,\tau}} = (\tau + D^{'}(\gamma)) - \frac{\text{FLPM}_{\alpha+1,\tau}}{\text{FLPM}_{\alpha,\tau}}$. By integrating each part of this last equality with respect to γ, we obtain:

$$\int_0^\tau (\tau - u)^\alpha \phi(u)du = e^{\alpha\tau + D(\gamma) - \int_0^\tau \frac{\text{FLPM}_{\alpha+1,\tau}}{\text{FLPM}_{\alpha,\tau}} d\gamma}$$

By setting $k(\tau) = -\int_0^\tau \frac{\text{FLPM}_{\alpha+1,\tau}}{\text{FLPM}_{\alpha,\tau}} d\gamma$, we get:

$$\int_0^\tau (\tau - u)^\alpha \phi(u)du = e^{\gamma\tau + D(\gamma) + k(\tau)} \tag{14}$$

Let us explicit the left member of relation (14).
A first integration by parts gives:

$$\int_0^\tau (\tau - u)^\alpha \phi(u)du = \alpha \int_0^\tau (\tau - u)^{\alpha-1} \phi(u)du$$

with $\Phi(0) = 0$, where Φ is the primitive function of ϕ and the credibility distribution of a nonnegative fuzzy variable.
A second integration by parts gives:

$$\alpha \int_0^\tau (\tau - u)^{\alpha-1} \phi(u)du = \alpha(\alpha - 1) \int_0^\tau (\tau - u)^{\alpha-2} \phi(u)du$$

By using inductive method, a αth integration by parts gives:

$$\int_0^\tau (\tau - u)^\alpha \phi(u)du = \alpha!\Phi(\tau)$$

By replacing this last result in relation (14), we get:

$$\alpha!\Phi(\tau) = e^{\gamma\tau + D(\gamma) + k(\tau)} \Leftrightarrow \Phi(\tau) = \frac{1}{\alpha!} e^{\gamma\tau + D(\gamma) + k(\tau)}.$$

So, we can set the function Φ as follows: $\Phi(u) = \frac{1}{\alpha!} e^{\gamma u + D(\gamma) + k(u)}$. The derivative of this function with respect to γ gives: $\phi(u) = \frac{\gamma + k^{'}(u)}{\alpha!} e^{\gamma u + D(\gamma) + k(u)}$

which can also be written as: $\phi(u) = e^{\gamma u + D(\gamma) + k(u) + \ln|\frac{\gamma + k^{'}(u)}{\alpha!}|}$. Finally, we take $K(u) = k(u) + \ln|\frac{\gamma + k^{'}(u)}{\alpha!}|$ and we get $\phi(u) = e^{\gamma u + D(\gamma) + K(u)}$. $\square$

We end this paragraph with the following useful result. For that, we need the following useful notation:
$\Phi(x^-) = \sup\{\Phi(y), y < x\}$ which means the credibility of getting a value of ξ not exceeding x.

Proposition 3. *Let ξ be a fuzzy variable, Φ its credibility distribution function, $\alpha \in \mathbb{R}_+^*$ and $\tau \in \mathbb{R}$.*

1.
$$FLPM_{\alpha,\tau}[\xi] = 0 \Leftrightarrow \Phi(\tau^-) = 0. \tag{15}$$

2.
$$\textit{If} \;\; \alpha = 1, \quad \textit{then} \;\; \Phi(\tau) = FLPM_{\alpha,\tau}[\xi]. \tag{16}$$

Proof: 1) ($\Rightarrow$) Assume that $\mathrm{FLPM}_{\alpha,\tau}[\xi] = 0$, then (8) implies $\forall r \in \mathbb{R}, r < \tau \implies \Phi(r) = 0$, that means, $\Phi(\tau^-) = \sup\{\Phi(r), r < \tau\} = 0$.
($\Leftarrow$) If $\Phi(\tau^-) = 0$, then the inequality $\Phi(r) \geq 0$ implies $\forall r \in \mathbb{R}, r < \tau \implies \Phi(r) = 0$. According to the relation (8), the previous implication leads to $\mathrm{FLPM}_{\alpha,\tau}[\xi] = 0$.

2) Assume that $\alpha = 1$.
According to the relation (8), we have: $\mathrm{FLPM}_{\alpha,\tau}[\xi] = \alpha \int_{-\infty}^{\tau} \phi(u)du = \alpha\Phi(\tau) = \Phi(\tau)$. $\square$

In the following Section, we introduce and analyze three dominance relations on fuzzy variables. For that, we proceed as follows:

1. We introduce and characterize the Mean-risk dominance relation based on FLPM on the set of fuzzy variables. We remark that this relation is not complete on the set of fuzzy variables.

2. We recall the first credibilistic dominance relation introduced recently by Li et Peng [13]. We characterize and determine some of its properties for fuzzy numbers.

3. We recall the second credibilistic dominance relation on fuzzy variables introduced by Peng et al. [20]. We introduce fuzzy counterparts of the two notions of Interval of Coincidence and Crossing Points of two random variables introduced by Osuna [19] and, we characterize crossing points for fuzzy numbers. We use these two notions to characterize the second credibilistic dominance relation.

4. We establish the relationship between the three relations.

3 Dominance relations of fuzzy variables

In the following Subsection, we study Mean-risk dominance for fuzzy variables.

3.1 Mean-risk dominance based on $\mathbf{FLPM}_{\alpha,\tau}$

We now define the fuzzy mean-risk dominance relation based on FLPM.

Definition 3. *Let $\alpha \in \mathbb{R}_+^*$ and $\tau \in \mathbb{R}$.*
The fuzzy mean-risk dominance is the binary relation on the set of fuzzy variables denoted by $\succeq_{\alpha,\tau}$ and defined as follows: For two fuzzy variables ξ_1 and ξ_2,

$$\xi_1 \succeq_{\alpha,\tau} \xi_2 \text{ if } \begin{cases} E[\xi_1] \geq E[\xi_2] \\ FLPM_{\alpha,\tau}[\xi_1] \leq FLPM_{\alpha,\tau}[\xi_2] \end{cases} \text{ with at least one strict inequality}. \tag{17}$$

Our first main result characterizes the dominance $\succeq_{\alpha,\tau}$ in the three following cases: (1) the two fuzzy variables have disjoint supports and τ is less than the minimum of the two supports, (2) the two fuzzy variables are symmetric and τ is between the minimum of the two supports and (3) one of the two fuzzy variables is a crisp one and the other one is a fuzzy variable with unbounded support on left and bounded on right at the threshold τ.

Notice that, intuitively speaking we can interpret the three results of this theorem as follows:

1. The first case means that $\succeq_{\alpha,\tau}$ is completely determined by expected returns when all possible returns for Φ_1 and Φ_2 lie at or above the target τ.

2. The second case means that if two distributions have equal means and one is certain to give a return as good as τ, then the former dominates the latter.

3. The third case reveals that the model may have a risk-seeking or "gambling" aspect when $\alpha < 1$ and all returns for Φ_1 and Φ_2 are at or below the target.

Theorem 1. *Assume that (17) holds. Then:*

1. *If* $\Phi_1(\tau^-) = \Phi_2(\tau^-) = 0$, *then* $\xi_1 \succeq_{\alpha,\tau} \xi_2$ *if and only if* $E[\xi_1] > E[\xi_2]$.

2. *If* $\begin{cases} E[\xi_1] = E[\xi_2] \\ \Phi_1(\tau^-) = 0 \\ \Phi_2(\tau^-) > 0 \end{cases}$ *then* $\xi_1 \succeq_{\alpha,\tau} \xi_2$.

3. *If* $E[\xi_1] = E[\xi_2] = \tau - r$ *(with* $r > 0$*),* Φ_1 *is a degenerate distribution that assigns credibility 1 to* $\tau - r$ *with* $r > 0$*, and* Φ_2 *is a non-degenerate distribution that has* $\Phi_2(\tau) = 1$*, then:*

$$\begin{cases} \xi_1 \succeq_{\alpha,\tau} \xi_2 \text{ if and only if } \alpha > 1 \\ \xi_2 \succeq_{\alpha,\tau} \xi_1 \text{ if and only if } \alpha < 1 \end{cases}. \tag{18}$$

To establish this proof, we recall the Jensen's Inequality for fuzzy variable introduced earlier by Liu [15] (Theorem 1.59, page 68):

" Jensen's Inequality: Let ξ be a fuzzy variable and $f : \mathbb{R} \to \mathbb{R}$ a strictly convex function. If $E[\xi]$ and $E[f(\xi)]$ are finite, then $f(E[\xi]) < E[f(\xi)]$."

We establish our proof:
Proof: 1) Let us assume that $\Phi_1(\tau^-) = \Phi_2(\tau^-) = 0$.
By the relation (15), we have $\text{FLPM}_{\alpha,\tau}[\xi_1] = \text{FLPM}_{\alpha,\tau}[\xi_2] = 0$.
($\Rightarrow$) We assume that $\xi_1 \succeq_{\alpha,\tau} \xi_2$.
If $E[\xi_1] \leq E[\xi_2]$, then by the fact that $\text{FLPM}_{\alpha,\tau}[\xi_1] = \text{FLPM}_{\alpha,\tau}[\xi_2] = 0$, there is not any strict inequality between the means or the fuzzy lower partial moments of the fuzzy variables ξ_1 and ξ_2; so by the definition of $\succeq_{\alpha,\tau}$, we have $\xi_1 \not\succeq_{\alpha,\tau} \xi_2$ and this contradicts the assumption. Therefore, we have: $E[\xi_1] > E[\xi_2]$.
($\Leftarrow$) We assume that $E[\xi_1] > E[\xi_2]$. Therefore, by the fact that $\text{FLPM}_{\alpha,\tau}[\xi_1] = \text{FLPM}_{\alpha,\tau}[\xi_2] = 0$, and the definition of $\succeq_{\alpha,\tau}$, we have $\xi_1 \succeq_{\alpha,\tau} \xi_2$.

2) Let us assume that $E[\xi_1] = E[\xi_2], \Phi_1(\tau^-) = 0, \Phi_2(\tau^-) > 0$.
By the relation (15), $\text{FLPM}_{\alpha,\tau}[\xi_1] = 0$ and $\text{FLPM}_{\alpha,\tau}[\xi_2] > 0$ and this leads to $\text{FLPM}_{\alpha,\tau}[\xi_2] > \text{FLPM}_{\alpha,\tau}[\xi_1]$. So, by the definition of $\succeq_{\alpha,\tau}$, we have: $\begin{cases} E[\xi_1] = E[\xi_2] \\ \text{FLPM}_{\alpha,\tau}[\xi_1] < \text{FLPM}_{\alpha,\tau}[\xi_2] \end{cases} \Rightarrow \xi_1 \succeq_{\alpha,\tau} \xi_2$.

