# Problem with critical Sobolev exponent and with weight

Rejeb Hadiji, Habib Yazidi

# Problem with critical Sobolev exponent and with weight

R.Hadiji and H.Yazidi
UFR des Sciences et Technologie, CNRS UMR 8050,
Université Paris 12 - Val-de-Marne,
61, avenue du Général de Gaulle,
94010 Créteil Cedex.
(hadiji@univ-paris12.fr, yazidi@univ-paris12.fr)

**Abstract**

We consider the problem: $-div(p\nabla u) = u^{q-1} + \lambda u$, $u > 0$ in $\Omega$, $u = 0$ on $\partial\Omega$. Where $\Omega$ is a bounded domain in $I\!R^n$, $n \geq 3$, $p : \bar{\Omega} \longrightarrow I\!R$ is a given positive weight such that $p \in H^1(\Omega) \cap C(\bar{\Omega})$, $\lambda$ is a real constant and $q = \frac{2n}{n-2}$. We study the effect of the behavior of $p$ near its minima and the impact of the geometry of domain on the existence of solutions for the above problem.

*Key Words:* Critical Sobolev exponent, variational methods.
*2000 Mathematics Subject Classification:* 35J20, 35J25, 35J60.

## 1 Introduction

In this paper we study the following problem:

$$(1.1) \qquad \begin{cases} -div(p(x)\nabla u) = u^{q-1} + \lambda u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega$ is a bounded domain in $I\!R^n$, $n \geq 3$, $p : \bar{\Omega} \longrightarrow I\!R$ is a given positive weight such that $p \in H^1(\Omega) \cap C(\bar{\Omega})$, $\lambda$ is a real constant and $q = \frac{2n}{n-2}$ is the critical exponent for the Sobolev embedding of $H_0^1(\Omega)$ into $L^q(\Omega)$.

In [BN], Brezis and Nirenberg treated the case where $p$ is constant. They proved, in particular, the existence of a solution of (1.1) for $0 < \lambda < \lambda_1$ if $n \geq 4$ and for $\lambda^* < \lambda < \lambda_1$ if $n = 3$, where $\lambda_1$ is the first eigenvalue of $-\Delta$ on $\Omega$ with zero Dirichlet boundary condition and $\lambda^*$ is a positive constant.

In this paper, we extend this result to the general case of where $p$ is not constant. The study of problem (1.1), shows that the existence of solutions depends, apart from parameter $\lambda$, on the behavior of p near its minima and on the geometry of the domain $\Omega$.

Set $p_0 = \min\{p(x),\, x \in \bar{\Omega}\}$, we suppose that $p^{-1}(\{p_0\}) \cap \Omega \neq \emptyset$ and let $a \in p^{-1}(\{p_0\}) \cap \Omega$. In the first part of this work, we study the effect of the behavior of $p$ near its minima on the existence of solution for our problem. The method that is mostly relied upon, apart from the identities of Pohozeav, is the adaptations to the new context of the arguments

developed in [BN].
We assume that, in a neighborhood of $a$, $p$ behaves like

$$p(x) = p_0 + \beta_k |x-a|^k + |x-a|^k \theta(x), \tag{1.2}$$

with $k > 0$, $\beta_k > 0$ and $\theta(x)$ tends to 0 when $x$ tends to $a$.
Note that the parameter $k$ will play an essential role in the study of our problem. Indeed, 2 appears as a critical value for $k$. More precisely the case $k > 2$ is treated by a classical procedure, however the case $0 < k \leq 2$ is less easily accessible. Therefore, in this case, we restrict ourself to the case where $p$ satisfies the additional condition

$$k\beta_k \leq \frac{\nabla p(x).(x-a)}{|x-a|^k} \quad \text{a.e } x \in \Omega. \tag{1.3}$$

Let us notice that if $p$ is sufficiently smooth, then condition (1.2) follows directly from Taylor's expansion of $p$ near $a$.
The fact that 2 is a critical value for $k$ appears clearly in dimension $n = 4$, therefore, in this dimension and with the aim of obtaining more explicit results, we assume moreover that $\theta$ satisfies $\int_{B(a,1)} \frac{\theta(x)}{|x-a|^4} dx < \infty$. Let us emphasize that this last condition is not necessary to prove the existence of solutions.
Moreover, in dimension $n = 3$, the problem is more delicate, then we treat it in a particular case; more precisely for $p(x) = p_0 + \beta_k |x-a|^k$, $k > 0$.
The first result of this paper is the following

**Theorem 1.1**
*Assume that $p \in H^1(\Omega) \cap C(\bar{\Omega})$ satisfies (1.2). Let $\lambda_1^{div}$ be the first eigenvalue of $-\mathrm{div}(p(x)\nabla .)$ on $\Omega$ with zero Dirichlet boundary condition, we have*
*1)If $n \geq 4$ and $k > 2$, then for every $\lambda \in ]0, \lambda_1^{div}[$ there exists a solution of (1.1).*
*2)If $n \geq 4$ and $k = 2$, then there exists a constant $\tilde{\gamma}(n) = \frac{(n-2)n(n+2)}{4(n-1)}\beta_2$ such that for every $\lambda \in ]\tilde{\gamma}(n), \lambda_1^{div}[$ there exists a solution of (1.1).*
*3)If $n = 3$ and $k \geq 2$, then there exists a constant $\gamma(k) > 0$ such that for every $\lambda \in ]\gamma(k), \lambda_1^{div}[$ there exists a solution of (1.1).*
*4)If $n \geq 3$, $0 < k < 2$ and $p$ satisfies the condition (1.3) then there exists $\lambda^* \in [\tilde{\beta}_k \frac{n^2}{4}, \lambda_1^{div}[$, where $\tilde{\beta}_k = \beta_k \min[(\mathrm{diam}\ \Omega)^{k-2}, 1]$, such that for any $\lambda \in ]\lambda^*, \lambda_1^{div}[$ problem (1.1) admits a solution.*
*5)If $n \geq 3$ and $k > 0$, then for every $\lambda \leq 0$ there is no minimizing solution of equation (1.1).*
*6)If $n \geq 3$ and $k > 0$, then there is no solution of problem (1.1) for every $\lambda \geq \lambda_1^{div}$.*

**Remark 1.1**
*In general, the intervals $]\tilde{\gamma}(n), \lambda_1^{div}[$ in 2) and $[\tilde{\beta}_k \frac{n^2}{4}, \lambda_1^{div}[$ in 4), may be empty. But there are some sufficient conditions for which the above intervals are nonempty:*
*1) If $p_0 > \dfrac{n(n-4)}{(n-1)(n-2)^2}\beta_2\,(\mathrm{diam}\ \Omega)^2$, then $\tilde{\gamma}(n) < \lambda_1^{div}$.*
*Notice that this condition is always true if $n$ is rather large.*

*2) If* $p_0 > \dfrac{\tilde{\beta}_k n^2}{(n-2)^2\,(\text{diam }\Omega)^2}$*, then* $\tilde{\beta}_k \frac{n^2}{4} < \lambda_1^{div}$.

The second part of this work is dedicated to the study of the effect of the geometry of the domain on the existence of solutions of our problem. More precisely, since for $\lambda = 0$ and $p \in H^1(\Omega) \cap C(\bar{\Omega})$ satisfying $\nabla p(x).(x-a) > 0$ a.e in $\Omega$, the problem (1.1) does not have a solution for a starshaped domain about $a$, we will modify the geometry of $\Omega$ in order to find a solution. Therefore, let $\Omega \subset I\!R^n$, $n \geq 3$ be a starshaped domain about $a$ and let $\varepsilon > 0$, we will study the existence of solution of the problem

$$(\mathbf{I}_\varepsilon) \quad \begin{cases} -div(p(x)\nabla u) = u^{q-1} & \text{in } \Omega_\varepsilon, \\ u > 0 & \text{in } \Omega_\varepsilon, \\ u = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases}$$

where $\Omega_\varepsilon = \Omega \setminus \bar{B}(a, \varepsilon)$.
For $p \equiv 1$ and $\lambda = 0$, the problem (1.1) has been first investigated in [C] and an interesting result of existence has been proved for domains with holes. In [BaC], this last result is extended to all domains having "nontrivial" topology (in a suitable sense). This nontrivially condition (which covers a large class of domains) is only sufficient for the solvability but not necessary as shown by some examples of contractible domains $\Omega$ for which (1.1) has solutions (see [D], [Di], [Pa]).
In other direction, [Le] shows that the solution of [C], on a domain with a hole of diameter $\varepsilon$ and center $x_0$, concentrates at the point $x_0$. In [H], the author generalized the result of [C] for the case where $u^q$ is replaced by $u^q + \mu u^\alpha$, where $\mu \in I\!R$ and $1 < \alpha < q$.
In this work, we consider the case where $p \in H^1(\Omega) \cap C(\bar{\Omega})$ and satisfying $\nabla p(x).(x-a) > 0$ a.e on $\Omega \setminus \{a\}$. The method we use in this part is an adaptation of those used in [C] and [H]. More particularly, we use the min-max techniques and a variant of the Ambrosetti-Rabinowitz theorem, see [AR].
The second result of this paper is the following

**Theorem 1.2**
*There exists* $\varepsilon_0 = \varepsilon_0(\Omega, p) > 0$ *such that for* $0 < \varepsilon < \varepsilon_0$ *the problem* $(I_\varepsilon)$ *has at least one solution in* $H_0^1(\Omega_\varepsilon)$.

The rest of this paper is divided into three sections. In Section 2 some preliminary results will be established. Section 3 and Section 4 are devoted respectively to the proof of Theorem 1.1 and the proof of Theorem 1.2.

## 2 Some preliminary results

We start by recalling some notations which will be frequently used throughout the rest of this paper. First, we define

$$S = \inf_{u \in H_0^1(\Omega), \|u\|_q = 1} \|\nabla u\|_2^2$$

that corresponds to the best constant for the Sobolev embedding $H_0^1(\Omega) \subset L^q(\Omega)$. Let us denote by $U_{a,\varepsilon}$ an extremal function for the Sobolev inequality

$$U_{a,\varepsilon}(x) = \frac{1}{(\varepsilon + |x-a|^2)^{\frac{n-2}{2}}}, \ x \in I\!R^n.$$

We set

$$u_{a,\varepsilon}(x) = \zeta(x) U_{a,\varepsilon}(x)\,,\ x \in I\!R^n, \tag{2.1}$$

where $\zeta \in C_0^\infty(\bar{\Omega})$ is a fixed function such that $0 \leq \zeta \leq 1$, and $\zeta \equiv 1$ in some neighborhood of $a$ included in $\Omega$.
We know from [BN] that

$$\|\nabla u_{a,\varepsilon}\|_2^2 = \frac{K_1}{\varepsilon^{\frac{n-2}{2}}} + O(1), \tag{2.2}$$

$$\| u_{a,\varepsilon} \|_q^2 = \frac{K_2}{\varepsilon^{\frac{n-2}{2}}} + O(\varepsilon) \tag{2.3}$$

and

$$\| u_{a,\varepsilon} \|_2^2 = \begin{cases} \frac{K_3}{\varepsilon^{\frac{n-4}{2}}} + O(1) \ if \ n \geq 5 \\ \frac{\omega_4}{2} |\log \varepsilon| + O(1) \ if \ n = 4 \end{cases} \tag{2.4}$$

where $K_1$ and $K_2$ are positive constants with $\frac{K_1}{K_2} = S$, $\omega_4$ is the area of $S^3$ and $K_3 = \displaystyle\int_{I\!R^n} \frac{1}{(1+|x|^2)^{n-2}} dx$.

