Stochastic Sensitivity Study for Optimal Credit allocation

Laurence Carassus, Simone Scotti

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Abstract: In this paper we present the detail computations involved in [2]. First we propose a quick presentation of the methodology developed by Bouleau (see [5]). Then, we apply this method to the problem of optimal credit allocation problem.

1 Introduction

This introduction is devoted to the presentation of error calculus in the sense of sensitivity with respect to a stochastic perturbation.

We shortly recall Gauss idea for error propagation. All the implied functions are assumed to be smooth enough. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $V = F(U_1, \ldots, U_n)$ be a real valued function of the $U_1, \ldots, U_n$ which are supposed to be erroneously measured. The U_i are seen as random variables with values very closed to their mean value $\mathbb{E}[U_i]$. The error of measure is given by $\sigma_{i,j} = \mathbb{E}\left[(U_i - \mathbb{E}[U_i])\,(U_j - \mathbb{E}[U_j])\right]$. Here the error is assumed to be small thus using a first order approximation, we obtain:

$$F(U_1, \ldots, U_n) - F(\mathbb{E}[U_1], \ldots, \mathbb{E}[U_n)]) = \sum_{i=1}^{n} (U_i - \mathbb{E}[U_i]) \frac{\partial F}{\partial U_i}(\mathbb{E}[U_1], \ldots, \mathbb{E}[U_n)]).$$

If we denote by $\sigma_V^2 = \mathbb{E}\left[F(U_1, \ldots, U_n) - F(\mathbb{E}[U_1], \ldots, \mathbb{E}[U_n)]\right]^2$,

$$\sigma_V^2 = \sum_{i,j=1}^{n} \sigma_{i,j} \frac{\partial F}{\partial U_i}(\mathbb{E}[U_1], \ldots, \mathbb{E}[U_n)]) \frac{\partial F}{\partial U_j}(\mathbb{E}[U_1], \ldots, \mathbb{E}[U_n)]) \tag{1}$$

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We see that the mean square error transmits with a first order differential calculus. Now we try to generalize Gauss equation (1) by considering that the measure of U_i is no more the constant value $\mathbb{E}[U_i]$ but a random variable. This random variable is defined on an (independant) copy of $(\Omega, \mathcal{F}, \mathbb{P})$ denoted by $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$. We called it $\widehat{U_i}$: in statistical language it is the estimator of U_i. Consider another real valued function G. We introduce the following random variables defined on the space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$:

$$\begin{aligned}
\text{Bias}(U_i) &= \mathbb{E}\left(U_i - \widehat{U_i}\right) = \mathbb{E}\left(U_i\right) - \widehat{U_i} && (2)\\
\text{Bias}(F(U_1,\ldots,U_n)) &= \mathbb{E}\left[F(U_1,\ldots,U_n) - F(\widehat{U_1},\ldots,\widehat{U_n})\right] && (3)\\
&= \mathbb{E}\left[F(U_1,\ldots,U_n)\right] - F(\widehat{U_1},\ldots,\widehat{U_n})\\
\text{MSE}(U_i,U_j) &= \mathbb{E}\left[\left(U_i - \widehat{U_i}\right)\left(U_j - \widehat{U_j}\right)\right] && (4)\\
\text{MSE}(F(U_1,\ldots,U_n),G(U_1,\ldots,U_n)) &= \mathbb{E}\left[\left(F(U_1,\ldots,U_n) - F(\widehat{U_1},\ldots,\widehat{U_n})\right)\left(G(U_1,\ldots,U_n) - G(\widehat{U_1},\ldots,\widehat{U_n})\right)\right].
\end{aligned}$$

Let $H \in \{F, G\}$, using a first order approximation we obtain:

$$H(U_1,\ldots,U_n) - H(\widehat{U_1},\ldots,\widehat{U_n}) = \sum_{i=1}^{n}(U_i - \widehat{U_i})\frac{\partial H}{\partial U_i}(\widehat{U_1},\ldots,\widehat{U_n}).$$

$$\text{MSE}(F(U_1,\ldots,U_n),G(U_1,\ldots,U_n)) = \sum_{i,j=1}^{n}\text{MSE}(U_i,U_j)\frac{\partial F}{\partial U_i}(\widehat{U_1},\ldots,\widehat{U_n})\frac{\partial G}{\partial U_i}(\widehat{U_1},\ldots,\widehat{U_n}). \quad (5)$$

For the bias we use a second order approximation and we get that:

$$\begin{aligned}
F(U_1,\ldots,U_n) - F(\widehat{U_1},\ldots,\widehat{U_n}) &= \sum_{i=1}^{n}(U_i - \widehat{U_i})\frac{\partial F}{\partial U_i}(\widehat{U_1},\ldots,\widehat{U_n}) +\\
&\quad \frac{1}{2}\sum_{i,j=1}^{n}(U_i - \widehat{U_i})(U_j - \widehat{U_j})\frac{\partial^2 F}{\partial U_i U_j}(\widehat{U_1},\ldots,\widehat{U_n})
\end{aligned}$$

Thus, one has:

$$\text{Bias}(F(U_1,\ldots,U_n)) = \sum_{i=1}^{n}\text{Bias}(U_i)\frac{\partial F}{\partial U_i}(\widehat{U_1},\ldots,\widehat{U_n}) + \frac{1}{2}\sum_{i,j=1}^{n}\text{MSE}(U_i,U_j)\frac{\partial^2 F}{\partial U_i U_j}(\widehat{U_1},\ldots,\widehat{U_n}). \quad (6)$$

The bias follows a second order differential calculus involving the mean square error.

The idea developed by Bouleau is to allow the errors to be random variables. The quadratic error on each U_i is given by $\Gamma(U_i, U_i)$ (shortly denoted by $\Gamma(U_i)$) where Γ is a bilinear operator associated to the MSE and the bias by $\mathcal{A}(U_i)$, where $\mathcal{A}$ is a linear operator. This representation would have the nice following property that if the sequence of pairs $(Y_n, \Gamma(Y_n))$ converges (in a certain sense) it should tends to $(Y, \Gamma(Y))$.

The rest of the paper is organized as follows. In Section 2, we present the mathematical setup for error calculus. Then in Section 3, we present the model for credit spreads, the expression of the P&L, the optimisation program and it solution. Finally, in section 4, we apply the machinery of error calculus to determine the variance and bias of the allocation.

2 Error Calculus

We now formalize the intuition exposed in the introduction. The axiomatization of random uncertainty propagation was introduced by Bouleau [4] as follows:

Definition 1 (Error structure)

An error structure is a term $\left(\widetilde{\Omega},\, \widetilde{\mathcal{F}},\, \widetilde{\mathbb{P}},\, \mathbb{D},\, \Gamma\right)$*, where*

- $\left(\widetilde{\Omega},\, \widetilde{\mathcal{F}},\, \widetilde{\mathbb{P}}\right)$ *is a probability space;*
- $\mathbb{D}$ *is a dense sub-vector space of* $L^2\left(\widetilde{\Omega},\, \widetilde{\mathcal{F}},\, \widetilde{\mathbb{P}}\right)$ *such that for any function* F *of class* C^1 *and globally Lipschitz (afterward denoted* $C^1 \cap Lip$*) and* $U \in \mathbb{D}$*, one has* $F(U) \in \mathbb{D}$*.*
- Γ *is a positive symmetric bilinear function from* $\mathbb{D} \times \mathbb{D}$ *into* $L^1\left(\widetilde{\Omega},\, \widetilde{\mathcal{F}},\, \widetilde{\mathbb{P}}\right)$ *satisfying the following functional calculus inspired by* (5)*: for any functions* F *and* G *of class* $\mathcal{C}^1 \cap Lip$ *and* $U, V \in \mathbb{D}^n$

$$\Gamma\left[F(U),\, G(V)\right] = \sum_{i,j=1}^{n} \frac{\partial F}{\partial U_i}(U)\, \frac{\partial G}{\partial U_j}(V)\, \Gamma[U,\, V] \;\; \widetilde{\mathbb{P}}\; a.s.; \tag{7}$$

