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## 3D-2D ASYMPTOTIC OBSERVATION FOR MINIMIZATION PROBLEMS ASSOCIATED WITH DEGENERATIVE ENERGY-COEFFICIENTS

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**ABSTRACT.** In this paper, a class of minimization problems, labeled by an index  $0 < h < 1$ , is considered. Each minimization problem is for a *free-energy*, motivated by the magnetics in 3D-ferromagnetic thin film, and in the context, the index  $h$  denotes the thickness of the observing film. The Main Theorem consists of two themes, which are concerned with the study of the solvability (existence of minimizers) and the 3D-2D asymptotic analysis for our minimization problems. These themes will be discussed under degenerative setting of the material coefficients, and such degenerative situation makes the energy-domain be variable with respect to  $h$ . In conclusion, assuming some restrictive conditions for the domain-variation, a definite association between our 3D-minimization problems, for very thin  $h$ , and a 2D-limiting problem, as  $h \searrow 0$ , will be demonstrated with helps from the theory of  $\Gamma$ -convergence.

**1. Introduction.** Let  $S \subset \mathbb{R}^2$  be a two-dimensional bounded domain with a smooth boundary, and let  $\Omega \subset \mathbb{R}^3$  be a three-dimensional cylindrical domain, given by  $\Omega := S \times (0, 1)$ . Let  $\alpha : \overline{\Omega} \rightarrow [0, \infty)$  be a given nonnegative and continuous function.

In this paper, let us imagine the situation that a ferromagnetic thin film is applied on a thin region  $\Omega^{(h)} := S \times (0, h)$  with a (small) thickness  $0 < h < 1$ . As a possible free-energy for the magnetic study in such situation, the following functional, denoted by  $\mathcal{E}_\alpha^{(h)}$ :

$$\mathcal{E}_\alpha^{(h)}(m) := \Psi_\alpha^{(h)}(m) + \int_{\Omega^{(h)}} \left( \varphi(m) + \frac{1}{2} \nabla \zeta \cdot m \right) d\mathcal{L}^3, \quad (1)$$

for any  $m = (m_1, m_2, m_3) \in L^2(\Omega^{(h)}; \mathbb{R}^3)$ ;

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subject to:

$$\operatorname{div}(-\nabla\zeta + \overline{0m}) = 0, \quad \text{in } \mathbb{R}^3, \quad (2)$$

$$|m| = m_s, \quad \mathcal{L}^3\text{-a.e. in } \Omega; \quad (3)$$

was proposed by Brown [7] (1963), where  $\Psi_\alpha$  is the lower semi-continuous envelopment of a functional:

$$\psi \in W^{1,2}(\Omega^{(h)}; \mathbb{R}^3) \cap L^2(\Omega^{(h)}; \mathbb{S}^2) \mapsto \int_{\Omega^{(h)}} \alpha |\nabla\psi|^2 d\mathcal{L}^3;$$

onto the space  $L^2(\Omega^{(h)}; \mathbb{R}^3)$ .

In (1), the value of  $\mathcal{E}_\alpha^{(h)}$  denotes an energy quantity, per unit volume in  $\Omega^{(h)}$ , and the variable  $m = (m_1, m_2, m_3)$  denotes the magnetization in the region  $\Omega^{(h)}$  of magnetic thin film. In this light, the minimizers of  $\mathcal{E}_\alpha^{(h)}$  are supposed to represent the most probable profile of the magnetization distribution applied on  $\Omega^{(h)}$ . Here, the given function  $\alpha$  is the so-called material coefficient, and this coefficient is supposed to be degenerative somewhere in  $\Omega$ .  $\varphi : \mathbb{R}^3 \rightarrow [0, \infty)$  is a given continuous function, which is involved in the magnetization anisotropy.

The function  $\zeta : \mathbb{R}^3 \rightarrow \mathbb{R}$  as in (1)-(2) denotes the magnetic field potential, and hence, it is prescribed as the solution of the simplified Maxwell equation (2). Here, the notation “ $\overline{0}$ ” denotes the zero-extension of functions. In addition to the above, let us note that the free-energy  $\mathcal{E}_\alpha^{(h)}$  is considered under the constrained condition (3), by a positive constant  $m_s$  of the magnetization saturation.

In this paper, we set:

$$\mathcal{L}^2(S) = 1 \quad (\text{and hence } \mathcal{L}^3(\Omega) = 1), \quad \text{and } m_s = 1;$$

for simplicity. On that basis, let us denote by  $T^{(h)}$  the scale transform, defined as:

$$T^{(h)} : x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mapsto (x_1, x_2, hx_3) \in \mathbb{R}^3;$$

to consider a rescaled minimization problem, denoted by (MP)<sup>(h)</sup>.

(MP)<sup>(h)</sup> Find a vectorial function  $m^{(h)} = (m_1^{(h)}, m_2^{(h)}, m_3^{(h)}) \in L^2(\Omega; \mathbb{R}^3)$  of three variables, which minimizes the following functional on  $L^2(\Omega; \mathbb{R}^3)$ :

$$\begin{aligned} \mathcal{F}_\alpha^{(h)}(m) &:= \Phi_\alpha^{(h)}(m) + \int_{\Omega} \varphi(m) d\mathcal{L}^3 \\ &\quad + \frac{1}{2} \int_{\Omega} \left( \nabla_{\mathbb{P}} \zeta \cdot m_{\mathbb{P}} + \frac{1}{h} \partial_3 \zeta m_3 \right) d\mathcal{L}^3, \end{aligned} \quad (4)$$

$$\text{for any } m = (m_1, m_2, m_3) \in L^2(\Omega; \mathbb{R}^3);$$

subject to:

$$\nabla_{\mathbb{P}} \cdot (-\nabla_{\mathbb{P}} \zeta + \overline{0m_{\mathbb{P}}}) + \frac{1}{h} \partial_3 \left( -\frac{1}{h} \partial_3 \zeta + \overline{0m_3} \right) = 0, \quad \text{in } \mathbb{R}^3; \quad (5)$$

where the subscript “ $\mathbb{P}$ ” denotes the restriction of the situation onto the two-dimensional plane  $\mathbb{R}^2$ , e.g.:

$$y_{\mathbb{P}} := (y_1, y_2), \quad \text{for } y = (y_1, y_2, y_3) \in \mathbb{R}^3,$$

$$\mu_{\mathbb{P}} := (\mu_1, \mu_2) \in L^2(\Omega; \mathbb{R}^2), \quad \text{for } \mu = (\mu_1, \mu_2, \mu_3) \in L^2(\Omega; \mathbb{R}^3),$$

and the distributional gradient

$$\nabla_{\mathbb{P}} \mu := \begin{pmatrix} \partial_1 \mu_1 & \partial_2 \mu_1 \\ \partial_1 \mu_2 & \partial_2 \mu_2 \\ \partial_1 \mu_3 & \partial_2 \mu_3 \end{pmatrix}, \quad \text{for } \mu = (\mu_1, \mu_2, \mu_3) \in L^2(\Omega; \mathbb{R}^3);$$

and  $\Phi_\alpha^{(h)}$  is the rescaled version of the lower semi-continuous envelopment  $\Psi_\alpha^{(h)}$  by  $T^{(h)}$ , and it is rigorously defined as:

$$\Phi_\alpha^{(h)}(m) := \begin{cases} \inf_{\{\psi^{(i)}\} \in \mathcal{Q}_\Omega(m)} \liminf_{i \rightarrow \infty} \int_\Omega \alpha^{(h)} \left( |\nabla_P \psi^{(i)}|^2 + \frac{1}{h^2} |\partial_3 \psi^{(i)}|^2 \right) d\mathcal{L}^3, \\ \text{if } |m| = 1, \mathcal{L}^3\text{-a.e. in } \Omega, \\ \infty, \text{ otherwise, for any } m \in L^2(\Omega; \mathbb{R}^3); \end{cases} \quad (6)$$

by using a composition:

$$\alpha^{(h)}(x) := (\alpha \circ T^{(h)})(x) = \alpha(x_1, x_2, hx_3), \quad \text{for all } x = (x_1, x_2, x_3) \in \bar{\Omega};$$

and a class of approximating functions:

$$\mathcal{Q}_\Omega(m) := \left\{ \{\psi^{(i)}\} \left| \begin{array}{l} \psi^{(i)} \in W^{1,2}(\Omega; \mathbb{R}^3) \cap L^2(\Omega; \mathbb{S}^2), \quad i = 1, 2, 3, \dots, \\ \text{and } \psi^{(i)} \rightarrow m \text{ in } L^2(\Omega; \mathbb{R}^3) \text{ as } i \rightarrow \infty \end{array} \right. \right\}; \quad (7)$$

for any  $m \in L^2(\Omega; \mathbb{S}^2)$ .

As is easily seen, the inverse transform  $(T^{(h)})^{-1}$  provides a bijective correspondence between the minimizers  $m^{(h)}$  of  $(\text{MP})^{(h)}$  and the minimizers  $m_{\text{org}}^{(h)} := m^{(h)} \circ (T^{(h)})^{-1}$  of the original free-energy  $\mathcal{E}_\alpha^{(h)}$ . Besides, let us note that the domains  $\text{Dom}(\mathcal{F}_\alpha^{(h)})$  of free-energies are not uniform, but variable with respect to  $0 < h < 1$ , and the variation is directly governed by the degenerating part of the coefficient:

$$A_0^{(h)} := (\alpha^{(h)})^{-1}(0) \subset \bar{\Omega}, \quad \text{for } 0 < h < 1.$$

Under very thin situation of the thickness  $h$ , it is naturally expected that the minimization problem  $(\text{MP})^{(h)}$  can be reduced to a simpler problem, considered in two-dimensional domain  $S$ . Such reduction will be realized through the limiting observation for  $(\text{MP})^{(h)}$  as  $h \searrow 0$ , and then, the binary function:

$$\begin{aligned} \alpha^\circ(x_1, x_2) &:= \alpha(x_1, x_2, 0) \text{ for any } (x_1, x_2) \in \bar{S}, \\ &\text{with the degenerating part } A_0^\circ := (\alpha^\circ)^{-1}(0); \end{aligned}$$

will be the material coefficient in the limiting problem. Actually, in the  $h$ -independent case of  $A_0^{(h)}$ , a number of like-minded study results, such as [1, 2, 3, 4, 5, 6, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18], were reported, from various viewpoints, and some of them concluded a definite association between the limiting profile of  $(\text{MP})^{(h)}$ , as  $h \searrow 0$ , and the following minimization problem, denoted by  $(\text{MP})^\circ$ .