We shall state some auxiliary results.
For $p \in C^1(\bar{\Omega})$ or $p \in H^1(\Omega) \cap C(\bar{\Omega})$ and $\nabla p(x).(x-a) \geq 0$ a.e $x \in \Omega$, we consider

$$\alpha(p) = \frac{1}{2} \inf_{u \in H_0^1(\Omega), u \neq 0} \frac{\int_\Omega \nabla p(x).(x-a) |\nabla u|^2 dx}{\int_\Omega |u|^2 dx}.$$

We easily see that $\alpha(p) \in [-\infty, +\infty[$, and we have the following result

**Proposition 2.1**
**1)** *If $p \in C^1(\Omega)$ and if there exists $b \in \Omega$ such that $\nabla p(b)(b-a) < 0$, then $\alpha(p) = -\infty$.*
**2)** *If $p \in H^1(\Omega) \cap C(\bar{\Omega})$ satisfying (1.2) and $\nabla p(x).(x-a) \geq 0$ a.e $x \in \Omega$, we have*
**2.a)** *If $k > 2$ and $p \in C^1(\Omega)$, then $\alpha(p) = 0$ for all $n \geq 3$.*
**2.b)** *If $0 < k \leq 2$ and $p$ satisfies condition (1.3) then for all $n \geq 3$ we have*

$$\frac{k}{2} \beta_k \left( \frac{n+k-2}{2} \right)^2 (\text{diam } \Omega)^{k-2} \leq \alpha(p).$$

**Proof.** We start by proving 1). Set $q(x) = \nabla p(x).(x-a)$, $\forall x \in \Omega$ and let $\varphi \in C_0^\infty(I\!R^n)$ such that $0 \leq \varphi \leq 1$ on $I\!R^n$, $\varphi \equiv 1$ on the ball $\{x, |x| < r\}$, and $\varphi \equiv 0$ outside the ball

$\{x, |x| < 2r\}$, where $r < 1$ is a positive constant .
Set $\varphi_j(x) = \varphi(j(x-b))$ for $j \in \mathbf{N}^*$. We have

$$\begin{aligned}\alpha(p) &\leq \frac{1}{2}\frac{\int_\Omega q(x)|\nabla\varphi_j(x)|^2dx}{\int_\Omega |\varphi_j|^2dx}\\ &\leq \frac{1}{2}\frac{\int_{B(b,\frac{2r}{j})} q(x)|\nabla\varphi_j(x)|^2dx}{\int_{B(b,\frac{2r}{j})} |\varphi_j|^2dx}.\end{aligned}$$

Using the change of variable $y = j(x-b)$, we get

$$\alpha(p) \leq \frac{j^2}{2}\frac{\int_{B(0,2r)} q(\frac{y}{j}+b)|\nabla\varphi(x)|^2dx}{\int_{B(0,2r)} |\varphi|^2dx}.$$

Applying the Dominated Convergence Theorem, we obtain

$$\alpha(p) \leq \frac{j^2}{2}\left[q(b)\frac{\int_{B(0,2r)} |\nabla\varphi(x)|^2dx}{\int_{B(0,2r)} |\varphi|^2dx} + o(1)\right].$$

Letting $j \to \infty$, we deduce the desired result.
Now we will prove 2.a).
Using (1.2) and since $p \in C^1(\Omega)$ in a neighborhood $V$ of $a$, we write

$$p(x) = p_0 + \beta_k|x-a|^k + \theta_1(x), \tag{2.5}$$

where $\theta_1 \in C^1(V)$ is such that

$$\lim_{x\to a}\frac{\theta_1(x)}{|x-a|^k} = 0. \tag{2.6}$$

Looking at (2.6), we deduce that there exists $0 < r < 1$, such that

$$\theta_1(x) \leq |x-a|^k \quad \forall x \in B(a, 2r). \tag{2.7}$$

Let $\varphi \in C_0^\infty(I\!R^n)$ be a function such that $0 \leq \varphi \leq 1$ on $I\!R^n$, $\varphi \equiv 1$ on the ball $\{x, |x| < r\}$, and $\varphi \equiv 0$ outside the ball $\{x, |x| < 2r\}$. Set $\varphi_j(x) = \varphi(j(x-a))$ for $j \in \mathbf{N}^*$, we have

$$0 \leq \alpha(p) \leq \frac{1}{2}\frac{\int_\Omega \nabla p(x).(x-a)|\nabla\varphi_j(x)|^2dx}{\int_\Omega |\varphi_j|^2dx}.$$

Using (2.5), we see that

$$0 \leq \alpha(p) \leq \frac{k\beta_k}{2}\frac{\int_{B(a,\frac{2r}{j})} |x-a|^k|\nabla\varphi_j(x)|^2dx}{\int_{B(a,\frac{2r}{j})} |\varphi_j|^2dx} + \frac{1}{2}\frac{\int_{B(a,\frac{2r}{j})} \nabla\theta_1(x).(x-a)|\nabla\varphi_j(x)|^2dx}{\int_{B(a,\frac{2r}{j})} |\varphi_j|^2dx}.$$

Performing the change of variable $y = j(x-a)$, and integrating by parts the second term of the right hand side, we obtain

$$0 \leq \alpha(p) \leq \frac{k\beta_k}{2j^{k-2}}\frac{\int_{B(0,2r)} |y|^k|\nabla\varphi(y)|^2dx}{\int_{B(0,2r)} |\varphi|^2dx} + \frac{j}{2}\frac{\int_{B(0,2r)} \theta_1(\frac{y}{j}+a)\nabla(y|\nabla\varphi(y)|^2)dx}{\int_{B(0,2r)} |\varphi|^2dx}.$$

Using (2.7), we write

$$0 \le \alpha(p) \le \frac{k\beta_k}{2j^{k-2}} \frac{\int_{B(0,2r)} |y|^k |\nabla\varphi(y)|^2 dx}{\int_{B(0,2r)} |\varphi|^2 dx} + \frac{1}{2j^{k-1}} \frac{\int_{B(0,2r)} |y|^k \nabla(|\nabla\varphi(y)|^2 y) dx}{\int_{B(0,2r)} |\varphi|^2 dx}.$$

Therefore, for $k > 2$ we deduce that $\alpha(p) = 0$, and this finishes the proof of this case. Now, in order to prove 2.b), we need to recall the following Hardy's inequality, see for example [CKN] or Theorem 330 in [HLP].

**Lemma 2.1**
*Let $t \in I\!R$ such that $t + n > 0$, we have $\forall u \in H_0^1(\Omega)$*

$$\int_\Omega |x|^t |u|^2 dx \le (\frac{2}{n+t})^2 \int_\Omega |x.\nabla u|^2 |x|^t dx.$$

*Moreover the constant $(\frac{2}{n+t})^2$ is optimal and is not achieved.*

Now we prove 2.b). Since $p$ satisfies (1.3), we have for all $u \in H_0^1(\Omega) \setminus \{0\}$,

$$\frac{\int_\Omega \nabla p(x).(x-a)|\nabla u(x)|^2 dx}{\int_\Omega |u(x)|^2 dx} \ge k\beta_k \frac{\int_\Omega |x-a|^k |\nabla u(x)|^2 dx}{\int_\Omega |u(x)|^2 dx}.$$

By applying the last Lemma for $0 < k = 2 + t \le 2$, we find

$$\frac{\int_\Omega \nabla p(x).(x-a)|\nabla u(x)|^2 dx}{\int_\Omega |u|^2 dx} \ge k\beta_k \left(\frac{n+k-2}{2}\right)^2 (diam\ \Omega)^{k-2}.$$

This implies that $\alpha(p) \ge \frac{k}{2}\beta_k(\frac{n+k-2}{2})^2 (diam\ \Omega)^{k-2}$. □

Let us give the following non-existence result

**Proposition 2.2**
*We assume that $\alpha(p) > -\infty$. There is no solution for (1.1) when $\lambda \le \alpha(p)$ and $\Omega$ is a starshaped domain about $a$.*

**Proof.** This follows from Pohozev's identity. Suppose that $u$ is a solution of (1.1). We first multiply (1.1) by $\nabla u(x).(x-a)$, next we integrate over $\Omega$ and we obtain

$$\int_\Omega u^{q-1} \nabla u(x).(x-a) dx = -\frac{n-2}{2} \int_\Omega |u(x)|^q dx, \tag{2.8}$$

$$\lambda \int_\Omega u \nabla u(x).(x-a) dx = -\frac{n}{2} \lambda \int_\Omega |u(x)|^2 dx \tag{2.9}$$

and

$$\begin{aligned} \int_\Omega -div(p(x)\nabla u)\nabla u(x).(x-a) dx &= -\frac{n-2}{2} \int_\Omega p(x)|\nabla u(x)|^2 dx \\ &- \frac{1}{2} \int_\Omega \nabla p(x).(x-a)|\nabla u(x)|^2 dx \\ &- \frac{1}{2} \int_{\partial\Omega} p(x)(x-a).\nu |\frac{\partial u}{\partial \nu}|^2 dx, \end{aligned} \tag{2.10}$$

where $\nu$ denotes the outward normal to $\partial\Omega$.
Combining (2.8), (2.9) and (2.10), we write
(2.11)

$$-\tfrac{n-2}{2}\int_\Omega p(x)|\nabla u(x)|^2dx - \frac{1}{2}\int_\Omega \nabla p(x).(x-a)|\nabla u(x)|^2dx =$$
$$-\frac{n-2}{2}\int_\Omega |u(x)|^q dx - \frac{n}{2}\lambda\int_\Omega |u(x)|^2dx.$$

On the other hand, we multiply (1.1) by $\frac{n-2}{2}u$ and we integrate by parts, we get

$$\frac{n-2}{2}\int_\Omega p(x)|\nabla u(x)|^2dx = \frac{n-2}{2}\int_\Omega |u(x)|^q dx + \frac{n-2}{2}\lambda\int_\Omega |u(x)|^2dx. \tag{2.12}$$

Combining (2.11) and (2.12), we obtain

$$\lambda\int_\Omega |u(x)|^2dx - \frac{1}{2}\int_\Omega \nabla p(x).(x-a)|\nabla u(x)|^2dx - \frac{1}{2}\int_{\partial\Omega} p(x)|\frac{\partial u}{\partial \nu}|^2(x-a).\nu dx = 0.$$

If $\Omega$ is starshaped about $a$, then $(x-a).\nu > 0$ on $\partial\Omega$, and

$$\lambda\int_\Omega |u(x)|^2dx - \frac{1}{2}\int_\Omega \nabla p(x).(x-a)|\nabla u(x)|^2dx > 0.$$

It follows that

$$\lambda > \frac{1}{2}\frac{\displaystyle\int_\Omega \nabla p(x).(x-a)|\nabla u(x)|^2dx}{\displaystyle\int_\Omega |u|^2dx}$$

and we obtain the desired result. □

## 3 Existence of solutions

Let $\Omega \in \mathbb{R}^n$, $n \geq 3$ be a bounded domain. In this section, we show that (1.1) possesses a solution of lower energy less than $p_0S$. We will use a minimization technique.
Set

$$Q_\lambda(u) = \frac{\int_\Omega p(x)|\nabla u(x)|^2dx - \lambda\int_\Omega |u(x)|^2dx}{\| u \|_q^2} \tag{3.1}$$

the functional associated to (1.1).
We define

$$S_\lambda(p) = \inf_{u\in H_0^1(\Omega), u\neq 0} Q_\lambda(u). \tag{3.2}$$

Let us remark that

$$S_\lambda(p) = \inf_{u\in H_0^1(\Omega), \|u\|_q=1}\int_\Omega p(x)|\nabla u(x)|^2dx - \lambda\int_\Omega |u(x)|^2dx.$$

The method used for the proof of Theorem 1.1 is the following : First we show that $S_\lambda(p) < p_0S$, we then prove that the infimum $S_\lambda(p)$ is achieved.
We have the following result

**Lemma 3.1**
*If $S_\lambda(p) < p_0 S$ for some $\lambda > 0$, then the infimum in (3.2) is achieved.*

**Proof.** Let $\{u_j\} \subset H_0^1(\Omega)$ be a minimizing sequence for (3.2) that is,

$$\|u_j\|_q = 1, \tag{3.3}$$

$$\int_\Omega p(x)|\nabla u_j(x)|^2 dx - \lambda \int_\Omega |u_j(x)|^2 dx = S_\lambda(p) + o(1) \quad \text{as } j \to \infty. \tag{3.4}$$

The sequence $u_j$ is bounded in $H_0^1(\Omega)$. Indeed, from (3.4), we have

$$\int_\Omega p(x)|\nabla u_j(x)|^2 dx = S_\lambda(p) + \lambda \int_\Omega |u_j(x)|^2 dx + o(1).$$

Using the embedding of $L^q(\Omega)$ into $L^2(\Omega)$, there exists a positive constant $C_1$ such that

$$\int_\Omega p(x)|\nabla u_j(x)|^2 dx \leq S_\lambda(p) + \lambda\, C_1 \|u_j\|_q^2 + o(1).$$

Using the fact that

$$\|u_j\|_q = 1,$$

we obtain

$$\int_\Omega p(x)|\nabla u_j(x)|^2 dx \leq S_\lambda(p) + \lambda\, C_1 + o(1).$$

Since $0 < p_0 \leq p(x)$ for every $x \in \Omega$, we deduce

$$\int_\Omega |\nabla u_j(x)|^2 dx \leq \frac{S_\lambda(p) + \lambda\, C_1}{p_0} + o(1).$$