3) Let us assume that Φ_1 is a degenerate distribution that assigns credibility 1 to $\tau - r$ with $r > 0$, and Φ_2 is a non-degenerate distribution that has $\Phi_2(\tau) = 1$ and $E[\xi_1] = E[\xi_2] = \tau - r$.
Let us set $f(y) = (\tau - y)^\alpha$ for $y \leq \tau$, and $r > 0$.
According to the fact that Φ_1 is a degenerate distribution function that assigns credibility 1 to $\tau - r$, we have the following equality $\int_{-\infty}^{\tau} (\tau - y)^\alpha d\Phi_1(y) =$

r^α. We also have: $f(E[\xi_1]) = r^\alpha$.
- If $0 < \alpha < 1$ then $-f$ is strictly convexe. By the Inequality of Jensens and the fact that $E[\xi_1] = E[\xi_2]$, we have: $E[-f(\xi_2)] = -\int_{-\infty}^{\tau}(\tau - y)^\alpha d\Phi_2(y) > -f(E[\xi_1])$, that means,
$\int_{-\infty}^{\tau}(\tau - y)^\alpha d\Phi_2(y) < r^\alpha$. Finally, we have:
$\int_{-\infty}^{\tau}(\tau - y)^\alpha d\Phi_2(y) < \int_{-\infty}^{\tau}(\tau - y)^\alpha d\Phi_1(y)$ and we conclude that $\xi_2 \succeq_{n,\tau} \xi_1$.
The converse case is obvious.
- If $\alpha > 1$ then f is strictly convexe. By the Inequality of Jensens and the fact that $E[\xi_1] = E[\xi_2]$, we have: $E[f(\xi_2)] = \int_{-\infty}^{\tau}(\tau - y)^\alpha d\Phi_2(y) > f(E[\xi_1])$, that means, $\int_{-\infty}^{\tau}(\tau - y)^\alpha d\Phi_2(y) > r^\alpha$. Finally, we have $\int_{-\infty}^{\tau}(\tau - y)^\alpha d\Phi_2(y) > \int_{-\infty}^{\tau}(\tau - y)^\alpha d\Phi_1(y)$ and we conclude that $\xi_1 \succeq_{\alpha,\tau} \xi_2$.
The converse case is obvious. □

Let us end this Paragraph by justifying that $\succeq_{\alpha,\tau}$ is not a complete relation on the set of fuzzy variables.

Remark 2. *If $\alpha = 1$, then according to the previous assumptions related to Φ_1 and Φ_2, $\Phi_1(\tau) = \Phi_2(\tau) = 1$ and by the relation (16), we have $FLPM_{\alpha,\tau}[\xi_1] = FLPM_{\alpha,\tau}[\xi_2] = 1$. Moreover, by the fact that $E[\xi_1] = E[\xi_2]$, neither ξ_1 nor ξ_2 satisfies one of the following relations: $\xi_1 \succeq_{\alpha,\tau} \xi_2$ and $\xi_2 \succeq_{\alpha,\tau} \xi_1$.*

In the following Subsection, we recall the first credibilistic dominance relation on the set of fuzzy variables. We characterize and determine some of its properties.

3.2 The First Credibilistic Dominance: $\succeq_1$

3.2.1 Definition and characterization

Definition 4. *(See Li et Peng [13], page 178, Definition 4) Let ξ_1 and ξ_2 be two fuzzy numbers and Φ_1, Φ_2 their credibility distribution functions respectively.*
The first credibilistic dominance is the binary relation on fuzzy variables denoted $\succeq_1$ and defined by: for all ξ_1 and ξ_2, $\xi_1 \succeq_1 \xi_2$ if $\forall r \in \mathbb{R}, \Phi_1(r) \leq \Phi_2(r)$.

From the previous definition, we deduce the strict component of $\succeq_1$ denoted by $\succ_1$ and its symmetric component denoted by $\sim_1$ defined as follows:

- $\xi_1 \succ_1 \xi_2$ if $\forall r \in \mathbb{R}, \Phi_1(r) < \Phi_2(r)$.
- $\xi_1 \sim_1 \xi_2$ if $\forall r \in \mathbb{R}, \Phi_1(r) = \Phi_2(r)$.

Consequently, we have $\xi_1 \succeq_1 \xi_2$, means that, $\xi_1 \succeq_1 \xi_2$ if ($\xi_1 \succ_1 \xi_2$ or $\xi_1 \sim_1 \xi_2$).

Our second main result characterizes $\succeq_1$ for two fuzzy numbers.

Theorem 2. *Let $\xi_1 = (a_1, b_1, c_1, d_1)$ and $\xi_2 = (a_2, b_2, c_2, d_2)$ be two fuzzy numbers. Then*

$$\xi_1 \succeq_1 \xi_2 \Leftrightarrow \begin{cases} a_1 \geq a_2 \\ b_1 \geq b_2 \\ c_1 \geq c_2 \\ d_1 \geq d_2 \end{cases} . \tag{19}$$

In other words, $\xi_1 \not\succeq_1 \xi_2$ if and only if ($a_1 < a_2$ or $b_1 < b_2$ or $c_1 < c_2$ or $d_1 < d_2$).
Figure 1 illustrates that the trapezoidal fuzzy number $\xi_2 = (a_2, b_2, c_2, d_2)$ dominates $\xi_1 = (a_1, b_1, c_1, d_1)$ by means of $\succeq_1$ while Figure 2 illustrates that neither dominates another by means of $\succeq_1$.

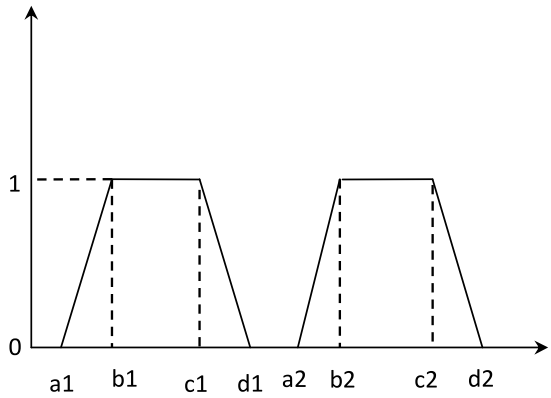

Figure 1: fuzzy number (a1, b1,c1,d1) dominated by (a2, b2,c2,d2) by means of $\geq$1

Proof: ($\Leftarrow$) Assume that $a_1 < a_2$ or $b_1 < b_2$ or $c_1 < c_2$ or $d_1 < d_2$ and let us prove that $\xi_1 \not\succeq_1 \xi_2$, that is, there exists some $r_0 \in \mathbb{R}$ such that $\Phi_1(r_0) > \Phi_2(r_0)$. We distinguish four cases:
1) Assume that $a_1 < a_2$. Let $r \in]a_1; a_2[$; $r > a_1 \Rightarrow \Phi_1(r) > \Phi_1(a_1) = 0$ and $r < a_2 \Rightarrow \Phi_2(r) = 0$, so we have $\Phi_1(r) > 0$ and $\Phi_2(r) = 0$. So, $\exists r \in \mathbb{R}/\Phi_1(r) > \Phi_2(r)$ which can be traduced by $\rceil(\forall r \in \mathbb{R}, \Phi_1(r) \leq \Phi_2(r))$.
2) Assume that $b_1 < b_2$. Let $r \in]b_1; b_2[$; $r > b_1 \Rightarrow \Phi_1(r) > \Phi_1(b_1) = \frac{1}{2}$ and $r < b_2 \Rightarrow \Phi_2(r) < \Phi_2(b_2) = \frac{1}{2}$, so we have $\Phi_1(r) > \Phi_2(r)$. So,

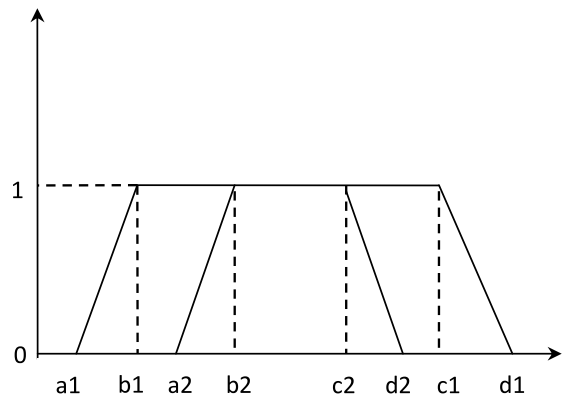

Figure 2: Incomparable fuzzy numbers by means of ≥ 1

$\exists r \in \mathbb{R}/\Phi_1(r) > \Phi_2(r)$ which can be traduced by $\rceil(\forall r \in \mathbb{R}, \Phi_1(r) \leq \Phi_2(r))$.
3) Assume that $c_1 < c_2$. Let $r \in]c_1; c_2[$; $r > c_1 \Rightarrow \Phi_1(r) > \Phi_1(c_1) = \frac{1}{2}$ and $r < c_2 \Rightarrow \Phi_2(r) < \Phi_2(c_2) = \frac{1}{2}$, so we have $\Phi_1(r) > \Phi_2(r)$. So, $\exists r \in \mathbb{R}/\Phi_1(r) > \Phi_2(r)$ which can be traduced by $\rceil(\forall r \in \mathbb{R}, \Phi_1(r) \leq \Phi_2(r))$.
4) Assume that $d_1 < d_2$. Let be $r \in]d_1; d_2[$; $r > d_1 \Rightarrow \Phi_1(r) = 1$ and $r < d_2 \Rightarrow \Phi_2(r) < \Phi_2(d_2) = 1$, so we have $\Phi_1(r) = 1$ and $\Phi_2(r) < 1$. So, $\exists r \in \mathbb{R}/\Phi_1(r) > \Phi_2(r)$ which can be traduced by $\rceil(\forall r \in \mathbb{R}, \Phi_1(r) \leq \Phi_2(r))$.
We conclude that $\xi_1 \not\succ_1 \xi_2$.

