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Recursive structures in the multispecies TASEP

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Abstract

We consider a multispecies generalization of the totally asymmetric simple exclusion process (TASEP) with the simple hopping rule: for the α and θ th-class particles ($\alpha < \theta$), the transition $\alpha\theta \rightarrow \theta\alpha$ occurs with a rate independent from the values α and θ . Ferrari and Martin (2007 *Ann. Prob.* **35** 807) obtained the stationary state of this model thanks to a combinatorial algorithm, which was subsequently interpreted as a matrix product representation by Evans *et al* (2009 *J. Stat. Phys.* **135** 217). This ‘matrix ansatz’ shows that the stationary state of the multispecies TASEP with N classes of particles (N -TASEP) can be constructed algebraically by the action of an operator on the $(N-1)$ -TASEP stationary state. Besides, Arita *et al* (2009 *J. Phys. A: Math Theor.* **45** 345002) analyzed the spectral structure of the Markov matrix: they showed that the set of eigenvalues of the N -TASEP contains those of the $(N-1)$ -TASEP and that the various spectral inclusions can be encoded in a hierarchical set-theoretic structure known as the Hasse diagram. Inspired by these works, we define nontrivial operators that allow us to construct eigenvectors of the N -TASEP by lifting the eigenvectors of the $(N-1)$ -TASEP. This goal is achieved by generalizing the matrix product representation and the Ferrari–Martin algorithm. In particular, we show that the matrix ansatz is not only a convenient tool to write the stationary state but in fact intertwines Markov matrices of different values of N .

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1. Introduction

Interacting particle systems are mathematical models used to study collective properties of N -body systems evolving with time [47]. Replacing the actual evolution rules (given by a classical or a quantum Hamiltonian) by a stochastic dynamics leads one to formulate many properties of the system in terms of probability theory and allows the use of powerful mathematical methods [36]. Hence, universal properties of nonequilibrium statistical physics are successfully explored through the investigation of stochastic processes, that are governed by simple dynamical rules at the microscopic level but that display a rich macroscopic behavior [43, 45]. When defining such models, some specific features of the original physical problem have to be discarded, but if the relevant and fundamental characteristics are retained (such as symmetry properties and conservation laws), it is expected that the large-scale and large-time behavior of the system will be described correctly [25]. Building a simple representation for complex phenomena is common procedure in statistical physics, leading to the emergence of paradigmatic models: the harmonic oscillator, the random walker, the Ising magnet and so on. These ‘beautiful models’ often display wonderful mathematical structures [9, 49].

In the field of nonequilibrium statistical mechanics, the asymmetric simple exclusion process (ASEP) has reached the status of a paradigm [14, 30, 37]. The ASEP is a lattice-gas model of interacting particles. Each of these particles is a random walker that hops from a site to one of the neighboring locations, but a move is allowed only if the target site is empty. This exclusion constraint mimics short-range interactions amongst particles. In order to drive this lattice gas out of equilibrium, non-vanishing currents have to be established in the system and this can be achieved by various means: by starting with a non-uniform initial condition that takes an infinite amount of time to relax, by coupling the system to external reservoirs that drive currents through the system (transport of particles, energy, heat, etc) or by introducing some intrinsic bias in the dynamics that favors motion in a privileged direction, each particle being an asymmetric random walker that drifts steadily along the direction of an external driving force. In particular, the case where moves are allowed in only one direction is said totally asymmetric (TASEP).

The ASEP has been invented several times and in different contexts due to its simplicity. It was probably first proposed as a prototype to describe the dynamics of ribosomes along RNA [38]. In the mathematical literature, Brownian motions with hard-core interactions were first studied by Spitzer [46] who coined the name *exclusion process*. The ASEP also appeared naturally in the description of transport processes in systems with strong geometric constraints such as macromolecules transiting through capillary vessels and that cannot overtake each other [34], or anisotropic conductors known as solid electrolytes where electrons hop from a vacant location to another and repel each other via Coulomb interaction [13]. Popular modern applications of the exclusion process include molecular motors that transport proteins within the cells along actin filaments [32] and, last but not least, the ASEP and its variants are ubiquitous in discrete models of traffic flow [42].

From the mathematical point of view, the ASEP is one of the simplest but nontrivial models for which the hydrodynamic limit can be rigorously proved. At large scales, the distribution of the particles of the ASEP emerges as a density field that evolves according to the Burgers equation with a vanishingly small viscosity [33, 47, 50]. The Burgers equation is the textbook prototype for shock formation: a smooth initial distribution can develop a singularity (a discontinuity) in finite time. A natural question that arises is whether this shock is an artifact of the hydrodynamic limit or if, under some specific conditions, the original ASEP does display some singularity at the microscopic scale [3, 4, 11, 12, 28]. This question was answered positively [16]: a shock does exist at the level of the particle system and its

width is of the order of the lattice size. However, defining precisely the position of the shock at the microscopic level requires a trick which is achieved by introducing a new type of particle called a *second-class particle*, which amounts to coupling two TASEP models which initially differ at a single location [35]. This second-class particle, denoted by 2, has the same dynamics as a normal (or first-class) particle, denoted by 1, but first-class particles treat it as a hole (denoted by 0). The local dynamical rules thus take the following simple form:

$$10 \rightarrow 01 \quad 12 \rightarrow 21 \quad \text{and} \quad 20 \rightarrow 02, \quad (1)$$

where all transitions occur with a same rate. It was proved rigorously in [20, 21] that the dynamics of the unique second-class particle mimics the motion of the shock in an infinite system. Alternatively, by considering a finite density of second-class particles in a periodic system, the shape of the microscopic fronts were exactly calculated [16].

A straightforward generalization of the two species case of first- and second-class particles is to the multispecies exclusion process where there is a hierarchy amongst N different species (or classes) of particles [1, 2, 39] (N -ASEP): a particle of class α views particles of classes $\beta > \alpha$ as holes and is viewed as a hole by particles of classes $\beta < \alpha$; in other words, ‘higher’ classes have lower overtaking priority, as will be defined precisely in the following section. The stationary state of this simplest multispecies generalization of the ASEP is highly nontrivial: it was first constructed by Ferrari and Martin [23, 24] in the TASEP case through a mapping to queueing processes. Their construction was inspired by earlier combinatorial results [5, 17, 22]. In [18, 40], the Ferrari–Martin algorithm was re-expressed as a matrix product representation in which the weight of a configuration in the N -ASEP is written as a linear combination of weights of configurations of the $(N-1)$ -ASEP. This matrix (product) ansatz hence allows one to derive steady-state properties of a given model knowing those of a *simpler system*.

More general spectral properties of the multispecies ASEP on a ring were investigated in [6] where models with different total number of species were related to each other in a different manner. In that work, the key observation was that particles of two consecutive classes cannot be distinguished by particles of other classes. Thus, identifying two consecutive classes n and $n + 1$ defines a natural projection from the N -ASEP onto the $(N-1)$ -ASEP. Because one can identify any two consecutive classes of particles, there are N different such projections. These mappings together with their commutation relations endow the set of all possible models with a poset structure represented by a Hasse diagram [6]. The existence of this structure leads to canonical inclusion relations and duality in the spectrum of the Markov matrix. In particular, through the identification mapping, the eigenvectors of the N -ASEP either vanish or project onto eigenvectors of the $(N-1)$ -ASEP. However, these projections blur some essential information (namely, the difference between the two consecutive classes n and $n + 1$) that cannot be retrieved easily. In particular, they do not allow one to build the eigenvectors of a model knowing those of a simpler system.

The above descriptions of the matrix ansatz and of the identification maps indicate that they operate in reverse directions and, hence, that they should be related. This also suggests that the matrix ansatz may have a range of applicability that exceeds stationary-state properties; the matrix ansatz should be useful also to build ‘excited states’ of the N -ASEP knowing those of the $(N-1)$ -ASEP. In this perspective, the matrix ansatz could be viewed as a ‘lifting procedure’ from the $(N-1)$ -ASEP to the N -ASEP that creates a new species of particles by separating a given class into two consecutive subclasses. Further, we have seen that there are N different choices for the identification maps from the N -ASEP to the $(N-1)$ -ASEP. Are there also N different matrix product representations that would correspond to splitting a

species n into two consecutive classes n and $n + 1$? Could there be N different generalizations of the Ferrari–Martin algorithm?

The objective of this work is to answer these questions in the TASEP case by reformulating them in the appropriate mathematical framework and stating them in rigorous terms. We will introduce a generalized matrix ansatz which constructs a lifting operator from the $(N - 1)$ -TASEP to the N -TASEP. This will provide with a new and much broader perspective about this technique and shed light on the recursive structures that underlie the N -TASEP dynamics. As a result, the problems stated above will be solved and families of quadratic algebras that encode these recursions will be constructed. The outline of this work is as follows. In section 2, we define the dynamical rules of the N -ASEP, describe the characteristics and some spectral properties of the Markov matrix, define identification operators between different systems (sectors) and formulate the fundamental problem addressed here. From section 3 onward, we restrict our consideration to the TASEP case, unless explicitly stated otherwise. In section 3, we recall the Ferrari–Martin algorithm of the stationary state of the N -TASEP and describe the associated matrix product representation. The main results are derived in section 4: we construct a set of generalized matrix ansatz and prove that this allows us to define a family of conjugation operators from the $(N - 1)$ -TASEP to the N -TASEP; each of these matrix ansatz leads to a different quadratic algebra and we find explicit representations for all these algebras; finally, we find a generalized Ferrari–Martin algorithm (GFMA) corresponding to each algebra. Concluding remarks are given in section 5. The appendices contain examples and technical details.

2. The N -ASEP: definition and properties

2.1. Definition of the model

The N -ASEP on the ring \mathbb{Z}_L with L sites is defined by the following dynamical rules. Each site $i \in \mathbb{Z}_L$ is assigned with a variable (local state) $k_i \in \{0, \dots, N - 1\}$ ($N \geq 0$). We introduce a stochastic process such that the nearest-neighbor pairs of local states $(\alpha, \beta) \in \{0, \dots, N - 1\}^2$ are interchanged with the following transition rates:

$$\alpha\beta \rightarrow \beta\alpha \quad \begin{cases} p & \text{if } \alpha < \beta, \\ q & \text{if } \alpha > \beta, \end{cases} \quad (2)$$

with p and q being real non-negative parameters, with the choice $q \leq p$ without loss of generality⁶. In particular, for $q = 0$, the model is totally asymmetric and is called the N -TASEP; for $0 < q < p$, the model is said to be partially asymmetric (N -PASEP); $q = p$ is the symmetric case (N -SSEP).

The dynamics is formulated in terms of the continuous-time master equation for the probability $P(k_1 \dots k_L; t)$ of finding a configuration $k_1 \dots k_L$ at time t :

$$\begin{aligned} \frac{d}{dt} P(k_1 \dots k_L; t) = & \sum_{i \in \mathbb{Z}_L} \mathbb{Q}(k_{i+1} - k_i) P(k_1 \dots k_{i-1} k_{i+1} k_i k_{i+2} \dots k_L; t) \\ & - \sum_{i \in \mathbb{Z}_L} \mathbb{Q}(k_i - k_{i+1}) P(k_1 \dots k_L; t), \end{aligned} \quad (3)$$

⁶ In the paper [6], the dynamical rule is defined as $\begin{cases} p & \text{if } \alpha > \beta, \\ q & \text{if } \alpha < \beta. \end{cases}$ We have to replace $p \leftrightarrow q$ or reverse the variables $k_i \rightarrow N + 2 - k_i$, if we wish to compare this work with [6]. There are studies on a variety of multispecies ASEP's (whose rules are expressed as $\alpha\beta \rightarrow \beta\alpha$ with rate $p_{\alpha\beta}$ in general), see [7, 10, 19, 27, 29, 31] for example.

where Θ is a step function defined as

$$\Theta(x) = \begin{cases} p & (x < 0), \\ 0 & (x = 0), \\ q & (x > 0). \end{cases} \quad (4)$$

This model can be regarded as an interacting multispecies particle system by interpreting the local state k_i as representing the site i occupied by a particle of the α th kind. In the conventional terminology, a particle of the α th kind for $1 \leq \alpha \leq N$ is called the ' α th-class particle'. We can view the local state $N+1$ as a vacant site (or a hole), which is often denoted by 0. However, for later convenience, it is better to use $N+1$ instead of 0 and we shall stick to that convention.