This gives the desired result.
Since $\{u_j\}$ is bounded in $H_0^1(\Omega)$ we may extract a subsequence still denoted by $u_j$, such that

$$u_j \rightharpoonup u \quad \text{weakly in } H_0^1(\Omega),$$

$$u_j \to u \quad \text{strongly in } L^2(\Omega),$$

$$u_j \to u \quad \text{a.e. on } \Omega,$$

with $\|u\|_q \leq 1$. Set $v_j = u_j - u$, so that

$$v_j \rightharpoonup 0 \quad \text{weakly in } H_0^1(\Omega)$$

$$v_j \to 0 \quad \text{strongly in } L^2(\Omega),$$

$$v_j \to 0 \quad \text{a.e. on } \Omega.$$

Using (3.3), the definition of $S$ and the fact that $\min\limits_{\bar\Omega} p(x) = p_0 > 0$, we have

$$\int_\Omega p(x)|\nabla u_j(x)|^2 dx \geq p_0 S.$$

From (3.4) it follows that $\lambda\|u\|_2^2 \geq p_0 S - S_\lambda(p) > 0$ and therefore $u \neq 0$. Using again (3.4) we obtain

$$\int_\Omega p(x)|\nabla u(x)|^2 dx + \int_\Omega p(x)|\nabla v_j(x)|^2 dx - \lambda \int_\Omega |u(x)|^2 dx = S_\lambda(p) + o(1), \tag{3.5}$$

since $v_j \rightharpoonup 0$ weakly in $H_0^1(\Omega)$. On the other hand, it follows from a result of [BL] that

$$\|u + v_j\|_q^q = \|u\|_q^q + \|v_j\|_q^q + o(1),$$

(which holds since $v_j$ is bounded in $L^q$ and $v_j \to 0$ a.e.). Thus, by (3.3), we have

$$1 = \|u\|_q^q + \|v_j\|_q^q + o(1)$$

and therefore

$$1 \leq \|u\|_q^2 + \|v_j\|_q^2 + o(1),$$

which leads to

$$1 \leq \|u\|_q^q + \frac{1}{p_0 S} \int_\Omega p(x)|\nabla v_j(x)|^2 dx + o(1). \tag{3.6}$$

We distinguish two cases:
(a) $S_\lambda(p) > 0$, which corresponds to $0 < \lambda < \lambda_1^{div}$,

(b) $S_\lambda(p) \leq 0$, which corresponds to $\lambda \geq \lambda_1^{div}$.
In case (a) we deduce from (3.6) that

$$S_\lambda(p) \leq S_\lambda(p)\|u\|_q^2 + (\frac{S_\lambda(p)}{p_0 S}) \int_\Omega p(x)|\nabla v_j(x)|^2 dx + o(1). \tag{3.7}$$

Combining (3.5) and (3.7) we obtain

$$\begin{aligned} \textstyle\int_\Omega p(x)|\nabla u(x)|^2 - \lambda|u(x)|^2 dx + \int_\Omega p(x)|\nabla v_j(x)|^2 dx \leq S_\lambda(p)\|u\|_q^2 \\ +(\frac{S_\lambda(p)}{p_0 S}) \int_\Omega p(x)|\nabla v_j(x)|^2 dx + o(1). \end{aligned}$$

Thus

$$\begin{aligned} \textstyle\int_\Omega p(x)|\nabla u(x)|^2 dx - \lambda \int_\Omega |u(x)|^2 dx \quad &\leq S_\lambda(p)\|u\|_q^2 \\ &+ \left[\frac{S_\lambda(p)}{p_0 S} - 1\right] \int_\Omega p(x)|\nabla v_j(x)|^2 dx + o(1). \end{aligned}$$

Since $S_\lambda(p) < p_0 S$, we deduce

$$\int_\Omega p(x)|\nabla u(x)|^2 dx - \lambda \int_\Omega |u(x)|^2 dx \leq S_\lambda(p)\|u\|_q^2, \tag{3.8}$$

this means that $u$ is a minimum of $S_\lambda(p)$.
In case (b), since $\|u\|_q^2 \leq 1$, we have $S_\lambda(p) \leq S_\lambda(p)\|u\|_q^2$. Again, we deduce (3.8) from (3.5). This concludes the proof of Lemma 3.1. □

To prove assertion 1) and 2) of Theorem 1.1 (case $k \geq 2$), we need the following

**Lemma 3.2**

*a) For $n \geq 4$, we have*

$$S_\lambda(p) < p_0 S \text{ for all } \lambda > 0 \text{ and for } k > 2.$$

*b) For $n = 4$ and $k = 2$, we have*

$$S_\lambda(p) < p_0 S \text{ for all } \lambda > 4\beta_2.$$

*c) For $n \geq 5$ and $k = 2$, we have*

$$S_\lambda(p) < p_0 S \text{ for all } \lambda > \frac{(n-2)n(n+2)}{4(n-1)}\beta_2.$$

*d) For $n = 3$ and $k \geq 2$, we have*

$$S_\lambda(p) < p_0 S \quad \text{for all } \lambda > \gamma(k) \text{ where } \gamma(k) \text{ is a positive constant.}$$

**Proof.** We shall estimate the ratio $Q_\lambda(u)$ defined in (3.1), with $u = u_{a,\varepsilon}$.
We claim that, as $\varepsilon \to 0$, we have
(3.9)

$$\varepsilon^{\frac{n-2}{2}} \int_\Omega p(x)|\nabla u_{a,\varepsilon}(x)|^2 dx \leq$$

$$\begin{cases} p_0 K_1 + O(\varepsilon^{\frac{n-2}{2}}) & \text{if } \begin{cases} n \geq 4 & \text{and} \\ n-2 < k, \end{cases} \\ p_0 K_1 + A_k \varepsilon^{\frac{k}{2}} + o(\varepsilon^{\frac{k}{2}}) & \text{if } \begin{cases} n \geq 4 & \text{and} \\ n-2 > k, \end{cases} \\ p_0 K_1 + \dfrac{(n-2)^2(\beta_{n-2}+M)\omega_n \varepsilon^{\frac{n-2}{2}}|\log\varepsilon|}{2} + o(\varepsilon^{\frac{n-2}{2}}|\log\varepsilon|) & \text{if } \begin{cases} n > 4 & \text{and} \\ k = n-2, \end{cases} \\ p_0 K_1 + 2\beta_2\omega_4 \varepsilon|\log\varepsilon| + o(\varepsilon|\log\varepsilon|) & \text{if } \begin{cases} n = 4 & \text{and} \\ k = 2, \end{cases} \end{cases}$$

with $K_1 = (n-2)^2 \int_{I\!R^n} \frac{|y|^2}{(1+|y|^2)^n} dy$, $s = \min(\frac{k}{2}, \frac{n-2}{2})$, $A_k = (n-2)^2 \beta_k \int_{I\!R^n} \frac{|x|^{k+2}}{(1+|x|^2)^n} dx$ and $M$ is a positive constant.

**Verification of (3.9)**

**1. Case $n \geq 4$ and $k > 0$, with $k \neq 2$ if $n = 4$.**

We have

$$\begin{aligned} \int_\Omega p(x)|\nabla u_{a,\varepsilon}(x)|^2 dx &= \int_\Omega \frac{p(x)|\nabla\zeta(x)|^2}{(\varepsilon+|x-a|^2)^{n-2}} dx + (n-2)^2 \int_\Omega \frac{p(x)|\zeta(x)|^2|x-a|^2}{(\varepsilon+|x-a|^2)^n} dx \\ &- 2(n-2) \int_\Omega \frac{p(x)\zeta(x)\nabla\zeta(x)(x-a)}{(\varepsilon+|x-a|^2)^{n-1}} dx. \end{aligned}$$

Since $\zeta \equiv 1$ on a neighborhood of $a$, we assume that $\varphi \equiv 1$ on $B(a,l)$ with $l$ is a small positive constant. Therefore we get $|\nabla\varphi|^2 \equiv 0$ on $B(a,l)$ and $\nabla\varphi(x).(x-a) = 0$ on $B(a,l)$.
Thus, we obtain

(3.10)
$$\begin{aligned}\int_\Omega p(x)|\nabla u_{a,\varepsilon}(x)|^2dx =& \int_{\Omega\setminus B(a,l)}\frac{p(x)|\nabla\zeta(x)|^2}{(\varepsilon+|x-a|^2)^{n-2}}dx + (n-2)^2\int_\Omega\frac{p(x)|\zeta(x)|^2|x-a|^2}{(\varepsilon+|x-a|^2)^n}dx\\ -& \quad 2(n-2)\int_{\Omega\setminus B(a,l)}\frac{p(x)\zeta(x)\nabla\zeta(x)(x-a)}{(\varepsilon+|x-a|^2)^{n-1}}dx.\end{aligned}$$

Therefore, applying the Dominated Convergence Theorem, (3.10) becomes

$$\int_\Omega p(x)|\nabla u_{a,\varepsilon}(x)|^2dx = (n-2)^2\int_\Omega\frac{p(x)|\zeta(x)|^2|x-a|^2}{(\varepsilon+|x-a|^2)^n}dx + O(1).$$

Using (1.2), a direct computation gives

$$\begin{aligned}\varepsilon^{\frac{n-2}{2}}\int_\Omega p(x)|\nabla u_{a,\varepsilon}(x)|^2dx &= (n-2)^2p_0\varepsilon^{\frac{n-2}{2}}\int_\Omega\frac{|x-a|^2}{(\varepsilon+|x-a|^2)^n}dx\\ &+ (n-2)^2\varepsilon^{\frac{n-2}{2}}\beta_k\int_\Omega\frac{|x-a|^{k+2}}{(\varepsilon+|x-a|^2)^n}dx\\ &+ (n-2)^2\varepsilon^{\frac{n-2}{2}}\int_\Omega\frac{|x-a|^{k+2}\theta(x)}{(\varepsilon+|x-a|^2)^n}dx\\ &+ (n-2)^2\varepsilon^{\frac{n-2}{2}}\int_\Omega\frac{|x-a|^{k+2}(\beta_k+\theta(x))(|\zeta(x)|^2-1)}{(\varepsilon+|x-a|^2)^n}dx\\ &+ O(\varepsilon^{\frac{n-2}{2}}).\end{aligned}$$

Using again the definition of $\zeta$, and applying the Dominated Convergence Theorem, we obtain

$$\begin{aligned}\varepsilon^{\frac{n-2}{2}}\int_\Omega p(x)|\nabla u_{a,\varepsilon}(x)|^2dx &= (n-2)^2p_0\varepsilon^{\frac{n-2}{2}}\int_\Omega\frac{|x-a|^2}{(\varepsilon+|x-a|^2)^n}dx\\ &+ (n-2)^2\varepsilon^{\frac{n-2}{2}}\beta_k\int_\Omega\frac{|x-a|^{k+2}}{(\varepsilon+|x-a|^2)^n}\\ &+ (n-2)^2\varepsilon^{\frac{n-2}{2}}\int_\Omega\frac{|x-a|^{k+2}\theta(x)}{(\varepsilon+|x-a|^2)^n}dx + O(\varepsilon^{\frac{n-2}{2}}).\end{aligned}$$

Here we will consider the following three subcases:
**1.1. If** $n-2>k$,

$$\varepsilon^{\frac{n-2}{2}}\int_\Omega p(x)|\nabla u_{a,\varepsilon}(x)|^2dx$$
$$= p_0(n-2)^2\varepsilon^{\frac{n-2}{2}}\left[\int_{I\!R^n}\frac{|x-a|^2}{(\varepsilon+|x-a|^2)^n}dx-\int_{I\!R^n\setminus\Omega}\frac{|x-a|^2}{(\varepsilon+|x-a|^2)^n}dx\right]$$

$$= (n-2)^2\varepsilon^{\frac{n-2}{2}}\left[\int_{I\!R^n}\frac{|x-a|^{k+2}(\beta_k+\theta(x))}{(\varepsilon+|x-a|^2)^n}-\int_{I\!R^n\setminus\Omega}\frac{|x-a|^{k+2}(\beta_k+\theta(x))}{(\varepsilon+|x-a|^2)^n}\right]$$
$$= O(\varepsilon^{\frac{n-2}{2}}).$$

Using a simple change of variable and applying the Dominated Convergence Theorem, we find

$$\varepsilon^{\frac{n-2}{2}}\int_\Omega p(x)|\nabla u_{a,\varepsilon}(x)|^2dx = p_0\int_{I\!R^n}\frac{|y|^2}{(1+|y|^2)^n}+(n-2)^2\varepsilon^{\frac{k}{2}}\int_{I\!R^n}\frac{|y|^{k+2}(\beta_k+\theta(a+\varepsilon^{\frac{1}{2}}y))}{(1+|y|^2)^n}dy$$
$$+ o(\varepsilon^{\frac{k}{2}}).$$

The fact that $\theta(x)$ tends to 0 when $x$ tends to $a$ gives that

$$\varepsilon^{\frac{n-2}{2}}\int_\Omega p(x)|\nabla u_{a,\varepsilon}(x)|^2dx = p_0K_1+A_k\varepsilon^{\frac{k}{2}}+o(\varepsilon^{\frac{k}{2}}),$$

with $K_1=(n-2)^2\int_{I\!R^n}\frac{|y|^2}{(1+|y|^2)^n}dy$ and $A_k=\beta_k\int_{I\!R^n}\frac{|y|^{k+2}}{(1+|y|^2)^n}dy$.