($\Rightarrow$) We use the method of contraposition.
Assume that that $a_1 \geq a_2$ and $b_1 \geq b_2$ and $c_1 \geq c_2$ and $d_1 \geq d_2$ and let us show that ξ_2 is dominated by ξ_1, that is, $\forall r \in \mathbb{R}, \Phi_1(r) \leq \Phi_2(r)$.
We consider the following cases:
- $\forall r \in]-\infty; a_2], r \leq a_2 < a_1$, so $\Phi_2(r) = \Phi_1(r) = 0$.
- $\forall r \in]a_2; a_1], a_2 < r \leq a_1$, and $\Phi_1(r) = 0, \Phi_2(r) > 0$. If $a_1 \geq b_2$, we have the same result else: by the fact that μ_1, μ_2 increase on $[a_1 \vee a_2, b_1 \wedge b_2]$, we have $\forall r \in]a_1; b_2]$, $\Phi_1(r) = \frac{1}{2}\mu_1(r)$, $\Phi_1(r) = \frac{1}{2}\mu_1(r)$. As $a_2 < a_1, b_2 < b_1$ and μ_1, μ_2 strictly increase on $[a_1, b_2]$ where the minimum value of ξ_1 (respectively of ξ_2) is reached on a_1 (respectively on a_2) and the maximum value of ξ_1 (respectively of ξ_2) is reached on b_1 (respectively on b_2) , we have $\mu_1(r) \leq \mu_2(r)$, which involves $\Phi_1(r) \leq \Phi_2(r)$.
- $\forall r \in]b_2; b_1], b_2 < r \leq b_1, \Phi_2(r) > \frac{1}{2}, \Phi_1(r) < \frac{1}{2}$ so $\Phi_2(r) > \Phi_1(r)$. If $b_1 \geq c_2$, we have the same result else:$\forall r \in]b_1; c_2], \Phi_2(r) = \Phi_1(r) = \frac{1}{2}$.
- $\forall r \in]c_2; c_1], c_2 < r \leq c_1, \Phi_2(r) > \Phi_1(r)$. If $c_1 \geq d_2$, we have the same result else: by the fact that μ_1, μ_2 decrease on $[c_1 \vee c_2, d_1 \wedge d_2]$, we have $\forall r \in]c_1; d_2]$,

$\Phi_1(r) = 1 - \frac{1}{2}\mu_1(r)$, $\Phi_1(r) = 1 - \frac{1}{2}\mu_1(r)$. As $c_2 < c_1, d_2 < d_1$ and μ_1, μ_2 strictly decrease on $[c_1, d_2]$ where the minimum value of ξ_1 (respectively of ξ_2) is reached on d_1 (respectively on d_2) and the maximum value of ξ_1 (respectively of ξ_2) is reached on c_1 (respectively on c_2) , we have $\mu_2(r) \leq \mu_1(r)$, which involves $\Phi_1(r) \leq \Phi_2(r)$.

- $\forall r \in]d_2; d_1], d_2 < r \leq d_1, \Phi_2(r) = 1, \Phi_1(r) > 1$.
- $\forall r \in]d_1; +\infty], \Phi_2(r) = \Phi_1(r) = 1$.

In all the cases, we have: $\Phi_1(r) \leq \Phi_2(r)$. $\square$

From the previous result, we deduce the characterization of $\sim_1$.

Corollary 2. *Let ξ_1 and ξ_2 be two fuzzy numbers. Then $\xi_1 \sim_1 \xi_2$ if and only if $\xi_1 = \xi_2$.*

Let us end this section by giving some properties on $\succeq_1$. The two first properties are satisfied for a given fuzzy number (a, b, c, d) and the other properties are satisfied on the particular family of trapezoidal fuzzy numbers.

3.2.2 Some properties of $\succeq_1$

Proposition 4. *Let ξ_i, ξ_j, ξ_k and ξ_l be four fuzzy numbers denoted respectively by $\forall t \in \{i, j, k, l\}$ $\xi_t = (a_t, b_t, c_t, d_t)$. We have the following properties:*

1. *If $\xi_i \succeq_1 \xi_j$ and $\xi_j \succeq_1 \xi_k$, then $\xi_i \succeq_1 \xi_k$.*
2. *If $\inf Supp(\xi_i) \geq \sup Supp(\xi_j)$, then $\xi_i \succeq_1 \xi_j$.*
3. *Furthermore, if $\forall t \in \{i, j, k, l\}$ $\xi_t = (a_t, b_t, c_t, d_t)$ is a trapezoidal fuzzy number, then*
   - *If $\xi_i \succeq_1 \xi_j$ then $-\xi_i \preceq_1 -\xi_j$.*
   - *If $\xi_i \succeq_1 \xi_j$ and $\xi_k \succeq_1 \xi_l$, then $\xi_i + \xi_k \succeq_1 \xi_j + \xi_l$.*
     In particular, if $\xi_j = \xi_k$, we have: $\xi_i + \xi_j \succeq_1 \xi_j + \xi_l$.
   - *If $\xi_i \succeq_1 \xi_j$, then $\forall \lambda \in \mathbb{R}^*, \begin{cases} \lambda\xi_i \succeq_1 \lambda\xi_j, & \text{if } \lambda > 0 \\ \lambda\xi_i \preceq_1 \lambda\xi_j, & \text{if } \lambda < 0 \end{cases}$.*

Proof: 1) Assume that $\xi_i \succeq_1 \xi_j$ and $\xi_j \succeq_1 \xi_k$. Let us prove that $\xi_i \succeq_1 \xi_k$. The two assertions $\xi_i \succeq_1 \xi_j$ and $\xi_j \succeq_1 \xi_k$ give $(a_i \geq a_j, b_i \geq b_j, c_i \geq c_j$ and $d_i \geq d_j)$ and, $(a_j \geq a_k, b_j \geq b_k, c_j \geq c_k$ and $d_j \geq d_k)$.
These two relations involve that $a_i \geq a_k, b_i \geq b_k, c_i \geq c_k, d_i \geq d_k$. Then $(a_i \geq a_k, b_i \geq b_k, c_i \geq c_k, d_i \geq d_k)$ and we deduce $\xi_i \succeq_1 \xi_k$.

2) Assume that $\inf Supp(\xi_i) \geq \sup Supp(\xi_j)$. Let us prove that $\xi_i \succ_1 \xi_j$.
$\inf Supp(\xi_i) \geq \sup Supp(\xi_j)$ means that $a_i \geq d_j$.
This last inequality also means that $d_i \geq c_i \geq b_i \geq a_i \geq d_j \geq c_j \geq b_j \geq a_j$ and we have: $a_i \geq a_j, b_i \geq b_j, c_i \geq c_j, d_i \geq d_j$. And this allows us to conclude that: $(a_i \geq a_j, b_i \geq b_j, c_i \geq c_j, d_i \geq d_j)$. Thus $\xi_i \succeq_1 \xi_j$.

3) Assume that $\forall t \in \{i, j, k, l\}$, $\xi_t = (a_t, b_t, c_t, d_t)$ is a trapezoidal fuzzy number. We distinguish three cases.

3-1) Assume that $\xi_i \succeq_1 \xi_j$. Let us show that $-\xi_i \preceq_1 -\xi_j$.
The assertion $\xi_i \succeq_1 \xi_j$ implies $(a_i \geq a_j, b_i \geq b_j, c_i \geq c_j, d_i \geq d_j)$ and this leads to $-a_i \leq -a_j, -b_i \leq -b_j, -c_i \leq -c_j, -d_i \leq -d_j$. By the Extension Principle of Zadeh, we have: $-\xi_j = (-a_j, -b_j, -c_j, -d_j)$ and $-\xi_i = (-a_i, -b_i, -c_i, -d_i)$. Since $(-a_i \leq -a_j, -b_i \leq -b_j, -c_i \leq -c_j, -d_i \leq -d_j)$, then $-\xi_i \preceq_1 -\xi_j$.

3-2) Assume that $\xi_i \succeq_1 \xi_j$ and $\xi_k \succeq_1 \xi_l$. Let us show that $\xi_i + \xi_k \succeq_1 \xi_j + \xi_l$.
The two assertions $\xi_i \succeq_1 \xi_j$ and $\xi_k \succeq_1 \xi_l$ imply $(a_i \geq a_j, b_i \geq b_j, c_i \geq c_j, d_i \geq d_j)$ and $(a_k \geq a_l, b_k \geq b_l, c_k \geq c_l, d_k \geq d_l)$.
These two relations involve that $a_i + a_k \geq a_j + a_l, b_i + b_k \geq b_j + b_l, c_i + c_k \geq c_j + c_l, d_i + d_k \geq d_j + d_l$. By the Extension Principle of Zadeh, we have: $\xi_i + \xi_k = (a_i + a_k, b_i + b_k, c_i + c_k, d_i + d_k)$ and $\xi_j + \xi_l = (a_j + a_l, b_j + b_l, c_j + c_l, d_j + d_l)$. Thus $\xi_i + \xi_k \succeq_1 \xi_j + \xi_l$.

3-3) Assume that $\xi_i \succeq_1 \xi_j$ and let λ be a real number different of zero.
The assertion $\xi_i \succeq_1 \xi_j$ implies $(a_i \geq a_j, b_i \geq b_j, c_i \geq c_j, d_i \geq d_j)$ and this leads to $\lambda a_i \leq \lambda a_j, \lambda b_i \leq \lambda b_j, \lambda c_i \leq \lambda c_j, \lambda d_i \leq \lambda d_j$ if $\lambda < 0$ and $\lambda a_i \geq \lambda a_j, \lambda b_i \geq \lambda b_j, \lambda c_i \geq \lambda c_j, \lambda d_i \geq \lambda d_j$ if $\lambda > 0$. By the Extension Principle of Zadeh, we have: $\lambda \xi_j = (\lambda a_j, \lambda b_j, \lambda c_j, \lambda d_j)$ and $\lambda \xi_i = (\lambda a_i, \lambda b_i, \lambda c_i, \lambda d_i)$ if $\lambda > 0$ and $\lambda \xi_j = (\lambda d_j, \lambda c_j, \lambda b_j, \lambda a_j)$ and $\lambda \xi_i = (\lambda d_i, \lambda c_i, \lambda b_i, \lambda a_i)$ if $\lambda < 0$.
We can conclude that $\begin{cases} \lambda \xi_i \succeq_1 \lambda \xi_j, & \text{if } \lambda > 0 \\ \lambda \xi_i \preceq_1 \lambda \xi_j, & \text{if } \lambda < 0 \end{cases}$. $\square$

In the following subsection, we recall the second credibilistic dominance relation on fuzzy variables introduced by Peng et al. [20]. We write fuzzy counterparts of the two notions of Interval of Coincidence and Crossing Points of two random variables introduced by Osuna [19] and, we characterize crossing points for fuzzy numbers. We use these two notions to characterize the second credibilistic dominance relation.