$(\text{MP})^\circ$  Find a vectorial function  $m^\circ = (m_1^\circ, m_2^\circ, m_3^\circ) \in L^2(S; \mathbb{R}^3)$  of two variables, which minimizes the following functional:

$$\begin{aligned} \mathcal{F}_\alpha^\circ(m) &:= \Phi_\alpha^\circ(m) + \int_S \varphi(m) d\mathcal{L}^2 + \frac{1}{2} \int_S |m_3|^2 d\mathcal{L}^2, \\ &\text{for any } m = (m_1, m_2, m_3) \in L^2(S; \mathbb{R}^3); \end{aligned} \quad (8)$$

where  $\Phi_\alpha^\circ$  is a convex function on  $L^2(S; \mathbb{R}^3)$ , defined as:

$$\Phi_\alpha^\circ(m) := \begin{cases} \inf_{\{\psi^{(i)}\} \in \mathcal{Q}_S(m)} \liminf_{i \rightarrow \infty} \int_S \alpha^\circ |\nabla \psi^{(i)}|^2 d\mathcal{L}^2, \\ \text{if } |m| = 1, \mathcal{L}^2\text{-a.e. in } S, \\ \infty, \text{ otherwise, for any } m \in L^2(S; \mathbb{R}^3); \end{cases} \quad (9)$$

by using a class of approximating sequences:

$$\mathcal{Q}_S(m) := \left\{ \{\psi^{(i)}\} \left| \begin{array}{l} \psi^{(i)} \in W^{1,2}(S; \mathbb{R}^3) \cap L^2(S; \mathbb{S}^2), \quad i = 1, 2, 3, \dots, \\ \text{and } \psi^{(i)} \rightarrow m \text{ in } L^2(S; \mathbb{R}^3) \text{ as } i \rightarrow \infty \end{array} \right. \right\}.$$

Now, the main theme of this paper is to verify whether analogous observation is available even under  $h$ -variable situation of  $A_0^{(h)}$  (or energy domains), or not. To this end, we here impose the following two conditions for the material coefficient  $\alpha$ :

- (a1)  $\mathcal{L}^3(A_0^{(h)}) = 0$ , for  $0 < h < 1$ ;
- (a2)  $\mathcal{L}^2(A_0^\circ) = 0$ , and  $\alpha^\circ(x_P) \leq \alpha(x)$ , for all  $x = (x_1, x_2, x_3) \in \bar{\Omega}$ .

Consequently, a certain positive answer for our theme will be demonstrated in the main theorem, stated as follows.

**Main Theorem.** (I) *Let us assume the condition (a1). Then, for any  $0 < h < 1$ , the problem  $(MP)^{(h)}$  admits at least one solution (minimizer)  $m^{(h)}$ .*  
 (II) *Under the conditions (a1)-(a2), there exist a sequence  $\{h_i \mid i = 1, 2, 3, \dots\} \subset (0, 1)$  and a function  $m^\circ \in L^2(S; \mathbb{R}^3)$  of two variables, such that:*  
 (i)  $h_i \searrow 0$ ,  $m^{(h_i)} \rightarrow m^\circ$  in  $L^2(\Omega; \mathbb{R}^3)$ ,  $\mathcal{F}_\alpha^{(h_i)}(m^{(h_i)}) \rightarrow \mathcal{F}_\alpha^\circ(m^\circ)$ , as  $i \rightarrow \infty$ ;  
 (ii) *the limit  $m^\circ$  solves the problem  $(MP)^\circ$ ;*

where  $\{m^{(h)} \mid 0 < h < 1\}$  is the sequence of minimizers  $m^{(h)}$ ,  $0 < h < 1$ , obtained in (I).

The content of this paper is as follows. In the next Section 2, some key-properties for the minimization problems  $(MP)^{(h)}$ ,  $0 < h < 1$ , and  $(MP)^\circ$  are briefly mentioned as preliminaries. In subsequent Section 3, the continuous dependence between the energy sequence  $\{\mathcal{F}_\alpha^{(h)} \mid 0 < h < 1\}$  and the energy  $\mathcal{F}_\alpha^\circ$ , as  $h \searrow 0$ , will be shown by means of the notion of  $\Gamma$ -convergence (cf. [9]). On that basis, the final Section 4 will be devoted to the proof of Main Theorem.

**Notation.** Throughout this paper, the Lebesgue measure is denoted by  $\mathcal{L}^n$ , for any observing dimension  $n \in \mathbb{N}$ .

For any abstract Banach space, the norm of  $X$  is denoted by  $|\cdot|_X$ . However, when  $X$  is an Euclidean space, the norm is simply denoted by  $|\cdot|$ . Besides, for any functional  $F : X \rightarrow (-\infty, \infty]$ , we denote by  $\text{Dom}(F)$  the domain of  $F$ , and for any  $r > 0$ , we denote by  $L(r; F)$  the sublevel set of  $F$ , more precisely:

$$\text{Dom}(F) := \{ \xi \in X \mid F(\xi) < \infty \} \quad \text{and} \quad L(r; F) := \{ \xi \in X \mid F(\xi) \leq r \}.$$

For any abstract Hilbert space  $H$ , the inner product of  $H$  is denoted by  $(\cdot, \cdot)_H$ . However, when  $H$  is an Euclidean space, the inner product between two vectors  $\xi, \eta \in H$  is simply denoted by  $\xi \cdot \eta$ .

**2. Preliminaries.** Let us start with summarizing the known-facts, concerned with the coupled Maxwell equation (5).

(Fact 1) (Summary of [18, Lemma 3.1]) Let us fix any  $0 < h < 1$ . Then, for any function  $m = (m_1, m_2, m_3) \in L^2(\Omega; \mathbb{R}^3)$ , the solution  $\zeta^{(h)}$  of the Maxwell equation (5) is prescribed in the scope of a Hilbert space:

$$V^{(h)} := \left\{ v \in H_{\text{loc}}^1(\mathbb{R}^3) \mid \nabla v \in L^2(\mathbb{R}^3; \mathbb{R}^3) \quad \text{and} \quad \int_{B_\Omega} v \, d\mathcal{L}^3 = 0 \right\};$$

endowed with a  $h$ -dependent inner product:

$$(u, v)_{V^{(h)}} := (\nabla_P u, \nabla_P v)_{L^2(\mathbb{R}^3; \mathbb{R}^{3 \times 2})} + \frac{1}{h^2} (\partial_3 u, \partial_3 v)_{L^2(\mathbb{R}^3; \mathbb{R}^3)}, \quad \text{for } u, v \in V^{(h)};$$

where  $B_\Omega$  is an (fixed) open ball containing  $\Omega$ . Then, the solution  $\zeta^{(h)} \in V^{(h)}$  is supposed to fulfill a weak formulation by the following variational identity:

$$\int_{\Omega} \left( (\nabla_{\mathbb{P}} \zeta^{(h)} - m_{\mathbb{P}}) \cdot \nabla_{\mathbb{P}} v + \frac{1}{h} \left( \frac{1}{h} \partial_3 \zeta^{(h)} - m_3 \right) \partial_3 v \right) d\mathcal{L}^3 = 0, \quad (10)$$

for any  $v \in V^{(h)}$ ;