- *the bilinear form* $\mathcal{E}[U,\, V] = \frac{1}{2}\widetilde{\mathbb{E}}\left[\Gamma[U,\, V]\right]$ *is closed, i.e.* $\mathbb{D}$ *equipped with the norm* $|U|_{\mathbb{D}} = \left(\widetilde{\mathbb{E}}\left[U^2\right] + \frac{1}{2}\mathcal{E}[U,\, U]\right)^{1/2}$ *is complete;*
- *the constant* 1 *belongs to* $\mathbb{D}$ *and* $\mathcal{E}[1, 1] = 0$

An error structure is a probability space equipped with a carré du champ operator Γ. We generally write $\Gamma[U]$ for $\Gamma[U,\, U]$. The Hille-Yosida theorem guarantees that there exists a semigroup and a generator $\mathcal{A}$ that are coherent with the Dirichlet form $\mathcal{E}$, see for instance Albeverio [1] and Fukushima et al. [7]. This generator $\mathcal{A} : \mathcal{DA} \to L^2\left(\widetilde{\Omega},\, \widetilde{\mathcal{F}},\, \widetilde{\mathbb{P}}\right)$ is a self-adjoint operator, its domain $\mathcal{DA}$ is included into $\mathbb{D}$. It is such that for all $U \in \mathcal{DA}$ and $V \in L^2\left(\widetilde{\Omega},\, \widetilde{\mathcal{F}},\, \widetilde{\mathbb{P}}\right)$

$$\mathcal{E}[U, V] = -\widetilde{\mathbb{E}}\left[\mathcal{A}[U]V\right]. \tag{8}$$

Moreover this operator satisfies, for $F \in \mathcal{C}^2 \cap Lip$, $U \in \mathcal{DA}$, $F(U) \in \mathcal{DA}$ and $\Gamma[U] \in L^2\left(\widetilde{\Omega},\, \widetilde{\mathcal{F}},\, \widetilde{\mathbb{P}}\right)$:

$$\mathcal{A}\left[F(U)\right] = \sum_{i=1}^{n} \frac{\partial F}{\partial U_i}(U)\mathcal{A}[U] + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 F}{\partial U_i U_j}(U)\, \Gamma[U] \;\; \widetilde{\mathbb{P}}\; a.s.. \tag{9}$$

This equation is similar to (6). From (8), $\mathcal{A}$ is a closed operator with respect to the norm $|\;|_{\mathbb{D}}$, in the sense that $\mathcal{DA}$ equipped with the norm $|\;|_{\mathbb{D}}$ is complete.

We introduce a basic example to show that the set of error structures is not empty and to give some further intuitions about the different operator introduced above.

Example 1 (Ornstein-Uhlenbeck error structure)

The Ornstein-Uhlenbeck structure is $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu, \mathbb{D}, \Gamma)$*, where* μ *is unidimensional centered Gaussian law and* $\mathbb{D} := H^{1,2}(\mu)$*, i.e. the first Sobolev space associated to* $L^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$*. Here we choose a particular semi-group and we compute the associated generator and Dirichlet form. To this end we introduce an auxiliary Ornstein-Hulenbeck process in a probability space* $(\Omega^B, \mathcal{F}^B, \mathbb{P}^B)$ *equipped with a Brownian motion* B*:*

$$dX_\epsilon = -\frac{1}{2} X_\epsilon d\epsilon + dB_\epsilon.$$

We denote by X_ϵ^x *the solution of the preceding equation starting at some* x*. We then define the semigroup as follows for all* $\omega \in \mathbb{R}$*:*

$$P_\epsilon[U](\omega) = \mathbb{E}^B U[X_\epsilon^x | x = \omega],$$

where $\mathbb{E}^B$ *is the expectation with respect to* $\mathbb{P}^B$*. Using Ito lemma, we have for* $U \in \mathbf{H}^{2,2}(\mu)$*:*

$$\begin{aligned}
X_\epsilon^x &= x e^{-\frac{1}{2}\epsilon} + \int_0^\epsilon e^{-\frac{1}{2}(\epsilon - s)} dB_s \\
U(X_\epsilon^x) &= U(x) - \frac{1}{2}\int_0^\epsilon U'(X_s^x) X_s^x ds + \frac{1}{2}\int_0^\epsilon U''(X_s^x) ds + \int_0^\epsilon U'(X_s^x) dB_s \qquad (10) \\
< U(X^x) >_\epsilon &= \int_0^\epsilon \left(U'(X_s^x)\right)^2 ds \qquad (11)
\end{aligned}$$

where $<>_\epsilon$ *denotes the quadratic variation operator. The generator* $\mathcal{A}$ *and the Dirichlet form* $\mathcal{E}$ *associated to the semi-group* P_ϵ *are given by (see [5] chapter II):*

$$\begin{aligned}
\mathcal{A}[U](\omega) &= \lim_{\epsilon \to 0} \frac{P_\epsilon[U](\omega) - U(\omega)}{\epsilon} = \frac{1}{2} U''(\omega) - \frac{1}{2}\omega U'(\omega) \qquad (12) \\
\mathcal{E}[U] &= \langle -\mathcal{A}[U], U \rangle_{L^2(\mu)} = \frac{1}{2}\int (\omega U'(\omega) - U''(\omega)) U(\omega) d\mu(\omega) \\
&= \frac{1}{2}\int (U'(\omega))^2 d\mu(\omega) \qquad (13)
\end{aligned}$$

where (13) *comes from integration by part formula (recall that* μ *is a centered gaussian law). As* $\mathcal{E}[U] = \frac{1}{2}\widetilde{\mathbb{E}}\left[\Gamma[U, U]\right]$*, we deduce from* (13) *that* $\Gamma(U) = (U')^2$ *for all* $U \in \mathbb{D}$*. From* (12)*, the related generator is* $\mathcal{A}[U] = \frac{1}{2}U'' - \frac{1}{2} Id \cdot U'$*, where* Id *denotes the identity operator. Here the domain* $\mathcal{DA} := \mathbf{H}^{2,2}(\mu)$*, i.e. the second Sobolev space.*

In this particular case, (7) *and* (9) *are easy to obtain:*

$$\begin{aligned}
\Gamma\left[F(U), G(V)\right](\omega) &= F'(U)(\omega) U'(\omega) G'(V)(\omega) V'(\omega) = F'(U)(\omega) G'(V)(\omega) \Gamma[U, V](\omega) \\
\mathcal{A}\left[F(U)\right](\omega) &= \frac{1}{2}\left(F''(U)(\omega)\left(U'(\omega)\right)^2 + F'(U)(\omega) U'(\omega)\right) - \frac{1}{2}\left(\omega F'(U)(\omega) U'(\omega)\right) = \\
&= F'(U)(\omega)\mathcal{A}[U](\omega) + \frac{1}{2} F''(U)(\omega)\Gamma[U](\omega)
\end{aligned}$$

We finish this example with the following remark. In the introduction, we have associated to the statistical notion of bias and mean squared error the operators $\mathcal{A}$ and Γ (see (3), (4) and (5)). In this example we can do the other way around: it is easy to show that the $\mathcal{A}$ is associated to a bias and Γ is associated to a mean square error in the following sense:

$$\begin{aligned} \epsilon\mathcal{A}[U](\omega) &= \mathbb{E}^B U[X_\epsilon^x - U(x)|x=\omega] \\ \epsilon\Gamma[U](\omega) &= \mathbb{E}^B U[(X_\epsilon^x - U(x))^2 |x=\omega] \end{aligned}$$

Afterwards, we will omit the explicit dependency on ω.