2.2. The Markov matrix

Let $\{| \rangle, \dots, |N+1\rangle\}$ be the basis of the single-site space C^{N+1} . The tensor product $k_1 k_L \dots k_L = | \rangle \otimes \dots \otimes |k_L\rangle \in (C^{N+1})^{\otimes L}$ corresponds to the configuration $k_1 \dots k_L$ on the ring. The probability vector at time t can be written as

$$|P(t)\rangle = \sum_{1 \leq k_i \leq N+1} P(k_1 \dots k_L; t) |k_1 \dots k_L\rangle. \quad (5)$$

In this language, the master equation (3) becomes

$$\frac{d}{dt} |P(t)\rangle = \mathbf{M}^{(N)} |P(t)\rangle, \quad (6)$$

where the (total) Markov matrix $\mathbf{M}^{(N)}$ is of the size $(N+1)^L \times (N+1)^L$. This Markov matrix can be written as the sum of local linear operators $\mathbf{M}_{\text{Loc } i, i+1}^{(N)}$ acting only on the i th and $(i+1)$ st components of the tensor product:

$$\mathbf{M}^{(N)} = \sum_{i \in \mathbb{Z}_L} \mathbf{M}_{\text{Loc } i, i+1}^{(N)}, \quad (7)$$

where the action of the local Markov matrices on a bond is given by

$$\mathbf{M}_{\text{Loc } \alpha, \beta}^{(N)} = \sum_{\alpha, \beta=1}^{N+1} (-\Theta(\alpha - \beta) |a\beta\rangle\langle a\beta| + \Theta(\alpha - \beta) |\beta a\rangle\langle a\beta|). \quad (8)$$

Note that off-diagonal elements of $\mathbf{M}^{(N)}$ are p, q or 0, and the diagonal elements are expressed as $px+qy$ with nonpositive integers x and y . The sum of entries in each column of $\mathbf{M}^{(N)}$ is 0, assuring the conservation of the total probability $\sum_{1 \leq k_i \leq N+1} P(k_1 \dots k_L; t)$. The master equation can be solved formally as

$$|P(t)\rangle = e^{t\mathbf{M}^{(N)}} |P(0)\rangle, \quad (9)$$

and thus the eigenvalues and the right eigenvectors of $\mathbf{M}^{(N)}$ give information for physical properties of the model. The Markov matrix has the eigenvalue 0, and we call the corresponding right eigenvectors stationary states. All the other eigenvalues have strictly negative real parts (Perron–Frobenius theorem [47]), which characterize the relaxation to the stationary states. We remark that the Markov matrix $\mathbf{M}^{(N)}$ is Hermitian only for $q = p$, where it becomes the Hamiltonian of the $sl(N+1)$ -invariant Heisenberg spin chain.

2.3. Particle conservation and the Hasse diagram structure

In view of the transition rule (2), the total Markov matrix obviously preserves the number of particles of each kind. Let $m_\alpha \geq 0$ denote the number of particles of the α th class (respectively holes), for $1 \leq \alpha \leq N$ (respectively $\alpha = N + 1$):

$$\underbrace{1 \cdots 1}_{m_1} \underbrace{2 \cdots 2}_{m_2} \cdots \underbrace{N+1 \cdots N+1}_{m_{N+1}}. \quad (10)$$

Then, the state space decomposes into sectors labeled by $\mathbf{m} = (m_1, \dots, m_{N+1})$ with the constraint $m_1 + \dots + m_{N+1} = L$:

$$(C^{N+1})^{\otimes L} = \bigoplus_{\mathbf{m}} V_{\mathbf{m}}. \quad (11)$$

The Markov matrix has a block diagonal structure, leaving each sector invariant

$$\mathcal{M}^{(N)} = \bigoplus_{\mathbf{m}} M_{\mathbf{m}}, \quad M_{\mathbf{m}} \in \text{End} V_{\mathbf{m}}, \quad (12)$$

where the square matrix $M_{\mathbf{m}}$ acts on the vector space $V_{\mathbf{m}}$ spanned by all configurations belonging to the sector \mathbf{m} :

$$V_{\mathbf{m}} = \bigoplus_{\#\{i|k_i=j\}=\mathbf{m}_j} C|k_1 \cdots k_L\rangle. \quad (13)$$

The dimension of $V_{\mathbf{m}}$ is given by $\dim V_{\mathbf{m}} = \frac{L!}{m_1! \cdots m_{N+1}!}$ and the total number of sectors for given L and N is $\sum_{\mathbf{m}} \dim V_{\mathbf{m}} = \sum_{\mathbf{m}} \frac{L!}{m_1! \cdots m_{N+1}!} = \sum_{\mathbf{m}} \binom{L}{\mathbf{m}} = 2^L$.

Let us call a sector $\mathbf{m} = (m_1, \dots, m_{N+1})$ a *basic sector* if $m_n > 0$ for all $n = 1, \dots, N+1$ (i.e. there exists at least one particle of each type). Note that there are no basic sectors for $N \geq L$. Thus, we shall always take $N \leq L-1$. For example, for $L=4$, the list of all basic sectors for different values of N is given by

$$\begin{aligned} N=3: & \quad (1, 1, 1, 1), \\ N=2: & \quad (2, 1, 1), (1, 2, 1), (1, 1, 2), \\ N=1: & \quad (3, 1), (2, 2), (1, 3), \\ N=0: & \quad (4). \end{aligned} \quad (14)$$

The number of basic sectors for given L and N is given by $\binom{L-1}{N}$. Let \mathbf{M} be the set of all labels for the basic sectors:

$$\mathbf{M} = \{(\mathbf{m}_1, \dots, \mathbf{m}_{N+1}) | 0 \leq N \leq L-1, m_i \in \mathbb{N}, m_1 + \dots + m_{N+1} = L\}. \quad (15)$$

We now introduce an alternative labeling of the basic sectors [6] that will be very useful in the following: let s_j be the total number of particles of classes $k \leq j$, i.e.

$$s_j = m_1 + m_2 + \dots + m_j. \quad (16)$$

We have $m_j = s_j - s_{j-1} > 0$ with the convention $s_0 = 0$, and thus each basic sector of the N -ASEP can be labeled by the set $s = \{s_1, \dots, s_N\}$ with $0 < s_1 < s_2 < \dots < s_N < L$. The set s is a subset of $\mathbf{A} = \{1, 2, \dots, L-1\}$, i.e. the set s is an element of $2^{\mathbf{A}}$, the power set (the set of all subsets) of \mathbf{A} . We can identify \mathbf{M} with $2^{\mathbf{A}}$ by the one-to-one correspondence (16). In the following, we shall use both labels equivalently: for instance, the invariant vector spaces (respectively the Markov matrices acting on them) will be denoted either by $V_{\mathbf{m}}$ or V_s (respectively $M_{\mathbf{m}}$ or M_s).

An example of the identification $\mathbf{M} \leftrightarrow \mathbf{S}$ for $L=4$ is given in figure 1. The set \mathbf{S} is equipped with a natural poset (partially ordered set) structure with respect to the inclusion

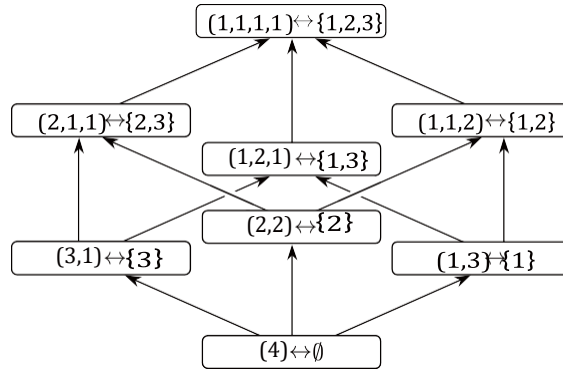


Figure 1. Basic sectors for $L = 4$ in the Hasse diagram. Each sector is labeled by an element of M and one of S .

\subseteq , which is encoded in the Hasse diagram [48]. In the present case, it is just the $(L-1)$ -dimensional hypercube, where each vertex corresponds to a sector. Every link of the hypercube becomes an arrow $t \rightarrow s$ meaning that $t \subset s$ and $\#s = \#t + 1$. The ‘maximal sector’ $\blacktriangle = \{1, 2, \dots, L\} \leftrightarrow \{1, 2, \dots, L\}$ corresponds to a unique sink and the ‘minimal sector’ \emptyset corresponds to a unique source, as shown in figure 1.

The following spectral properties in this Hasse diagram were proved in [6].

- *Spectral inclusion.* Let $\text{Spec}(s)$ be the multiset of the eigenvalues of M_s with the multiplicity of an element representing the degree of its degeneracy. Then,

$$\text{Spec}(s) \supset \text{Spec}(t) \quad (17)$$

for any pair of sectors $s \supset t$. In particular, $\text{Spec}(\blacktriangle)$ contains the eigenvalues of the Markov matrix M_s of all the sectors s . An example of this spectral inclusion for $L = 4$ is displayed in figure 2.

- *Spectral duality.* Using the spectral inclusion theorem, we can classify the eigenvalues $E \in \text{Spec}(s)$ into two types: (i) eigenvalues that already exist in lower sectors $u \subset s$ and (ii) eigenvalues that appear at s . Let us call eigenvalues of the second type *genuine eigenvalues*. More explicitly, the genuine spectrum $\text{Spec}^\circ(s)$ of the sector s is defined as

$$\text{Spec}^\circ(\emptyset) := \text{Spec}(\emptyset) = \{0\}, \quad \text{Spec}^\circ(s) := \text{Spec}(s) \setminus \bigcup_{u \subset s} \text{Spec}(u). \quad (18)$$

Then, the following duality relation was proved:

$$\text{Spec}^\circ(s) = -L(p+q) - \text{Spec}^\circ(\bar{s}), \quad (19)$$

where the sector $\bar{s} = \blacktriangle \setminus s$ is furthest from the sector s in the Hasse diagram.

These two spectral relations were derived by introducing a family of linear operators connecting different sectors, that we now review.

2.4. The identification operators ϕ_{ts} and the conjugation property

Let us consider two (basic) sectors $s = \{s_1 < \dots < s_N\}$ and $t = s \setminus \{s_{n_1}, \dots, s_{n_u}\}$. We introduce a linear operator $\phi_{ts} : V_s \rightarrow V_t$, whose action on the basis vectors is given by

$$|k_1 \dots k_L\rangle \in V_s \mapsto |k'_1 \dots k'_L\rangle \in V_t, \quad \text{with } x^r = x - \#\{i | n_i < x\}. \quad (20)$$

Note that $\phi_{ts} : V_s \rightarrow V_t$ is surjective. For $s = t$, ϕ_{ss} is the identity operator for any sector s .