Moreover, taking more one function  $\tilde{m} \in L^2(\Omega; \mathbb{R}^3)$ , arbitrarily, and taking another solution  $\tilde{\zeta}^{(h)}$  of (10) when  $m = \tilde{m}$ , it follows that:

$$|\zeta^{(h)} - \tilde{\zeta}^{(h)}|_{V^{(h)}} \leq |m - \tilde{m}|_{L^2(\Omega; \mathbb{R}^3)}. \quad (11)$$

Hence, the variational problem (10) is well-posed.

(Fact 2) (Summary of [15, Proposition 4.1]) Let us set:

$$\begin{cases} F_{\text{mag}}^{(h)}(m) := \frac{1}{2} \int_{\Omega} \left( \nabla_{\mathbb{P}} \zeta^{(h)} \cdot m_{\mathbb{P}} + \frac{1}{h} \partial_3 \zeta^{(h)} m_3 \right) d\mathcal{L}^3, \\ F_{\text{mag}}^{\circ}(m) := \frac{1}{2} \int_{\Omega} |m_3|^2 d\mathcal{L}^3, \end{cases} \quad (12)$$

for any  $m = (m_1, m_2, m_3) \in L^2(\Omega; \mathbb{R}^3)$ ;

by using the solution  $\zeta^{(h)}$  of the variational identity (10). On that basis, let us assume that  $\{\tilde{m}^{(h)} \mid 0 < h < 1\} \subset L^2(\Omega; \mathbb{R}^3)$ , and  $\tilde{m}^{(h)} \rightarrow \tilde{m}$  in  $L^2(\Omega; \mathbb{R}^3)$  as  $h \searrow 0$ , for some  $\tilde{m} = (\tilde{m}_1, \tilde{m}_2, \tilde{m}_3) \in L^2(\Omega; \mathbb{R}^3)$ . Then:

$$F_{\text{mag}}^{(h)}(\tilde{m}^{(h)}) \rightarrow F_{\text{mag}}^{\circ}(\tilde{m}), \quad \text{as } h \searrow 0.$$

Next, let us look toward the key-properties of the lower semi-continuous envelopments  $\Phi_{\alpha}^{(h)}$ ,  $0 < h < 1$ , and  $\Phi_{\alpha}^{\circ}$ .

**Lemma 2.1.** (I) For any  $0 < h < 1$ , the functional  $\Phi_{\alpha}^{(h)}$  is a maximal functional in the class of l.s.c. functionals on  $L^2(\Omega; \mathbb{R}^3)$ , supporting the functional:

$$\psi \in W^{1,2}(\Omega; \mathbb{R}^3) \cap L^2(\Omega; \mathbb{S}^2) \mapsto \int_{\Omega} \alpha^{(h)} \left( |\nabla_{\mathbb{P}} \psi|^2 + \frac{1}{h^2} |\partial_3 \psi|^2 \right) d\mathcal{L}^3.$$

Moreover:

$$(i-1) \quad \Phi_{0,\alpha}^{(h)}(m) := \int_{\Omega \setminus A_0^{(h)}} \alpha^{(h)} \left( |\nabla_{\mathbb{P}} m|^2 + \frac{1}{h^2} |\partial_3 m|^2 \right) d\mathcal{L}^3 \leq \Phi_{\alpha}^{(h)}(m),$$

for any  $m \in W_{\text{loc}}^{1,2}(\Omega \setminus A_0^{(h)}; \mathbb{R}^3) \cap L^2(\Omega; \mathbb{S}^2)$ ;

$$(i-2) \quad W^{1,2}(\Omega; \mathbb{R}^3) \cap L^2(\Omega; \mathbb{S}^2) \subset \text{Dom}(\Phi_{\alpha}^{(h)}) \subset \text{Dom}(\Phi_{0,\alpha}^{(h)}) \subset W_{\text{loc}}^{1,2}(\Omega \setminus A_0^{(h)}; \mathbb{R}^3),$$

and  $\Phi_{0,\alpha}^{(h)} = \Phi_{\alpha}^{(h)}$  on  $W^{1,2}(\Omega; \mathbb{R}^3) \cap L^2(\Omega; \mathbb{S}^2)$ ;

(i-3) for any  $m \in \text{Dom}(\Phi_{\alpha}^{(h)})$ , there exists a sequence  $\{\mu^{(i)} \mid i = 1, 2, 3, \dots\} \subset W^{1,2}(\Omega; \mathbb{R}^3) \cap L^2(\Omega; \mathbb{S}^2)$  such that

$$\mu^{(i)} \rightarrow m \text{ in } L^2(\Omega; \mathbb{R}^3) \text{ and } \Phi_{0,\alpha}^{(h)}(\mu^{(i)}) \rightarrow \Phi_{\alpha}^{(h)}(m), \text{ as } i \rightarrow \infty.$$