**1.2. If** $n-2<k$,

$$\varepsilon^{\frac{n-2}{2}}\int_\Omega p(x)|\nabla u_{a,\varepsilon}(x)|^2dx = p_0K_1+(n-2)^2\beta_k\varepsilon^{\frac{n-2}{2}}\int_\Omega\frac{|x-a|^{k+2}}{(\varepsilon+|x-a|^2)^n}dx$$
$$+ (n-2)^2\varepsilon^{\frac{n-2}{2}}\int_\Omega\frac{|x-a|^{k+2}\theta(x)}{(\varepsilon+|x-a|^2)^n}dx+O(\varepsilon^{\frac{n-2}{2}}).$$

Since $\Omega$ is a bounded domain, there exists some positive constant $R$ such that $\Omega\subset B(a,R)$ and thus

$$\varepsilon^{\frac{n-2}{2}}\int_\Omega p(x)|\nabla u_{a,\varepsilon}(x)|^2dx= p_0K_1+O(\varepsilon^{\frac{n-2}{2}})$$
$$+(n-2)^2\varepsilon^{\frac{n-2}{2}}\left[\int_{B(a,R)}\frac{|x-a|^{k+2}(\beta_k+\theta(x))}{(\varepsilon+|x-a|^2)^n}dx-\int_{B(a,R)\setminus\Omega}\frac{|x-a|^{k+2}(\beta_k+\theta(x)}{(\varepsilon+|x-a|^2)^n}dx\right].$$

By a simple change of variable, we get

$$\varepsilon^{\frac{n-2}{2}}\int_\Omega p(x)|\nabla u_{a,\varepsilon}(x)|^2dx = p_0K_1+(n-2)^2\varepsilon^{\frac{n-2}{2}}\int_{B(0,R)}\frac{|y|^{k+2}(\beta_k+\theta(a+y))}{(\varepsilon+|y|^2)^n}dy$$
$$+ O(\varepsilon^{\frac{n-2}{2}}).$$

Using the definition of $\theta$ given by (1.2), there exists a positive constant $M$ such that

$$\varepsilon^{\frac{n-2}{2}} \int_\Omega p(x)|\nabla u_{a,\varepsilon}(x)|^2 dx \leq p_0 K_1 + (n-2)^2 \varepsilon^{\frac{n-2}{2}} (\beta_k + M) \int_{B(0,R)} \frac{|y|^{k+2}}{(\varepsilon + |y|^2)^n} dy + O(\varepsilon^{\frac{n-2}{2}}).$$

Applying the Dominated Convergence Theorem we deduce that

$$\varepsilon^{\frac{n-2}{2}} \int_\Omega p(x)|\nabla u_{a,\varepsilon}(x)|^2 dx \leq p_0 K_1 + O(\varepsilon^{\frac{n-2}{2}})$$

and this completes the proof of (3.9) in this case.

**1.2. If** $k = n-2$,

$$\begin{aligned}\varepsilon^{\frac{n-2}{2}} \int_\Omega p(x)|\nabla u_{a,\varepsilon}(x)|^2 dx &= p_0 K_1 + (n-2)^2 \beta_{n-2} \varepsilon^{\frac{n-2}{2}} \int_\Omega \frac{|x-a|^n}{(\varepsilon + |x-a|^2)^n} dx \\ &+ (n-2)^2 \varepsilon^{\frac{n-2}{2}} \int_\Omega \frac{|x-a|^n \theta(x)}{(\varepsilon + |x-a|^2)^n} dx + O(\varepsilon^{\frac{n-2}{2}}).\end{aligned}$$

Since $\Omega$ is a bounded domain, there exists some positive constant $R$ such that $\Omega \subset B(a,R)$ and thus

$$\begin{aligned}&\varepsilon^{\frac{n-2}{2}} \int_\Omega p(x)|\nabla u_{a,\varepsilon}(x)|^2 dx = p_0 K_1 + O(\varepsilon^{\frac{n-2}{2}}) \\ &+(n-2)^2 \varepsilon^{\frac{n-2}{2}} \left[ \int_{B(a,R)} \frac{|x-a|^n (\beta_{n-2} + \theta(x))}{(\varepsilon + |x-a|^2)^n} dx - \int_{B(a,R)\setminus\Omega} \frac{|x-a|^n (\beta_{n-2} + \theta(x))}{(\varepsilon + |x-a|^2)^n} dx \right].\end{aligned}$$

Hence

$$\varepsilon^{\frac{n-2}{2}} \int_\Omega p(x)|\nabla u_{a,\varepsilon}(x)|^2 dx = p_0 K_1 + (n-2)^2 \varepsilon^{\frac{n-2}{2}} \int_{B(a,R)} \frac{|x-a|^n (\beta_{n-2} + \theta(x))}{(\varepsilon + |x-a|^2)^n} dx + O(\varepsilon^{\frac{n-2}{2}}).$$

Using the definition of $\theta$ given by (1.2), there exists a positive constant $M$ such that

(3.11)
$$\varepsilon^{\frac{n-2}{2}} \int_\Omega p(x)|\nabla u_{a,\varepsilon}(x)|^2 dx \leq p_0 K_1 + (n-2)^2 (\beta_{n-2} + M) \varepsilon^{\frac{n-2}{2}} \int_{B(a,R)} \frac{|x-a|^n}{(\varepsilon + |x-a|^2)^n} dx + O(\varepsilon^{\frac{n-2}{2}}).$$

On the other hand, an easy computation gives

$$\begin{aligned}\varepsilon^{\frac{n-2}{2}} \int_{B(a,R)} \frac{|x-a|^n}{(\varepsilon + |x-a|^2)^n} dx &= \omega_n \varepsilon^{\frac{n-2}{2}} \int_0^R \frac{r^{2n-1}}{(\varepsilon + r^2)^n} dr \\ &= \frac{\omega_n}{2n} \varepsilon^{\frac{n-2}{2}} \int_0^R \frac{((\varepsilon + r^2)^n)'}{(\varepsilon + r^2)^n} dr + O(\varepsilon^{\frac{n-2}{2}})\end{aligned}$$

and

$$\varepsilon^{\frac{n-2}{2}} \int_{B(a,R)} \frac{|x-a|^n}{(\varepsilon + |x-a|^2)^n} dx = \frac{\omega_n}{2} \varepsilon^{\frac{n-2}{2}} |\log \varepsilon| + o(\varepsilon^{\frac{n-2}{2}} |\log \varepsilon|). \tag{3.12}$$

Inserting (3.12) into (3.11) we obtain

$$\varepsilon^{\frac{n-2}{2}}\int_\Omega p(x)|\nabla u_{a,\varepsilon}(x)|^2dx \le p_0K_1 + \frac{(n-2)^2(\beta_{n-2}+M)\omega_n}{2}\varepsilon^{\frac{n-2}{2}}|\log\varepsilon| + o(\varepsilon^{\frac{n-2}{2}}|\log\varepsilon|).$$

**2)Case** $n=4$ **and** $k=2$.

As we have announced in the introduction, we assume in this case the following additional condition on $\theta$: $\int_{\mathcal{B}(a,1)}\frac{\theta(x)}{|x-a|^4}dx<\infty$. We have

$$\begin{aligned}\int_\Omega p(x)|\nabla u_{a,\varepsilon}|^2dx &= \int_\Omega \frac{p(x)|\nabla\zeta(x)|^2}{(\varepsilon+|x-a|^2)^2}dx + 4\int_\Omega\frac{p(x)|\zeta(x)|^2|x-a|^2}{(\varepsilon+|x-a|^2)^4}dx\\ &- 4\int_\Omega\frac{p(x)\zeta(x)\nabla\zeta(x)(x-a)}{(\varepsilon+|x-a|^2)^3}dx.\end{aligned}$$

Using (1.2) and the fact that $\zeta\equiv 1$ near $a$, it follows that

$$\begin{aligned}\int_\Omega p(x)|\nabla u_{a,\varepsilon}|^2dx &= 4p_0\int_\Omega\frac{|\zeta(x)|^2|x-a|^2}{(\varepsilon+|x-a|^2)^4}dx + 4\beta_2\int_\Omega\frac{|\zeta(x)|^2|x-a|^4}{(\varepsilon+|x-a|^2)^4}dx\\ &+ 4\int_\Omega\frac{|x-a|^4\theta(x)}{(\varepsilon+|x-a|^2)^4}dx + O(1),\\ &= \frac{4p_0}{\varepsilon}\int_{I\!R^n}\frac{|y|^2}{(1+|y|^2)^4}dy + 4\int_\Omega\frac{|x-a|^4(\beta_k+\theta(x))}{(\varepsilon+|x-a|^2)^4}dx + O(1).\end{aligned}$$

Since $\int_{B(a,1)}\frac{\theta(x)}{|x-a|^4}dx<\infty$, we obtain

$$\begin{aligned}\int_\Omega\frac{|x-a|^4\theta(x)}{(\varepsilon+|x-a|^2)^4}dx &= \int_\Omega\frac{\theta(x)}{|x-a|^4}dx + o(1)\\ &= O(1).\end{aligned}$$

Consequently

$$\int_\Omega p(x)|\nabla u_{a,\varepsilon}|^2dx = \frac{4p_0}{\varepsilon}\int_{I\!R^n}\frac{|y|^2}{(1+|y|^2)^4}dy + 4\beta_k\int_\Omega\frac{|x-a|^4}{(\varepsilon+|x-a|^2)^4}dx + O(1).$$

Let $R_i>0$, $i=1,2$ such that

$$\int_{|x-a|\le R_1}\frac{|x-a|^4}{(\varepsilon+|x-a|^2)^4}dx \le \int_\Omega\frac{|x-a|^4}{(\varepsilon+|x-a|^2)^4}dx \le \int_{|x-a|\le R_2}\frac{|x-a|^4}{(\varepsilon+|x-a|^2)^4}dx.$$

We see that

$$\begin{aligned}\int_{|x-a|\le R}\frac{|x-a|^4}{(\varepsilon+|x-a|^2)^4}dx &= \omega_4\int_0^R\frac{r^7}{(\varepsilon+r^2)^4}dr,\\ &= \frac{1}{8}\omega_4\int_0^R\frac{((\varepsilon+r^2)^4)'}{(\varepsilon+r^2)^4}dr - \omega_4\int_0^R\frac{r\varepsilon^3+3r^3\varepsilon^2+3\varepsilon r^4}{(\varepsilon+r^2)^4}dr,\\ &= \frac{1}{2}\omega_4|\log\varepsilon| - \omega_4\int_0^{\frac{R}{\varepsilon^{\frac12}}}\frac{t+3t^3+3t^5}{(1+t^2)^4}dt + O(1),\\ &= \frac{1}{2}\omega_4|\log\varepsilon| + O(1).\end{aligned}$$

Hence, we have

$$\int_\Omega p(x)|\nabla u_{a,\varepsilon}|^2dx = \frac{p_0K_1}{\varepsilon} + 2\beta_2\omega_4|\log\varepsilon| + O(1),$$

where $K_1 = \int_{I\!R^n} \frac{|y|^2}{(1+|y|^2)^4}dy$. This completes the proof of (3.9).

Let us come back to the proof of Lemma 3.2.

It is convenient to rewrite (3.9) as

(3.13)

$$\varepsilon^{\frac{n-2}{2}}\int_\Omega p(x)|\nabla u_{a,\varepsilon}|^2dx \le \begin{cases} p_0K_1 + o(\varepsilon) & \text{if } n\ge 5, \text{ and } k>2\ ,\\ p_0K_1 + A_2\varepsilon + o(\varepsilon) & \text{if } n\ge 5, \text{ and } k=2\ ,\\ p_0K_1 + A_k\varepsilon^{\frac{k}{2}} + o(\varepsilon^{\frac{k}{2}}) & \text{if } n\ge 4, \text{ and } k<2\ ,\\ p_0K_1 + o(\varepsilon) & \text{if } n=4, \text{ and } k>2\ ,\\ p_0K_1 + 2\omega_4\beta_2\varepsilon|\log\varepsilon| + o(\varepsilon|\log\varepsilon|) & \text{if } n=4, \text{ and } k=2\ . \end{cases}$$

Combining (3.13), (2.3) and (2.4), we obtain

(3.14)

$$S_\lambda(p) \le Q_\lambda(u_{a,\varepsilon}) \le \begin{cases} p_0S - \lambda\frac{K_3}{K_2}\varepsilon + o(\varepsilon) & \text{if } n\ge 5, \text{ and } k>2\ ,\\ p_0S - (\lambda - C)\frac{K_3}{K_2}\varepsilon + o(\varepsilon) & \text{if } n\ge 5, \text{ and } k=2\ ,\\ p_0S + A_k\varepsilon^{\frac{k}{2}} + o(\varepsilon^{\frac{k}{2}}) & \text{if } n\ge 4, \text{ and } k<2\ ,\\ p_0S - \lambda\frac{\omega_4}{2K_2}\varepsilon|\log\varepsilon| + o(\varepsilon|\log\varepsilon|) & \text{if } n=4, \text{ and } k>2\ ,\\ p_0S - \frac{\omega_4}{2K_2}[\lambda - 4\beta_2]\varepsilon|\log\varepsilon| + o(\varepsilon|\log\varepsilon|) & \text{if } n=4, \text{ and } k=2\ , \end{cases}$$

with $C = \frac{A_2}{K_3} = \frac{\beta_2(n-2)n(n+2)}{4(n-1)}$.

Assertions a), b) and c) of Lemma 3.2 follow directly for $\varepsilon$ small enough.