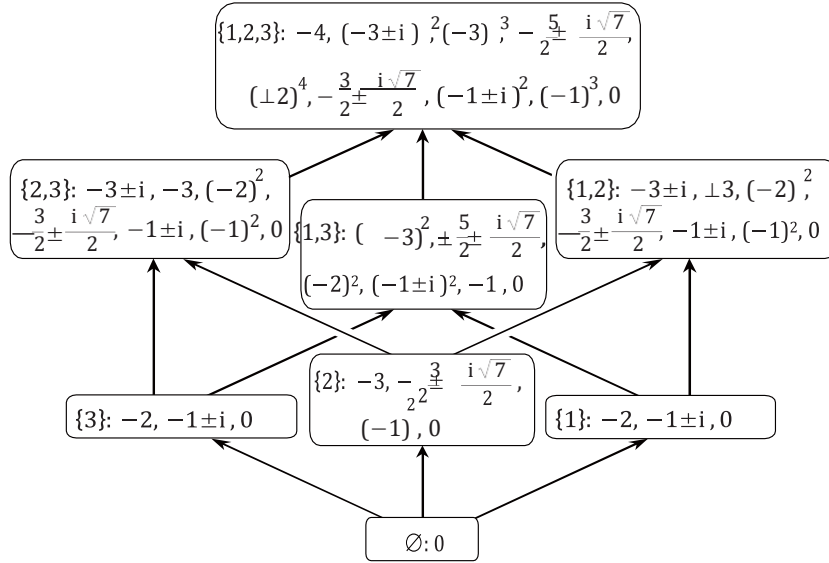


Figure 2. $\text{Spec}(s)$ for $L = 4$ with $p = 1$ and $q = 0$ (TASEP case). Each superscript denotes the multiplicity. For example, the Markov matrix $M_{\{1,3\}}$ has eigenvalue 2 and its degree of degeneracy is 2.

To understand the general definition (20), we give an example $s = \{2, 3, 5, 8\} \supset t = \{2, 5\}$ with $L = 9$. These two sectors correspond to the following compositions of the system into different types of particles:

$$\begin{array}{cccccccc}
 & 2 & 3 & & 5 & & 8 & \\
 s: & 1 & 1 & 2 & 3 & 3 & 4 & 4 & 4 & 5 \\
 t: & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 3
 \end{array} \quad (21)$$

According to these lists, we define ϕ_{ts} to be the operator replacing (or identifying) the local states according to the rules $3 \rightarrow 2$, $4 \rightarrow 3$, $5 \rightarrow 3$ (keeping 1 and 2 unchanged) within all the ket vectors $|k_1 \cdots k_L\rangle$ in V_s . For example,

$$\phi_{ts}|345214431\rangle = |233213321\rangle. \quad (22)$$

More generally, for a pair of sectors $t \subseteq s$, consider a chain $s_0 \supset s_1 \supset \cdots \supset s_u$ of sectors such that $s_0 = s$, $s_u = t$ and $\#s_j = \#s_{j+1} + 1$ for all $0 \leq j < u$. Then, the decomposition

$$\phi_{ts} = \phi_{ts_{u-1}} \phi_{s_{u-1}s_{u-2}} \cdots \phi_{s_2s_1} \phi_{s_1s} \quad (23)$$

holds and is independent of the choice of the intermediate sectors [6].

2.5. The conjugation property

A crucial property of the identification operators is that they provide a conjugation between different sectors. More precisely, for any two sectors such that $s \supseteq t$ the following diagram

commutes:

$$\begin{array}{ccc}
 & M_s & \\
 V_s & \xrightarrow{\quad} & V_s \\
 \phi_{ts} \downarrow & & \downarrow \phi_{ts} \\
 V_t & \xrightarrow{\quad} & V_t \\
 & M_t &
 \end{array} \quad (24)$$

which means

$$M_t \phi_{ts} = \phi_{ts} M_s. \quad (25)$$

We call this type of relation a *conjugation relation*, relating the dynamics governed by M_s and that governed by M_t . The identification operator ϕ_{ts} intertwines the two Markov matrices M_s and M_t , and we call ϕ_{ts} a *conjugation matrix*. Relation (25) was the key [6] in proving the spectral properties (17) and (19). Each right eigenvector $|E\rangle$ with eigenvalue E in the sector s can be projected down by ϕ_{ts} to the sector t , and $\phi_{ts}|E\rangle$ is also a right eigenvector with eigenvalue E in the sector s (under the assumption $\phi_{ts} \neq 0$). In particular, the stationary state of a sector s is mapped by the identification operator to the stationary state of any sector t such that $t \subseteq s$.

2.6. Looking for an inverse conjugation relation

The identification operator ϕ_{ts} always maps an upper (more complex) sector s into a lower (simpler) sector t . This implies that identification matrix ϕ_{ts} cannot help us to construct a right eigenvector in the sector s knowing the eigenvector with the same eigenvalue in a smaller sector t . It would be very useful if we could *lift* information from lower sectors to upper sectors in the Hasse diagram. One of the motivations of this work can be formulated as follows. Can we construct a matrix ψ_{st} which lifts up right eigenvectors from a lower sector t to an upper sector s ? This would be possible if we could define an operator ψ_{st} from V_t to V_s such that the following diagram commutes:

$$\begin{array}{ccc}
 & M_s & \\
 V_s & \xrightarrow{\quad} & V_s \\
 \psi_{st} \uparrow & & \uparrow \psi_{st} \\
 & M_t & \\
 V_t & \xrightarrow{\quad} & V_t
 \end{array} \quad (26)$$

Equivalently, for any sectors $t \subseteq s$ the conjugation matrix ψ_{st} satisfies

$$\psi_{st} M_t = M_s \psi_{st}. \quad (27)$$

Note that the directions of the vertical arrows are opposite as compared to those in the previous diagram (24), which is a crucial difference. Equivalently, comparing equation (27) with equation (25), we observe that the order of the products is opposite. The property (27) has the following important consequence: let $|E_t\rangle$ be any eigenvector of M_t with eigenvalue E . Then

$$M_s (\psi_{st} |E_t\rangle) = \psi_{st} (M_t |E_t\rangle) = E (\psi_{st} |E_t\rangle), \quad (28)$$

which means that $\psi_{st} |E_t\rangle$ is an eigenvector of M_s with the same eigenvalue E (if $\psi_{st} |E_t\rangle \neq 0$). In other words, the map ψ_{st} allows us to lift eigenvectors from a smaller sector to a larger sector and provides a constructive information from a simpler system to a more complex one (whereas the identification operator ϕ_{ts} erases information). The existence of ψ_{st} is a very nontrivial property of the model. In the following section, we review the construction of the stationary state, which will be generalized later to define ψ_{st} .

3. The stationary state of the N -TASEP

From this section on, we set $p = 1$ and $q = 0$, i.e. we consider the TASEP case, unless explicitly stated otherwise. In the N -TASEP dynamics, each particle can hop to its right nearest-neighbor site if the target site is empty or occupied by a higher class particle.

A stationary state is a right eigenvector of the Markov matrix corresponding to the eigenvalue 0. We denote it by $|\tilde{P}\rangle = \sum_{\tau} \tilde{P}(\tau) |\tau\rangle$:

$$0 = \mathcal{M}^{(N)} |\tilde{P}\rangle. \quad (29)$$

For each sector m , the Markov matrix has a unique stationary state $|\tilde{P}_m\rangle$ up to an overall constant factor. For basic sectors $s \leftrightarrow m$, we write $|\tilde{P}_s\rangle = |\tilde{P}_m\rangle$. From the ‘grand-canonical’ stationary state $|\tilde{P}\rangle$, which is a solution of equation (29), we extract the stationary state of each sector, by restricting the components of $|\tilde{P}\rangle$ to that sector. We emphasize that the multispecies TASEP on a ring does not satisfy the detailed-balance condition and exhibits non-vanishing currents in its stationary state, which is one of the major characteristics of nonequilibrium systems⁷.

The stationary state of the N -TASEP is nontrivial as soon as $N \geq 2$. The stationary state of the 2-TASEP was constructed by using a matrix product representation in [16]. However, this technique did not seem easily generalizable to higher values of N (see [39] for an attempt for the case $N = 3$). The solution to this problem came from two completely different directions. In [5] it was shown that the stationary state provided by the matrix product representation for the 2-TASEP could be interpreted in terms of weights in a binary tree. On the other hand, in [22], these weights were rewritten in terms of Dyck paths which also appear as trajectories of a queueing process and therefore the 2-TASEP was reinterpreted as a queueing process. This fact was generalized to the N -TASEP which was mapped into a system of coupled queueing processes [24]; this construction leads to the Ferrari–Martin algorithm for the stationary state of the N -TASEP that we review in the following subsection.

3.1. Review of Ferrari and Martin’s construction

We now reformulate the algorithm found in [24] that constructs the stationary state of the N -TASEP, in terms of the convention adopted in this work. The basic idea is to obtain the stationary state of N -TASEP from that of $(N-1)$ -TASEP. The following algorithm is valid for any sector $s = \{s_1 < \dots < s_N\}$. In figure 3, we provide an explicit example for a ring of size $L = 9$ and with $s = \{2, 5, 6\}$ (i.e. $m_1 = 2$, $m_2 = 5$, $m_3 = 1$ and $m_4 = 3$).

In order to relate a configuration of the $(N-1)$ -TASEP to a configuration of the N -TASEP, we consider two lines, each of them corresponding to a lattice of size L .

- (i) On the upper line, we set s_N black boxes b and $(L - s_N)$ white boxes w arbitrarily, that we write as $c_1 \dots c_L$ (with $c_i = b, w$).

On the lower line, there is a configuration $k_1 \dots k_L$ of the $(N-1)$ -TASEP corresponding to the sector $s \setminus \{s_N\}$; thus, on the lower line, there are $(s_v - s_{v-1})$ v th-class particles ($1 \leq v \leq N-1$, $s_0 = 0$) and $(L - s_{N-1})$ N th-class particles (i.e. empty sites).

- (ii-1) Let $i_1^{(1)}, \dots, i_{s_1}^{(1)}$ be the positions of the s_1 first-class particles on the lower line. For

the first first-class particle located at $i_1^{(1)}$, find the nearest black box $c_{i_r} = b$ with $i_r \leq i_1^{(1)}$ and put a particle of type 1 on it. If there is no such black box, put the particle 1 on the rightmost black box. For the second first-class particle located at $i_2^{(1)}$, find the nearest

⁷ However, it is important to keep in mind that boundary conditions are absolutely crucial in nonequilibrium physics: indeed, in a closed segment with *reflecting* boundaries, the detailed-balance condition is satisfied [10].

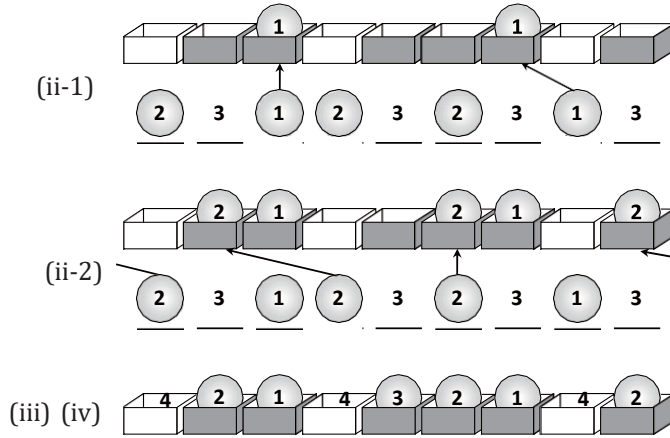


Figure 3. The Ferrari–Martin algorithm that constructs a configuration of N -TASEP from that of $(N-1)$ -TASEP. This figure provides the specific example: $F(wbbwbbbwb, 231232313) = 421432142$.