Remark 1 (Statistical interpretation)

It is possible to push further the analogy between statists and theory of Dirichlet forms used to compute errors. In [6] it is shown that one can constructed confidence interval for some random variable U using $\mathcal{A}[U]$ and $\Gamma[U]$. This is achieve by choosing an error structure linked to Fisher's information matrix.

The main drawback of the carré du champ operator Γ is its bi-linearity, which makes computations awkward to perform. An easy way to overcome this drawback is to introduce a new operator, called the gradient, see Bouleau and Hirsch [3], section II.6. We recall the definition of gradient operator associated to Γ.

Proposition 1 (Gradient operator)

From now we assume that the space $\mathbb{D}$ is separable. Let $\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}, \mathbb{D}, \Gamma\right)$ be an error structure. Let $\left(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}}\right)$ be an (independent) copy of $\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}\right)$ and $\mathcal{H} = L^2\left(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}}\right)$ be an auxiliary Hilbert space equipped with scalar product $< X, Y >_{\mathcal{H}} = \overline{\mathbb{E}}(XY)$, where $\overline{\mathbb{E}}$ is the expectation computed under $\overline{\mathbb{P}}$. Let $L^2\left(\left(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}}\right), \widetilde{\mathbb{P}}\right)$ or shortly $L^2(\widetilde{\mathbb{P}}, \mathcal{H})$ the space of $L^2\left(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}}\right)$ valued random variables equipped with the scalar product $< A, B >_{L^2(\widetilde{\mathbb{P}}, \mathcal{H})} = \widetilde{\mathbb{E}}[< A, B >_{\mathcal{H}}] = \widetilde{\mathbb{E}}\overline{\mathbb{E}}(AB)$. Then there exists a linear operator, called gradient and denoted by $(\,)^\# : \mathbb{D} \to L^2(\widetilde{\mathbb{P}}, \mathcal{H})$, with the following two properties:

$$\forall\, U\, V \in \mathbb{D},\ \Gamma[U,V] = < U^\#, V^\# >_{\mathcal{H}} = \overline{\mathbb{E}}\left(U^\# V^\#\right) \tag{14}$$

$$\forall\, U \in \mathbb{D}^n,\ F \in \mathcal{C}^1 \cap Lip, (F(U_1, \ldots, U_n))^\# = \sum_{i=1}^n \left(\frac{\partial F}{\partial x_i} \circ U\right) U_i^\# \tag{15}$$

The gradient operator is a useful tool when computing Γ because it is linear, whereas the carré du champ operator is bilinear.

Let $U \in \mathbb{D}$, by (14) and Definition 1 (item 4) $\widetilde{\mathbb{E}}\overline{\mathbb{E}}[U^\#] = \widetilde{\mathbb{E}}\Gamma[U] = 2\mathcal{E}(U,U)$ and the gradient operator is closed in the sense that $\mathbb{D}$ equipped with the norm $|\,|_{\mathbb{D}}$ is complete.

In this paper, we will use frequently the two following straightforward lemmas:

Lemma 1 (Chain rules for a product of random variables)

Let U and $V \in \mathbb{D}$ then

$$\begin{aligned} (UV)^{\#} &= UV^{\#} + VU^{\#} && (16) \\ \Gamma[UV] &= U^2\Gamma[V] + V^2\Gamma[U] + 2UV\Gamma[U,V] && (17) \end{aligned}$$

Moreover, if U and $V \in \mathcal{DA}$, then

$$\mathcal{A}[UV] = U\mathcal{A}[V] + V\mathcal{A}[U] + \Gamma[U,V] \quad (18)$$

Proof: (16) follows from (15), (17) follows from (7) and (18) follows from (9). □

Lemma 2 (Expectation)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, we will denote $\mathbb{E}$ the conditional expectation w.r.t. $\mathbb{P}$. Let $U \in \mathbb{D} \subset L^2\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}\right)$ and $V \in \mathcal{DA}$ then one has

$$\begin{aligned} \mathbb{E}[U^{\#}] &= U^{\#} \\ \mathbb{E}[\mathcal{A}[V]] &= \mathcal{A}[V]. \end{aligned}$$

Let $U, V \in \mathbb{D}$ then one has

$$\mathbb{E}[\Gamma[U,V]] = \Gamma[U,V].$$

Let $V \in L^2\left(\Omega \times \widetilde{\Omega}, \mathcal{F} \otimes \widetilde{\mathcal{F}}, \mathbb{P} \otimes \widetilde{\mathbb{P}}\right)$ such that $V \in \mathbb{D}$ and $\mathbb{E}[V] \in \mathbb{D}$, one has:

$$(\mathbb{E}[V])^{\#} = \mathbb{E}[V^{\#}].$$

Let $V \in L^2\left(\Omega \times \widetilde{\Omega}, \mathcal{F} \otimes \widetilde{\mathcal{F}}, \mathbb{P} \otimes \widetilde{\mathbb{P}}\right)$ such that $V \in \mathbb{D}\mathcal{A}$ and $\mathbb{E}[V] \in \mathbb{D}\mathcal{A}$, one has:

$$\mathcal{A}[\mathbb{E}[V]] = \mathbb{E}[\mathcal{A}[V]].$$

Proof: Let $U \in L^1\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}\right)$ then $\mathbb{E}[U] = U$. So the first three equalities follow from the fact that $U^{\#} \in L^2(\widetilde{\mathbb{P}}, \mathcal{H})$, $\mathcal{A}[V] \in L^2\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}\right)$ and $\Gamma[U,V] \in L^1\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}\right)$. For the forth equation we use the fact that the gradient operator is closed and we can exchange the gradient operator and the integral sign. The proof of this fact proceeds by an approximation of the integral by a sum, then we apply the gradient operator and finally we take the limit using the closeness of the gradient operator, see for instance Bouleau [4] section VI.2. Lemma VI.8. The proof of the last equation is similar. □

We observe that the error theory based on Dirichlet forms restricts its analysis to the study of the first two orders of error propagation, i.e. the bias and the variance. This fact is justified

by the lack of information on the parameter uncertainties, generally given by the Fischer information matrix, that is often quite limited. The study of higher orders is a very difficult problem for both mathematical and practical reasons. From the mathematical point of view, it would be necessary to study chain rules of higher orders, involving skewness and kurtosis, and to prove that the related operators are closed in a suitable space. However, the crucial problem remains to have sufficiently accurate estimates for the higher order uncertainties. This statistical obstacle cannot be overcome easily. Therefore, it seems reasonable to restrict the study to the two first orders.

3 Financial setup

We consider a continuous time financial model with K credit issuers and one synthetic asset referred to as the benchmark. It represents the global evolution of the credit market. Each credit issuer is characterized by its spread over the risk free rate. The uncertainty is represented by a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions. First, we define on this space a $\mathbb{F}$-adapted, continuous-time, two-state valued Markov chain, $Y = (Y_t)_{t\geq 0}$. The state space of the Markov chain is equal to $\{g, b\}$.