Now we prove d) of Lemma 3.2 (case $n=3$ and $k\ge 2$). We will estimate the ratio

$$Q_\lambda(u) = \frac{\int_\Omega p(x)|\nabla u|^2dx - \lambda\|u\|_2^2}{\|u\|_q^2}$$

with

$$u(x) = u_{\varepsilon,a}(r) = \frac{\zeta(r)}{(\varepsilon + r^2)^{\frac{1}{2}}}, r=|x|,\ \varepsilon>0,$$

where $\zeta$ is a fixed smooth function satisfying $0\le\zeta\le 1$, $\zeta = 1$ in $\{x,\ |x-a| < \frac{R}{2}\}$ and $\zeta = 0$ in $\{x,\ |x-a|\ge R\}$, where $R$ is a positive constant such that $B(a,R)\subset\Omega$.

We claim that, as $\varepsilon\to 0$,

(3.15)

$$\int p(x)|\nabla u_{a,\varepsilon}(x)|^2dx = \frac{p_0K_1}{\varepsilon^{\frac{1}{2}}} + \omega_3\int_0^R(p_0 + \beta_kr^k)|\zeta'(r)|^2dr + \omega_3k\int_0^R|\zeta|^2r^{k-2}dr + o(1).$$

And from [BN], we already have

$$\|\nabla u_{a,\varepsilon}\|_2^2 = \frac{K_1}{\varepsilon^{\frac{1}{2}}} + \omega_3\int_0^R|\zeta'(r)|^2dr + O(\varepsilon^{\frac{1}{2}}), \tag{3.16}$$

$$\|u_{a,\varepsilon}\|_6^2 = \frac{K_2}{\varepsilon^{\frac{1}{2}}} + O(\varepsilon^{\frac{1}{2}}), \tag{3.17}$$

$$\|u_{a,\varepsilon}\|_2^2 = \omega_3 \int_0^R \zeta^2(r)dr + O(\varepsilon^{\frac{1}{2}}), \tag{3.18}$$

where $K_1$ and $K_2$ are positive constants such that $\frac{K_1}{K_2} = S$ and $\omega_3$ is the area of $S^2$ .

**Verification of (3.15).**

Using (1.2), (3.16) and the fact that $\zeta = 0$ in $\{x, |x-a| \geq R\}$, we write

$$\begin{aligned}
\int p(x)|\nabla u_{a,\varepsilon}(x)|^2 dx &= \frac{p_0 K_1}{\varepsilon^{\frac{1}{2}}} + \omega_3 p_0 \int_0^R |\zeta'(r)|^2 dr \\
&+ \omega_3 \beta_k \int_0^R \left[\frac{|\zeta'(r)|^2}{\varepsilon + r^2} - \frac{2r\zeta(r)\zeta'(r)}{(\varepsilon + r^2)^2} + \frac{r^2\zeta^2(r)}{(\varepsilon + r^2)^3}\right] r^{k+2} dr \\
&+ O(\varepsilon^{\frac{1}{2}}).
\end{aligned}$$

The fact that $\zeta = 1$ in $\{x, |x-a| < \frac{R}{2}\}$, $\zeta'(0) = 0$ and $\zeta(R) = 0$ gives

$$-2\int_0^R \frac{\zeta(r)\zeta'(r)r^{k+3}}{(\varepsilon + r^2)^2} dr = (k+3)\int_0^R \frac{|\zeta(r)|^2 r^{k+2}}{(\varepsilon + r^2)^2} dr - 4\int_0^R \frac{|\zeta(r)|^2 r^{k+4}}{(\varepsilon + r^2)^3} dr.$$

Consequently

$$\begin{aligned}
\int_\Omega p(x)|\nabla u_{a,\varepsilon}(x)|^2 dx &= \frac{p_0 K_1}{\varepsilon^{\frac{1}{2}}} + \omega_3 p_0 \int_0^R |\zeta'(r)|^2 dr + \omega_3 \beta_k \int_0^R \frac{|\zeta'(r)|^2 r^{k+2}}{\varepsilon + r^2} dr \\
&- 3\omega_3 \beta_k \int_0^R \frac{|\zeta(r)|^2 r^{k+4}}{(\varepsilon + r^2)^3} dr + (k+3)\omega_3 \beta_k \int_0^R \frac{|\zeta(r)|^2 r^{k+2}}{(\varepsilon + r^2)^2} dr \\
&+ O(\varepsilon^{\frac{1}{2}}).
\end{aligned}$$

Applying the Dominated Convergence Theorem, we get the desired result.

Combining (3.15), (3.17) and (3.18), we obtain

$$\begin{aligned}
Q_\lambda(u_{a,\varepsilon}) &= p_0 S + \omega_3 \left[\int_0^R (p_0 + \beta_k r^k)|\zeta'(r)|^2 dr + k\beta_k \int_0^R |\zeta(r)|^2 r^{k-2} dr - \lambda \int_0^R \zeta^2(r) dr\right] \frac{\varepsilon^{\frac{1}{2}}}{K_2} \\
&+ O(\varepsilon),
\end{aligned}$$

thus,

$$\begin{aligned}
Q_\lambda(u_{a,\varepsilon}) &= p_0 S + \frac{\omega_3 \int_0^R \zeta^2(r)dr}{K_2} \left[\frac{\int_0^R (p_0 + \beta_k r^k)|\zeta'(r)|^2 dr + k\int_0^R |\zeta(r)|^2 r^{k-2} dr}{\int_0^R |\zeta(r)|^2 dr} - \lambda\right] \varepsilon^{\frac{1}{2}} \\
&+ O(\varepsilon).
\end{aligned} \tag{3.19}$$

Set $D(k,\zeta) = \dfrac{\int_0^R (p_0 + \beta_k r^k)|\zeta'(r)|^2 dr + k\int_0^R |\zeta(r)|^2 r^{k-2} dr}{\int_0^R |\zeta(r)|^2 dr}$ and $\gamma(k) = \inf_H D(k,\zeta)$

where $H$ is defined by

$H = \{\zeta \in C_0^\infty(\bar\Omega),\, 0 \le \zeta \le 1,\ \zeta = 1 \text{ in } \{x,\, |x-a| < \frac{R}{2}\} \text{ and } \zeta = 0 \text{ in } \{x,\, |x-a| \ge R\}\}$.
This finishes the proof of Lemma 3.2. $\square$

Now, we go back to proof of assertion 3) in Theorem 1.1 (case $0 < k < 2$).
First of all, let us emphasize that if the domain $\Omega$ is starshaped about $a$, the assertion 3) is more interesting. Indeed, it gives a better estimate of the least value of the parameter $\lambda$ over which there is a solution to problem (1.1).
In the case of a non-starshaped domain, combining the fact that $S_0(p) = p_0 S$ with the properties of $S_\lambda(p)$ (see the proof of lemma 3.4), we have that there exists $\lambda^* \in [0, \lambda_1^{div}[$ such that for all $\lambda \in ]\lambda^*, \lambda_1^{div}[$, the problem (1.1) has a solution. Note that we have no other information on $\lambda^*$.
Therefore, throughout the rest of this proof, we assume that the domain $\Omega$ is starshaped about $a$.
We need two Lemmas. Let us start by the following

**Lemma 3.3**
*Assume $0 < k \le 2$. Then there exists a constant $\tilde\beta_k = \beta_k \min[(\text{diam } \Omega)^{k-2}, 1]$ such that*

$$(3.20) \qquad S_\lambda(p) = p_0 S \ \textit{ for every } \ \lambda \in ]-\infty, \tilde\beta_k \frac{n^2}{4}]$$

*and the infimum of $S_\lambda(p)$ is not achieved for every $\lambda \in ]-\infty, \tilde\beta_k \frac{n^2}{4}[$.*

**Proof.** We know from (3.14) that

$$S_\lambda(p) \le Q_\lambda(u_{a,\varepsilon}) \le p_0 S + A_k \varepsilon^{\frac{k}{2}} + o(\varepsilon^{\frac{k}{2}}) \quad \text{with } A_k \text{ is a positive constant,}$$

thus

$$S_\lambda(p) \le p_0 S.$$

On the other hand, we know from Lemma 2.2 and Proposition 2.1, that for $0 < k \le 2$, for every $\lambda \le \frac{k}{2}\beta_k(\frac{n+k-2}{2})^2\,(diam\ \Omega)^{k-2}$, problem (1.1) has no solution. So we exclude the case $S_\lambda(p) < p_0 S$, otherwise, Lemma 3.1 will yield in a contradiction.
We conclude that for $0 < k \le 2$, we have

$$(3.21) \qquad S_\lambda(p) = p_0 S \quad \text{for every} \quad \lambda \le \frac{k}{2}\beta_k(\frac{n+k-2}{2})^2\,(diam\ \Omega)^{k-2}.$$

Now, we consider $\tilde p$ defined by

$$(3.22) \qquad \begin{cases} \tilde p(x) & = p(x) & \forall x \in \Omega \setminus B(a,r), \\ \tilde p(x) & = p_0 + \beta_k |x-a|^2 & \forall x \in B(a, \frac{r}{2}), \\ p(x) & \ge \tilde p(x) & \forall x \in B(a,r) \setminus B(a, \frac{r}{2}), \end{cases}$$

where $r < 1$ is a positive constant.
Since $0 < k \le 2$, we have $|x-a|^k \ge |x-a|^2$ for every $x \in B(a,r)$ and $p(x) \ge \tilde p(x)$ in $\Omega$.
Let $u \in H_0^1(\Omega)$ with $\|u\|_q = 1$, then

$$\int_\Omega p(x)|\nabla u(x)|^2 dx - \lambda \int_\Omega |u(x)|^2 dx \ge \int_\Omega \tilde p(x)|\nabla u(x)|^2 dx - \lambda \int_\Omega |u(x)|^2 dx,$$

thus,

(3.23)
$$\int_\Omega p(x)|\nabla u(x)|^2dx - \lambda\int_\Omega |u(x)|^2dx \geq \int_\Omega (p_0 + \frac{1}{2}(\tilde{p}(x) - p_0))|\nabla u(x)|^2dx$$
$$-\lambda\int_\Omega |u(x)|^2dx + \frac{1}{2}\int_\Omega (\tilde{p}(x) - p_0)|\nabla u(x)|^2dx.$$

Set $\tilde{\tilde{p}}(x) = p_0 + \frac{1}{2}(\tilde{p}(x) - p_0)$.
From (1.3) we deduce that

$$p(x) - p_0 \geq \beta_k|x-a|^k \text{ a.e in } \Omega. \tag{3.24}$$

Using (3.22) and (3.24), a simple computation gives $\tilde{p}(x) - p_0 \geq \tilde{\beta}_k|x-a|^2$ a.e in $\Omega$, with $\tilde{\beta}_k = \beta_k \min[(diam\ \Omega)^{k-2}, 1]$.
Applying Lemma 2.1, we find

$$\int_\Omega (\tilde{p}(x) - p_0)|\nabla u(x)|^2dx \geq \tilde{\beta}_k\frac{n^2}{4}\int_\Omega |u(x)|^2dx.$$

Inequality (3.23) becomes for every $u \in H_0^1(\Omega)$,

$$\int_\Omega p(x)|\nabla u|^2dx - \lambda\int_\Omega |u|^2dx \geq \int_\Omega \tilde{\tilde{p}}(x)|\nabla u|^2dx - \left(\lambda - \tilde{\beta}_k\frac{n^2}{8}\right)\int_\Omega |u|^2dx.$$

Thus, we find

$$S_\lambda(p) \geq \inf_{\|u\|_q^2=1}\left[\int_\Omega \tilde{\tilde{p}}(x)|\nabla u|^2dx - (\lambda - \tilde{\beta}_k\frac{n^2}{8})\int_\Omega |u|^2dx\right].$$

On the other hand $\lambda - \tilde{\beta}_k\frac{n^2}{8} \leq \frac{1}{2}\tilde{\beta}_k\frac{n^2}{4}$ since $\lambda \leq \tilde{\beta}_k\frac{n^2}{4}$, so by (3.21), we conclude that

$$\inf_{\|u\|_q=1}\left[\int_\Omega \tilde{\tilde{p}}(x)|\nabla u|^2dx - (\lambda - \tilde{\beta}_k\frac{n^2}{8})\int_\Omega |u|^2dx\right] = p_0S,$$

hence, (3.20) follows.
Now, we are able to prove that the infimum in (3.20) is not achieved. Suppose by contradiction that it is achieved by some $u_0$. Let $\delta$ such that $\tilde{\beta}_k\frac{n^2}{4} \geq \delta > \lambda$. Using $u_0$ as a test function for $S_\delta$, we obtain

$$S_\delta(p) \leq \frac{\int_\Omega p(x)|\nabla u_0|^2dx - \delta\int_\Omega |u_0|^2dx}{\|u_0\|_q^2} < \frac{\int_\Omega p(x)|\nabla u_0|^2dx - \lambda\int_\Omega |u_0|^2dx}{\|u_0\|_q^2}$$

and thus $S_\delta(p) < S_\lambda(p) = p_0S$. This is a contradiction since $S_\delta(p) = p_0S$ for $\delta \leq \tilde{\beta}_k\frac{n^2}{4}$.
□

The second Lemma on which the proof of assertion 3) in Theorem 1.1 is based is the following