- unoccupied black box $c_{i^r} = b$ with $i^r \leq i_1^{(1)}$ and put a particle of type 1 on it. If there is no such black box, put the particle of type 1 on the rightmost unoccupied black box. Iterate this procedure s_1 times, i.e. find the nearest unoccupied black box $c_{i^r} = b$ with $i^r \leq i_1^{(1)}$ for the 4th first-class particle located at $i_1^{(1)}$ and put a particle of type 1 on it or on the rightmost unoccupied black box if i^r does not exist.
- (ii-2) Now we consider the second-class particles on the lower line. Recall that there are $m_2 = s_2 - s_1$ of them, and set their positions as $i_1^{(2)}, \dots, i_{s_2-s_1}^{(2)}$. There are $(s_N - s_1)$ unoccupied black boxes remaining on the upper line. We must iterate the following procedure $(s_2 - s_1)$ times: find the nearest unoccupied black box $c_{i^r} = b$ with $i^r \leq i_1^{(2)}$ for the 4th second-class particle ($1 \leq 4 \leq s_2 - s_1$), and put a particle of type 2 on it or on the rightmost unoccupied black box if i^r does not exist.
- (ii-v) In the same way, we go on for the third-, fourth-, \dots , $(N-1)$ st-class particles: for the v th-class particles, there are $(s_v - s_{v-1})$ v th-class particles on the lower line, with positions $\{i_1^{(v)}, \dots, i_{s_v-s_{v-1}}^{(v)}\}$. There are $(s_N - s_{v-1})$ unoccupied black boxes remaining on the upper line. Iterate the following procedure $(s_v - s_{v-1})$ times: find the nearest unoccupied black box $c_{i^r} = b$ with $i^r \leq i_1^{(v)}$ for the 4th v th-class particle ($1 \leq 4 \leq s_v - s_{v-1}$), and put a particle of type v on it or on the rightmost unoccupied black box if i^r does not exist.
- (iii) Now, there are $(s_N - s_{N-1})$ unoccupied black boxes that remain. Put particles of type N on them.
- (iv) Regarding the $(L - s_N)$ white boxes as an empty sites, i.e. as particles of type $N+1$, we have thus constructed a well-defined configuration $F(c_1 \dots c_L, k_1 \dots k_L)$ of the N -TASEP on the upper line, belonging to the sector s , starting from a configuration $k_1 \dots k_L$ of the $(N-1)$ -TASEP on the lower line that was in the sector $s \setminus \{s_N\}$. (Note that an N represents an empty site in the sector $s \setminus \{s_N\}$, whereas $N+1$ does in s .)

Note that *different* configurations of $(N-1)$ -TASEP on the lower line can lead to a *same* configuration of the N -TASEP on the upper line: for example,

$$F(bbwwb, 32133) = F(bbwwb, 31323) = 21443. \quad (30)$$

It was proved in [24] that the stationary weight of a given configuration in the sector s is given (up to normalization) by the sum of the weights of all configurations in $s \setminus \{s_N\}$ that are

related to it through this construction. Equivalently, we have

$$\bar{P}_s = \sum_{c_1 \cdots c_L, k_1 \cdots k_L} F(c_1 \cdots c_L, k_1 \cdots k_L) (k_1 \cdots k_L | \bar{P}_{s \setminus \{s_N\}}). \quad (31)$$

Here, the summation \sum runs over $c_1 \cdots c_L$ and $k_1 \cdots k_L$ with $\#\{i | c_i = b\} = s_N$ and $k_1 \cdots k_L$ belonging to the sector $s \setminus \{s_N\}$.

3.2. Matrix product representation for the stationary state

The Ferrari–Martin algorithm was restated as a matrix product representation in [18]. The basic idea of the matrix product representation is to express the stationary probability as the trace of a product of matrices over a suitable algebra. This technique, initially invented in [15] for the (one-species) ASEP with open boundaries, has been generalized to many stochastic interacting particle systems including discrete-time updates, a second nearest-neighbor interaction and non-conservative dynamics (see [10] for an exhaustive review). For the N -TASEP on Z_L , the stationary weight of a configuration $j_1 \cdots j_L$ is given by the trace of an L -fold matrix product:

$$\bar{P}(j_1 \cdots j_L) = \text{Tr} X_{j_1}^{(N)} \cdots X_{j_L}^{(N)}, \quad (32)$$

or equivalently as

$$|P\rangle = \text{Tr} \left(\left(X_1^{(N)} \right)^{\otimes L} \right). \quad (33)$$

In [40], an explicit solution for the operators $X_j^{(N)}$'s was found. It is given by the following tensor product recursions:

$$X_j^{(N)} = \sum_{K=1}^N a_{JK}^{(N,N)} \otimes X_K^{(N-1)} \quad \text{for } 1 \leq j \leq N+1, \quad (34)$$

with $Y \otimes X_1^{(1)} = Y \otimes X_2^{(1)} = Y$ (i.e. $X_1^{(1)} = X_2^{(1)} = 1$). The operators $a_{JK}^{(N,N)}$ are given in the following table:

$J \setminus K$	$1 \cdots N-1$	N
$1 \cdots N-1$	$A^{\otimes(J-1)} \otimes \delta \otimes 1^{\otimes(K-J-1)} \otimes \epsilon \otimes 1^{\otimes(N-K-1)}$	$A^{\otimes(J-1)} \otimes \delta \otimes 1^{\otimes(N-J-1)}$
N	0	$A^{\otimes(N-1)}$
$N+1$	$1^{\otimes(K-1)} \otimes \epsilon \otimes 1^{\otimes(N-K-1)}$	$1^{\otimes(N-1)}$

We read $\delta \otimes 1^{\otimes(-1)} \otimes \epsilon = 1$ and $\delta \otimes \epsilon^{\otimes x} \otimes \epsilon = 0$ for $x \leq -2$. The operators δ , ϵ and A are the fundamental building blocks that are ubiquitous in the matrix ansatz technique [10]. These three operators generate a quadratic algebra and satisfy the following relations:

$$\delta \epsilon = 1, \quad \delta A = 0, \quad A \epsilon = 0. \quad (36)$$

A common representation of this algebra is given by the infinite-dimensional matrices

$$\delta = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots \end{pmatrix}, \quad \epsilon = \begin{pmatrix} 1 & 0 & & \\ & 1 & 0 & \\ & & \ddots & \ddots \end{pmatrix}, \quad A = \begin{pmatrix} & & & \\ & 0 & 0 & \\ & & \ddots & \ddots \end{pmatrix}. \quad (37)$$

Remark. The Ferrari–Martin algorithm cannot be easily defined for the PASEP because the directionality plays a crucial role in the algorithm. Nevertheless, the matrix product representation can readily be generalized to the PASEP case ($p = 1$ and $q \neq 0$) as follows. In the table in (35), we replace the operators δ , ϵ and A by the ‘ q -deformed’ operators δ_q , ϵ_q and A_q that generate a quadratic algebra with the following relations:

$$\delta_q \epsilon_q - q \epsilon_q \delta_q = (1 - q)1, \quad \delta_q A_q = q A \delta_q, \quad A_q \epsilon_q = q \epsilon_q A_q. \quad (38)$$

Then, the matrix product representation (33) with this deformation provides the stationary state of the N -PASEP, as was shown in [40].

3.3. Interpretation of the matrix ansatz as a linear mapping

The recursion relation (34) implies that each stationary weight of the N -ASEP can be expressed as a linear combination of weights of the $(N - 1)$ -ASEP [40]. More precisely, using equation (34) we can write

$$|\bar{P}^{(N)}\rangle = W^{(N,N)} |\bar{P}^{(N-1)}\rangle, \quad (39)$$

where the matrix $W^{(N,N)}$ is defined in terms of its elements as

$$(j_1 \cdots j_L | W^{(N,N)} | k_1 \cdots k_L) = \text{Tr } a_{j_1 k_1}^{(N)} \cdots a_{j_L k_L}^{(N)} \quad (40)$$

for $1 \leq j_i \leq N + 1$ and $1 \leq k_i \leq N$. Moreover the matrix $W^{(N,N)}$ has the following property that we shall call *sector specificity*: suppose that the configuration $j_1 \cdots j_L$ belongs to a basic sector $s = \{s_1 < \cdots < s_N\}$ and that the element $(j_1 \cdots j_L | W^{(N,N)} | k_1 \cdots k_L)$ is nonzero, then the configuration $k_1 \cdots k_L$ belongs to the sector $s \setminus \{s_N\}$. Conversely, if $j_1 \cdots j_L$ belongs to s and $k_1 \cdots k_L$ does not belong to $s \setminus \{s_N\}$, then $(j_1 \cdots j_L | W^{(N,N)} | k_1 \cdots k_L) = 0$. We will prove this property in appendix B. This property allows us to consider the mapping $\psi_{s, s \setminus \{s_N\}}: V_{s \setminus \{s_N\}} \rightarrow V_s$, defined as the restriction of $W^{(N,N)}$ to the sectors s and $s \setminus \{s_N\}$. This mapping provides us with a construction of the stationary state of the basic sector s by lifting up that of $s \setminus \{s_N\}$:

$$|\bar{P}_s\rangle = \psi_{s, s \setminus \{s_N\}} |\bar{P}_{s \setminus \{s_N\}}\rangle. \quad (41)$$

Using this equation repeatedly, we have

$$|\bar{P}_s\rangle = \psi_{s, s \setminus \{s_N\}} \psi_{s \setminus \{s_N\}, s \setminus \{s_{N-1}, s_N\}} \cdots \psi_{\{s_1, s_2\} \setminus \{s_1\}, \emptyset} |\bar{P}_{\emptyset}\rangle = |1 \cdots 1\rangle, \quad (42)$$

where $|1 \cdots 1\rangle$ is the only configuration of the 0-TASEP on a ring of size L (the configuration in which all sites are empty). Hence, the stationary state of the basic sector $s = \{s_1, \dots, s_N\}$ is constructed along the way

$$\emptyset \rightarrow \{s_1\} \rightarrow \{s_1, s_2\} \rightarrow \cdots \rightarrow s \setminus \{s_{N-1}, s_N\} \rightarrow s \setminus \{s_N\} \rightarrow s \quad (43)$$

in the Hasse diagram.

On the other hand, using (23), we can project $|\bar{P}_s\rangle$ down to the minimal sector via arbitrary intermediate sectors, i.e. for any $\{n_1, \dots, n_N\} = \{1, \dots, N\}$,

$$\phi_{\emptyset s_{N-1}} \phi_{s_{N-1} s_{N-2}} \cdots \phi_{s_2 s_1} \phi_{s_1 s} |\bar{P}_s\rangle = \text{constant} |1 \cdots 1\rangle, \quad (44)$$

where $s_x = s \setminus \{s_{n_1}, \dots, s_{n_x}\}$. Comparing equations (42) and (44), we observe that the ψ mappings play a role opposite to that of the ϕ 's. The ψ 's are thus good candidates to be solutions to the conjugation relation (27). This property will be proved in the following section.