Let $(W(t))_{t\geq 0} = (W_0(t), \ldots, W_K(t))_{t\geq 0}$ be a standard $(\mathbb{F}, \mathbb{P})$-Brownian motion of dimension $K+1$. Let us define the $K+1$-dimensional stochastic process $(X(t))_{t\geq 0}$, with $X(t) := (X_k(t))_{0\leq k\leq K}$. The process X_0 represents the spread of the benchmark, the drift of which μ_0 is assumed to be a measurable function of the Markov chain Y:

$$\begin{aligned} \frac{dX_0(t)}{X_0(t)} &= \mu_0(Y_t)dt + \sigma_0 dW_0(t), \\ X_0(0) &= x_0 \end{aligned}$$

For all $k \in \{1, \ldots, K\}$, $\Theta_k = (\Theta_k(t))_{t\geq 0}$ are $\mathbb{F}$-adapted processes (thus influenced by Y), representing the unknown drift of X_k and:

$$\begin{aligned} \frac{dX_k(t)}{X_k(t)} &= \Theta_k(t)dt + \sigma_k \left(\rho_k dW_0(t) + \sqrt{1-\rho_k^2} \sum_{j=1}^{K} L_{k,j} dW_j(t) \right) \\ X_k(0) &= x_k \end{aligned}$$

where $L := [L_{i,j}]$ is a $K \times K$ lower triangular matrix, such that $C := L \cdot L'$ is a (non-degenerated) correlation matrix. It means that $C := (C_{i,j})_{1\leq i,j\leq K}$ is a symmetric, semi-definite positive matrix with unit diagonal coefficients. We also denote $Z_k^{\perp}(t) = \sum_{j=1}^{K} L_{k,j} W_j(t)$. It is a standard $(\mathbb{F}, \mathbb{P})$-Brownian motion of dimension 1.

We introduce $\mathbb{G} := \{\mathcal{G}_t\}_{t\geq 0}$ and $\mathbb{G}^0 := \{\mathcal{G}_t^0\}_{t\geq 0}$, the right continuous, complete filtrations generated respectively by the following processes:

$$\begin{aligned} \mathcal{G}_t &= \sigma\{X_0(s), X_1(s), \ldots, X_K(s) | s \leq t\} \\ \mathcal{G}_t^0 &= \sigma\{X_0(s) | s \leq t\} \end{aligned}$$

Now we state the following assumptions which will prevail throughout the paper.

Assumption 1 *We assume that W_0 and Y are independent. We assume that the processes $\mu_0(Y)$ and Θ_k are uniformly bounded and measurable. Finally $\sigma_k > 0$ for $k = 0, \ldots, K$ and $-1 < \rho_k < 1$ for $k = 1, \ldots, K$.*

Assumption 2 *Let $p_t := \mathbb{P}\left[\{Y_t = b\} \mid \mathcal{G}_t^0\right]$, for any $t \geq 0$. We assume that*

$$p_t = \mathbb{P}\left[\{Y_t = b\} \mid \mathcal{G}_t\right].$$

In [2], we have proved the following proposition.

Proposition 2 *Under Assumptions 1 and 2,*

$$\begin{aligned}
\frac{dX_0(t)}{X_0(t)} &= \mathbb{E}\left\{\mu_0(Y_t) \mid \mathcal{G}_t^0\right\} dt + \sigma_0 d\widehat{W}_0(t) \\
\frac{dX_k(t)}{X_k(t)} &= \left(\mathbb{E}\left\{\mu_0(Y_t) \mid \mathcal{G}_t^0\right\} + e_k(t)\right) dt + \sigma_k \left(\rho_k d\widehat{W}_0(t) + \sqrt{1-\rho_k^2} d\widehat{Z}_k^{\perp}(t)\right), \quad (19)
\end{aligned}$$

where

$$\begin{aligned}
e_k(t) &= \mathbb{E}\left\{\Theta_k(t) | \mathcal{G}_t\right\} - \mathbb{E}\{\mu_0(Y_t) | \mathcal{G}_t^0\} \\
\widehat{W}_0(t) &= W_0(t) + \frac{1}{\sigma_0} \int_0^t \left(\mu_0(Y_s) - \mathbb{E}\left\{\mu_0(Y_s) \mid \mathcal{G}_s^0\right\}\right) ds \\
\widehat{Z}_k^{\perp}(t) &= Z_k^{\perp}(t) + \frac{1}{\sigma_k \sqrt{1-\rho_k^2}} \int_0^t \left(\Theta_k(s) - \mathbb{E}\left\{\mu_0(Y_s) \mid \mathcal{G}_s^0\right\} - e_k(t)\right) ds \\
&\quad - \frac{\rho_k}{\sqrt{1-\rho_k^2}} \int_0^t \frac{\mu_0(Y_s) - \mathbb{E}\{\mu_0(Y_s) | \mathcal{G}_s^0\}}{\sigma_0} ds.
\end{aligned}$$

$\widehat{W}_0$ is a $(\mathbb{G}^0, \mathbb{P})$-Brownian motion and for $k = 1, \ldots, K$, $\widehat{Z}_k^{\perp}$ is a $(\mathbb{G}, \mathbb{P})$-Brownian motion. Moreover $\mathbb{E}\left\{\mu_0(Y_t) \mid \mathcal{G}_t^0\right\} = (\mu_0(b) - \mu_0(g))p_t + \mu_0(g)$, where $(p_t)_{t \geq 0}$ is solution of

$$dp_t = \left[-\left(\lambda_b + \lambda_g\right) p_t + \lambda_b\right] dt + \frac{\mu_0(b) - \mu_0(g)}{\sigma_0} p_t(1-p_t) d\widehat{W}_0(t).$$

The $e_k(t)$ are interpreted as the $\mathcal{G}_t$-adapted views of the economic agent. Let

$$\epsilon_k = \frac{1}{T} \int_0^T e_k(t) dt. \quad (20)$$

Then from (19) and (20), we obtain that

$$X_k(T) = x_k e^{\int_0^T \left(\mathbb{E}\left\{\mu_0(Y_t) \mid \mathcal{G}_t^0\right\}\right) dt + T\epsilon_k - \frac{\sigma_k}{2} T + \sigma_k \int_0^T \left(\rho_k d\widehat{W}_0(t) + \sqrt{1-\rho_k^2} d\widehat{Z}_k^{\perp}(t)\right)}, \quad (21)$$

In [2] the ϵ_k have been in a first time assumed to be constant. Now they are estimated with some uncertainty and we will perform error calculus on them.

We assume that it is possible to invest in the benchmark and on each credit issuer through a debt product - a bond or a CDS - the price of which at time $t \geq 0$ is given by $P^{(k)}(t, X_k(t))$, $k \in \{0, \ldots, K\}$. Moreover, let us denote by $Cap(t)$ the deterministic capitalisation factor at time t for the risk free rate.

Assumption 3 *The mappings $P^{(k)}(\cdot, \cdot)$ from $\mathbb{R}^+ \times \mathbb{R}^+$ to $(0, \infty)$ are, at least, twice continuously differentiable.*

We will denote by $\dot{P}_1^{(k)}(\cdot, \cdot)$ its first order derivative with respect to the first variable, by $\dot{P}_2^{(k)}(\cdot, \cdot)$ its first order derivative with respect to the second variable, and by $\ddot{P}_2^{(k)}(\cdot, \cdot)$ its second order derivative with respect to the second variable.