**Lemma 3.4**
*There exists $\lambda^* \in [\tilde{\beta}_k\frac{n^2}{4}, \lambda_1^{div}[$, such that for all $\lambda \in ]\lambda^*, \lambda_1^{div}[$ we have*

$$S_\lambda(p) < p_0S.$$

**Proof.**
The proof is based on a study of some properties of the function $\lambda \mapsto S_\lambda(p)$. We have $S_{\lambda_1^{div}}(p) = 0$. Indeed let $\varphi_1$ be the eigenfunction of $div(p\nabla.)$ corresponding to $\lambda_1^{div}$, we have

$$S_{\lambda_1^{div}} \leq \frac{\int p(x)|\nabla \varphi_1|^2 dx - \lambda_1^{div} \int |\varphi_1|^2 dx}{(\int |\varphi_1|^q dx)^{\frac{2}{q}}} = 0.$$

Moreover, $\lambda \mapsto S_\lambda(p)$ is continuous and $S_{\tilde{\beta}_k \frac{n^2}{4}}(p) = p_0 S$. Then according to the Mean Value Theorem, there exists $\beta \in ]\tilde{\beta}_k \frac{n^2}{4}, \lambda_1^{div}[$ such that $0 < S_\beta(p) < p_0 S$. But the function $\lambda \mapsto S_\lambda(p)$ is decreasing hence $\forall \lambda \in [\beta, \lambda_1^{div}[$ we have $S_\lambda(p) < p_0 S$, and the Lemma follows at once. □

Now we have all the necessary ingredients for the proof of Theorem 1.1.
**Proof of Theorem 1.1 concluded**: Concerning the proof of 1), 2), 3) and 4), let $u \in H_0^1(\Omega)$ be given by Lemma 3.1, that is,

$$\|u\|_q = 1 \text{ and } \int_\Omega p(x)|\nabla u(x)|^2 dx - \lambda \int_\Omega |u(x)|^2 dx = S_\lambda(p).$$

We may as well assume that $u \geq 0$. Since $u$ is a minimizer for (3.2) there exists a Lagrange multiplier $\mu \in I\!R$ such that

$$-div(p\nabla u) - \lambda u = \mu u^{q-1} \text{ on } \Omega.$$

In fact, $\mu = S_\lambda(p)$, and $S_\lambda(p) > 0$ since $\lambda < \lambda_1^{div}$. It follows that $\gamma u$ satisfies (1.1) for some appropriate constant $\gamma > 0$ ($\gamma = (S_\lambda(p))^{\frac{1}{q-2}}$), note that $u > 0$ on $\Omega$ by the strong maximum principle.
Now we prove the assertion 5) of Theorem 1.1. From (3.14) and since $\lambda \leq 0$ we have

$$p_0 S \leq S_\lambda(p) \leq Q_\lambda(u_{a,\varepsilon}) \leq p_0 S + o(1).$$

Hence $S_\lambda(p) = p_0 S$ and the infimum is not achieved, indeed we suppose that $S_\lambda(p)$ is achieved by some function $u \in H_0^1(\Omega)$, in that case

$$S_\lambda(p) = \int_\Omega p(x)|\nabla u(x)|^2 dx - \lambda \int_\Omega |u(x)|^2 dx, \text{ with } \|u\|_q = 1.$$

Using the fact that $S$ is not attained and since $\lambda \leq 0$, we deduce

$$p_0 S < p_0 \int_\Omega |\nabla u(x)|^2 dx \leq S_\lambda(p) = p_0 S,$$

then we obtain a contradiction.
Finally we prove assertion 6) in Theorem 1.1. Let $\varphi_1$ be the eigenfunction corresponding to $\lambda_1^{div}$ with $\varphi_1 > 0$ on $\Omega$. Suppose that $u$ is a solution of (1.1). We have

$$\begin{aligned} -\int_\Omega div(p(x)\nabla u(x))\varphi_1(x) dx &= \lambda_1^{div} \int_\Omega u(x)\varphi_1(x) dx \\ &= \int_\Omega u^{q-1}(x)\varphi_1(x) dx + \lambda \int_\Omega u(x)\varphi_1(x) dx, \end{aligned}$$

thus

$$\lambda_1^{div} \int_\Omega u(x)\varphi_1(x)dx > \lambda \int_\Omega u(x)\varphi_1(x)dx$$

and

$$\lambda_1^{div} > \lambda.$$

This completes the proof of Theorem 1.1.

## 4 The effect of the geometry of the domain

Let $\Omega \subset I\!\!R^n$, $n \geq 3$, be a bounded domain. We study the equation

$$(4.1) \qquad \begin{cases} -div(p(x)\nabla u) = u^{q-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

where $q = \frac{2n}{n-2}$ and $p : \bar{\Omega} \longrightarrow I\!\!R$ is a positive weight belonging to $C(\bar{\Omega}) \cap H_0^1(\Omega)$.
We assume in this section that $p$ is such that $\nabla p(x).(x-a) \geq 0$ a.e $x \in \Omega$ and we set $p_0 = p(a)$.
Let us start by the following non-existence result

**Lemma 4.1**
*There is no solution of (4.1) if $\Omega$ is a starshaped domain about $a$.*

**Proof.** This follows from Pohozaev's identity.
Suppose that $u$ is a solution of (4.1), we have (see Lemma 2.2 Section 2 for $\lambda = 0$),

$$(4.2) \qquad \int_\Omega \nabla p(x).(x-a)|\nabla u(x)|^2 dx + \int_{\partial\Omega} p(x)[(x-a).\nu]|\frac{\partial u}{\partial \nu}|^2 dx = 0.$$

Note that $(x-a).\nu > 0$ a.e on $\partial\Omega$ since $\Omega$ is starshaped about $a$.
Since $\nabla p(x).(x-a) \geq 0$ a.e $x \in \Omega$, we deduce from (4.2) that $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, and then by (4.1) we have

$$\int_\Omega u^{q-1}(x)dx = -\int_\Omega div(p(x)\nabla u(x))dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} dx = 0,$$

thus

$$u \equiv 0.$$

□

Suppose that $\Omega$ is starshaped about $a$. In view of Lemma 4.1, we will modify the geometry of $\Omega$ in order to find a solution of problem 4.1. For a $\varepsilon > 0$ small enough, we set $\Omega_\varepsilon = \Omega \setminus \bar{B}(a, \varepsilon)$.
We investigate the problem (4.1) in the new domain $\Omega_\varepsilon$, and, throughout the rest of this

paper, we shall denote this new problem by $(I_\varepsilon)$ .
Since $p$ is a continuous function, then $\forall\ \theta > 0$, $\exists\, r_0 > 0$ such that $\forall \sigma \in \Sigma$, where $\Sigma$ designates the unit sphere of $I\!R^n$, we have $|p(a + r_0\sigma) - p_0| < \frac{\theta}{2S^{\frac{n}{2}}}$.
Throughout the rest of this Section, $\theta > 0$ is fixed, small enough, and $r_0 > 0$ is given as the previous definition.
We recall the main result of this section which we have already stated by theorem 1.2 in the introduction

**Theorem 4.1**
*There exists $\varepsilon_0 = \varepsilon_0(\Omega, p) \le r_0$ such that for every $0 < \varepsilon < \varepsilon_0$, the problem $(I_\varepsilon)$ has at least one solution in $H^1_0(\Omega_\varepsilon)$.*

In order to prove the Theorem 4.1, we need to apply the following result, see [AR],

**Theorem A 1**
*Let $E$ be a $C^1$ function defined on a Banach space $X$, and let $K$ a compact metric space. We denote by $K^*$ a nonempty subset of $K$, closed, different from $K$ and we fix $f^* \in C(K^*, X)$.*
*We define $\mathcal{P} = \{f \in C(K, X)/f = f^* on\ K^*\}$ and $c = \inf_{f\in\mathcal{P}} \sup_{t\in K} E(f(t))$*
*Suppose that for every $f$ of $\mathcal{P}$, we have*

$$\max_{t\in K} E(f(t)) > \max_{t\in K^*} E(f(t)),$$

*then there exists a sequence $(u_j) \subset X$ such that $E(u_j) \longrightarrow c$ and $E'(u_j) \longrightarrow 0$ in $X^*$.*

We consider the functional

$$E(u) = \frac{1}{2}\int_{\Omega_\varepsilon} p(x)|\nabla u(x)|^2 dx - \frac{1}{q}\int_{\Omega_\varepsilon} |u(x)|^q dx.$$

In addition to Theorem A 1, the proof of Theorem 4.1 requires the following result (see [B] and Proposition 2.1 in [S])

**Theorem A 2**
*Suppose that for some sequence $(u_j) \subset H^1_0(\Omega_\varepsilon)$ we have $E(u_j) \to c \in ]\frac{1}{n}(p_0 S)^{\frac{n}{2}}, \frac{2}{n}(p_0 S)^{\frac{n}{2}}[$ and $dE(u_j) \to 0$ in $H^{-1}(\Omega_\varepsilon)$. Then $(u_j)$ contains a strongly convergent subsequence.*

Now, we return to the proof of Theorem 4.1.
We shall need the following functions:

$$\Gamma \quad : H^1_0(\Omega_\varepsilon) \longrightarrow I\!R, \quad \Gamma(u) = \int_{\Omega_\varepsilon} p(x)|\nabla u(x)|^2 dx - \int_{\Omega_\varepsilon} |u(x)|^q dx.$$
$$F \quad : H^1_0(\Omega_\varepsilon) \longrightarrow I\!R^n, \quad F(u) = (p_0 S)^{-\frac{n}{2}} \int_{\Omega_\varepsilon} x p(x)|\nabla u(x)|^2 dx.$$

We have the following result

**Lemma 4.2**
*For every neighborhood $V$ of $\bar{\Omega}_\varepsilon$ there exists $\eta > 0$ such that if $u \neq 0$, $\Gamma(u) = 0$ and $E(u) \leq \frac{1}{n}(p_0 S)^{\frac{n}{2}} + 2\eta$, then $F(u) \in V$.*

**Proof.** We proceed by contradiction. We assume that there exists $V$ a compact neighborhood of $\bar{\Omega}_\varepsilon$ not containing $a$, such that $\forall j \in \mathbf{N}^*$, we have

$$\begin{aligned} u_j &\neq 0, \\ \Gamma(u_j) &= 0, \\ E(u_j) &\leq \frac{1}{n}(p_0 S)^{\frac{n}{2}} + \frac{1}{j}, \\ F(u_j) &\notin V. \end{aligned}$$

Since $\Gamma(u_j) = 0$, we see that

$$\int_{\Omega_\varepsilon} p(x)|\nabla u_j|^2 dx = \int_{\Omega_\varepsilon} |u_j|^q dx$$

and

$$\int_{\Omega_\varepsilon} p(x)|\nabla u_j|^2 dx = \left( \frac{\int_{\Omega_\varepsilon} p(x)|\nabla u_j|^2 dx}{\left(\int_{\Omega_\varepsilon} |u_j|^q dx\right)^{\frac{2}{q}}} \right)^{\frac{n}{2}}.$$

Consequently

$$E(u_j) = \frac{1}{n} \int_{\Omega_\varepsilon} p(x)|\nabla u_j(x)|^2 dx.$$

Using the definition of $u_j$, the fact that $p_0 = \min_{\bar{\Omega}} p(x)$ and the definition of $S$, we write

$$\frac{1}{n}(p_0 S)^{\frac{n}{2}} \leq \frac{1}{n} \left( \frac{p_0 \int_{\Omega_\varepsilon} |\nabla u_j|^2 dx}{\left(\int_{\Omega_\varepsilon} |u_j|^q dx\right)^{\frac{2}{q}}} \right)^{\frac{n}{2}} \leq E(u_j) \leq \frac{1}{n}(p_0 S)^{\frac{n}{2}} + \frac{1}{j}$$

and we deduce

$$\int_{\Omega_\varepsilon} p(x)|\nabla u_j(x)|^2 dx = (p_0 S)^{\frac{n}{2}} + o(1).$$

Applying the Theorem 2 in [C], (see also Lemma I.1 and Lemma I.4 in [L]), for a subsequence of $(u_j)_j$ still denoted by $(u_j)_j$, there exists $x_0 \in \bar{\Omega}_\varepsilon$ such that

$$p(x)|\nabla u_j|^2 \longrightarrow (p_0 S)^{\frac{n}{2}} \delta_{x_0} \; (j \to \infty),$$

where the above convergence is understood for the weak topology of bounded measures on $\bar{\Omega}_\varepsilon$ and where $\delta_{x_0}$ is the Dirac measure at $x_0$.
As a consequence, $F(u_j) \in \bar{\Omega}_\varepsilon \subset V$, and this contradicts the hypothesis. $\square$

Let $R_0 > 0$ such that $B(a, 2R_0) \subset \Omega$.
For $k \in \mathbf{N}^*$, let $\varphi_k \in C^\infty(I\!R^n, [0,1])$ such that

$$\begin{cases} \varphi_k(x) = 0 & \text{if } |x-a| \leq \frac{1}{4k^2} \text{ and if } |x-a| \geq 2R_0, \\ \varphi_k(x) = 1 & \text{if } \frac{1}{2k^2} \leq |x-a| \leq R_0. \end{cases}$$