We emphasize that the matrix ansatz that we have considered above allows us to construct the stationary state only along a very specific path in the Hasse diagram: in the sector $s \setminus \{s_N\}$, there are $m_1 = s_1$ particles of class 1, $m_2 = s_2 - s_1$ particles of class 2, ..., $m_{N-1} = s_{N-1} - s_{N-2}$

and $m_N = L - s_{N-1}$ particles of class N . In the sector s , the number m_j of particles of class j with $1 \leq j \leq N-1$ is the same, but there are new $m'_N = s_N - s_{N-1}$ particles of class N and $m'_{N+1} = L - s_N$ particles of class $N+1$, with $m'_{N+1} + m'_N = m_N$. Hence, when the sector $s \setminus \{s_N\}$ is lifted up to s , a new species is created by splitting the particles of class N (that have the lowest priority) into two subspecies of classes N and $N+1$. In the Ferrari–Martin algorithm, this means that *a new species is created from the holes*. However, in the Hasse diagram, there exist different paths between two connected but non-adjacent sectors. This observation suggests that the matrix ansatz and the Ferrari–Martin algorithm should be generalized in order to construct the stationary state via arbitrary intermediate sectors.

In the following section, we show that the matrix ansatz not only provides a way to write the stationary state but also allows one to define mappings that satisfy the conjugation relation (27)⁸. Therefore, this technique provides a tool to lift eigenvectors by intertwining the dynamics corresponding to different values of N . Besides, we shall also construct intertwining operators between arbitrary sectors, by defining and using more general quadratic algebras than those considered previously.

4. Conjugation matrices from a generalized matrix ansatz

In this section, we derive a general conjugation relation between N -TASEP models with different values of N by using a generalized matrix ansatz which allows us to create a new class of particles by splitting any intermediate species into two subspecies.

4.1. Generalized quadratic algebra

We define a family of rectangular matrices $\{a^{(N,n)}\}_{1 \leq n \leq N}$ of size $(N+1) \times N$, indexed by the integer n . The elements $a^{JK,n} = (J|a^{(N,n)}|K)$ of the matrix $a^{(N,n)}$ are operators given in the following table:

$J \setminus K$	$1 \quad \cdots \quad n-1$	n	$n+1 \quad \cdots \quad N$
1 \vdots $n-1$	$A^{\otimes(J-1)} \otimes \delta \otimes 1^{\otimes(K-J-1)}$ $\otimes \epsilon \otimes 1^{\otimes(N-K-1)}$	$A^{\otimes(J-1)} \otimes$ $\delta \otimes 1^{\otimes(N-J-1)}$	$A^{\otimes(J-1)} \otimes \delta \otimes 1^{\otimes(K-J-2)}$ $\otimes \delta \otimes 1^{\otimes(N-K)}$
n	0	$A^{\otimes(n-1)} \otimes$ $1^{\otimes(N-n)}$	$A^{\otimes(n-1)} \otimes 1^{\otimes(K-n-1)}$ $\otimes \delta \otimes 1^{\otimes(N-K)}$
$n+1$	$1^{\otimes(K-1)} \otimes \epsilon \otimes$ $1^{\otimes(n-K-1)} \otimes A^{\otimes(N-n)}$	$1^{\otimes(n-1)} \otimes$ $A^{\otimes(N-n)}$	0
$n+2$ \vdots $N+1$	$1^{\otimes(K-1)} \otimes \epsilon \otimes 1^{\otimes(J-K-3)}$ $\otimes \epsilon \otimes A^{\otimes(N-J+1)}$	$1^{\otimes(J-3)} \otimes$ $\epsilon \otimes A^{\otimes(N-J+1)}$	$1^{\otimes(K-2)} \otimes \delta \otimes 1^{\otimes(J-K-2)}$ $\otimes \epsilon \otimes A^{\otimes(N-J+1)}$

(45)

where we read $\delta \otimes 1^{\otimes(-1)} \otimes \epsilon = 1$ and $\delta \otimes 1^{\otimes x} \otimes \epsilon = 0$ for $x \leq -2$. The fundamental operators δ , ϵ and A satisfy the relations given in equation (36). In general, each element of

⁸ This interpretation of the matrix ansatz as an intertwining operator between different dynamics was already used in a recent study of a one-species TASEP with open boundaries and with annihilation [8]. There, conjugation matrices with respect to the system size were constructed, which allowed them to calculate the normalization factor and certain correlation functions (see [51] for a related approach).

$a^{(N,n)}$ is either 0 or an $(N-1)$ -fold tensor product of $1, A, \delta$ or ϵ . Some examples are given in appendix A. Note that for the case $n = N$, one retrieves the operators that were given in table (35).

Let us define

$$W^{(N,n)} = \text{Tr}[(a^{(N,n)})^{\otimes L}]. \quad (46)$$

We shall show that the following relation is satisfied:

$$M^{(N)} W^{(N,n)} = W^{(N,n)} M^{(N-1)}, \quad (47)$$

In other words, for any value of n , the matrix $W^{(N,n)}$ allows us to embed the system with $(N-1)$ classes of particles into the system with N classes of particles. Using $W^{(N,n)}$, we shall be able to construct sector specific conjugation operators that intertwine the dynamics between any two basic sectors along any path in the Hasse diagram.

We now derive equation (47). The method used is an extension of the hat matrix technique, that was developed to prove various matrix product representations [10, 26]. Suppose that, for each value of n , there exists an operator valued $(N+1) \times N$ matrix $\hat{a}^{(N,n)}$ such that the following identity, that we shall call the *hat relation*, is satisfied:

$$M_{\text{Loc}}^{(N)} (a^{(N,n)} \otimes a^{(N,n)}) - (a^{(N,n)} \otimes a^{(N,n)}) M_{\text{Loc}}^{(N-1)} = a^{(N,n)} \otimes \hat{a}^{(N,n)} - \hat{a}^{(N,n)} \otimes a^{(N,n)}. \quad (48)$$

Then, equation (47) is a consequence of this relation. Indeed, from relation (48), we obtain, taking the L -fold tensor product,

$$\begin{aligned} & \sum_{i \in Z_L} M_{\text{Loc}}^{(N)} \big|_{i, i+1} (a^{(N,n)})^{\otimes L} - (a^{(N,n)})^{\otimes L} \big|_{i \in Z_L} M_{\text{Loc}}^{(N-1)} \big|_{i, i+1} \\ &= \sum_{i \in Z_L} (a^{(N,n)})^{\otimes i} \otimes \hat{a}^{(N,n)} \otimes (a^{(N,n)})^{\otimes (L-i-1)} \\ &- \sum_{i \in Z_L} (a^{(N,n)})^{\otimes (i-1)} \otimes \hat{a}^{(N,n)} \otimes (a^{(N,n)})^{\otimes (L-i)} = 0. \end{aligned} \quad (49)$$

Taking the trace of this relation on the space on which the operators $a^{(N,n)}$ act, and noting that local Markov matrices sum up to total Markov matrices, we obtain equation (47). We emphasize that the hat matrices $\hat{a}^{(N,n)}$ are used in the proof but do not appear in the final

result (47). To summarize, the conjugation relation (47) follows from the hat relation (48) and in order to show that this latter relation exists, we need to specify the operators $\hat{a}^{(N,n)}$. We claim that the hat relation (48) is satisfied for the choice

$$\hat{a}^{(N,n)} = d_n a^{(N,n)} \quad \text{with} \quad d_n = \text{diag} \left(\frac{1}{c}, \frac{1}{c}, 0, \dots, 0 \right). \quad (50)$$

$\begin{matrix} c & & & & \\ & n & & & \\ & & -1 & & \\ & & & c & \\ & & & & N+1-n \end{matrix}$

This explicit expression of $\hat{a}^{(N,n)}$ leads to closed quadratic relations between the elements a_{JK} $J \in \{1, \dots, N\}$ $K \in \{1, \dots, N+1\}$ (note that for simplicity, we write a_{JK} instead of $a^{(N,n)}$). Indeed, using the expressions of the local Markov matrices, we have

$$(JJ^r | M_{\text{Loc}}^{(N)} | a^{(N,n)} \otimes a^{(N,n)} | KK^r) = \begin{cases} -a_{JK} a_{J^r K^r} & (J < J^r), \\ 0 & (J = J^r), \\ a_{J^r K} a_{JK^r} & (J > J^r), \end{cases} \quad (51)$$

$$(JJ^r | a^{(N,n)} \otimes a^{(N,n)} | M_{\text{Loc}}^{(N-1)} | KK^r) = \begin{cases} -a_{JK} a_{J^r K^r} + a_{JK^r} a_{J^r K} & (K < K^r), \\ 0 & (K \geq K^r). \end{cases} \quad (52)$$

Besides, using equation (50), we can calculate each element of the right-hand side of (48) as

$$(JJ^r | a^{(N,n)} \otimes \hat{a}^{(N,n)} - \hat{a}^{(N,n)} \otimes a^{(N,n)} | KK^r) = \begin{cases} -a_{JK} a_{J^r K^r} & (J \leq n < J^r), \\ a_{JK} a_{J^r K^r} & (J > n \geq J^r), \\ 0 & (\text{otherwise}). \end{cases} \quad (53)$$

Substituting equations (51), (52) and (53) into the hat equation (48) leads to the following closed quadratic algebra generated by the operators a_{JK} :

	I ($K < K^r$)	II ($K \geq K^r$)
A ($J \leq n < J^r$)	$a_{JK} a_{J^r K^r} = a_{JK^r} a_{J^r K}$	–
B $J < J^r \leq n$ or $n < J < J^r$	$a_{JK^r} a_{J^r K} = 0$	$a_{JK} a_{J^r K^r} = 0$
C ($J = J^r$)	$a_{JK} a_{JK^r} = a_{JK^r} a_{JK}$	–
D ($J > n \geq J^r$)	$a_{J^r K} a_{JK^r} = a_{JK^r} a_{J^r K}$	$a_{JK} a_{J^r K^r} = a_{J^r K} a_{JK^r}$
E $J > J^r > n$ or $n \geq J > J^r$	$a_{JK} a_{J^r K^r} + a_{J^r K} a_{JK^r} = a_{JK^r} a_{J^r K}$	$a_{J^r K} a_{JK^r} = 0$

(54)

(Note that the cases A-II and C-II do not give any nontrivial relation, and E-II for $K > K^r$ is equivalent to B-I.) Showing that the matrix $W^{(N,n)}$ (46) satisfies the conjugation relation (47) therefore reduces to checking that $a_{JK}^{(N,n)}$'s defined by table (45) actually give a representation for (54). Thus the proof of the conjugation relation (47) reduces to a purely mechanical procedure. We have checked this for several values of (N, n) by using Mathematica.

We emphasize that the key ingredient is the generalized hat relation (48) together with the ansatz (50) for the hat matrices $a^{(N,n)}$ which allows one to define a quadratic algebra. The fact that the ansatz depends on the integer n , with $1 \leq n \leq N$, provides a family of quadratic algebras indexed by n . We also note that the conjugation relation (47) shows that the quadratic algebras defined in table (54) provide representations for the stationary state of the N -TASEP: this is a much more concise (albeit more abstract) proof than the one given in [40].