(II) The functional  $\Phi_{\alpha}^{\circ}$  is a maximal functional in the class of l.s.c. functionals on  $L^2(S; \mathbb{R}^3)$ , supporting the functional:

$$\psi \in W^{1,2}(S; \mathbb{R}^3) \cap L^2(S; \mathbb{S}^2) \mapsto \int_S \alpha^{\circ} |\nabla \psi|^2 d\mathcal{L}^2.$$

Moreover:

- (ii-1)  $\Phi_{0,\alpha}^\circ(m) := \int_{S \setminus A_0^\circ} \alpha^\circ |\nabla m|^2 d\mathcal{L}^2 \leq \Phi_\alpha^{(h)}(m)$ , for any  $m \in W_{\text{loc}}^{1,2}(S \setminus A_0^\circ; \mathbb{R}^3) \cap L^2(S; \mathbb{S}^2)$ ;
- (ii-2)  $W^{1,2}(S; \mathbb{R}^3) \cap L^2(S; \mathbb{S}^2) \subset \text{Dom}(\Phi_\alpha^\circ) \subset \text{Dom}(\Phi_{0,\alpha}^\circ) \subset W_{\text{loc}}^{1,2}(S \setminus A_0^\circ; \mathbb{R}^3)$ , and  $\Phi_{0,\alpha}^\circ = \Phi_\alpha^\circ$  on  $W^{1,2}(S; \mathbb{R}^3)$
- (ii-3) for any  $m \in \text{Dom}(\Phi_\alpha^{(h)})$ , there exists a sequence  $\{\mu^{(i)} \mid i = 1, 2, 3, \dots\} \subset W^{1,2}(S; \mathbb{R}^3) \cap L^2(S; \mathbb{S}^2)$  such that
- $$\mu^{(i)} \rightarrow m \text{ in } L^2(S; \mathbb{R}^3) \text{ and } \Phi_{0,\alpha}^\circ(\mu^{(i)}) \rightarrow \Phi_\alpha^\circ(m), \text{ as } i \rightarrow \infty.$$

*Proof.* This lemma follows, directly, from the definition formulas (6) and (9).  $\square$

**Remark 1.** (Key-properties for free-energies) By virtue of (4), (8) and Lemma 2.1, the functional  $\mathcal{F}_\alpha^{(h)}$  (resp.  $\mathcal{F}_\alpha^\circ$ ) turns out to be l.s.c. in  $L^2(\Omega; \mathbb{R}^3)$  (resp. in  $L^2(S; \mathbb{R}^3)$ ), and  $\text{Dom}(\mathcal{F}_\alpha^{(h)}) = \text{Dom}(\Phi_\alpha^{(h)})$ , for  $0 < h < 1$  (resp.  $\text{Dom}(\mathcal{F}_\alpha^\circ) = \text{Dom}(\Phi_\alpha^\circ)$ ). Furthermore, under the conditions (a1)-(a2), as in introduction, the variation of energy-domains  $\text{Dom}(\mathcal{F}_\alpha^{(h)})$ , with respect to  $h$ , will be restrictive in the sense that  $\text{Dom}(\mathcal{F}_\alpha^{(h)})$  will be included in  $W_{\text{loc}}^{1,2}(\Omega \setminus (A_0^\circ \times (0, 1)); \mathbb{R}^3)$ , uniformly, for all  $0 < h < 1$ .

Taking into account of Lemma 2.1, Remark 1 and [16, Corollary 2], we can derive the following corollary.

**Corollary 1.** (Compactness) *Let us assume the condition (a1), as in introduction. Then, for any  $0 < h < 1$  and any  $r > 0$ , the sublevel sets  $L(r; \Phi_\alpha^{(h)})$  and  $L(r; \mathcal{F}_\alpha^{(h)})$  are compact in  $L^2(\Omega; \mathbb{R}^3)$ . Additionally, if we assume the conditions (a1)-(a2), as in introduction, then for any  $r > 0$ , the sublevel sets  $L(r; \Phi_\alpha^\circ)$  and  $L(r; \mathcal{F}_\alpha^\circ)$  are compact in  $L^2(S; \mathbb{R}^3)$ , and the unions*

$$\mathcal{U}_\Phi(r) := \bigcup_{0 < h < 1} L(r; \Phi_\alpha^{(h)}) \text{ and } \mathcal{U}_\mathcal{F}(r) := \bigcup_{0 < h < 1} L(r; \mathcal{F}_\alpha^{(h)});$$

are relatively compact in  $L^2(\Omega; \mathbb{R}^3)$ .

*Proof.* Let us assume (a1), and let us fix any  $0 < h < 1$  and any  $r > 0$ . Here, taking the solution  $\zeta^{(h)}$  as the test function of (10), we have:

$$F_{\text{mag}}^{(h)}(m) = \frac{1}{2} |\zeta^{(h)}|_{V^{(h)}}^2 \geq 0, \text{ for any } m \in L^2(\Omega; \mathbb{R}^3). \quad (13)$$

Subsequently, we see from (4), (13) and (i-1) of Lemma 2.1 that:

$$L(r; \mathcal{F}_\alpha^{(h)}) \subset L(r; \Phi_\alpha^{(h)}) \subset L(r; \Phi_{0,\alpha}^{(h)}).$$

Since the compactness of  $L(r; \Phi_{0,\alpha}^{(h)})$  in  $L^2(\Omega; \mathbb{R}^3)$  is already concluded in [16, Corollary 2], we can say that its closed subsets  $L(r; \Phi_\alpha^{(h)})$  and  $L(r; \mathcal{F}_\alpha^{(h)})$  are also so.