We consider the family of functions

$$u_t^\sigma(x) = \left[\frac{1-t}{(1-t)^2 + |x-a-t\sigma|^2}\right]^{\frac{n-2}{2}},$$

where $t \in [0,1[$, $\sigma \in \Sigma$ and where $\Sigma$ denotes the unit sphere of $I\!R^n$.
We see easily that $\int_{I\!R^n} |\nabla u_t^\sigma|^2 dx$ and $\int_{I\!R^n} |u_t^\sigma|^q dx$ are independent of $t \in [0,1[$ and of $\sigma \in \Sigma$. We also have

$$\int_{I\!R^n} |\nabla u_t^\sigma(x)|^2 dx = S \left(\int_{I\!R^n} |u_t^\sigma(x)|^q dx\right)^{\frac{2}{q}}.$$

We set

$$v_{t,k}^\sigma(x) = \frac{(1-t)^{\frac{n-2}{2}} k^{\frac{n-2}{2}} \varphi_k(x)}{((1-t)^2 + |k(x-a-tr_0\sigma)|^2)^{\frac{n-2}{2}}},$$

we remark that $v_{t,k}^\sigma \in H_0^1(\Omega_\varepsilon)$. For $r > 0$, let $g(r) = E(rv_{t,k}^\sigma)$, then
$rg'(r) = \Gamma(rv_{t,k}^\sigma)$, $g(r) \to -\infty$, when $r \to +\infty$, $g(0) = 0$ and $g(r) > 0$ for $r > 0$ small enough.
We conclude, from the above, that $g$ reaches its maximum at

$$r = \left[\frac{\int_{\Omega_\varepsilon} p(x)|\nabla v_{t,k}^\sigma|^2 dx}{\int_{\Omega_\varepsilon} |v_{t,k}^\sigma|^q dx}\right]^{\frac{1}{q-2}} > 0.$$

We set $w_{t,k}^\sigma = rv_{t,k}^\sigma$. We have

**Lemma 4.3**
*The following two statements are true:*

**a)**$\forall \delta > 0$, $\exists k_0 \geq 1$ *such that* $(\forall\, k \geq k_0)$ *then*
$(\forall \sigma \in \Sigma$ *and* $\forall t \in [0,1[,\ E(w_{t,k}^\sigma) \leq \frac{1}{n}(p_0 S)^{\frac{n}{2}} + \delta)$
**b)**$\forall \alpha > 0$, $\exists \mu > 0$ *such that* $(\mu < t < 1)$ *then*
$(\forall \sigma \in \Sigma$ *and* $\forall k \geq 1,\ E(w_{t,k}^\sigma) \leq \frac{1}{n}(p_0 S)^{\frac{n}{2}} + \alpha)$
*and* $|F(w_{t,k}^\sigma) - (a + r_0\sigma)| \leq \alpha$.

**Proof.** Before proving this Lemma, let us remark that the function $v_{t,k}^\sigma$ corresponds to the function $u_{a,\varepsilon}$ defined in the beginning of this paper, so for more details of calculus we refer to section 2.
We start by proving the assertion a). Let $t \in [0,1[$, we have

$$\begin{aligned} E(w_{t,k}^\sigma) &= \frac{1}{2}\int_{\Omega_\varepsilon} p(x)|\nabla w_{t,k}^\sigma|^2 dx - \frac{1}{q}\int_{\Omega_\varepsilon} |w_{t,k}^\sigma|^q dx, \\ &= \frac{r^2}{2}\int_{\Omega_\varepsilon} p(x)|\nabla v_{t,k}^\sigma|^2 dx - \frac{r^q}{q}\int_{\Omega_\varepsilon} |v_{t,k}^\sigma|^q dx. \end{aligned}$$

Using the definition of $r$, the definition of $\varphi_k$ and applying the Dominated Convergence Theorem, we obtain, as $k \to \infty$,

$$E(w_{t,k}^{\sigma}) = \frac{1}{n}\left[\frac{k^n(n-2)^2(1-t)^{n-2}\int_{\{\frac{1}{2k^2}\le|x-a|\le R_0\}} p(x)\frac{|k(x-a-tr_0\sigma)|^2}{((1-t)^2+|k(x-a-tr_0\sigma)|^2)^n}dx}{\left[k^n(n-2)^2(1-t)^n\int_{\{\frac{1}{2k^2}\le|x-a|\le R_0\}}\frac{1}{((1-t)^2+|k(x-a-tr_0\sigma)|^2)^n}dx\right]^{\frac{2}{q}}}\right]^{\frac{n}{2}}$$
$$+\ o(1).$$

By the following change of variable $y = \frac{k(x-a-tr_0\sigma)}{1-t}$, we see that

$$E(w_{t,k}^{\sigma}) = \frac{1}{n}\left[\frac{(n-2)^2\int_{\{\frac{1}{2k(1-t)}-\frac{tr_0}{1-t}\le|y|\le\frac{kR_0}{1-t}+\frac{tr_0}{1-t}\}} p(\frac{y(1-t)}{k}+a+tr_0\sigma)\frac{|y|^2}{(1+|y|^2)^n}dy}{\left[\int_{\{\frac{1}{2k(1-t)}-\frac{tr_0}{1-t}\le|y|\le\frac{kR_0}{1-t}+\frac{tr_0}{1-t}\}}\frac{1}{(1+|y|^2)^n}dy\right]^{\frac{2}{q}}}\right]^{\frac{n}{2}}$$
$$+\ o(1).$$

Applying again the Dominated Convergence Theorem, we deduce, as $k \to \infty$, that

$$\begin{aligned} E(w_{t,k}^{\sigma}) &= \frac{1}{n}\left[\frac{(n-2)^2p(a+tr_0\sigma)\int_{I\!R^n}\frac{|y|^2}{(1+|y|^2)^n}dy}{\left[\int_{I\!R^n}\frac{1}{((1-t)^2+|y|^2)^n}dy\right]^{\frac{2}{q}}}\right]^{\frac{n}{2}} + o(1), \\ &= \frac{1}{n}(p(a+tr_0\sigma))^{\frac{n}{2}}S^{\frac{n}{2}} + o(1). \end{aligned}$$

Now, using the definition of $r_0$, a simple computation shows that $\forall\delta > 0$, $\exists k_0 \geq 1$ such that $\forall k \geq k_0$, we have

$$E(w_{t,k}^{\sigma}) \leq \frac{1}{n}(p_0 S)^{\frac{n}{2}} + \delta,$$

which finishes the proof of a).

Now we return to the proof of b), let $k \in \mathbf{N}^*$, we have

$$\begin{aligned} E(w_{t,k}^{\sigma}) &= \frac{1}{2}\int_{\Omega_\varepsilon} p(x)|\nabla w_{t,k}^{\sigma}|^2dx - \frac{1}{q}\int_{\Omega_\varepsilon}|w_{t,k}^{\sigma}|^q dx \\ &= \frac{r^2}{2}\int_{\Omega_\varepsilon} p(x)|\nabla v_{t,k}^{\sigma}|^2dx - \frac{r^q}{q}\int_{\Omega_\varepsilon}|v_{t,k}^{\sigma}|^q dx. \end{aligned}$$

Looking at the definition of $\varphi_k$ and $r$, we easily see, as $t \to 1$, that

$$E(w_{t,k}^{\sigma}) = \frac{1}{n}\left[\frac{k^n(n-2)^2(1-t)^{n-2}\int_{I\!R^n} p(x)\frac{|k(x-a-tr_0\sigma)|^2}{((1-t)^2+|k(x-a-tr_0\sigma)|^2)^n}dx}{\left[k^n(1-t)^n\int_{I\!R^n}\frac{1}{((1-t)^2+|k(x-a-tr_0\sigma)|^2)^n}dx\right]^{\frac{2}{q}}}\right]^{\frac{n}{2}} + O((1-t)^{n-2}).$$

By the change of variable $y = \frac{k(x-a-tr_0\sigma)}{1-t}$, we get

$$E(w_{t,k}^{\sigma}) = \frac{1}{n}\left[\frac{(n-2)^2\int_{I\!R^n} p(\frac{(1-t)y}{k}+a+tr_0\sigma)\frac{|y|^2}{(1+|y|^2)^n}dy}{\left[\int_{I\!R^n}\frac{1}{(1+|y|^2)^n}dy\right]^{\frac{2}{q}}}\right]^{\frac{n}{2}} + O((1-t)^{n-2}).$$

Applying the Dominated Convergence Theorem, we obtain

$$E(w^\sigma_{t,k}) = \frac{1}{n}\left[\frac{(n-2)^2 p(a+r_0\sigma)\int_{I\!R^n}\frac{|y|^2}{(1+|y|^2)^n}dy}{\left[\int_{I\!R^n}\frac{1}{(1+|y|^2)^n}dy\right]^{\frac{2}{q}}}\right]^{\frac{n}{2}} + O((1-t)^{n-2}),$$

$$= \frac{1}{n}(p(a+r_0\sigma))^{\frac{n}{2}}S^{\frac{n}{2}} + O((1-t)^{n-2}).$$

Using the definition of $r_0$, a simple computation shows that $\forall \alpha > 0$, $\exists \mu > 0$ such that $\forall\, \mu < t < 1$, we have

$$E(w^\sigma_{t,k}) \le \frac{1}{n}(p_0 S)^{\frac{n}{2}} + \alpha.$$

On the other hand

$$F(w^\sigma_{t,k}) = (p_0 S)^{-\frac{n}{2}}\int_{I\!R^n} xp(x)|\nabla w^\sigma_{t,k}(x)|^2 dx,$$

$$= (p_0 S)^{-\frac{n}{2}} r^2 \int_{I\!R^n} xp(x)|\nabla v^\sigma_{t,k}(x)|^2 dx.$$

By the definition of $v^\sigma_{t,k}$ and $r$, we write

$$F(w^\sigma_{t,k}) = (p_0 S)^{-\frac{n}{2}}\left[\frac{(1-t)^{n-2}(n-2)^2\int_{I\!R^n} p(x)\frac{|k(x-a-tr_0\sigma)|^2}{((1-t)^2+|k(x-a-tr_0\sigma)|^2)^n}dx}{(1-t)^n\int_{I\!R^n}\frac{1}{((1-t)^2+|k(x-a-tr_0\sigma)|^2)^n}dx}\right]^{\frac{2}{q-2}} \times$$

$$(1-t)^{n-2}k^n(n-2)^2\int_{I\!R^n} x\,p(x)\frac{|k(x-a-tr_0\sigma)|^2}{((1-t)^2+|k(x-a-tr_0\sigma)|^2)^n}dx + o(1-t).$$

The change of variable $y = \frac{k(x-a-tr_0\sigma)}{1-t}$ gives

$$F(w^\sigma_{t,k}) = (p_0 S)^{-\frac{n}{2}}\left[\frac{(n-2)^2\int_{I\!R^n} p\left(\frac{(1-t)y}{k}+a+tr_0\sigma\right)\frac{|y|^2}{(1+|y|^2)^n}dx}{\int_{I\!R^n}\frac{1}{(1+|y|^2)^n}dx}\right]^{\frac{2}{q-2}} \times$$

$$(n-2)^2\int_{I\!R^n}\frac{(\frac{(1-t)y}{k}+a+tr_0\sigma)\,p(\frac{(1-t)y}{k}+a+tr_0\sigma)\,|y|^2}{(1+|y|^2)^n}dx + o(1-t).$$

Applying the Dominated Convergence Theorem, we deduce that

$$F(w^\sigma_{t,k}) = (p_0 S)^{-\frac{n}{2}}(p(a+r_0\sigma))^{\frac{n}{2}}\left[\frac{(n-2)^2\int_{I\!R^n}\frac{|y|^2}{(1+|y|^2)^n}dy}{\left[\int_{I\!R^n}\frac{1}{(1+|y|^2)^n}dy\right]^{\frac{2}{q}}}\right]^{\frac{q}{q-2}}(a+r_0\sigma) + o(1-t),$$

$$= (p_0 S)^{-\frac{n}{2}}(p(a+r_0\sigma))^{\frac{n}{2}}S^{\frac{n}{2}}(a+r_0\sigma) + o(1-t).$$

Using the definition of $r_0$ we get the desired result. $\square$

**Consequences**

Let $V$ be a compact neighborhood of $\bar{\Omega}_\varepsilon$ not containing $a$. Let $0 < \eta < r_0$ small enough, which corresponds to $V$ as in Lemma 4.2, verifying $r_0\sigma + \xi \ne a$ for $|\sigma| = 1$ and $|a-\xi| \le \eta$.