4.2. Sector specificity

We consider a basic sector $s = \{s_1, \dots, s_N\}$ and let $C(s)$ be the set of all configurations of s . We define $\psi_{s, s \setminus \{s_{n^r}\}}^{(N,n)}: V_{s \setminus \{s_{n^r}\}} \rightarrow V_s$ by

$$(j_1 \cdots j_L | \psi_{s, s \setminus \{s_{n^r}\}}^{(N,n)} | k_1 \cdots k_L) = \text{Tr } a_{j_1 k_1}^{(N,n)} \cdots a_{j_L k_L}^{(N,n)} \quad (55)$$

for $1 \leq n, n^r \leq N$, $j_1 \cdots j_L \in C(s)$ and $k_1 \cdots k_L \in C(s \setminus \{s_{n^r}\})$. In fact, $\psi_{s, s \setminus \{s_{n^r}\}}^{(N,n)}$ is nothing but a sub-matrix of $W^{(N,n)}$. Noting that $M_s V_s \subset V_s$ and $M_{s \setminus \{s_{n^r}\}} V_{s \setminus \{s_{n^r}\}} \subset V_{s \setminus \{s_{n^r}\}}$, we deduce from equation (47) that

$$M_s \psi_{s, s \setminus \{s_{n^r}\}}^{(N,n)} = \psi_{s, s \setminus \{s_{n^r}\}}^{(N,n)} M_{s \setminus \{s_{n^r}\}}, \quad (56)$$

which is the conjugation relation between s and $s \setminus \{s_{n^r}\}$. The following property implies that the conjugation matrix $\psi_{s, s \setminus \{s_{n^r}\}}^{(N,n)}$ vanishes unless $n^r = n$. More precisely, we have

(i) If $k_1 \cdots k_L$ ($1 \leq k_L \leq N$) does *not* belong to $C(s \setminus \{s_n\})$, then

$$\text{Tr } a_{j_1 k_1}^{(N,n)} \cdots a_{j_L k_L}^{(N,n)} = 0. \quad (57)$$

(ii) Equivalently,

$$\text{if } \text{Tr } a_{j_1 k_1}^{(N,n)} \cdots a_{j_L k_L}^{(N,n)} \neq 0, \quad \text{then } k_1 \cdots k_L \in C(s \setminus \{s_n\}). \quad (58)$$

This statement will be proved in appendix B. Note that the sort sequences of the sectors \mathbf{s} and $\mathbf{s} \setminus \{\mathbf{s}_n\}$ can be represented as

$$\begin{array}{ccccccc}
S_1 & & S_n - S_{n-1} & & S_{n+1} - S_n & & L - S_n \\
- & - & - & - & - & - & - \\
1 & \cdots & 1 & 2 & \cdots & n-1 & n \\
\text{Ic} & & \text{Ic} & & \text{Ic} & & \text{Ic} \\
n & \cdots & n & n+1 & \cdots & n+1 & n+2 \cdots N \\
N & N+1 & \cdots & N+1 & \cdots & N+1 & \cdots
\end{array} \quad (59)$$

$$c_{\underbrace{1 \cdots n-1}_{S_1}} \underbrace{c_{\underbrace{\cdots n}_{S_{n+1}-S_{n-1}}} \cdots c_{\underbrace{n+1 \cdots N-1}_{S_{n+1}-S_{n-1}}}}_{S_{n+1}-S_{n-1}} c_{\underbrace{N \cdots N}_{L-S_n}}, \quad (60)$$

respectively. The index n specifies which kind of particles splits when the sector $s \setminus \{s_n\}$ is lifted to s . We simply write

$$\psi_{S, S \setminus \{s_n\}} = \psi_{S, S \setminus \{s_n\}}^{(N, n)}. \quad (61)$$

We recall that n appears explicitly in (50) for the hat matrix $\hat{\mathbf{a}}^{(N,n)}$, leading to the quadratic algebra generated by the matrix elements of $\mathbf{a}^{(N,n)}$. It is important to note that the sector specification property depends on representation (45).

We emphasize that the statements (57) or (58) do not guarantee that $\psi_{s, s \setminus \{s_n\}}$ is non-vanishing (i.e. meaningful). However, based on exact calculations for small system sizes using Mathematica, we shall conjecture the stronger property that the mapping $\psi_{s, s \setminus \{s_n\}}$ is injective.

4.3. Uniqueness

Conjugation operators T from $V_{\mathbf{s} \setminus \{s_n\}}$ to $V_{\mathbf{s}}$ that satisfy $\mathbf{M}_{\mathbf{s}} T = T \mathbf{M}_{\mathbf{s} \setminus \{s_n\}}$ are not unique. Indeed, because of the spectral inclusion $\text{Spec}(s) \supset \text{Spec}(s \setminus \{s_n\})$, there exist, in principle, at least $\dim V_{\mathbf{s} \setminus \{s_n\}}$ such conjugation operators [8]. We conjecture, however, that the conjugation matrix $\psi_{\mathbf{s}, \mathbf{s} \setminus \{s_n\}}$ is unique if the following additional constraint is imposed.

Uniqueness conjecture. The solution to $\mathbf{M}_s \boldsymbol{\psi}_{s, \mathcal{S} \setminus \{s_n\}} = \boldsymbol{\psi}_{s, \mathcal{S} \setminus \{s_n\}} \mathbf{M}_{\mathcal{S} \setminus \{s_n\}}$ is unique up to an overall constant factor when the following constraint is imposed:

$$(j_1 \cdots j_L | \psi_{s, s \setminus \{s_n\}} | k_1 \cdots k_L) = 0, \quad (62)$$

for all configurations $j_1, \dots, j_L \in \mathcal{C}(s)$ and $k_1, \dots, k_L \in \mathcal{C}(s \setminus \mathcal{S}_n)$ such that $\frac{1}{2} \leq \frac{j_i}{k_i}$ such that $k_i + 1 \leq j_i \leq n$ or $n + 1 \leq j_i \leq k_i$.

The conjugation matrix $\psi_{\mathbf{s}, \mathbf{s} \setminus \{s_n\}}$ constructed by using the operators given in (45) satisfies this condition (see example (A.3))⁹.

4.4. Conjugation relation for $s \supset t$

Now we turn to the construction of the conjugation matrix between Markov matrices of arbitrary sectors s and t such that $s \supset t$. Let us set $s = \{s_1 < \dots < s_N\}$ and $t \setminus s_{n_1} = \{s_2, \dots, s_{n_2}\}$. Using the conjugation relation (56) between nearest-neighbor pairs repeatedly, we achieve the conjugation relation

$$\psi_{\text{st}} M_t = M_s \psi_{\text{st}}, \quad (63)$$

where

$$\psi_{\text{st}} = \psi_{s_1} \psi_{s_1 s_2} \cdots \psi_{s_{l-2} s_{l-1}} \psi_{s_{l-1} t} \quad (64)$$

with $s_x = s \setminus \{s_{n_1}, \dots, s_{n_x}\} = t \cup \{s_{n_{x+1}}, \dots, s_{n_u}\}$ (for simplicity, we have omitted the superscripts in the ψ mappings as in equation (61)). Each $\psi_{s_{x-1} s_x}$ is constructed by using $a^{(N+1-x, y)}$ with $y \in \overline{n_x} \setminus \{z \in \overline{n_x} : n_z < n_x\}$. *A priori*, this definition depends on the order chosen to enumerate the set $s \setminus t$. However, we conjecture that ψ_{st} is independent from the choice of the path in the Hasse diagram.

⁹ Note that the uniqueness conjecture does not claim that representation (45) for the hat relation (48) is unique.

4.4.1. Commutativity. The path independence in the Hasse diagram can be summarized by the following statement, which we have checked for several values of (N, n) by using Mathematica.

Commutativity conjecture. For any sector $s = \{s_1 < \dots < s_N\}$ and for $1 \leq x < y \leq N$,

$$\psi_{s, s \setminus \{s_x\}} \psi_{s \setminus \{s_x\}, s \setminus \{s_x, s_y\}} = \psi_{s, s \setminus \{s_y\}} \psi_{s \setminus \{s_y\}, s \setminus \{s_x, s_y\}}. \quad (65)$$

This conjecture implies that for two sectors $t \subset s$, with $s \setminus t = \{s_{n_1}, \dots, s_{n_u}\}$, ψ_{st} is independent of the path in the Hasse diagram. In other words, for any reordering such that $\{\tilde{n}_1, \dots, \tilde{n}_u\} = \{n_1, \dots, n_u\}$

$$\psi_{ss_1} \psi_{s_1 s_2} \dots \psi_{s_{u-2} s_{u-1}} \psi_{s_{u-1} t} = \psi_{s \tilde{s}_1} \psi_{\tilde{s}_1 \tilde{s}_2} \dots \psi_{\tilde{s}_{u-2} \tilde{s}_{u-1}} \psi_{\tilde{s}_{u-1} t} \quad (66)$$

with $s_x = s \setminus \{s_{n_1}, \dots, s_{n_x}\}$ and $\tilde{s}_x = s \setminus \{s_{\tilde{n}_1}, \dots, s_{\tilde{n}_x}\}$.

Remark. In particular, we note that the stationary state $|\bar{P}_s\rangle$ of each sector $s = \{s_1 < \dots < s_N\}$ can be obtained as

$$|\bar{P}_s\rangle = \psi_{ss_1} \psi_{s_1 s_2} \dots \psi_{s_{N-2} s_{N-1}} \psi_{s_{N-1} \emptyset} |1 \dots 1\rangle, \quad (67)$$

where we can choose the intermediate sectors $s_x = s_{n_1}, \dots, s_{n_x}$ arbitrarily. Since the stationary state is unique in each sector, the compositions with different sets of intermediate sectors must be the same up to a constant factor. This observation supports the commutativity conjecture.

Example. We take $s = \{2, 3, 5, 7, 11, 13\}$ and $t = \{7, 13\}$ so that $s \setminus t = \{2, 3, 5, 11\}$. If one chooses $s_{n_1} = 11$, $s_{n_2} = 5$ and $s_{n_3} = 3$, then the intermediate conjugation matrices are constructed from $a^{(6,5)}$, $a^{(5,3)}$ and $a^{(4,2)}$, respectively, and this leads to the conjugation matrix ψ_{st} as follows:

$$\psi_{st} = \psi_{\{2,3,5,7,11,13\}, \{2,3,5,7,13\}} \psi_{\{2,3,5,7,13\}, \{2,3,7,13\}} \psi_{\{2,3,7,13\}, \{2,7,13\}}. \quad (68)$$

Taking $s_{n_1} = 5$, $s_{n_2} = 3$ and $s_{n_3} = 11$, we obtain the following conjugation matrix:

$$\tilde{\psi}_{st} = \psi_{\{2,3,5,7,11,13\}, \{2,3,7,11,13\}} \psi_{\{2,3,7,11,13\}, \{2,7,11,13\}} \psi_{\{2,7,11,13\}, \{2,7,13\}}, \quad (69)$$

where the intermediate conjugation matrices are constructed by $a^{(6,3)}$, $a^{(5,2)}$ and $a^{(4,3)}$, respectively. The commutativity conjecture implies that $\psi_{st} = \tilde{\psi}_{st}$.

4.5. Generalized Ferrari–Martin algorithm

The algebraic construction that we have presented for all values of n , with $1 \leq n \leq N$, can be turned into an algorithm to calculate the stationary weights of the N -TASEP, that generalizes the original Ferrari–Martin algorithm [24]. Figure 4 provides an example for $(N, n) = (5, 3)$.

Given two sectors $s = \{s_1 < \dots < s_N\}$ and $s \setminus \{s_n\}$, the matrix product representation of the conjugation matrix $\psi_{s, s \setminus \{s_n\}}$ leads to the following generalization of the Ferrari–Martin algorithm.

- (i) Set s_n black boxes and $(L - s_n)$ white boxes arbitrarily as $c_1 \dots c_L$ ($c_i = b, w$) on the upper line, and a configuration $k_1 \dots k_L$ of the sector $s \setminus \{s_n\}$ on the lower line. There are

$$(s_v - s_{v-1}) \text{ } v\text{-th-class particles} \quad (\text{for } 1 \leq v \leq n-1, s_0 = 0), \quad (70)$$

$$(s_{n+1} - s_{n-1}) \text{ } n\text{-th-class particles}, \quad (71)$$

$$(s_{v+1} - s_v) \text{ } v\text{-th-class particles} \quad (\text{for } n+1 \leq v \leq N, s_{N+1} = L) \quad (72)$$

on the lower line.