The P&L, at time $t \leq T$, of a buy and hold position on the asset $k \in \{0, \ldots, K\}$ is given by

$$\mathrm{P\&L}_k(t, X_k(t)) = P^{(k)}\left(t, X_k(t)\right) - P^{(k)}\left(0, x_k\right) \times Cap(t). \tag{22}$$

In [2], we have considered two alternative portfolio representations. The first one uses the allocation on the assets to outperform the benchmark (benchmarked allocation): this corresponds to the case $\zeta = 1$. The second one is a simple allocation on the K assets with no benchmark reference (total return allocation) and correspond to the case $\zeta = 0$. For any $\pi \in \mathbb{R}^K$:

$$G(\pi, t, X(t)) := \sum_{k=1}^{K} \mathrm{P\&L}_k(t, X_k(t)) \times \pi_k - \zeta \mathrm{P\&L}_0(t, X_0(t)) \tag{23}$$

Define for any $(k, j) \in \{0, \ldots, K\}^2$,

$$Cov[k, j] := \mathbb{E}\left\{\mathrm{P\&L}_k(t)\mathrm{P\&L}_j(t)\right\} - M_k M_j,$$

where for any $k \in \{0, \ldots, K\}$, $M_k := \mathbb{E}\left\{\mathrm{P\&L}_k(t)\right\}$. Note that for ease of notation we drop the time indexation. We also set

$$M := (M_k)_{1 \leq k \leq K}, \;\; Cov := (Cov[i, j])_{(i,j) \in \{1,\ldots,K\}^2} \text{ and } Cov[0] := (Cov[0, k])_{k \in \{1,\ldots,K\}}.$$

Then $\mathbb{V}\{G(\pi, T)\} = \pi' \cdot Cov \cdot \pi - 2\zeta\pi' \cdot Cov[0] + \zeta^2 Cov[0, 0]$ and $\mathbb{E}\{G(\pi, T)\} = \pi' \cdot M - \zeta M_0$. So the mean-variance program $(\mathcal{P})$ solved by the investor can be written

$$(\mathcal{P}): \begin{cases} \min\limits_{\pi \in \mathbb{R}^K} \dfrac{1}{2}\left(\pi' \cdot Cov \cdot \pi - \zeta\pi' \cdot Cov[0]\right\} \\ \text{s.t. } \pi' \cdot M \geq r + \zeta M_0 \\ \text{and } \pi' \cdot \mathbb{I} = 1 \end{cases}$$

where $\mathbb{I}$ is the element of $\mathbb{R}^K$ with all its components equal to 1 and $r > 0$ is the return budget constraint.

Before solving the optimization program, we define the following quantities:

$$\begin{array}{lll} z_1 := \mathbb{I}' \cdot Cov^{-1} \cdot \mathbb{I} & z_2 := M' \cdot Cov^{-1} \cdot M & z_3 := M' \cdot Cov^{-1} \cdot \mathbb{I} \\ z_4 := \mathbb{I}' \cdot Cov^{-1} \cdot Cov[0] & z_5 := M' \cdot Cov^{-1} \cdot Cov[0] & z_6 := 1 - \zeta z_4 \\ z_7 = r + \zeta M_0 - \zeta z_5 & & z_{10} := (z_1 z_2 - z_3^2)^{-1} \end{array} \tag{24}$$

Proposition 3 *Assume that the following condition holds*

$$z_1 z_7 > z_3 z_6, \tag{25}$$

and that M is not co-linear to $\mathbb{I}$. Then, the solution of $(\mathcal{P})$ is given by

$$\pi^* = Cov^{-1} \cdot (\zeta Cov[0] + \mu M - \nu \mathbb{I}) . \tag{26}$$

where

$$\mu = \frac{z_7 z_1 - z_6 z_3}{z_1 z_2 - z_3^2} \quad and \quad \nu = \frac{z_7 z_3 - z_6 z_2}{z_1 z_2 - z_3^2},$$

Remark 2 *In [2] we express* (26) *as a separation in two funds expressions (see (18) in Proposition 2). Here π^* is express with the Lagrange multipliers for ease of computation.*

4 Error Calculus on the Optimal Allocation

We define the following quantities, for $1 \leq k \leq K$ and $0 \leq j \leq K$:

$$\begin{array}{rcl} \phi_k & := & T\mathbb{E}^{\mathbb{P}} \left\{ X_k(T) \dot{P}_2^{(k)}(T, X_k(T)) \right\} \\ \psi_k & := & T^2\mathbb{E}^{\mathbb{P}} \left\{ X_k^2(T) \ddot{P}_2^{(k)}(T, X_k(T)) \right\} \\ \Phi_{k,j} & := & T\mathbb{E}^{\mathbb{P}} \left\{ X_k(T) \dot{P}_2^{(k)}(T, X_k(T)) P\&L_j(T, X_j(T)) \right\} \\ \Psi_{k,j} & := & T^2\mathbb{E}^{\mathbb{P}} \left\{ X_k^2(T) \ddot{P}_2^{(k)}(T, X_k(T)) P\&L_j(T, X_j(T)) \right\} \\ \Upsilon_{kj} & := & T^2\mathbb{E}^{\mathbb{P}} \left\{ X_k(T) X_j(T) \dot{P}_2^{(k)}(T, X_k(T)) \dot{P}_2^{(j)}(T, X_j(T)) \right\} \end{array}$$

The first result provides the sensitivity analysis for the expected returns M.

Proposition 4 (Sensitivity of the expected value of P&L) *For any $1 \leq k, j \leq K$,*

we have

$$
\begin{aligned}
P\&L_k^{\#}(T, X_k) &= \dot{P}_2^{(k)}(T, X_k(T))TX_k(T)\epsilon_k^{\#} && \text{(27a)}\\
M_k^{\#} &= \phi_k \epsilon_k^{\#} && \text{(27b)}\\
\Gamma[M_k, M_j] &= \phi_k \phi_j \Gamma[\epsilon_k, \epsilon_j] && \text{(27c)}\\
\Gamma[P\&L_k(T, X_k), P\&L_j(T, X_j)] &= \dot{P}_2^{(k)}(T, X_k(T))\dot{P}_2^{(j)}(T, X_j(T)) && \text{(27d)}\\
&\quad \times T^2 X_k(T) X_j(T) \Gamma[\epsilon_k, \epsilon_j]\\
\mathcal{A}[P\&L_k(T, X_k)] &= \dot{P}_2^{(k)}(T, X_k(T))TX_k(T)\mathcal{A}[\epsilon_k] && \text{(27e)}\\
&\quad + \frac{1}{2}\dot{P}_2^{(k)}(T, X_k(T))T^2 X_k(T)\Gamma[\epsilon_k]\\
&\quad + \frac{1}{2}\ddot{P}_2^{(k)}(T, X_k(T))T^2 X_k^2(T)\Gamma[\epsilon_k]\\
\mathcal{A}[M_k] &= \phi_k \mathcal{A}[\epsilon_k] + \left(\frac{1}{2}T\phi_k + \frac{1}{2}\psi_k\right)\Gamma[\epsilon_k] && \text{(27f)}
\end{aligned}
$$

Proof: From (21) and (15), we have $X_k^{\#}(T) = TX_k(T)\epsilon_k^{\#}$. Then (15) again gives (27a). Equation (27b) comes from Lemma 2. For (27c) we use (27b) and (14). Moreover, (27d) follows from (14) and (27a). Finally a direct application of bias chain rule (9) gives (27e) and applying Lemma 2 we find Equation (27f). □

We state the following Proposition about the sensitivity of the variance covariance matrix.