By Lemma 4.3, there exists $k_0 \geq 1$ such that :

$$E(w^\sigma_{t,k_0}) \leq \frac{2}{n}(p_{{}_0}S)^{\frac{n}{2}} - \eta,\ \forall \sigma \in \Sigma,\ \forall t \in [0,1[. \tag{4.3}$$

**Remark 4.1**
*We choose $\varepsilon_0 = \varepsilon_0(\Omega, p) \leq \frac{1}{4k_0^2}$ small enough and such that $\forall 0 < \varepsilon < \varepsilon_0$ we have $\{x\ |x - a| \leq \varepsilon\} \not\subset V$.*

We fix $\lambda > 1$, large enough such that $E(\lambda w^\sigma_{t,k_0}) < 0$, $\forall \sigma \in \Sigma$, $\forall t \in [0,1[$. In order to apply Theorem A 1, we define the sets $K$, $K^*$ and the function $f^*$ as
$K = [0,1] \times \bar{B}(a, r_0)$,
$K^* = \partial K = [0,1] \times \partial \bar{B}(a, r_0) \cup \{0,1\} \times \bar{B}(a, r_0)$ and
$f^* : K \to H^1_0(\Omega_\varepsilon)$,
$f^*(s, tr_0\sigma) = \lambda s w^\sigma_{t,k_0}$.
The conclusion of Theorem 4.1 follows from the next

**Lemma 4.4**
*We have*

$$\sup_K E(f) \geq \frac{1}{n}(p_{{}_0}S)^{\frac{n}{2}} + 2\eta,\ \forall f \in \mathcal{P}.$$

We postpone the proof of Lemma 4.4 and we complete the proof of Theorem 4.1. From (4.3) we have

$$\max_{r \geq 0} E(rv^\sigma_{t,k_0}) = E(w^\sigma_{t,k_0}) \leq \frac{2}{n}(p_{{}_0}S)^{\frac{n}{2}} - \eta \quad \forall \sigma \in \Sigma, \quad \forall t \in [0,1[.$$

From assertion b) of Lemma 4.3 there exists $\mu > 0$, we fix $t_0 \in ]\mu, 1[$ such that

$$\max_{r \geq 0} E(rv^\sigma_{t_0,k_0}) = E(w^\sigma_{t_0,k_0}) \leq \frac{1}{n}(p_{{}_0}S)^{\frac{n}{2}} + \eta, \quad \forall \sigma \in \Sigma.$$

then

$$\max_{\partial K} E(f^*) \leq \frac{1}{n}(p_{{}_0}S)^{\frac{n}{2}} + \eta \quad \text{ and } \quad \sup_K E(f^*) < \frac{2}{n}(p_{{}_0}S)^{\frac{n}{2}}.$$

So, by Lemma 4.4,

$$\sup_K E(f) \geq \frac{1}{n}(p_{{}_0}S)^{\frac{n}{2}} + 2\eta > \frac{1}{n}(p_{{}_0}S)^{\frac{n}{2}} + \eta \geq \sup_{\partial K} E(f^*)$$

and

$$c = \inf_{f \in \mathcal{P}} \sup_{t \in K} E(f) \in ]\frac{1}{n}(p_{{}_0}S)^{\frac{n}{2}}, \frac{2}{n}(p_{{}_0}S)^{\frac{n}{2}}[.$$

Applying Theorem A 1 and Theorem A 2, we obtain the conclusion of Theorem 4.1.

**Proof of Lemma 4.4**. We argue by contradiction. Suppose that there exists $f \in C(K, H^1_0(\Omega_\varepsilon))$ with $f = f^*$ on $\partial K$, and $E(f(s,\xi)) \leq \frac{1}{n}(p_{{}_0}S)^{\frac{n}{2}} + 2\eta$, $\forall (s,\xi) \in K$.
We consider the function $G : K \longrightarrow I\!R^{n+1}$, defined by

$$G(s,\xi) = (s, F(f(s,\xi))).$$

We will prove that

$$\deg(G, K, (\lambda^{-1}, a)) = 1. \tag{4.4}$$

The map $H : [0,1] \times K \longrightarrow \mathbb{R}^{n+1}$, defined by
$H(t,s,\xi) = tG(s,\xi) + (1-t)(s,\xi) = (s, tF(f(s,\xi)) + (1-t)\xi)$
is a homotopy between $G$ and $Id_K$, where $Id_K$ is the Identity application of $K$.
To get (4.4), we start by checking that $(\lambda^{-1}, a) \notin H(t, \partial K)$.
If not, there exists $(s,\xi) \in \partial K$ such that $H(t,s,\xi) = (\lambda^{-1}, a)$, as a consequence $s = \lambda^{-1}$ and $a = tF(f(\lambda^{-1}, \xi)) + (1-t)\xi = t(F(w^\sigma_{t_0,k_0}) - \xi) + \xi$.
Since $s = \lambda^{-1} \in ]0,1[$, we have $\xi \in \partial \bar{B}(a, r_0)$. But, since $|F(w^\sigma_{t_0,k_0}) - (a + r_0\sigma)| < \eta$ $\forall \sigma \in \Sigma$ (see Lemma 4.3), the fact that $t(F(w^\sigma_{t_0,k_0}) - \xi) + \xi = a$, $\xi \in \partial \bar{B}(a, r_0)$ leads to a contradiction. Then, we deduce that $(\lambda^{-1}, a) \notin H(t, \partial K)$ and consequently $\forall t \in [0,1]$, $\deg(H(t,.), K, (\lambda^{-1}, a))$ is well defined.
We consider the following sets:
$K^+ = \{(s,\xi) \in K \mid \Gamma(f(s,\xi)) > 0\} \cup (0,\xi)$, $K^- = \{(s,\xi) \in K \mid \Gamma(f(s,\xi)) < 0\}$ and $K^0 = \{(s,\xi) \in K \mid \Gamma(f(s,\xi)) = 0\}$.
If $(s,\xi) \in \partial K$ then we have $f(s,\xi) = f^*(s,\xi) = \lambda s w^\sigma_{t_0,k_0}$ and

$$\begin{aligned}
\Gamma(f(s,\xi)) &= (s\lambda)^2 \int_{\Omega_\varepsilon} p(x) |\nabla w^\sigma_{t_0,k_0}(x)|^2 dx - (s\lambda)^q \int_{\Omega_\varepsilon} |w^\sigma_{t_0,k_0}(x)|^q dx \\
\Gamma(f(s,\xi)) &= [(s\lambda)^2 - (s\lambda)^q] \int_{\Omega_\varepsilon} p(x) |\nabla w^\sigma_{t_0,k_0}(x)|^2 dx.
\end{aligned}$$

Since $\int_{\Omega_\varepsilon} p(x)|\nabla w^\sigma_{t_0,k_0}(x)|^2 dx > 0$, we see that

$$\text{If} \quad (s,\xi) \in \partial K \quad \text{and if} \quad 0 \le s < \lambda^{-1}, \quad \text{then} \quad (s,\xi) \in K^+ \tag{4.5}$$

$$\text{If} \quad (s,\xi) \in \partial K \quad \text{and if} \quad \lambda^{-1} < s \le 1, \quad \text{then} \quad (s,\xi) \in K^- \tag{4.6}$$

$$(\lambda^{-1}, \xi) \in K^0, \quad \forall \xi \in \partial \bar{B}(a, r_0). \tag{4.7}$$

Let $(s,\xi) \in K^0$, we have $\Gamma(f(s,\xi)) = 0$. Moreover, since $E(f(s,\xi)) \le \frac{1}{n}(p_0 S)^{\frac{n}{2}} + 2\eta$, looking at Lemma 4.2, we deduce that

$$F(f(s,\xi)) \in V.$$

Consequently $\forall (s,\xi) \in K^0$, $F(f(s,\xi)) \neq a$ since $a \notin V$.
Hence $(\lambda^{-1}, a) \notin G(K^0) = G(K \setminus (K^+ \cup K^-))$, then

$$\deg(G, K^+, (\lambda^{-1}, a)) + \deg(G, K^-, (\lambda^{-1}, a)) = \deg(G, K, (\lambda^{-1}, a)). \tag{4.8}$$

On the other hand, since $(\lambda^{-1}, a) \notin H(t, \partial K)$ $\forall t \in [0,1]$ we have

$$\deg(H(1,.), K, (\lambda^{-1}, a)) = \deg(H(0,.), K, (\lambda^{-1}, a)).$$

Using the fact that $H(0,.) = G$, $H(1,.) = Id_K$ and $\deg(Id_K, K, (\lambda^{-1}, a)) = 1$, we deduce (4.4).
Now, we will prove that

$$\deg(G, K^+, (\lambda^{-1}, a)) = 0 \tag{4.9}$$

$$\deg(G, K^-, (\lambda^{-1}, a)) = 0. \tag{4.10}$$

Fix $R > \lambda^{-1}$ and let $y \in \mathbb{R}^{n+1}$ such that $|y| \geq R$ then $y \notin G(K)$.
We define the path $r(t) = (tR + (1-t)\lambda^{-1}, a)$, for $t \in [0,1]$.
We claim that $r(t) \notin G(\partial K^+)$ $\forall t \in [0,1]$.
If not, there exists $(s,\xi) \in \partial K^+$ with $(Rt + (1-t)\lambda^{-1}, a) = (s, F(f(s,\xi)))$. Hence $s = tR + (1-t)\lambda^{-1} \geq \lambda^{-1}$ and $a = F(f(s,\xi))$. But $\forall (s,\xi) \in K^0$, we have $F(f(s,\xi)) \neq a$, then $(s,\xi) \notin K^0$. Hence $(s,\xi) \in \partial K \cap K^+$, (4.5) implies that $s < \lambda^{-1}$ and this contradicts the fact that $s \geq \lambda^{-1}$. Thus $r(t) \notin G(\partial K^+)$ $\forall t \in [0,1]$. Hence $\deg(G, K^+, r(t))$ is well defined and is independent of $t$.
Since $(R,a) \notin G(K)$ we obtain

$$\deg(G, K^+, (R,a)) = 0.$$

Using the fact that

$$\deg(G, K^+, r(t)) = \deg(G, K^+, (R,a)) \quad \forall t \in [0,1],$$

we deduce (4.9).
Similarly, we prove (4.10) by using the path $q(t) = (-tR + (1-t)\lambda^{-1}, a)$, $t \in [0,1]$. We have that $\deg(G, K^-, q(t))$ is independent of $t$. Using the fact that $(-R,a) \notin G(K)$, we conclude that

$$\deg(G, K^-, (\lambda^{-1}, a)) = \deg(G, K^-, (-R,a)) = 0.$$

From (4.4), (4.8), (4.9) and (4.10) we obtain a contradiction, and Lemma 4.4 is proved.

## References


[AR] A. Ambrosetti and P. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal. **14**, 1973, p. 349-381.

[BaC] A. Bahri and J. M. Coron, On a nonlinear elliptic equation involving the critical Sobolev exponent: The effect of the topology of the domain, Comm. Pure Appl. Math. **41**, 1988, p. 253-294.

[B] H. Brezis, Elliptic equations with limiting Sobolev exponents. The impact of topology, Comm. Pure Appl. Math. **39**, 1986, p. S17-S39.

[BN] H. Brezis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, Comm. Pure Appl. Math. **36**, 1983, p. 437-477.

[BL] H. Brezis and E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, Proc. Amer. Math. Soc. **88**, 1983, p. 486-490.

[C] J. M. Coron, Topologie et cas limite des injections de Sobolev, C.R. acad. Sc., Paris. **299**, 1985, p. 209-211.

[CKN] L. Caffarelli, R. Kohn and L. Nirenberg, First order interpolation inequalities with weights, Compositio Math. **53**, 1984, p. 259-275.

[D] E. N. Dancer, A note on an equation with critical exponent, Bull. London Math. Soc. **20**, n°. 6, 1988, p. 600-602.

[Di] W. Y. Ding, Positive solutions of $\Delta u + u^{\frac{n+2}{n-2}} = 0$ on contractible domains, J. Partial Differential Equations **2**, n°. 4, 1989, p. 83-88.

[H] R. Hadiji, Solutions positives de l'equation $-\Delta u = u^p + \mu u^q$ dans un domaine à trou, Annales de la faculté des Sciences de Toulouse. **XI**, n°. 3, 1990, p. 55-71.

[HLP] G. H. Hardy, J. E. Littlewood and G. Polya: *Inequalities*. Cambridge University Press, 1952.

[L] P. L. Lions, The concentration-compactness principle in the calculus of variations. The limit case, part 1, Rev. Mat. Iberoamericana **1**, n°. 1, 1985, p. 145-201.

[Le] R. Lewandowski, Little holes and convergence of solutions of $-\Delta u = u^{\frac{N+2}{N-2}}$, Nonlinear Analysis, Theory, Methods and Applications **14**, n°. 10, 1990, p. 873-888.

[Pa] D. Passaseo, Multiplicity of positive solutions of nonlinear elliptic equations with critical Sobolev exponent in some contractible domains, Manuscripta Math. **65**, n°. 2, 1989, p. 147-165.

[P] R. S. J. Pohozaev, Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$, Soviet Math. Doklady **6**, 1965, p. 1408-1411.

[S] M. Struwe, A global compactness result for elliptic boundary value problems involving limiting nonlinearities, Math. Z. **187**, 1984, p. 511-517.