Just as in the above, the assertion for the sublevel sets  $L(r; \Phi_\alpha^\circ)$  and  $L(r; \mathcal{F}_\alpha^\circ)$  (resp. for the unions  $\mathcal{U}_\Phi(r)$  and  $\mathcal{U}_\mathcal{F}(r)$ ) can be concluded, with the helps from the condition (a2) and [16, Corollary 2] (resp. [16, Theorem 3.4]).  $\square$

**3. Continuous dependence of energies.** The objective in this section is summarized in the following theorem, concerned with continuous dependence ( $\Gamma$ -convergence) of lower semi-continuous envelopments, as  $h \searrow 0$ .

**Theorem 3.1.** ( *$\Gamma$ -convergence from  $\Phi_\alpha^{(h)}$  to  $\Phi_\alpha^\circ$  as  $h \searrow 0$ )* Let us assume the conditions (a1)-(a2), as in introduction. Then, the sequence  $\{\Phi_\alpha^{(h)} \mid 0 < h < 1\}$  of the lower semi-continuous envelopments converges to  $\Phi_\alpha^\circ$ , in the sense of  $\Gamma$ -convergence, as  $h \searrow 0$ . More precisely, referring to [9] (or [1, Lemma 2.3]), the above assertion is equivalent to:

- ( $\gamma 1$ )  $\liminf_{h \searrow 0} \Phi_\alpha^{(h)}(\tilde{m}^{(h)}) \geq \Phi_\alpha^\circ(\tilde{m}^\circ)$ , if  $\{\tilde{m}^{(h)} \mid 0 < h < 1\} \subset L^2(\Omega; \mathbb{R}^3)$ ,  $\tilde{m}^\circ \in L^2(\Omega; \mathbb{R}^3)$ , and  $\tilde{m}^{(h)} \rightarrow \tilde{m}^\circ$  in  $L^2(\Omega; \mathbb{R}^3)$  as  $h \searrow 0$ ;
- ( $\gamma 2$ ) for any  $\hat{m}^\circ \in \text{Dom}(\Phi_\alpha^\circ)$ , there exists a sequence  $\{\hat{m}^{(h)} \mid 0 < h < 1\} \subset L^2(\Omega; \mathbb{R}^3)$ , such that  $\hat{m}^{(h)} \rightarrow \hat{m}^\circ$  in  $L^2(\Omega; \mathbb{R}^3)$  and  $\Phi_\alpha^{(h)}(\hat{m}^{(h)}) \rightarrow \Phi_\alpha^\circ(\hat{m}^\circ)$ , as  $h \searrow 0$ .

This theorem is proved by relying on some classes of open sets, mentioned in the following lemma.

**Lemma 3.2.** (*Open coverings for non-degenerate parts*) There exists a sequence  $\{\bar{\alpha}_\ell^\circ \mid \ell = 1, 2, 3, \dots\} \subset (0, 1)$  and a covering  $\{S_\ell \mid \ell = 1, 2, 3, \dots\} \subset S \setminus A_0^\circ$  of  $S \setminus A_0^\circ$  with smooth boundaries  $\partial S_\ell$  ( $\ell = 1, 2, 3, \dots$ ), such that:

$$\begin{cases} \emptyset \neq S_1 \subset \subset \dots \subset S_\ell \subset \subset \dots \subset S \setminus A_0^\circ = \bigcup_{\ell=0}^{\infty} S_\ell, \\ \bar{\alpha}_1^\circ > \dots > \bar{\alpha}_\ell^\circ > \dots > 0 = \lim_{\ell \rightarrow \infty} \bar{\alpha}_\ell^\circ, \quad \text{and } \alpha^\circ \geq \bar{\alpha}_\ell^\circ \text{ on } S_\ell, \text{ for } \ell = 1, 2, 3, \dots \end{cases}$$

Hence, if we assume the conditions (a1)-(a2) as in introduction, then a sequence  $\{\Omega_\ell^\dagger\} := \{S_\ell \times (0, 1) \mid \ell = 1, 2, 3, \dots\}$  turns out to be a covering of an open set  $\Omega^\dagger := \Omega \setminus (A_0^\circ \times (0, 1))$ , with Lipschitz boundaries  $\partial \Omega_\ell^\dagger$  ( $\ell = 1, 2, 3, \dots$ ), such that:

$$\begin{cases} \emptyset \neq \Omega_1^\dagger \subset \subset \dots \subset \Omega_\ell^\dagger \subset \subset \dots \subset \Omega^\dagger = \bigcup_{\ell=0}^{\infty} \Omega_\ell^\dagger \subset \Omega \setminus A_0^{(h)}, \quad \mathcal{L}^3(\Omega \setminus \Omega^\dagger) = 0, \\ \alpha^{(h)} \geq \alpha^\circ \geq \bar{\alpha}_\ell^\circ > 0 \text{ on } \Omega_\ell^\dagger, \text{ for } \ell = 1, 2, 3, \dots \text{ and } 0 < h < 1. \end{cases} \quad (14)$$

*Proof of Lemma 3.2.* This lemma is a direct consequence of the line of arguments, discussed in [16, Lemma 4.1, Remark 4-5].  $\square$

*Proof of Theorem 3.1.* First, we verify the assertion ( $\gamma 1$ ). Then, it is enough to consider only the case when  $\liminf_{h \searrow 0} \Phi_\alpha^{(h)}(\tilde{m}^{(h)}) < \infty$ , since another case is trivial.