Proposition 5 (Sensitivity of the variance covariance matrix) *For any* $1 \leq k, j \leq K$*, we have*

$$
\begin{aligned}
Cov_{kj}^{\#} &= \delta_{k,j}^C \epsilon_k^{\#} + \delta_{j,k}^C \epsilon_j^{\#} && \text{(28)}\\
Cov[0]_j^{\#} &= \delta_{j,0}^C \epsilon_j^{\#} && \text{(29)}\\
\Gamma[Cov_{kj}, Cov_{il}] &= \delta_{k,j}^C \delta_{i,l}^C \Gamma[\epsilon_k, \epsilon_i] + \delta_{j,k}^C \delta_{i,l}^C \Gamma[\epsilon_j, \epsilon_i] && \text{(30)}\\
&\quad + \delta_{k,j}^C \delta_{l,i}^C \Gamma[\epsilon_k, \epsilon_l] + \delta_{j,k}^C \delta_{l,i}^C \Gamma[\epsilon_j, \epsilon_l]\\
\mathcal{A}[Cov_{kj}] &= \delta_{k,j}^C \mathcal{A}[\epsilon_k] + \delta_{j,k}^C \mathcal{A}[\epsilon_j] && \text{(31)}\\
&\quad + \alpha_{k,j}^C \Gamma[\epsilon_k] + \alpha_{j,k}^C \Gamma[\epsilon_j] + \beta_{kj}^C \Gamma[\epsilon_k, \epsilon_j]\\
\mathcal{A}[Cov[0]_k] &= \delta_{k,0}^C \mathcal{A}[\epsilon_k] + \alpha_{k,0}^C \Gamma[\epsilon_k] && \text{(32)}
\end{aligned}
$$

where we set:

$$
\begin{aligned}
\delta_{k,j}^C &:= \Phi_{k,j} - \phi_k M_j\\
\alpha_{k,j}^C &:= \frac{1}{2}\left[T\Phi_{k,j} + \Psi_{k,j} - (T\phi_k + \psi_k) M_j\right]\\
\beta_{kj}^C &:= \Upsilon_{kj} - \phi_k \phi_j
\end{aligned}
$$

with $1 \leq k \leq K$ *and* $0 \leq j \leq K$*.*

Proof: This result is a direct consequence of Lemmas 1, 2 and Proposition 4. We recall that as X_0 does not depend on $\{\epsilon_k\}_{k=1,\dots K}$ it is unaffected by drift estimation uncertainty. □

Our optimal allocation depends on the inverse of the variance covariance matrix, see (26).

Proposition 6 (Sensitivity of the variance covariance matrix) *For any $1 \leq i,l,m,n \leq K$, we have*

$$\left(Cov_{il}^{-1}\right)^{\#} = \sum_k \delta_{k,il}^{Cov^{-1}} \epsilon_k^{\#} \tag{33a}$$

$$\Gamma[Cov_{il}^{-1}, Cov_{mn}^{-1}] = \sum_{kj} \delta_{k,il}^{Cov^{-1}} \delta_{j,mn}^{Cov^{-1}} \Gamma[\epsilon_k, \epsilon_j] \tag{33b}$$

$$\mathcal{A}[Cov_{il}^{-1}] = \sum_k \left(\delta_{k,il}^{Cov^{-1}} \mathcal{A}[\epsilon_k] + \alpha_{k,il}^{Cov^{-1}} \Gamma[\epsilon_k]\right) + \sum_{kj} \beta_{kj,il}^{Cov^{-1}} \Gamma[\epsilon_k, \epsilon_j] \tag{33c}$$

where

$$\begin{aligned}
\delta_{k,il}^{Cov^{-1}} &:= -\sum_m \left(Cov_{ik}^{-1} Cov_{ml}^{-1} + Cov_{im}^{-1} Cov_{kl}^{-1}\right) \delta_{k,m}^{C} \\
\alpha_{k,il}^{Cov^{-1}} &:= -\sum_m \left(Cov_{ik}^{-1} Cov_{ml}^{-1} + Cov_{im}^{-1} Cov_{kl}^{-1}\right) \alpha_{k,m}^{C} \\
\beta_{kj,il}^{Cov^{-1}} &:= -Cov_{ik}^{-1} Cov_{jl}^{-1} \beta_{kj}^{C} - \sum_m \left(Cov_{ik}^{-1} \delta_{k,m}^{C} \delta_{j,ml}^{Cov^{-1}} + Cov_{im}^{-1} \delta_{j,m}^{C} \delta_{k,jl}^{Cov^{-1}}\right)
\end{aligned}$$

Proof: Note that $Cov\ Cov^{-1} = \mathbb{I}$, where $\mathbb{I}$ (the identity matrix) is unaffected by the uncertainty on the coefficients of Cov. Using (16), we find $0 = \mathbb{I}^{\#} = Cov^{\#} Cov^{-1} + Cov (Cov^{-1})^{\#}$. Using (28), (33a) follows. (33b) follows from (33a) and (14). From (18), we get that

$$0 = \mathcal{A}[\mathbb{I}] = \mathcal{A}[CovCov^{-1}] = \mathcal{A}[Cov]\, Cov^{-1} + Cov\, \mathcal{A}[Cov^{-1}] + \Gamma[Cov, Cov^{-1}].$$

Thus, using (14) we get that

$$\mathcal{A}[Cov^{-1}] = -Cov^{-1}\, \mathcal{A}[Cov]\, Cov^{-1} - Cov^{-1}\, \overline{\mathbb{E}}\left(Cov^{\#} (Cov^{-1})^{\#}\right)$$

and we conclude using (28), and (33a). □

We turn now to study the sensitivity of z_i, see relations (24), we remark that only the sensitivities of z_1 to z_5 need to be computed since z_6 and z_7 are linear combination of z_1 to z_5.

Proposition 7 (Sensitivity of z_i) *For $a = 1, \dots, 5$, we have*

$$\begin{aligned}
z_a^{\#} &= \sum_k \delta_k^{z_a} \epsilon_k^{\#} \\
\mathcal{A}[z_a] &= \sum_k \delta_k^{z_a} \mathcal{A}[\epsilon_k] + \sum_k \alpha_k^{z_a} \Gamma[\epsilon_k] + \sum_{kj} \beta_{kj}^{z_a} \Gamma[\epsilon_k, \epsilon_j]
\end{aligned}$$

where all sum are taken from 1 *to* K *and*

$$\delta_k^{z_1} := \sum_{il} \delta_{k,il}^{Cov^{-1}} \qquad \alpha_k^{z_1} := \sum_{il} \alpha_{k,il}^{Cov^{-1}} \qquad \beta_{kj}^{z_1} := \sum_{il} \beta_{kj,il}^{Cov^{-1}}$$