As in the original Ferrari–Martin algorithm, put particles of the v th class from $v = 1$ to $v = n-1$, according to rule (ii-v).

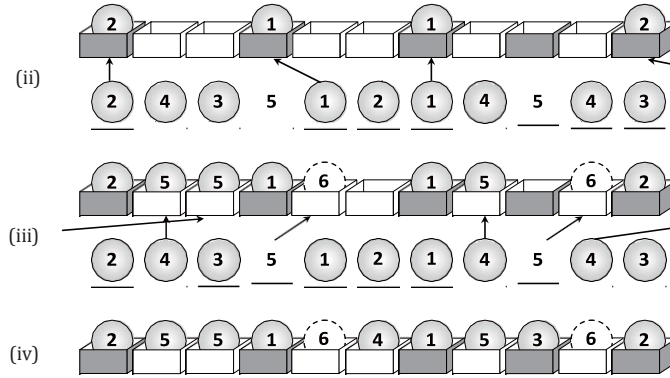


Figure 4. The GFMA that constructs a configuration of N -TASEP from that of $(N-1)$ -TASEP. This figure provides the example $F(bwwbwwbwwb, 24351214543) = 25516415362$.

- (ii-v) There are $(s_n - s_{n-1})$ unoccupied black boxes. Let $i_1, \dots, i_{s_n - s_{n-1}}$ be the positions of the v th-class particles. Iterate the following procedure $(s_n - s_{n-1})$ times; find the nearest unoccupied black box $c_{i^*} = b$ with $i^* \leq i_4$ for the 4th v ($= k_{i_4}$), and put the particle v on it or on the rightmost unoccupied black box if i^* does not exist. Put particles of the $(v+1)$ st class from $v = N$ to $v = n+1$, according to the rule (iii-v) which is opposite to (ii).
- (iii-v) There are $(s_{v+1} - s_n)$ unoccupied white boxes. Let $i_1, \dots, i_{s_{v+1} - s_n}$ be the positions of the v th-class particles. Iterate the following procedure $(s_{v+1} - s_n)$ times; find the nearest unoccupied white box $c_{i^*} = w$ with $i^* \geq i_4$ for the 4th v ($= k_{i_4}$), and put the particle $v+1$ on it or on the leftmost unoccupied white box if i^* does not exist. (Note that for $v = N$, we 'put empty sites' (i.e. particles of the $(N+1)$ st class).)
- (iv) There are $(s_n - s_{n-1})$ unoccupied black boxes and $(s_{n+1} - s_n)$ unoccupied white boxes. Put n 's and $(n+1)$'s on them, respectively. We have a configuration $F(c_1 \dots c_L, k_1 \dots k_L)$ on the upper line.

This GFMA constructs the same conjugation matrix:

$$\psi_{s, s \setminus \{s_n\}} = |F(c_1 \dots c_L, k_1 \dots k_L)(k_1 \dots k_L|, \quad (73)$$

where $|$ runs over $c_1 \dots c_L$ and $k_1 \dots k_L$ with $\#\{i | c_i = b\} = s_n$ and $k_1 \dots k_L$ belonging to the sector $s \setminus \{s_n\}$.

This algorithm allows us to close the loop between this work and the previous articles [18, 24, 40]. In these previous works, the original Ferrari–Martin algorithm, in which empty sites played a very special role, was used to construct a quadratic algebra to represent the stationary state. Here, we have found a family of matrix product representations that allow to split any given species into two subspecies (so that empty sites do not play a distinguished role anymore).

To conclude this section, we precisely show the equivalence (73) between the matrix product representation and the GFMA. We first note that each nonzero element of $a_{JK}^{(N,n)}$ (45) has the form

$$a_{JK}^{(N,n)} = a_{JK,1}^{(N,n)} \otimes \dots \otimes a_{JK,n-1}^{(N,n)} \otimes a_{JK,n+1}^{(N,n)} \otimes \dots \otimes a_{JK,N}^{(N,n)} \quad \text{with } a \in \{1, A, \delta, \epsilon\}. \quad (74)$$

(Note the shift in the subscripts that occurs for $v > n$.) Let

$$A = \sum_{\mu \geq 0} C|\mu\rangle, \quad (|\mu| = (0, \dots, 0, 1, 0, \dots), \quad |\mu\rangle = (|\mu|^T)_{(\mu+1)\text{st}} \quad (75)$$

be the space on which the matrix $a_{JK,v}^{(N,n)}$ acts (thus $a_{JK}^{(N,n)}$ acts on $A^{\otimes(N-1)}$). The fundamental matrices A , δ and κ act on $|\mu\rangle$ as

$$A|\mu\rangle = \begin{cases} 0 & (\mu \geq 1), \\ |0\rangle & (\mu = 0), \end{cases} \quad \delta|\mu\rangle = \begin{cases} |\mu-1\rangle & (\mu \geq 1), \\ 0 & (\mu = 0), \end{cases} \quad \kappa|\mu\rangle = |\mu+1\rangle. \quad (76)$$

The form (74) implies that its trace is again decomposed as

$$\text{Tr } a_{j_1 k_1, v}^{(N,n)} \cdots a_{j_L k_L, v}^{(N,n)} = \bigcup_{\substack{1 \leq v \leq N \\ v \neq n}} \begin{matrix} j_1 k_1, v & j_L k_L, v & 0 & j_i k_i \\ (N,n) & (N,n) & & \end{matrix} \quad \text{at least one } a = 0. \quad (77)$$

Furthermore, we find

$$\text{Tr } a_{j_1 k_1, v}^{(N,n)} \cdots a_{j_L k_L, v}^{(N,n)} = \sum_{\substack{1 \leq v \leq N \\ v \neq n}} ((\mu_v | a_{j_1 k_1, v}^{(N,n)} \cdots a_{j_L k_L, v}^{(N,n)} | \mu_v)) \quad (78)$$

$$= \begin{cases} ((\mu_v | a_{j_1 k_1, v}^{(N,n)} \cdots a_{j_L k_L, v}^{(N,n)} | \mu_v)) = 1 & (*), \\ 0 & (\text{otherwise}). \end{cases} \quad (79)$$

The symbol $*$ denotes the case where there exists $(\mu_1, \dots, \mu_{n-1}, \mu_{n+1}, \dots, \mu_N)$ such that

$$a_{j_1 k_1, v}^{(N,n)} \cdots a_{j_L k_L, v}^{(N,n)} | \mu_v) = | \mu_v), \quad a_{j_1 k_1, v}^{(N,n)} \cdots a_{j_L k_L, v}^{(N,n)} | \mu_v) = 0 \quad (\mu_v \neq \mu_v). \quad (80)$$

The set of numbers $(\mu_1, \dots, \mu_{n-1}, \mu_{n+1}, \dots, \mu_N)$ which satisfies this condition is unique if it exists. This uniqueness is true for basic sectors. On the other hand, we draw a vertical line at each bond in the GFMA as shown in figure 5. The correspondence between the algorithm and the action of the operators can be understood by regarding $|\mu\rangle$ as the number μ of arrows $v \rightarrow v$ ($v < n$) or $v \rightarrow v+1$ ($v > n$) crossing each vertical line: δ decreases the number μ of arrows, κ increases μ , A tests whether $\mu = 0$ or not and the identity operator 1 indeed does nothing. Then we find that, for given configurations $j_1 \cdots j_L$ and $k_1 \cdots k_L$ of N -species and $(N-1)$ -species sectors, the matrix product $a_{j_1 k_1}^{(N,n)} \cdots a_{j_L k_L}^{(N,n)}$ gives a unique pattern of arrows (if $*$ is satisfied). This means that $j_1 \cdots j_L$ can be obtained by using the GFMA from $k_1 \cdots k_L$:

$$F(c_1 \cdots c_L, k_1 \cdots k_L) = j_1 \cdots j_L \quad \text{with} \quad c_i = b(j_i \leq n), \quad w(j_i \geq n+1). \quad (81)$$

This relation is true because representation (45) obeys the rule of the GFMA. (For instance, the v th element of $a_{j_i k_i}^{(N,n)}$ for $j_i, k_i < n$ and $v \leq j_i - 1$ is A , which means that j_i can be put on the i th site of the upper line only when no arrow $v \rightarrow v$ crosses the vertical line between sites i and $i+1$.) Thus, this graphical construction shows that the matrix product representation and the GFMA are equivalent.

Figure 5 provides an example of the correspondence in the case $(N, n) = (3, 2)$: for $j_1 \cdots j_{10} = 3211414433$ and $k_1 \cdots k_{10} = 1233321212$, the actions of $a_{j_1 k_1, 1}^{(3,2)}$ and $a_{j_i k_i, 3}^{(3,2)}$ give

trajectories of the numbers of arrows $1 \rightarrow 1$ and $3 \rightarrow 4$, respectively. Namely, the condition $*$ is satisfied for these configurations: one can show

$$a_{j_1 k_1, 1}^{(3,2)} \cdots a_{j_{10} k_{10}, 1}^{(3,2)} |1\rangle = |1\rangle, \quad a_{j_1 k_1, 1}^{(3,2)} \cdots a_{j_{10} k_{10}, 1}^{(3,2)} | \mu_1^f) = 0 \quad (\mu_1^f \neq 1), \quad (82)$$

$$a_{j_1 k_1, 3}^{(3,2)} \cdots a_{j_{10} k_{10}, 3}^{(3,2)} |0\rangle = |0\rangle, \quad a_{j_1 k_1, 3}^{(3,2)} \cdots a_{j_{10} k_{10}, 3}^{(3,2)} | \mu_3^f) = 0 \quad (\mu_3^f \neq 0). \quad (83)$$

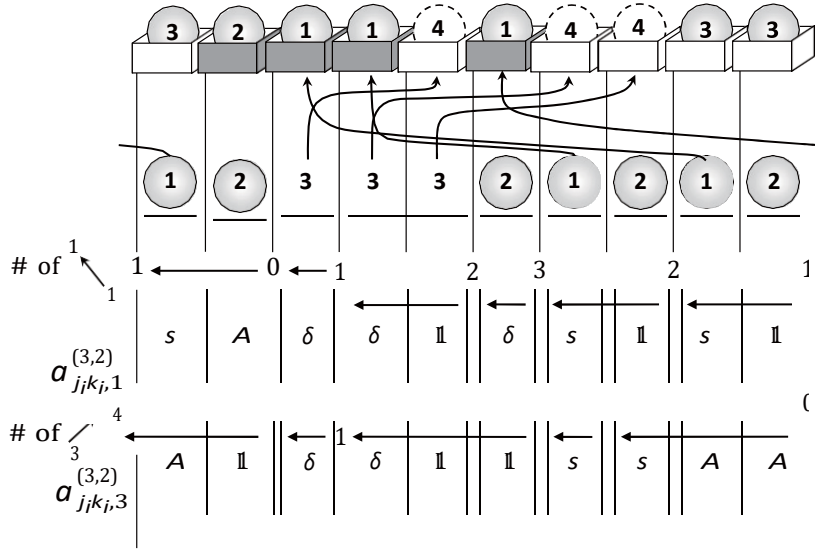


Figure 5. Correspondence between representation (45) and the GFMA.

5. Concluding remarks

The multispecies exclusion process exhibits rich combinatorial properties that can be encoded in a recursive structure known as the Hasse diagram. The relevance of this diagram can be understood by using identification maps: if one blurs the difference between particles that belong to two adjacent classes, a simpler system is obtained that inherits the properties of the original system. Successive identifications allow one to reduce the initial multispecies system to the one-species system. In this procedure, information is irreversibly lost.