On account of (i-3) in Lemma 2.1, we find a sequence  $\{\tilde{h}_i \mid i = 1, 2, 3, \dots\} \subset (0, 1)$  and a sequence  $\{\tilde{\mu}^{(i)} \mid i = 1, 2, 3, \dots\} \subset W^{1,2}(\Omega; \mathbb{R}^3) \cap L^2(\Omega; \mathbb{S}^2)$ , such that:

$$\begin{cases} \tilde{h}_i \searrow 0 \text{ and } \tilde{\mu}^{(i)} \rightarrow \tilde{m}^\circ \text{ in } L^2(\Omega; \mathbb{R}^3), \text{ as } i \rightarrow \infty, \\ \lim_{i \rightarrow \infty} \Phi_{0,\alpha}^{(\tilde{h}_i)}(\tilde{\mu}^{(i)}) = \liminf_{h \searrow 0} \Phi_\alpha^{(h)}(\tilde{m}^{(h)}). \end{cases}$$

Here, in the light of (14) and (i-1) in Lemma 2.1,

$$\begin{aligned} |\partial_3 \tilde{m}^\circ|_{L^2(\Omega_\ell^\dagger; \mathbb{R}^3)}^2 &\leq \liminf_{i \rightarrow \infty} |\partial_3 \tilde{\mu}^{(i)}|_{L^2(\Omega_\ell^\dagger; \mathbb{R}^3)}^2 \\ &\leq \frac{1}{\bar{\alpha}_\ell^\circ} \sup_{i \geq 1} \Phi_{0,\alpha}^{(\tilde{h}_i)}(\tilde{\mu}^{(i)}) \lim_{i \rightarrow \infty} \tilde{h}_i^2 = 0, \quad \ell = 1, 2, 3, \dots, \end{aligned} \quad (15)$$

which implies  $\partial_3 \check{m}^\circ = 0$ ,  $\mathcal{L}^3$ -a.e. in  $\Omega^\dagger$ . Hence, the limit  $\check{m}^\circ$  can be regarded to belong to the class  $L^2(S \setminus A_0^\circ; \mathbb{R}^3)$  ( $= L^2(S; \mathbb{R}^3)$ ) of binary functions. Subsequently, let us set:

$$\check{\mu}^{(i)}(x_1, x_2) := \check{\mu}^{(i)}(x_1, x_2, c_i), \text{ for } \mathcal{L}^2\text{-a.e. } (x_1, x_2) \in S \text{ and } i = 1, 2, 3, \dots;$$

by using a collection  $\{c_i \mid i = 1, 2, 3, \dots\} \subset (0, 1)$  of constants, such that:

$$\int_S \alpha^{(\check{h}_i)}(x_1, x_2, c_i) |\nabla_{\mathbb{P}} \check{\mu}^{(i)}(x_1, x_2, c_i)|^2 d\mathcal{L}^2 \leq \int_\Omega \alpha^{(\check{h}_i)} |\nabla_{\mathbb{P}} \check{\mu}^{(i)}|^2 d\mathcal{L}^3, \quad i = 1, 2, 3, \dots.$$

Then, with the helps from (15) and Fubini's theorem, it is computed that:

$$\begin{aligned} & \lim_{i \rightarrow \infty} |\check{\psi}^{(i)} - \check{\mu}^{(i)}|_{L^2(\Omega; \mathbb{R}^3)}^2 \\ &= \lim_{i \rightarrow \infty} \left( |\check{\psi}^{(i)} - \check{\mu}^{(i)}|_{L^2(\Omega_\ell^\dagger; \mathbb{R}^3)}^2 + |\check{\psi}^{(i)} - \check{\mu}^{(i)}|_{L^2(\Omega \setminus \Omega_\ell^\dagger; \mathbb{R}^3)}^2 \right) \\ &\leq \lim_{i \rightarrow \infty} \int_{\Omega_\ell^\dagger} \int_0^1 |\partial_3 \check{\mu}^{(i)}|^2 d\mathcal{L}^1 d\mathcal{L}^3 + 4\mathcal{L}^3(\Omega \setminus \Omega_\ell^\dagger) \\ &\leq \lim_{i \rightarrow \infty} |\partial_3 \check{\mu}^{(i)}|_{L^2(\Omega_\ell^\dagger)}^2 + 4\mathcal{L}^3(\Omega \setminus \Omega_\ell^\dagger) = 4\mathcal{L}^3(\Omega \setminus \Omega_\ell^\dagger), \quad \text{for } \ell = 1, 2, 3, \dots \end{aligned}$$

It implies that:

$$\check{\psi}^{(i)} \rightarrow \check{m}^\circ \text{ in } L^2(\Omega; \mathbb{R}^3), \text{ as } i \rightarrow \infty; \quad (16)$$

since  $\mathcal{L}^3(\Omega \setminus \Omega_\ell^\dagger) \rightarrow 0$  as  $\ell \rightarrow \infty$ . Taking into account of (a2), (16), (ii-2) in Lemma 2.1 and the lower semi-continuity of  $\Phi_\alpha^\circ$ , we deduce that:

$$\begin{aligned} \liminf_{h \searrow 0} \Phi_\alpha^{(h)}(\check{m}^{(h)}) &= \lim_{i \rightarrow \infty} \Phi_\alpha^{(\check{h}_i)}(\check{\mu}^{(i)}) \geq \liminf_{i \rightarrow \infty} \int_\Omega \alpha^{(\check{h}_i)} |\nabla_{\mathbb{P}} \check{\mu}^{(i)}|^2 d\mathcal{L}^3 \\ &\geq \liminf_{i \rightarrow \infty} \int_S \alpha^{(\check{h}_i)}(x_1, x_2, c_i) |\nabla \check{\psi}^{(i)}(x_1, x_2)|^2 d\mathcal{L}^2 \geq \liminf_{i \rightarrow \infty} \Phi_\alpha^\circ(\check{\psi}^{(i)}) \geq \Phi_\alpha^\circ(\check{m}^\circ). \end{aligned}$$

Thus, the assertion ( $\gamma 1$ ) is concluded.