3.3 The Second Credibilistic Dominance: $\succeq_2$

3.3.1 Definition, crossing points and characterization

Definition 5. *(Peng et al. [20], page 299, Definition 10) Let ξ_1 and ξ_2 be two fuzzy numbers with Φ_1, Φ_2 their respective cumulative credibility distribution functions, ϕ_1 and ϕ_2 their respective density function with $\phi_1 \neq \phi_2$.*
$\xi_1 \succeq_2 \xi_2$ *if* $\begin{cases} \forall t \in \mathbb{R}, \int_{-\infty}^{t} [\Phi_2(r) - \Phi_1(r)]dr \geq 0 \\ \exists t_0 \in \mathbb{R}, \int_{-\infty}^{t_0} [\Phi_2(r) - \Phi_1(r)]dr > 0 \end{cases}$.

We note that $\int_{-\infty}^{t} [\Phi_2(r) - \Phi_1(r)]dr$ represents a balance of areas between Φ_1 and Φ_2 that means the difference of areas resulting from integrating each function from $-\infty$ to t, with the following order: the area below Φ_2 minus the area below Φ_1.

In the following, we will characterize the second credibilistic dominance $\succeq_2$ by writing the fuzzy counterpart of second order stochastic dominance's characterization for random variables proposed recently by Osuna [19]. Therefore, we introduce, analogously as did Osuna [19] for random variables (see Definition 3.1 P 760), the two notions of Interval of coincidence and crossing points for two fuzzy variables.

The Intervals of coincidence of two fuzzy variables is the half open interval, open at the right, where the two curves of their distributions functions coincide. For example, in Figure 3, the two straight lines entitled I.C. are the two intervals of coincidence of Φ_1 and Φ_2. Formally, we have:

Definition 6. *Intervals of coincidence (IC)*
The half-open interval $[a, b)$, with $a < b$ is an interval of coincidence (IC) for Φ_1 and Φ_2 if $\Phi_1(t) = \Phi_2(t)$ for all $t \in [a, b)$.

From this definition we can deduce that any value t_0 belongs to an interval of coincidence if there exists some $\epsilon > 0$ such that the interval $[t_0, t_0+\epsilon)$ is IC.

We now introduce two types of crossing points for fuzzy variables, namely, crossing point of type I and crossing point of type II. Analogously to Definition 3.2 of page 760 in Osuna [19], the crossing point of type II of ξ_1 and ξ_2 is the point where the two curves of their distribution functions intersect and the curve which strictly minimizes before that point strictly majorizes after that point. The crossing point of type I of ξ_1 and ξ_2 is the upper bound of a given interval of coincidence (point where the two curves of their distribution functions coincide before it and are distinct after it). Formally, we have the following Definition:

Definition 7. *Crossing points (CP)*

1. *If t_0 does not belong to an IC, but $[a, t_0)$ is an IC of type I and $a = \min t$ such that $[a, t_0)$ is an IC, t_0 corresponds to a CP if there exists some $\epsilon > 0$ such that for all $s \in]0, \epsilon[$, we have*

$$\left\{ \begin{array}{l} \Phi_1(a-s) \neq \Phi_2(a-s) \\ \Phi_1(t_0+s) \neq \Phi_2(t_0+s) \\ \left(\begin{array}{c} \Phi_1(a-s) - \Phi_2(a-s) < 0 \text{ and } \Phi_1(t_0+s) - \Phi_2(t_0+s) > 0 \\ \text{or} \\ \Phi_1(a-s) - \Phi_2(a-s) > 0 \text{ and } \Phi_1(t_0+s) - \Phi_2(t_0+s) < 0 \end{array} \right) \end{array} \right. .$$

2. *Any other value t_0 corresponds to a CP or type II if there exists some $\epsilon > 0$ such that for all $s \in]0, \epsilon[$, we have*

$$\left\{ \begin{array}{l} \Phi_1(t_0-s) \neq \Phi_2(t_0-s) \\ \Phi_1(t_0+s) \neq \Phi_2(t_0+s) \\ \left(\begin{array}{c} \Phi_1(t_0-s) - \Phi_2(t_0-s) < 0 \text{ and } \Phi_1(t_0+s) - \Phi_2(t_0+s) > 0 \\ \text{or} \\ \Phi_1(a-s) - \Phi_2(a-s) > 0 \text{ and } \Phi_1(t_0+s) - \Phi_2(t_0+s) < 0 \end{array} \right) \end{array} \right. .$$

3. *Convention: (a) if t_0 belongs to an IC, it does not correspond to a CP; (b) let $m_1 = \min\{t/\Phi_1(t) > 0\}$ and $m_2 = \min\{t/\Phi_2(t) > 0\}$, and let $t_1 = \min(m_1, m_2)$: the interval $(-\infty, t_1[$ is an IC and t_1 does not correspond to a CP.*

Our third main result establishes a characterization of the second credibilistic dominance.

Theorem 3. *Let ξ_1 and ξ_2 be two fuzzy variables, Φ_1 and Φ_2 their respective absolutely continuous credibility distribution and $t_{01}, ..., t_{0k}$ be their k $(k \geq 1)$ crossing points (ordered so increasing) such that $t_{01} > \min\{\min\{t : \Phi_1(t) > 0\}, \min\{t : \Phi_2(t) > 0\}\}$. Then*
$\xi_1 \succeq_2 \xi_2$ if and only if

$$\left\{ \begin{array}{l} \forall i = \{1, ..., k\}, \int_{-\infty}^{t_{0i}} [\Phi_2(r) - \Phi_1(r)]dr \geq 0 \\ \left(\begin{array}{c} \int_{-\infty}^{+\infty} [\Phi_2(r) - \Phi_1(r)]dr = 0 \quad \text{and } \exists t_{0h} \in \{t_{01}, ..., t_{0k}\}, \int_{-\infty}^{t_{0h}} [\Phi_2(r) - \Phi_1(r)]dr > 0 \\ \text{or} \\ \int_{-\infty}^{\infty} [\Phi_2(r) - \Phi_1(r)]dr > 0 \end{array} \right) \end{array} \right. .$$

Proof: According to the fact that the cumulative distribution functions (with respect to the credibility distribution) Φ_1 and Φ_2 have the same properties as the classical stochastic ones, we can refer to the proof proposed in Osuna [19], Theorem 4.1, page 761. □

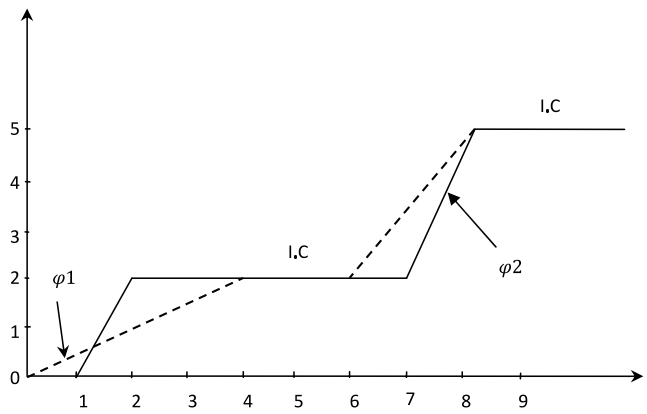

Figure 3: : Intervals of coincidence (I.C.) of two fuzzy variables based on their distribution functions

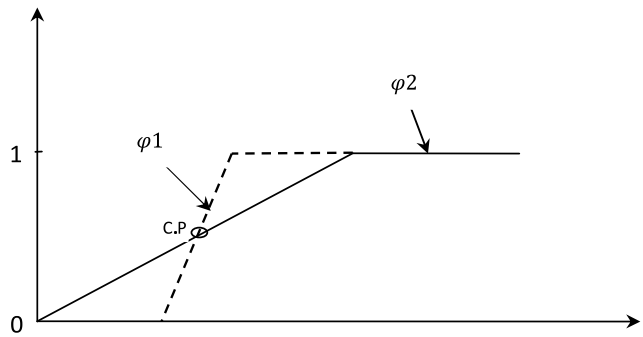

Figure 4: Crossing point (CP) of two fuzzy variables based on their distribution functions

3.3.2 Characterization of Crossing Points of two fuzzy numbers

The following result characterizes crossing points of two fuzzy numbers in the following six cases: (i) the three first cases are illustrated by Figure 5 and (ii) the three last cases allow us to find crossing points when the core of one of at least one of the fuzzy number is reduced to a single point, that is a crisp number.

Proposition 5. *Let $\xi_i = (a_i, b_i, c_i, d_i)$ and $\xi_j = (a_j, b_j, c_j, d_j)$ be two fuzzy numbers satisfying $[a_i \vee a_j, b_i \wedge b_j] \neq \emptyset$, $[b_i \vee b_j, c_i \wedge c_j] \neq \emptyset$ and $[c_i \vee c_j, d_i \wedge d_j] \neq \emptyset$. Let μ_i and μ_j be their respective membership functions, Φ_i and Φ_j be their respective credibility distribution functions. Then we have:*

1. $\left(\exists \epsilon > 0, \exists r_0 \in \mathbb{R}, \forall s \in]0, \epsilon[, \begin{cases} \mu_i(r_0 - s) < \mu_j(r_0 - s), \mu_i(r_0 + s) > \mu_j(r_0 + s) \\ r_0 - s, r_0 + s \in [a_i \vee a_j, b_i \wedge b_j] \end{cases} \right)$
$\Leftrightarrow r_0$ *is a crossing point of type II.*

2. $\left(\exists \epsilon > 0, \exists r_0 \in \mathbb{R}, \forall s \in]0, \epsilon[, \begin{cases} \mu_i(r_0 - s) < \mu_j(r_0 - s), \mu_i(r_0 + s) > \mu_j(r_0 + s) \\ r_0 - s, r_0 + s \in [c_i \vee c_j, d_i \wedge d_j] \end{cases} \right)$
$\Leftrightarrow r_0$ *is a crossing point of type II.*

3. $([b_i, c_i] \subseteq [b_j, c_j]$ *and* $[a_i, d_i] \subseteq [a_j, d_j], b_i \neq c_i, b_j \neq c_j) \Leftrightarrow c_i$ *is a crossing point of type I and* $b_i = \min\{t/[t, c_i)$ *is I.C*$\}$.

4. $([a_i, d_i] \subseteq [a_j, d_j], b_i = c_i, b_j \neq c_j, b_i \in [b_j, c_j]) \Leftrightarrow c_i$ *is a crossing point of type II .*

5. $([a_i, d_i] \subseteq [a_j, d_j], b_i \neq c_i, b_j = c_j, b_j \in [b_i, c_i]) \Leftrightarrow c_j$ *is a crossing point of type II.*

6. $([a_i, d_i] \subseteq [a_j, d_j], b_i = c_i = b_j = c_j, a_i \neq a_j, d_i \neq d_j) \Leftrightarrow c_j$ *is a crossing point of type II.*

To establish our result, we need the following Lemma.