In this paper, we have shown that the matrix product representation, which was originally used as a method for representing the stationary state, allows one to define conjugation operators that relate systems with different values of N in the TASEP case. Therefore, the matrix ansatz provides a method to lift information from a simple system to a more complex one: it allows one to calculate not only the stationary state but some excited states as well, that describe how the system relaxes toward its steady state. More precisely, each link in the Hasse diagram corresponds to a lifting operator and is associated with a different quadratic algebra. Hence, the N -TASEP leads to families of algebras, connected through compatibility relations. We believe that this feature is general and that the investigation initiated here could be extended in the following directions.

(i) The N -TASEP is an integrable model and its Markov matrix can be viewed as one special member of a family of commuting transfer matrices for the Perk–Schultz model [41, 44]. One natural question is to study if the matrix product representation for such vertex models allows one to define conjugation operators.

(ii) We also emphasize that we have studied here the N -TASEP only, where particles hop in one definite direction. It is natural to expect that our results could be extended to the N -PASEP. A naïve guess would be to start with the TASEP solution (45) and make the following replacements:

$$\delta \rightarrow \delta_q, \quad \zeta \rightarrow \zeta_q, \quad A \rightarrow A_q, \quad (84)$$

where δ_q , ϵ_q and A_q satisfy the q -deformed quadratic relations (38). However, this guess is correct only for $n = 1$ and $n = N$ but wrong for $1 < n < N$. The basic mathematical reason is that the q -deformed quadratic relations (38) are not stable by tensor product unless $q = 0$: a different approach seems to be required for solving the N -PASEP in full generality.

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Appendix A. An explicit example

Here we write down the operator-valued rectangular matrix $a^{(N,n)}$ (45) for a few values of (N, n) :

$$a^{(2,1)} = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 1 & \delta \\ 2 & A & 0 \\ 3 & \epsilon & 1 \end{array}, \quad a^{(2,2)} = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 1 & \delta \\ 2 & 0 & A \\ 3 & \epsilon & 1 \end{array}, \quad (\text{A.1})$$

$$a^{(3,1)} = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 \otimes 1 & \delta \otimes 1 & 1 \otimes \delta \\ 2 & A \otimes A & 0 & 0 \\ 3 & \epsilon \otimes A & 1 \otimes A & 0 \\ 4 & 1 \otimes \epsilon & \delta \otimes \epsilon & 1 \otimes 1 \end{array}, \quad a^{(3,2)} = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 \otimes 1 & \delta \otimes 1 & \delta \otimes \delta \\ 2 & 0 & A \otimes 1 & A \otimes \delta \\ 3 & \epsilon \otimes A & 1 \otimes A & 0 \\ 4 & \epsilon \otimes \epsilon & 1 \otimes \epsilon & 1 \otimes 1 \end{array}, \quad (\text{A.2})$$

$$a^{(3,3)} = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 \otimes 1 & \delta \otimes \epsilon & \delta \otimes 1 \\ 2 & 0 & A \otimes 1 & A \otimes \delta \\ 3 & 0 & 0 & A \otimes A \\ 4 & \epsilon \otimes 1 & 1 \otimes \epsilon & 1 \otimes 1 \end{array},$$

$$a^{(6,3)} = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 & \delta \cdot \epsilon \cdot 1 \cdot 1 \cdot 1 & \delta \cdot 1 \cdot 1 \cdot 1 \cdot 1 & \delta \cdot 1 \cdot \delta \cdot 1 \cdot 1 & \delta \cdot 1 \cdot 1 \cdot \delta \cdot 1 & \delta \cdot 1 \cdot 1 \cdot 1 \cdot \delta \\ 2 & 0 & A \cdot 1 \cdot 1 \cdot 1 \cdot 1 & A \cdot \delta \cdot 1 \cdot 1 \cdot 1 & A \cdot \delta \cdot \delta \cdot 1 \cdot 1 & A \cdot \delta \cdot 1 \cdot \delta \cdot 1 & A \cdot \delta \cdot 1 \cdot 1 \cdot \delta \\ 3 & 0 & 0 & A \cdot A \cdot 1 \cdot 1 \cdot 1 & A \cdot A \cdot \delta \cdot 1 \cdot 1 & A \cdot A \cdot 1 \cdot \delta \cdot 1 & A \cdot A \cdot 1 \cdot 1 \cdot \delta \\ 4 & \epsilon \cdot 1 \cdot A \cdot A \cdot A & 1 \cdot \epsilon \cdot A \cdot A \cdot A & 1 \cdot 1 \cdot A \cdot A \cdot A & 0 & 0 & 0 \\ 5 & \epsilon \cdot 1 \cdot \epsilon \cdot A \cdot A & 1 \cdot \epsilon \cdot \epsilon \cdot A \cdot A & 1 \cdot 1 \cdot \epsilon \cdot A \cdot A & 1 \cdot 1 \cdot 1 \cdot A \cdot A & 0 & 0 \\ 6 & \epsilon \cdot 1 \cdot 1 \cdot \epsilon \cdot A & 1 \cdot \epsilon \cdot 1 \cdot \epsilon \cdot A & 1 \cdot 1 \cdot 1 \cdot \epsilon \cdot A & 1 \cdot 1 \cdot \delta \cdot \epsilon \cdot A & 1 \cdot 1 \cdot 1 \cdot 1 \cdot A & 0 \\ 7 & \epsilon \cdot 1 \cdot 1 \cdot 1 \cdot \epsilon & 1 \cdot \epsilon \cdot 1 \cdot 1 \cdot \epsilon & 1 \cdot 1 \cdot 1 \cdot 1 \cdot \epsilon & 1 \cdot 1 \cdot \delta \cdot 1 \cdot \epsilon & 1 \cdot 1 \cdot 1 \cdot \delta \cdot \epsilon & 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \end{array}, \quad (\text{A.3})$$

Appendix B. Proof of the statement (58)

We shall use the following property: let $b_i \in \{1, A, \delta, \epsilon\}$ ($1 \leq i \leq L$).

$$\#\{i|b_i = \delta\} \quad \#\{i|b_i = \epsilon\} \Rightarrow \text{Tr}(b_1 \cdots b_L) = 0. \quad (\text{B.1})$$

We also note the decomposition (77).

We first consider the case $(N, n) = (3, 2)$ as an example. Noting the properties (B.1) and (77), we find the following necessary condition such that $\text{Tr} a_{j_1 k_1}^{(3,2)} \cdots a_{j_L k_L}^{(3,2)} \neq 0$: every $a_{j_i k_i}^{(3,2)}$ is nonzero and

$$\#\{i|a_{j_i k_i, v}^{(3,2)} = \delta\} = \#\{i|a_{j_i k_i, v}^{(3,2)} = \epsilon\} \quad (\text{B.2})$$

for $v = 1, 3$. For given 3- and 2-species configurations $j_1 \cdots j_L$ and $k_1 \cdots k_L$, we have

$$\#\{i|j_i = 1\} = \#\{i|a_{j_i k_i, 1}^{(3,2)} = \delta\} + \#\{i|j_i = k_i = 1\}, \quad (\text{B.3})$$

$$\#\{i|j_i = 4\} = \#\{i|a_{j_i k_i, 3}^{(3,2)} = \epsilon\} + \#\{i|j_i = k_i + 1 = 4\}, \quad (\text{B.4})$$

$$\#\{i|k_i = 1\} = \#\{i|a_{j_i k_i, 1}^{(3,2)} = \epsilon\} + \#\{i|j_i = k_i = 1\}, \quad (\text{B.5})$$

$$\#\{i|k_i = 3\} = \#\{i|a_{j_i k_i, 3}^{(3,2)} = \delta\} + \#\{i|j_i - 1 = k_i = 3\}, \quad (\text{B.6})$$

assuming that every $a_{j_i k_i}^{(3,2)} \neq 0$. (See the explicit form for $a^{(3,2)}$ (A.2).) Condition (B.2) and relations (B.3)–(B.6) imply that

$$\#\{i|j_i = 1\} = \#\{i|k_i = 1\}, \quad \#\{i|j_i = 4\} = \#\{i|k_i = 3\}. \quad (\text{B.7})$$

This consequence means that if $j_1 \cdots j_L \in \mathcal{C}(\{s_1, s_2, s_3\})$ and $\text{Tr} a_{j_1 k_1}^{(3,2)} \cdots a_{j_L k_L}^{(3,2)} \neq 0$ then $k_1 \cdots k_L \in \mathcal{C}(\{s_1, s_3\})$.

In the GFMA, condition (B.2) corresponds to the conservation of the number of arrows when going around the ring (80). As in figure 5, for $j_1 \cdots j_L = 3211414433$ and $k_1 \cdots k_L = 1233321212$, we actually observe

$$\#\{i|a_{j_i k_i, 1}^{(3,2)} = \delta\} = \#\{i|a_{j_i k_i, 1}^{(3,2)} = \epsilon\} = 3, \quad (\text{B.8})$$

$$\#\{i|a_{j_i k_i, 3}^{(3,2)} = \delta\} = \#\{i|a_{j_i k_i, 3}^{(3,2)} = \epsilon\} = 2. \quad (\text{B.9})$$

For the general case, a necessary condition for $\text{Tr} a_{j_1 k_1}^{(N,n)} \cdots a_{j_L k_L}^{(N,n)} \neq 0$ is that every

$a_{j_i k_i}^{(N,n)}$ is nonzero and

$$\#\{i|a_{j_i k_i, v}^{(N,n)} = \delta\} = \#\{i|a_{j_i k_i, v}^{(N,n)} = \epsilon\} \quad \text{for } \forall v \in \{1, \dots, n-1, n+1, \dots, N\}. \quad (\text{B.10})$$

We can assume that every $a_{j_i k_i}^{(N,n)}$ is nonzero. From definition (45), we find

$$\begin{aligned} \#\{i|j_i = v\} &= \#\{i|a_{j_i k_i, v}^{(N,n)} = \delta\} + \#\{i|j_i = k_i = v\} \quad (1 \leq v \leq n-1), \\ \#\{i|k_i = v\} &= \#\{i|a_{j_i k_i, v}^{(N,n)} = \epsilon\} + \#\{i|j_i = k_i + 1 = v\} \quad (n+2 \leq v \leq N+1), \end{aligned} \quad (\text{B.11})$$

$$\begin{aligned} \#\{i|j_i = v\} &= \#\{i|a_{j_i k_i, v}^{(N,n)} = \delta\} + \#\{i|j_i - 1 = k_i = v\} \quad (n+1 \leq v \leq N). \end{aligned} \quad (\text{B.12})$$

Consequently, we have

$$\#\{i|j_i = v\} = \#\{i|k_i = v\} \quad \text{for } 1 \leq v \leq n-1, \quad (\text{B.13})$$

$$\#\{i|j_i = v\} = \#\{i|k_i = v-1\} \quad \text{for } n+2 \leq v \leq N+1, \quad (\text{B.14})$$

which exactly means that if $j_1 \cdots j_L \in C(s)$ ($s = \{s_1, \dots, s_N\}$) and $\text{Tr } a_{j_1 k_1}^{(N,n)} \cdots a_{j_L k_L}^{(N,n)} = 0$, then $k_1 \cdots k_L \in s \setminus \{s_n\}$. (See the figures of the sort sequences (59) and (60).) We emphasize, however, that even if $j_1 \cdots j_L \in C(s)$ and $k_1 \cdots k_L \in C(s \setminus \{s_n\})$, the matrix element $\text{Tr } a_{j_1 k_1}^{(N,n)} \cdots a_{j_L k_L}^{(N,n)}$ can still be equal to zero.

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