$$\begin{aligned}
\delta_k^{z_2} &:= \sum_{il} M_i \delta_{k,il}^{Cov^{-1}} M_l + 2 \sum_i \phi_k \, Cov_{ki}^{-1} M_i \\
\alpha_k^{z_2} &:= \sum_{il} M_i \alpha_{k,il}^{Cov^{-1}} M_l + (T\phi_k + \psi_k) \sum_i M_i Cov_{ik}^{-1} \\
\beta_{kj}^{z_2} &:= \sum_{il} M_i \beta_{kj,il}^{Cov^{-1}} M_l + 2 \sum_i \phi_j M_i \delta_{k,ij}^{Cov^{-1}} + Cov_{jk}^{-1} \phi_j \phi_k \\
\delta_k^{z_3} &:= \sum_i \phi_k \, Cov_{ki}^{-1} + \sum_{il} \delta_{k,il}^{Cov^{-1}} \, M_i \\
\alpha_k^{z_3} &:= \sum_{il} M_i \alpha_{k,il}^{Cov^{-1}} + \sum_i \frac{1}{2} (T\phi_k + \psi_k) Cov_{ki}^{-1} \\
\beta_{kj}^{z_3} &:= \sum_{il} M_i \beta_{kj,il}^{Cov^{-1}} + \sum_i \phi_k \delta_{j,ki}^{Cov^{-1}} \\
\delta_k^{z_4} &:= \sum_{il} \delta_{k,il}^{Cov^{-1}} Cov[0]_l + \sum_i \delta_{k,0}^{C} Cov_{ik}^{-1} \\
\alpha_k^{z_4} &:= \sum_{il} \alpha_{k,il}^{Cov^{-1}} Cov[0]_l + \sum_i \alpha_{k,0}^{C} Cov_{ik}^{-1} \\
\beta_{kj}^{z_4} &:= \sum_{il} \beta_{kj,il}^{Cov^{-1}} Cov[0]_l + + \sum_i \delta_{j,0}^{C} \delta_{k,ij}^{Cov^{-1}} \\
\delta_k^{z_5} &:= \sum_{il} M_i \delta_{k,il}^{Cov^{-1}} Cov[0]_l + \sum_i M_i Cov_{ik}^{-1} \delta_{k,0}^{C} + \sum_i \phi_k Cov_{ki}^{-1} Cov[0]_i \\
\alpha_k^{z_5} &:= \sum_{il} M_i \alpha_{k,il}^{Cov^{-1}} Cov[0]_l + \sum_i M_i Cov_{ik}^{-1} \alpha_{k,0}^{C} + \frac{1}{2} (T\phi_k + \psi_k) \sum_i Cov_{ki}^{-1} Cov[0]_i \\
\beta_{kj}^{z_5} &:= \sum_{il} M_i \beta_{kj,il}^{Cov^{-1}} Cov[0]_l + \sum_i M_i \delta_{j,0}^{C} \delta_{k,ij}^{Cov^{-1}} + \sum_i \delta_{k,ji}^{Cov^{-1}} \phi_j Cov[0]_i + Cov_{jk}^{-1} \phi_j \delta_{k,0}^{C}
\end{aligned}$$

Proof: The proof is similar to the proof of Proposition 6 and is based on Propositions 4, 5, 6 and Lemma 1 (recall also the definitions (24)). □

Remark 3 *From (24),* $z_6^{\#} = -\zeta z_4^{\#}$ *and* $\mathcal{A}[z_6] = -\zeta \mathcal{A}[z_4]$ *and we set*

$$\delta_k^{z_6} := -\zeta \delta_k^{z_4} \qquad \alpha_k^{z_6} := -\zeta \alpha_k^{z_4} \qquad \beta_{kj}^{z_6} := -\zeta \beta_{kj}^{z_4}$$

Similarly, $z_7^{\#} = -\zeta z_5^{\#}$ *and* $\mathcal{A}[z_7] = -\zeta \mathcal{A}[z_5]$ *and we set*

$$\delta_k^{z_7} := -\zeta \delta_k^{z_5} \qquad \alpha_k^{z_7} := -\zeta \alpha_k^{z_5} \qquad \beta_{kj}^{z_7} := -\zeta \beta_{kj}^{z_5}$$

The following corollary comes directly from Proposition 7 together with (9).

Corollary 1 (Sensitivity of μ and ν) *We have*

$$\begin{aligned}
\mu^{\#} &= \sum_k \delta_k^{\mu} \epsilon_k^{\#} \\
\nu^{\#} &= \sum_k \delta_k^{\nu} \epsilon_k^{\#} \\
\mathcal{A}[\mu] &= \sum_k \delta_k^{\mu} \mathcal{A}[\epsilon_k] + \sum_k \alpha_k^{\mu} \Gamma[\epsilon_k] + \sum_{k,j} (\beta_{kj}^{\mu} + \chi_{kj}^{\mu}) \Gamma[\epsilon_k, \epsilon_j] \\
\mathcal{A}[\nu] &= \sum_k \delta_k^{\nu} \mathcal{A}[\epsilon_k] + \sum_k \alpha_k^{\nu} \Gamma[\epsilon_k] + \sum_{kj} (\beta_{kj}^{\nu} + \chi_{kj}^{\nu}) \Gamma[\epsilon_k, \epsilon_j]
\end{aligned}$$

where

$$\begin{aligned}
\delta_k^{\mu} &:= \sum_{a=1,\dots,5} \frac{\partial \mu}{\partial z_a} \delta_k^{z_a} \qquad \alpha_k^{\mu} := \sum_{a=1,\dots,5} \frac{\partial \mu}{\partial z_a} \alpha_k^{z_a} \qquad \beta_{kj}^{\mu} := \sum_{a=1,\dots,5} \frac{\partial \mu}{\partial z_a} \beta_{kj}^{z_a} \\
\delta_k^{\nu} &:= \sum_{a=1,\dots,5} \frac{\partial \nu}{\partial z_a} \delta_k^{z_a} \qquad \alpha_k^{\nu} := \sum_{a=1,\dots,5} \frac{\partial \nu}{\partial z_a} \alpha_k^{z_a} \qquad \beta_{kj}^{\nu} := \sum_{a=1,\dots,5} \frac{\partial \nu}{\partial z_a} \beta_{kj}^{z_a} \\
\chi_{kj}^{\mu} &:= \frac{1}{2} \sum_{a,b=1,\dots,5} \frac{\partial^2 \mu}{\partial z_a \partial z_b} \delta_k^{z^a} \delta_j^{z^b} \qquad \chi_{kj}^{\nu} := \frac{1}{2} \sum_{a,b=1,\dots,5} \frac{\partial^2 \nu}{\partial z_a \partial z_b} \delta_k^{z^a} \delta_j^{z^b}
\end{aligned}$$

All derivatives are listed in appendix for sake of completeness.

We are now in position to state the result giving the sensitivity analysis of the optimal allocation π^*.

Theorem 1 (Sensitivity of the optimal strategy) *Let π^* be the solution of Program $(\mathcal{P})$, given by Proposition 3. We get that:*

$$\begin{aligned}
(\pi_i^*)^{\#} &= \sum_k \delta_{k,i}^{\pi} \epsilon_k^{\#} \\
\Gamma[\pi_i^*, \pi_l^*] &= \sum_{kj} \delta_{k,i}^{\pi} \delta_{j,l}^{\pi} \Gamma[\epsilon_k, \epsilon_j] \\
\Gamma[\pi_i^*, M_l] &= \sum_k \delta_{k,i}^{\pi} \phi_l \Gamma[\epsilon_k, \epsilon_l] \\
\mathcal{A}[\pi_i^*] &= \sum_k \delta_{k,i}^{\pi} \mathcal{A}[\epsilon_k] + \sum_k \alpha_{k,i}^{\pi} \Gamma[\epsilon_k] + \sum_{kj} \beta_{kj,i}^{\pi} \Gamma[\epsilon_k, \epsilon_j]
\end{aligned}$$