Lemma 1. *Let* $\xi_i = (a_i, b_i, c_i, d_i)$ *and* $\xi_j = (a_j, b_j, c_j, d_j)$ *be two fuzzy numbers satisfying* $[a_i \vee a_j, b_i \wedge b_j] \neq \emptyset, [b_i \vee b_j, c_i \wedge c_j] \neq \emptyset$ *and* $[c_i \vee c_j, d_i \wedge d_j] \neq \emptyset$. *Let* μ_i *and* μ_j *be their respective membership functions and,* Φ_i *and* Φ_j *be their respective credibility distribution functions. Then we have:*

1. $\left(\exists \epsilon > 0, \exists r_0 \in \mathbb{R}, \forall s \in]0, \epsilon[, \begin{cases} \mu_i(r_0 - s) < \mu_j(r_0 - s), \mu_i(r_0 + s) > \mu_j(r_0 + s) \\ r_0 - s, r_0 + s \in [a_i \vee a_j, b_i \wedge b_j] \end{cases} \right)$
$\Leftrightarrow \Phi_i(r_0 - s) < \Phi_j(r_0 - s), \Phi_i(r_0 + s) > \Phi_j(r_0 + s)$.

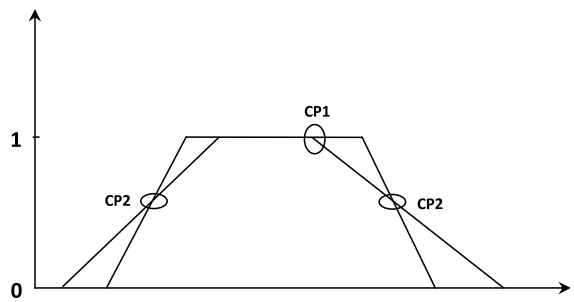

Figure 5: Crossings points of type I and type II of two fuzzy numbers based on their membership functions

2. $\left(\exists \epsilon > 0, \exists r_0 \in \mathbb{R}, \forall s \in]0, \epsilon[, \left\{ \begin{array}{l} \mu_i(r_0 - s) < \mu_j(r_0 - s), \mu_i(r_0 + s) > \mu_j(r_0 + s) \\ r_0 - s, r_0 + s \in [c_i \vee c_j, d_i \wedge d_j] \end{array} \right. \right)$
$\Leftrightarrow \Phi_i(r - s) > \Phi_j(r - s), \Phi_i(r + s) < \Phi_j(r + s)$.

3. $\forall r \in \mathbb{R}, (r \in [b_i \vee b_j, c_i \wedge c_j]) \Rightarrow \Phi_i(r) = \Phi_j(r)$.

Proof of the Lemma: Let us recall that Φ is given by (5). We suppose that: $[a_i \vee a_j, b_i \wedge b_j] \neq \emptyset, [b_i \vee b_j, c_i \wedge c_j] \neq \emptyset, [c_i \vee c_j, d_i \wedge d_j] \neq \emptyset$.
1) Let us suppose that $\exists \epsilon > 0, \exists r_0 \in \mathbb{R}$ and let be $s : 0 < s < \epsilon, \mu_i(r_0 - s) < \mu_j(r - s), \mu_i(r_0 + s) > \mu_j(r_0 + s)$ and $r_0 - s, r_0 + s \in [a_i \vee a_j, b_i \wedge b_j]$, then $\mu_i(r_0 - s) < \mu_j(r_0 - s) \Rightarrow \Phi_i(r_0 - s) = \frac{1}{2}\mu_i(r_0 - s) < \Phi_j(r_0 - s) = \frac{1}{2}\mu_j(r_0 - s)$ and
$\mu_i(r_0 + s) > \mu_j(r_0 + s) \Rightarrow \Phi_i(r_0 + s) = \frac{1}{2}\mu_i(r_0 + s) > \Phi_j(r_0 + s) = \frac{1}{2}\mu_j(r_0 + s)$.
The converse case is proved in the same manner.

2) Let us suppose that $\exists \epsilon > 0, \exists r_0 \in \mathbb{R}$ and let be $s : 0 < s < \epsilon, \mu_i(r_0 - s) < \mu_j(r_0 - s), \mu_i(r + s) > \mu_j(r_0 + s)$ and $r_0 - s, r_0 + s \in [c_i \vee c_j, d_i \wedge d_j]$, then $\mu_i(r_0 - s) < \mu_j(r_0 - s) \Rightarrow \Phi_i(r_0 - s) = 1 - \frac{1}{2}\mu_i(r_0 - s) > \Phi_j(r_0 - s) = 1 - \frac{1}{2}\mu_j(r_0 - s)$ and
$\mu_i(r_0 + s) > \mu_j(r_0 + s) \Rightarrow \Phi_i(r_0 + s) = 1 - \frac{1}{2}\mu_i(r_0 + s) < \Phi_j(r_0 + s) = 1 - \frac{1}{2}\mu_j(r_0 + s)$.
The converse case is proved in the same manner.

3) If $r \in [b_i \vee b_j, c_i \wedge c_j]$, then $\Phi_i(r) = \Phi_j(r) = \frac{1}{2}$. $\square$

Remark 3. *We have an analogous result if* $\exists \epsilon > 0, \exists r_0 \in \mathbb{R}, \forall s \in]0, \epsilon[, \mu_i(r_0 - s) > \mu_j(r_0 - s)$ *and* $\mu_i(r_0 + s) < \mu_j(r_0 + s)$.

We now give the proof of our Proposition.
Proof of the Proposition: 1) Let us suppose that $\exists \epsilon > 0, \exists r_0 \in \mathbb{R}$ and let be $s : 0 < s < \epsilon$ such that $\mu_i(r_0 - s) < \mu_j(r_0 - s)$ and $\mu_i(r_0 + s) > \mu_j(r_0 + s)$, with $r_0 - s, r_0 + s \in [a_i \vee a_j, b_i \wedge b_j$. According to the Lemma 1, we have $\Phi_i(r_0 - s) < \Phi_j(r_0 - s)$ and $\Phi_i(r_0 + s) > \Phi_j(r_0 + s)$ and by Definition 7, we can conclude that r_0 is a crossing point of type II.
We prove the converse case by the same manner.

2) We use the same method as in 1.
3) Let us show that c_i is a crossing point of type I.
$[b_i, c_i] \subseteq [b_j, c_j] \Rightarrow [b_i \vee b_j, c_i \wedge c_j] = [b_i, c_i]$ and by Lemma 1 and Definition 5, we have:
$b_i = \min\{t/[t, c_i) is I.C\}$.
Now, let us find $\epsilon > 0$ such that $\forall s : 0 < s < \epsilon, \Phi_i(b_i - s) < \Phi_j(b_i - s)$ and

$\Phi_i(c_i+s) > \Phi_j(c_i+s)$.
i) If $b_i \neq b_j$ and $c_i \neq c_j$
then we set $\epsilon = (b_i - b_j) \wedge (c_j - c_i)$ and we easily check that $\epsilon > 0$ according to the fact that $[b_i, c_i] \subset [b_j, c_j]$ and $b_i \neq b_j, c_i \neq c_j$. We have two cases:
<u>1st case</u>: $b_i - b_j < c_j - c_i$
We have $\epsilon = b_i - b_j$, and $b_i - \epsilon = b_j, c_i + \epsilon = c_i + b_i - b_j$.
$\Phi_i(b_i - \epsilon) - \Phi_j(b_i - \epsilon) = \Phi_i(b_j) - \Phi_j(b_j) < 0$; indeed, on one hand, $\Phi_i(b_i - s) < \Phi_i(b_i) = \frac{1}{2}$ because $b_i - s < b_i$, Φ_i increases; on the other hand $\Phi_j(b_j - s) > \Phi_j(b_j - \epsilon) = \Phi_j(b_j) = \frac{1}{2}$.
Furthermore, $\Phi_i(c_i + s) > \Phi_i(c_i) = \frac{1}{2}$ because Φ_i increases and
$\Phi_j(c_i + s) < \Phi_j(c_j) = \frac{1}{2}$ because $c_i + s < c_i + \epsilon < c_i + c_j - c_i = c_j$ and F_j increases.
<u>2nd case</u>: $c_j - c_i < b_i - b_j$
We have $\epsilon = c_j - c_i$, and $c_i + \epsilon = c_j, b_i - \epsilon = b_i - c_j + c_i$.
$\Phi_i(b_i - s) - \Phi_j(b_i - s) < 0$ because: $\Phi_i(b_i - s) < \Phi_i(b_i) = \frac{1}{2}$ and $b_i - \epsilon = 2b_i - b_j > b_j, (b_i > b_j)$ so $\Phi_j(b_i - s) > \Phi_j(b_i - \epsilon) > \Phi_j(b_j) = \frac{1}{2}$ as Φ_j increases and $b_i - \epsilon > b_j$.
Furthermore, $\Phi_i(c_i + s) - \Phi_j(c_i + s) > 0$; indeed, $c_i + s < c_i + \epsilon = c_j$, so $\Phi_j(c_i + s) < \Phi_j(c_j) = \frac{1}{2}$. On the other hand Φ_i increases and $\Phi_i(c_i + s) > \Phi_i(c_i) = \frac{1}{2}$.
ii) If $b_i = b_j$ and $c_i \neq c_j$
Then $\epsilon = c_j - c_i$ and we easily conclude as in i).
iii) If $c_i = c_j$ and $b_i \neq b_j$
Then $\epsilon = b_i - b_j$ and we easily conclude as in i).
iv) If $c_i = c_j$ and $b_i = b_j$.
Then we take $\epsilon = (b_j - a_j) \wedge (d_j - c_j)$.
It is easy to check that for all $s : 0 < s < \epsilon, b_j - s \in [a_j, b_i[$ and $c_j + s \in [c_i, d_j[$. $(c_i = c_j, b_i = b_j) \Rightarrow [b_j, c_j] = [b_i, c_i]$; thus the support of ξ_i is included in the support of ξ_j and their cores coincide that means μ_j and μ_i coincide only in $[b_j, c_j]$, and this allow us to say that $\forall s \in [a_j, b_i[, \mu_j(s) > \mu_i(s)$ and $\forall s \in [c_i, d_j[, \mu_i(s) < \mu_j(s)$.
Furthermore, $\forall s \in [a_j, b_i[, \Phi_j(s) > \Phi_i(s)$ by the fact that $\mu_j(s) > \mu_i(s)$ and $\forall s \in [c_i, d_j[, \Phi_j(s) < \Phi_i(s)$ by the fact that $\mu_i(s) < \mu_j(s)$; these last inequalities lead us to $\Phi_i(c_j + s) > \Phi_j(c_j + s), \Phi_i(b_j - s) < \Phi_j(b_j + s)$.
The converse case can be proved easily.

4) By taking $\epsilon = \min(b_i - b_j, c_j - b_i)$, we can easily check that: $\forall s : 0 < s < \epsilon, \Phi_i(c_i - s) < \Phi_j(c_i - s), \Phi_i(c_i + s) > \Phi_j(c_i + s)$.
The converse case is obvious.

5) By taking $\epsilon = \min(b_j - b_i, c_i - b_j,)$ we can easily check that: $\forall s : 0 < s < \epsilon, \Phi_j(c_j - s) < \Phi_i(c_j - s), \Phi_j(c_j + s) > \Phi_i(c_j + s)$.

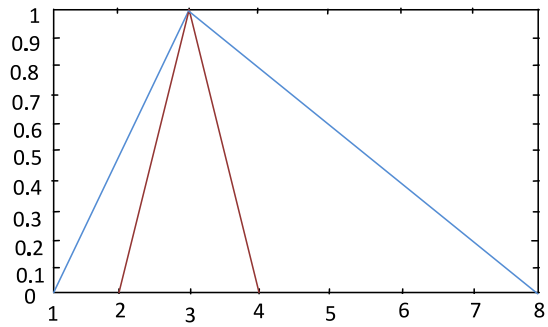

Figure 6: Incomparable fuzzy numbers by means of ≥ 2

The converse case is obvious.

6) By taking $\epsilon = \max(a_i - a_j, d_j - d_i)$, we can easily check that: $\forall s : 0 < s < \epsilon, \Phi_j(c_j - s) > \Phi_i(c_j - s), \Phi_j(c_j + s) < \Phi_i(c_j + s)$.
The converse case is obvious. $\square$

Remark 4.

- *We have an analogous result if* $\exists \epsilon > 0, \exists r_0, \forall s \in]0, \epsilon[, \mu_i(r_0 - s) > \mu_j(r_0 - s)$ *and* $\mu_i(r_0 + s) < \mu_j(r_0 + s)$.
- *The binary relation* $\succeq_2$ *on the set of fuzzy numbers is not complete. Let us take the triangular fuzzy numbers* $\xi_1 = (1, 3, 8)$ *and* $\xi_2 = (2, 3, 4)$ *drawn in Figure 6 and,* Φ_1 *and* Φ_2 *are their respective credibility distributions.*
  By Proposition 5, we can prove that the only crossing point is obtained at $r_0 = 3$. *Then, we have:*
  $\int_{-\infty}^{3} [\Phi_1(r) - \Phi_2(r)]dr = \frac{1}{4} > 0, \int_{-\infty}^{+\infty} [\Phi_1(r) - \Phi_2(r)]dr = \frac{-1}{5} < 0$ *and by Theorem 3 we conclude that* $\xi_1 \not\succeq_2 \xi_2$ *and* $\xi_2 \not\succeq_2 \xi_1$.

We establish the relationship between the three relations.

3.4 Relations between $\succeq_{\alpha,\tau}, \succeq_1$ and $\succeq_2$.

The following result shows that $\succeq_1$ is stronger than $\succeq_{\alpha,\tau}$ and $\succeq_2$.

Proposition 6. *Let* ξ_1 *and* ξ_2 *be two fuzzy variables. Then*

$$\xi_1 \succeq_1 \xi_2 \Rightarrow \begin{cases} (\forall \alpha \in \mathbb{R}_+^*, \forall \tau \in \mathbb{R}, \xi_1 \succeq_{\alpha,\tau} \xi_2) \\ \xi_1 \succeq_2 \xi_2 \end{cases} .$$

Proof: Let ξ_1 and ξ_2 be two fuzzy variables with uncertainty distributions Φ_1 and Φ_2 respectively, α and τ some given reals.
1) Assume that $\xi_1 \succeq_1 \xi_2$ and we prove that $\xi_1 \succeq_{\alpha,\tau} \xi_2$.
$\xi_1 \succeq_1 \xi_2 \Rightarrow \forall r \in \mathbb{R}, \Phi_1(r) \leq \Phi_2(r)$, that is,

$$\forall r \in \mathbb{R}, Cr\{\xi_1 \leq r\} \leq Cr\{\xi_2 \leq r\} \tag{20}$$

and

$$\forall r \in \mathbb{R}, Cr\{\xi_1 \geq r\} \geq Cr\{\xi_2 \geq r\} \tag{21}$$

According to the definition of $\succeq_1$.
In the other hand, we have: $E[\xi_1] = \int_0^{+\infty} Cr\{\xi_1 \geq r\}\ dr\ -\ \int_{-\infty}^0 Cr\{\xi_1 \leq r\}\ dr$ and $E[\xi_2] = \int_0^{+\infty} Cr\{\xi_2 \geq r\}\ dr\ -\ \int_{-\infty}^0 Cr\{\xi_2 \leq r\}\ dr$
According to (20) and (21), we conclude that $E[\xi_1] \geq E[\xi_2]$.
In the same manner, according to Remark 1 we have: $\mathrm{FLPM}_{\alpha,\tau}[\xi_1] = \alpha \int_{-\infty}^{\tau} (\tau - x)^{\alpha-1} Cr\{\xi_1 \leq x\} dx$ and $\mathrm{FLPM}_{\alpha,\tau}[\xi_2] = \alpha \int_{-\infty}^{\tau} (\tau - x)^{\alpha-1} Cr\{\xi_2 \leq x\} dx$
These last relations allow us to conclude that $\mathrm{FLPM}_{\alpha,\tau}[\xi_1] < \mathrm{FLPM}_{\alpha,\tau}[\xi_2]$.
Finally, we can conclude that $\xi_1 \succeq_{\alpha,\tau} \xi_2$.
2) Since $\forall r \in \mathbb{R}, \Phi_1(r) \leq \Phi_2(r)$ then $\forall t \in \mathbb{R}, \int_{-\infty}^t \Phi_2(r) - \Phi_1(r)\ \ dr \geq 0$. We can easily obtain the proof. $\square$

Now we use the following example to justify that the converse of the two previous implications are not true.

Example 1. *Let us consider the triangular fuzzy numbers* $\xi_1 = (1, 3, 5)$ *and* $\xi_2 = (2, 3, 4)$.

- *By Proposition 5, we can prove that the only crossing point is obtained at* $r_0 = 3$. *Then, we have:*
  $\int_{-\infty}^3 [\Phi_1(r) - \Phi_2(r)] dr = \frac{1}{4} > 0, \int_{-\infty}^{+\infty} [\Phi_1(r) - \Phi_2(r)] dr = 0$ *and by Theorem 3 we conclude that* $\xi_2 \succeq_2 \xi_1$.
  But by Proposition 6, $\xi_2 \not\succeq_1 \xi_1$.

- *By using this same example, we have:*
  $E[\xi_1] = E[\xi_2] = 3$, $FLPM_{2,3}[\xi_1] = \frac{2}{3}$ *and* $FLPM_{2,3}[\xi_2] = \frac{1}{6}$. *And* $FLPM_{2,3}[\xi_2] < FLPM_{2,3}[\xi_1]$ *and hence* $\Phi_2 \succeq_{2,3} \Phi_1$.
  But by Proposition 6, $\xi_2 \not\succeq_1 \xi_1$.

We end with the two following examples which specify that there is not a link between $\succeq_2$ and $\succeq_{\alpha,\tau}$ (See Figure 7).

Example 2. *1. Let us consider the triangular fuzzy numbers* $\xi_1 = (1.5, 4, 5)$ *and* $\xi_2 = (2, 3, 4)$.
By Proposition 5, we can prove that the only crossing point is obtained at $r_0 = 4$. *Then, we have:*
$\int_{-\infty}^{4}[\Phi_2(r) - \Phi_1(r)]dr = \frac{1}{10} > 0, \int_{-\infty}^{+\infty}[\Phi_2(r) - \Phi_1(r)]dr = \frac{-11}{5} < 0$ *and by Theorem 3, we conclude that* $\xi_1 \not\succeq_2 \xi_2$.
But, $E[\xi_1] = 3.625, E[\xi_2] = 3$, *i.e.,* $E[\xi_1] > E[\xi_2]$, $FLPM_{2,4}[\xi_1] \approx 1.042$ *and* $FLPM_{2,4}[\xi_2] = \frac{4}{3}$, *ie* $FLPM_{2,4}[\xi_1] < FLPM_{2,4}[\xi_2]$ *so* $\xi_1 \succeq_{2,4} \xi_2$.

2. Let us consider the triangular fuzzy numbers $\xi_3 = (1, 4, 5)$ *and* $\xi_4 = (2, 3, 4)$ *and,* Φ_3 *and* Φ_4 *are their respective credibility distribution functions.*
By Proposition 5, we can prove that the only crossing point is obtained at $r_0 = 4$. *Then, we have:*
$\int_{-\infty}^{4}[\Phi_3(r) - \Phi_4(r)]dr = \frac{1}{12} > 0, \int_{-\infty}^{+\infty}[\Phi_3(r) - \Phi_4(r)]dr = \frac{7}{3} > 0$ *and by Theorem 3, we conclude that* $\xi_2 \succeq_2 \xi_1$.
But, $E[\xi_3] = \frac{7}{2}$ *and* $E[\xi_4] = 3$. *Thus* $E[\xi_3] > E[\xi_4]$, $FLPM_{2,4}[\xi_3] = \frac{3}{2}$ *and* $FLPM_{2,4}[\xi_4] = \frac{4}{3}$. *Then* $FLPM_{2,4}[\xi_4] < FLPM_{2,4}[\xi_3]$ *and* $\xi_4 \not\succeq_{2,4} \xi_3$.

4 Concluding remarks

In this paper, we introduce Fuzzy lower partial moment (FLPM) based on credibility measure and propose some of its properties. We also introduce the n-th order fuzzy kappa index (FK) which a fuzzy risk adjusted performance measure that can be useful to classified fuzzy variables (e.g. portfolio returns performance). We complete our credibility fuzzy dominance with a general analysis of the mean-risk dominance based on FLPM, the first and second

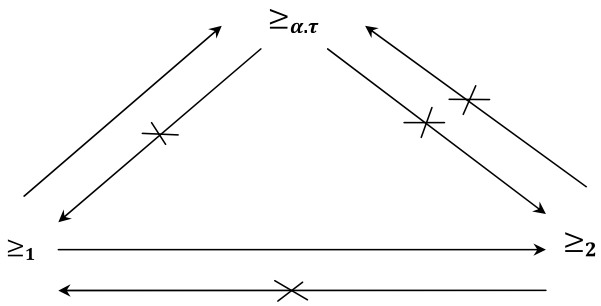

Figure 7: Links between the three fuzzy dominances

order dominance of fuzzy variables. We also characterize the obtained fuzzy mean-risk dominance. Because in some situations, it is not absolutely possible to compare two or multiple fuzzy variable, we introduce and characterize the crossing points between two fuzzy variables distributions and propose a theorem that relates them to second order fuzzy dominance.