where

$$\begin{aligned}
\delta^{\pi}_{k,i} &:= \sum_l \delta^{Cov^{-1}}_{k,il} \left(\zeta Cov[0]_l + \mu M_l - \nu\right) && (37a)\\
&\quad + \sum_l Cov^{-1}_{il}\left(\delta^{\mu}_k M_l - \delta^{\nu}_k\right) + Cov^{-1}_{ik}\left(\zeta\delta^{C}_{k,0} + \mu\phi_k\right)\\
\alpha^{\pi}_{k,i} &:= \sum_l \alpha^{Cov^{-1}}_{k,il}\left(\zeta Cov[0]_l + \mu M_l - \nu\right) + \sum_l Cov^{-1}_{il}\left(M_l\alpha^{\mu}_k - \alpha^{\nu}_k\right) && (37b)\\
&\quad + Cov^{-1}_{ik}\left[\zeta\alpha^{C}_{k,0} + \frac{1}{2}\mu\left(T\phi_k + \psi_k\right)\right]\\
\beta^{\pi}_{kj,i} &:= \sum_l \beta^{Cov^{-1}}_{kj,il}\left(\zeta Cov[0]_l + \mu M_l - \nu\right) + \sum_l \delta^{Cov^{-1}}_{k,il}\left(\delta^{\mu}_j M_l - \delta^{\nu}_j\right) && (37c)\\
&\quad + \sum_l Cov^{-1}_{il}\left(M_l\chi^{\mu}_{kj} + M_l\beta^{\mu}_{kj} - \chi^{\nu}_{kj} - \beta^{\nu}_{kj}\right) + \delta^{Cov^{-1}}_{k,ij}\left(\zeta\delta^{C}_{j,0} + \mu\phi_j\right)\\
&\quad + Cov^{-1}_{ik}\phi_k\delta^{\mu}_j
\end{aligned}$$

Proof: Again the proof is similar to the previous ones. We apply Propositions 4, 5, 6, 7, Corollary 1, Lemma 1 together with (9) and (14). □

Let us provide a corollary the same analysis for the optimal return.

Corollary 2 *With the notations of Theorem 1, let R^* be the optimal return defined by $R^* := M'\pi^*$. Then we get that:*

$$\begin{aligned}
(R^*)^{\#} &= \sum_k \left(\phi_k\pi^*_k + \sum_i \delta^{\pi}_{k,i}M_i\right)\epsilon^{\#}_k\\
\Gamma[R^*] &= \sum_{k,j}\left(\phi_k\pi^*_k + \sum_i \delta^{\pi}_{k,i}M_i\right)\left(\phi_j\pi^*_j + \sum_i \delta^{\pi}_{j,i}M_i\right)\Gamma[\epsilon_k,\epsilon_j]\\
\mathcal{A}[R^*] &= \sum_k\left(\phi_k\pi^*_k + \sum_i \delta^{\pi}_{k,i}M_i\right)\mathcal{A}[\epsilon_k] + \sum_k\left(\sum_i \alpha^{\pi}_{k,i}M_i + \frac{1}{2}(T\phi_k + \psi_k)\pi^*_k\right)\Gamma[\epsilon_k]\\
&\quad + \sum_{k,j}\left(\delta^{\pi}_{k,j}\phi_j + \sum_i \beta^{\pi}_{kj,i}M_i\right)\Gamma[\epsilon_k,\epsilon_j]
\end{aligned}$$

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A Explicit derivatives of ν and μ

$$\begin{aligned}
\frac{\partial \nu}{\partial z_1} &= -\nu z_2 z_{10} \\
\frac{\partial \nu}{\partial z_2} &= -\mu z_3 z_{10} \\
\frac{\partial \nu}{\partial z_3} &= \nu z_3 z_{10} + \mu z_2 z_{10} \\
\frac{\partial \nu}{\partial z_4} &= \zeta z_2 z_{10} \\
\frac{\partial \nu}{\partial z_5} &= -\zeta z_3 z_{10}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mu}{\partial z_1} &= -\nu z_3 z_{10} \\
\frac{\partial \mu}{\partial z_2} &= -\mu z_1 z_{10} \\
\frac{\partial \mu}{\partial z_3} &= \nu z_1 z_{10} + \mu z_3 z_{10} \\
\frac{\partial \mu}{\partial z_4} &= \zeta z_3 z_{10} \\
\frac{\partial \mu}{\partial z_5} &= -\zeta z_1 z_{10}
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2\nu}{\partial z_1^2} &= 2\nu z_2^2 z_{10}^2 \\
\frac{\partial^2\nu}{\partial z_1 \partial z_2} &= (\nu z_1 + \mu z_3) z_2 z_{10}^2 - \nu z_{10} \\
\frac{\partial^2\nu}{\partial z_1 \partial z_3} &= -(3\nu z_2 + \mu z_2) z_2 z_{10}^2 \\
\frac{\partial^2\nu}{\partial z_1 \partial z_4} &= -\zeta z_2^2 z_{10}^2 \\
\frac{\partial^2\nu}{\partial z_1 \partial z_5} &= \zeta z_2 z_3 z_{10}^2 \\
\frac{\partial^2\nu}{\partial z_2^2} &= 2\mu z_1 z_3 z_{10}^2 \\
\frac{\partial^2\nu}{\partial z_2 \partial z_3} &= -(\nu z_1 + 3\mu z_3) z_3 z_{10}^2 - \mu z_{10} \\
\frac{\partial^2\nu}{\partial z_2 \partial z_4} &= -\zeta z_3^2 z_{10}^2 \\
\frac{\partial^2\nu}{\partial z_2 \partial z_5} &= \zeta z_1 z_3 z_{10}^2 \\
\frac{\partial^2\nu}{\partial z_3^2} &= [\nu(3z_3^2 + z_1 z_2) + 4\mu z_2 z_3] z_{10}^2 + \nu z_{10} \\
\frac{\partial^2\nu}{\partial z_3 \partial z_4} &= 2\zeta z_2 z_3 z_{10}^2 \\
\frac{\partial^2\nu}{\partial z_3 \partial z_5} &= -\zeta(z_3^2 + z_1 z_2) z_{10}^2 \\
\frac{\partial^2\nu}{\partial z_4^2} &= \frac{\partial^2\nu}{\partial z_5^2} = \frac{\partial^2\nu}{\partial z_4 \partial z_5} = 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2\mu}{\partial z_1^2} &= 2\nu z_2 z_3 z_{10}^2 \\
\frac{\partial^2\mu}{\partial z_1 \partial z_2} &= (\mu z_3 + \nu z_1) z_3 z_{10}^2 \\
\frac{\partial^2\mu}{\partial z_1 \partial z_3} &= -(3\nu z_3 + \mu z_2) z_3 z_{10}^2 - \nu z_{10} \\
\frac{\partial^2\mu}{\partial z_1 \partial z_4} &= -\zeta z_2 z_3 z_{10}^2 \\
\frac{\partial^2\mu}{\partial z_1 \partial z_5} &= \zeta z_3^2 z_{10}^2 \\
\frac{\partial^2\mu}{\partial z_2^2} &= 2\mu z_1^2 z_{10}^2 \\
\frac{\partial^2\mu}{\partial z_2 \partial z_3} &= -(3\mu z_3 + \nu z_1) z_1 z_{10}^2 \\
\frac{\partial^2\mu}{\partial z_2 \partial z_4} &= -\zeta z_1 z_3 z_{10}^2 \\
\frac{\partial^2\mu}{\partial z_2 \partial z_5} &= \zeta z_1^2 z_{10}^2 \\
\frac{\partial^2\mu}{\partial z_3^2} &= [2\mu(z_3^2 + z_1 z_2) + 4\nu z_1 z_3] z_{10}^2 \\
\frac{\partial^2\mu}{\partial z_3 \partial z_4} &= \zeta z_{10}(1 + 2 z_3^2 z_{10}) \\
\frac{\partial^2\mu}{\partial z_3 \partial z_5} &= -2\zeta z_1 z_3 z_{10}^2 \\
\frac{\partial^2\mu}{\partial z_4^2} &= \frac{\partial^2\mu}{\partial z_5^2} = \frac{\partial^2\mu}{\partial z_4 \partial z_5} = 0
\end{aligned}
$$