5 Appendix on some examples of FLPM

5.1 Some examples of FLPM and of Fuzzy Kappa Index of a trapezoidal fuzzy number

In the following example, we compute from Corollary 1 the FLPMs and the Kappa index for a trapezoidal fuzzy number $\xi = (a, b, c, d)$ for some values of α and τ.

Example 3. *1. For $\alpha = 0$ we have:*

$$FLPM_{0,\tau}[\xi] = \begin{cases} 0 \ if \ \tau < a \\ \frac{(\tau-a)}{2(b-a)} \ if \ a \leq \tau < b \\ \frac{1}{2} \ if \ b \leq \tau < c \\ \frac{1}{2} + \frac{(\tau-c)}{2(d-c)} \ if \ c \leq \tau < d \\ 1 \ if \ \tau \geq d \end{cases} .$$

The credibility of loss $FLPM_{0,\tau}$ coincides with the cumulative distribution of credibility Φ.

2. For $\alpha = 1$ we have:

$$FLPM_{1,\tau}[\xi] = \begin{cases} 0 \ if \ \tau < a \\ \frac{(\tau-a)^2}{4(b-a)} \ if \ a \leq \tau < b \\ \frac{[(\tau-a)^2-(\tau-b)^2]}{4(b-a)} \ if \ b \leq \tau < c \\ \frac{[(\tau-a)^2-(\tau-b)^2]}{4(b-a)} + \frac{(\tau-c)^2}{4(d-c)} \ if \ c \leq \tau < d \\ \frac{[(\tau-a)^2-(\tau-b)^2]}{4(b-a)} + \frac{[(\tau-c)^2-(\tau-d)^2]}{4(d-c)} \ if \ \tau \geq d \end{cases}$$

and

$$FK_{1,\tau}[\xi] = \begin{cases} \frac{(b-a)(a+b+c+d-4\tau)}{(\tau-a)^2} \ if \ a \leq \tau < b \\ \frac{(b-a)(a+b+c+d-4\tau)}{[(\tau-a)^2-(\tau-b)^2]} \ if \ b \leq \tau < c \\ \frac{(a+b+c+d-4\tau)}{[\frac{[(\tau-a)^2-(\tau-b)^2]}{(b-a)}+\frac{(\tau-c)^2}{(d-c)}]} \ if \ c \leq \tau < d \\ \frac{(a+b+c+d-4\tau)}{[\frac{[(\tau-a)^2-(\tau-b)^2]}{(b-a)}+\frac{[(\tau-c)^2-(\tau-d)^2]}{(d-c)}]} \ if \ \tau \geq d \end{cases} .$$

The following table gives Fuzzy Kappa index of first order for seven triangular fuzzy numbers when the threshold $\tau \in \{\frac{e}{10}, \frac{e}{5}, \frac{e}{4}, \frac{e}{3}, \frac{e}{2}\}$.

τ	$\frac{e}{10}$	$\frac{e}{5}$	$\frac{e}{4}$	$\frac{e}{3}$	$\frac{e}{2}$
$FK_{1,\tau}[(-0.3, 1.8, 2.3)]$	54.6	27.9	20.8	13.3	5.8
$FK_{1,\tau}[(-0.4, 2.0, 2.2)]$	42.5	23.3	17.9	11.8	5.4
$FK_{1,\tau}[(-0.5, 1.9, 2.7)]$	29.2	18.0	14.1	9.6	4.6
$FK_{1,\tau}[(-0.6, 2.2, 2.8)]$	28.1	17.0	13.5	9.3	4.5
$FK_{1,\tau}[(-0.7, 2.4, 2.7)]$	25.2	15.5	12.4	8.7	4.3
$FK_{1,\tau}[(-0.8, 2.5, 3.0)]$	22.6	14.1	11.4	8.0	4.1
$FK_{1,\tau}[(-0.6, 1.8, 3.0)]$	23.0	14.2	11.3	7.9	3.9

3. *For* $\alpha = 2$ *we have:*

$$FLPM_{2,\tau}[\xi] = \begin{cases} 0 \ \textit{if } \tau < a \\ \frac{(\tau-a)^3}{6(b-a)} \ \textit{if } a \leq \tau < b \\ \frac{[(\tau-a)^3-(\tau-b)^3]}{6(b-a)} \ \textit{if } b \leq \tau < c \\ \frac{[(\tau-a)^3-(\tau-b)^3]}{6(b-a)} + \frac{(\tau-c)^3}{6(d-c)} \ \textit{if } c \leq \tau < d \\ \frac{[(\tau-a)^3-(\tau-b)^3]}{6(b-a)} + \frac{[(\tau-c)^3-(\tau-d)^3]}{6(d-c)} \ \textit{if } \tau \geq d \end{cases}.$$

and

$$FK_{2,\tau}[\xi] = \begin{cases} \frac{\sqrt{6}(b-a)^{\frac{1}{2}}(a+b+c+d-4\tau)}{4(\tau-a)^{\frac{3}{2}}} \ \textit{if } a \leq \tau < b \\ \frac{\sqrt{6}(b-a)^{\frac{1}{2}}(a+b+c+d-4\tau)}{4[(\tau-a)^3-(\tau-b)^3]^{\frac{1}{2}}} \ \textit{if } b \leq \tau < c \\ \frac{(a+b+c+d-4\tau)}{4[\frac{[(\tau-a)^3-(\tau-b)^3]}{6(b-a)}+\frac{(\tau-c)^3}{6(d-c)}]^{\frac{1}{2}}} \ \textit{if } c \leq \tau < d \\ \frac{(a+b+c+d-4\tau)}{4[\frac{[(\tau-a)^3-(\tau-b)^3]}{6(b-a)}+\frac{[(\tau-c)^3-(\tau-d)^3]}{6(d-c)}]^{\frac{1}{2}}} \ \textit{if } \tau \geq d \end{cases}.$$

- *For* $\alpha = 2$, *we have the following table which gives Fuzzy Kappa index of the second order for the same seven triangular fuzzy numbers.*

τ	$\frac{e}{10}$	$\frac{e}{5}$	$\frac{e}{4}$	$\frac{e}{3}$	$\frac{e}{2}$
$FK_{2,\tau}[(-0.3, 1.8, 2.3)]$	15.3	9.0	7.1	4.9	2.4
$FK_{2,\tau}[(-0.4, 2.0, 2.2)]$	12.3	7.6	6.1	4.4	2.3
$FK_{2,\tau}[(-0.5, 1.9, 2.7)]$	9.7	6.3	5.2	3.7	2.0
$FK_{2,\tau}[(-0.6, 2.2, 2.8)]$	9.0	6.0	4.9	3.6	1.9
$FK_{2,\tau}[(-0.7, 2.4, 2.7)]$	8.1	5.5	4.6	3.4	1.89
$FK_{2,\tau}[(-0.8, 2.5, 3.0)]$	7.4	5.1	4.2	3.22	1.80
$FK_{2,\tau}[(-0.6, 1.8, 3.0)]$	7.8	5.3	4.4	3.28	1.81

- *For $\alpha = 2$ and $\tau = e$ (e is the mean), we obtain semi-variances of the seven triangular numbers (see Sadefo Kamdem et al. [21]).*

τ	e
$FLPM_{2,\tau}[(-0.3, 1.8, 2.3)]$	*0.74*
$FLPM_{2,\tau}[(-0.4, 2.0, 2.2)]$	*0.96*
$FLPM_{2,\tau}[(-0.5, 1.9, 2.7)]$	*0.96*
$FLPM_{2,\tau}[(-0.6, 2.2, 2.8)]$	*1.3*
$FLPM_{2,\tau}[(-0.7, 2.4, 2.7)]$	*1.6*
$FLPM_{2,\tau}[(-0.8, 2.5, 3.0)]$	*1.82*
$FLPM_{2,\tau}[(-0.6, 1.8, 3.0)]$	*0.96*

4. *For $\alpha = 3$ we have:*

$$FLPM_{3,\tau}[\xi] = \begin{cases} 0 \ \textit{if } \tau < a \\ \frac{(\tau-a)^4}{8(b-a)} \ \textit{if } a \leq \tau < b \\ \frac{[(\tau-a)^4-(\tau-b)^4]}{8(b-a)} \ \textit{if } b \leq \tau < c \\ \frac{[(\tau-a)^4-(\tau-b)^4]}{8(b-a)} + \frac{(\tau-c)^4}{8(d-c)} \ \textit{if } c \leq \tau < d \\ \frac{[(\tau-a)^4-(\tau-b)^4]}{8(b-a)} + \frac{[(\tau-c)^4-(\tau-d)^4]}{8(d-c)} \ \textit{if } \tau \geq d \end{cases}$$

and

$$FK_{3,\tau}[\xi] = \begin{cases} \frac{8^{\frac{1}{3}}(b-a)^{\frac{1}{3}}(a+b+c+d-4\tau)}{4(\tau-a)^{\frac{4}{3}}} \ \textit{if } a \leq \tau < b \\ \frac{8^{\frac{1}{3}}(b-a)^{\frac{1}{3}}(a+b+c+d-4\tau)}{4[(\tau-a)^4-(\tau-b)^4]^{\frac{1}{3}}} \ \textit{if } b \leq \tau < c \\ \frac{(a+b+c+d-4\tau)}{4[\frac{[(\tau-a)^4-(\tau-b)^4]}{8(b-a)}+\frac{(\tau-c)^4}{8(d-c)}]^{\frac{1}{3}}} \ \textit{if } c \leq \tau < d \\ \frac{(a+b+c+d-4\tau)}{4[\frac{[(\tau-a)^4-(\tau-b)^4]}{8(b-a)}+\frac{[(\tau-c)^4-(\tau-d)^4]}{8(d-c)}]^{\frac{1}{3}}} \ \textit{if } \tau \geq d \end{cases}$$

The following tabular gives Fuzzy Kappa index of the third order for seven triangular fuzzy numbers when the threshold $\tau \in \{\frac{e}{10}, \frac{e}{5}, \frac{e}{4}, \frac{e}{3}, \frac{e}{2}\}$.

τ	$\frac{e}{10}$	$\frac{e}{5}$	$\frac{e}{4}$	$\frac{e}{3}$	$\frac{e}{2}$
$FK_{3,\tau}[(-0.3, 1.8, 2.3)]$	9.6	5.9	4.7	3.4	1.7
$FK_{3,\tau}[(-0.4, 2.0, 2.2)]$	7.8	5.0	4.1	3.0	1.6
$FK_{3,\tau}[(-0.5, 1.9, 2.7)]$	6.4	4.3	3.59	2.6	1.49
$FK_{3,\tau}[(-0.6, 2.2, 2.8)]$	5.9	4.0	3.4	2.5	1.45
$FK_{3,\tau}[(-0.7, 2.4, 2.7)]$	5.37	3.7	3.17	2.4	1.38
$FK_{3,\tau}[(-0.8, 2.5, 3.0)]$	4.9	3.5	2.9	2.2	1.32
$FK_{3,\tau}[(-0.6, 1.8, 3.0)]$	5.30	3.6	3.1	2.3	1.34

5. *For $\alpha = 4$ we have:*

$$FLPM_{4,\tau}[\xi] = \begin{cases} 0 \ if \ \tau < a \\ \frac{(\tau-a)^5}{10(b-a)} \ if \ a \leq \tau < b \\ \frac{[(\tau-a)^5-(\tau-b)^5]}{10(b-a)} \ if \ b \leq \tau < c \\ \frac{[(\tau-a)^5-(\tau-b)^5]}{10(b-a)} + \frac{(\tau-c)^5}{10(d-c)} \ if \ c \leq \tau < d \\ \frac{[(\tau-a)^5-(\tau-b)^5]}{10(b-a)} + \frac{[(\tau-c)^5-(\tau-d)^5]}{10(d-c)} \ if \ \tau \geq d \end{cases}$$

and

$$FK_{4,\tau}[\xi] = \begin{cases} \frac{10^{\frac{1}{4}}(b-a)^{\frac{1}{4}}(a+b+c+d-4\tau)}{4(\tau-a)^{\frac{5}{4}}} \ if \ a \leq \tau < b \\ \frac{10^{\frac{1}{4}}(b-a)^{\frac{1}{4}}(a+b+c+d-4\tau)}{4[(\tau-a)^5-(\tau-b)^5]^{\frac{1}{4}}} \ if \ b \leq \tau < c \\ \frac{(a+b+c+d-4\tau)}{4[\frac{[(\tau-a)^5-(\tau-b)^5]}{10(b-a)}+\frac{(\tau-c)^5}{10(d-c)}]^{\frac{1}{4}}} \ if \ c \leq \tau < d \\ \frac{(a+b+c+d-4\tau)}{4[\frac{[(\tau-a)^5-(\tau-b)^5]}{10(b-a)}+\frac{[(\tau-c)^5-(\tau-d)^5]}{10(d-c)}]^{\frac{1}{4}}} \ if \ \tau \geq d \end{cases}$$

- *For $\alpha = 4$, the following table gives Fuzzy Kappa index of the fourth order of the seven triangular fuzzy numbers.*

τ	$\frac{e}{10}$	$\frac{e}{5}$	$\frac{e}{4}$	$\frac{e}{3}$	$\frac{e}{2}$
$FK_{4,\tau}[(-0.3, 1.8, 2.3)]$	7.5	4.7	3.8	2.7	1.4
$FK_{4,\tau}[(-0.4, 2.0, 2.2)]$	6.1	4.0	3.3	2.4	1.3
$FK_{4,\tau}[(-0.5, 1.9, 2.7)]$	5.1	3.51	2.9	2.2	1.25
$FK_{4,\tau}[(-0.6, 2.2, 2.8)]$	4.7	3.3	2.8	2.1	1.21
$FK_{4,\tau}[(-0.7, 2.4, 2.7)]$	4.2	3.0	2.5	1.9	1.15
$FK_{4,\tau}[(-0.8, 2.5, 3.0)]$	3.9	2.8	2.4	1.8	1.11
$FK_{4,\tau}[(-0.6, 1.8, 3.0)]$	4.2	3.0	2.5	1.9	1.14

- *For $\alpha = 4$ and $\tau = e$ (e is the mean), we obtain the semi-kurtosis of the seven triangular fuzzy numbers (see Sadefo Kamdem, Tassak et Fono [21]).*

τ	e
$FLPM_{4,\tau}[(-0.3, 1.8, 2.3)]$	*3.24*
$FLPM_{4,\tau}[(-0.4, 2.0, 2.2)]$	*5.53*
$FLPM_{4,\tau}[(-0.5, 1.9, 2.7)]$	*5.53*
$FLPM_{4,\tau}[(-0.6, 2.2, 2.8)]$	*10.24*
$FLPM_{4,\tau}[(-0.7, 2.4, 2.7)]$	*15.39*
$FLPM_{4,\tau}[(-0.8, 2.5, 3.0)]$	*19.77*
$FLPM_{4,\tau}[(-0.6, 1.8, 3.0)]$	*5.53*

5.2 Some examples of Fuzzy Lower partial Moments of a fuzzy portfolio return

From (9), it is straightforward to obtain the following proposition:

Remark 5. *1. Let $\xi_p = (\gamma_a, \gamma_b, \gamma_c, \gamma_d)$ be a trapezoidal fuzzy return of a portfolio. Then the fuzzy lower partial moment of the portfolio return ξ_p is given by:*

$$FLPM_{\alpha,\tau}[\xi] = \begin{cases} 0 \ if\ \tau < \gamma_a(x) \\ \frac{(\tau-\gamma_a(x))^{n+1}}{2(n+1)(\gamma_b(x)-\gamma_a(x))} \ if\ \gamma_a(x) \leq \tau < \gamma_b(x) \\ \frac{[(\tau-\gamma_a(x))^{n+1}-(\tau-\gamma_b(x))^{n+1}]}{2(n+1)(\gamma_b(x)-\gamma_a(x))} \ if\ \gamma_b(x) \leq \tau < \gamma_c(x) \\ \frac{[(\tau-\gamma_a(x))^{n+1}-(\tau-\gamma_b(x))^{n+1}]}{2(n+1)(\gamma_b(x)-\gamma_a(x))} + \frac{(\tau-\gamma_c(x))^{n+1}}{2(n+1)(\gamma_d(x)-\gamma_c(x))} \ if\ \gamma_c(x) \leq \tau < \gamma_d(x) \\ \frac{[(\tau-\gamma_a(x))^{n+1}-(\tau-\gamma_b(x))^{n+1}]}{2(n+1)(\gamma_b(x)-\gamma_a(x))} + \frac{[(\tau-\gamma_c(x))^{n+1}-(\tau-\gamma_d(x))^{n+1}]}{2(n+1)(\gamma_d(x)-\gamma_c(x))} \ if\ \tau \geq \gamma_d(x) \end{cases} \tag{22}$$

where $\gamma_z(x) = \sum_{i=1}^{n} x_i z_i$ for $z = a, b, c, d \in \mathbb{R}$ with $a < b < c < d$.

2. When $\tau = E[\xi]$, we can deduce that:
-$FLPM_{2,\tau}[\xi]$ is the semi-variance of the portfolio return ξ
-$FLPM_{4,\tau}[\xi]$ is the semi-kurtosis of the portfolio return ξ. For more details, see Sadefo Kamdem et al. [21]).

Example 4. *1. $\tau = \frac{e}{10}$,*

$$FK_{4,\tau} = \frac{10^{\frac{1}{4}}(2.1x_1+2.4x_2+2.4x_3+2.8x_4+3.1x_5+3.3x_6+2.4x_7)^{\frac{1}{4}}(\frac{9(5.6x_1+5.8x_2+6x_3+6.6x_4+6.8x_5+7.2x_6+6x_7)}{10})}{4(\frac{-39\sum_{i=1}^{7} a_i x_i + 2\sum_{i=1}^{7} b_i x_i + \sum_{i=1}^{7} c_i x_i}{40})^{\frac{5}{4}}}$$

2. $\tau = \frac{e}{5}$,

$$FK_{4,\tau} = \frac{10^{\frac{1}{4}}(2.1x_1+2.4x_2+2.4x_3+2.8x_4+3.1x_5+3.3x_6+2.4x_7)^{\frac{1}{4}}(\frac{4(5.6x_1+5.8x_2+6x_3+6.6x_4+6.8x_5+7.2x_6+6x_7)}{5})}{4(\frac{-19\sum_{i=1}^{7} a_i x_i + 2\sum_{i=1}^{7} b_i x_i + \sum_{i=1}^{7} c_i x_i}{20})^{\frac{5}{4}}}$$

3. $\tau = \frac{e}{4}$,

$$FK_{4,\tau} = \frac{10^{\frac{1}{4}}(2.1x_1+2.4x_2+2.4x_3+2.8x_4+3.1x_5+3.3x_6+2.4x_7)^{\frac{1}{4}}(\frac{3(5.6x_1+5.8x_2+6x_3+6.6x_4+6.8x_5+7.2x_6+6x_7)}{4})}{4(\frac{-15\sum_{i=1}^{7} a_i x_i+2\sum_{i=1}^{7} b_i x_i+\sum_{i=1}^{7} c_i x_i}{16})^{\frac{5}{4}}}$$

4. $\tau = \frac{e}{3}$,

$$FK_{4,\tau} = \frac{10^{\frac{1}{4}}(2.1x_1+2.4x_2+2.4x_3+2.8x_4+3.1x_5+3.3x_6+2.4x_7)^{\frac{1}{4}}(\frac{2(5.6x_1+5.8x_2+6x_3+6.6x_4+6.8x_5+7.2x_6+6x_7)}{3})}{4(\frac{-11\sum_{i=1}^{7} a_i x_i+2\sum_{i=1}^{7} b_i x_i+\sum_{i=1}^{7} c_i x_i}{12})^{\frac{5}{4}}}$$

5. $\tau = \frac{e}{2}$,

$$FK_{4,\tau} = \frac{10^{\frac{1}{4}}(2.1x_1+2.4x_2+2.4x_3+2.8x_4+3.1x_5+3.3x_6+2.4x_7)^{\frac{1}{4}}(\frac{(5.6x_1+5.8x_2+6x_3+6.6x_4+6.8x_5+7.2x_6+6x_7)}{2})}{4(\frac{-7\sum_{i=1}^{7} a_i x_i+2\sum_{i=1}^{7} b_i x_i+\sum_{i=1}^{7} c_i x_i}{8})^{\frac{5}{4}}}$$

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