Next, we verify the assertion ( $\gamma 2$ ). Let us take any  $\hat{m}^\circ \in \text{Dom}(\Phi_\alpha^\circ)$ . Then, construction of the required sequence  $\{\hat{m}^{(h)} \mid 0 < h < 1\}$  will be on the basis of a sequence  $\{\hat{\mu}^{(i)} \mid i = 1, 2, 3, \dots\} \subset W^{1,2}(S; \mathbb{R}^3) \cap L^2(S; \mathbb{R}^3)$ , which will be obtained as the approximating sequence, as in (ii-3) of Lemma 2.1, when  $m = \hat{m}^\circ$ . Here, noting that  $\alpha^{(h)} \rightarrow \alpha^\circ$  in  $C(\bar{\Omega})$  as  $h \searrow 0$ , there exists a sequence  $\{\hat{h}_i \mid i = 1, 2, 3, \dots\} \subset (0, 1)$ , such that:

$$\begin{cases} \bullet \hat{h}_1 > \dots > \hat{h}_i > \dots > 0 = \lim_{i \rightarrow \infty} \hat{h}_i, \\ \bullet 0 \leq \Phi_{0,\alpha}^{(h)}(\hat{\mu}^{(i)}) - \Phi_{0,\alpha}^\circ(\hat{\mu}^{(i)}) = \int_\Omega (\alpha^{(h)} - \alpha^\circ) |\nabla_{\mathbb{P}} \hat{\mu}^{(i)}|^2 d\mathcal{L}^3 < \frac{1}{2^i}, \\ \quad \text{for any } 0 < h < \hat{h}_i, \quad i = 1, 2, 3, \dots \end{cases}$$

On that basis, the finding sequence  $\{\hat{m}^{(h)}\}$  will be constructed by putting:

$$\hat{m}^{(h)} := \hat{\mu}^{(i)} \text{ in } L^2(\Omega; \mathbb{R}^3), \text{ if } \hat{h}_{i+1} \leq h < \hat{h}_i, \quad i = 1, 2, 3, \dots;$$

with an optional setting  $\hat{m}^{(h)} := \hat{\mu}^{(1)}$  in  $L^2(\Omega; \mathbb{R}^3)$ , for  $\hat{h}_1 \leq h < 1$ .  $\square$

The above Theorem 3.1 actually implies the  $\Gamma$ -convergence of free-energies, stated as follows.

**Corollary 2.** ( *$\Gamma$ -convergence from  $\mathcal{F}_\alpha^{(h)}$  to  $\mathcal{F}_\alpha^\circ$  as  $h \searrow 0$ ) Let us assume the conditions (a1)-(a2). Then, the sequence  $\{\mathcal{F}_\alpha^{(h)} \mid 0 < h < 1\}$  of free-energies converges to the limiting one  $\mathcal{F}_\alpha^\circ$ , in the sense of  $\Gamma$ -convergence, as  $h \searrow 0$ .*

*Proof of Corollary 2.* This corollary is immediately concluded, by taking into account of Theorem 3.1 and (Fact 2) in Section 2.  $\square$

**Remark 2.** On account of (13) and (Fact 2), we will see that the sequence  $\{\hat{m}^{(h)} \mid 0 < h < 1\}$  as in ( $\gamma 2$ ) of Theorem 3.1 will realize the convergence:

$$\mathcal{F}_\alpha^{(h)}(\hat{m}^{(h)}) \rightarrow \mathcal{F}_\alpha^\circ(\hat{m}^\circ) \text{ as } h \searrow 0.$$

**4. Proof of Main Theorem.** We divide this section into two subsections, for the respective assertions (I) and (II) of Main Theorem.

**4.1. Proof of (I) of Main Theorem.** The proof of this assertion will be a slight modification of the argument, discussed in [16, Section 5.1]. In fact, under (a1), and under the fixed setting of  $0 < h < 1$ , we can take the so-called *minimizing sequence*  $\{m_*^{(i)} \mid i = 1, 2, 3, \dots\} \subset \text{Dom}(\mathcal{F}_\alpha^{(h)})$  that is supposed to satisfy:

$$\mathcal{F}_\alpha^{(h)}(m_*^{(i)}) \searrow F_*^{(h)} := \inf_{m \in L^2(\Omega; \mathbb{R}^3)} \mathcal{F}_\alpha^{(h)}(m) \text{ as } i \rightarrow \infty.$$

Here, on account of (13), (Fact 1) and Corollary 1, a convergence subsequence  $\{m_*^{(i_k)} \mid k = 1, 2, 3, \dots\} \subset \{m_*^{(i)}\}$  will be found with the limit  $m_* \in L^2(\Omega; \mathbb{R}^3)$ , and it will be seen that:

$$\begin{cases} m_*^{(i_k)} \rightarrow m_* \text{ in } L^2(\Omega; \mathbb{R}^3), & \varphi(m_*^{(i_k)}) \rightarrow \varphi(m_*) \text{ in } L^1(\Omega), \\ F_{\text{mag}}^{(h)}(m_*^{(i_k)}) \rightarrow F_{\text{mag}}^{(h)}(m_*), \end{cases} \text{ as } k \rightarrow \infty.$$

In response to the above, we infer from the lower semi-continuity of  $\mathcal{F}_\alpha^{(h)}$  that the limit  $m_*$  is one of minimizers of (MP)<sup>(h)</sup>.  $\square$

**4.2. Proof of (II) of Main Theorem.** Let us assume the conditions (a1)-(a2), and let us take a sequence  $\{m^{(h)} \mid 0 < h < 1\}$  of minimizers of  $\mathcal{F}_\alpha^{(h)}$ ,  $0 < h < 1$ .

Let us set  $\nu^* := [1, 0, 0] \in \mathbb{S}^2$ . Then, for the variational identity (10) when  $m \equiv \nu^*$ , taking the solution itself as the test function  $v$  in (10) yields that  $F_{\text{mag}}^{(h)}(\nu^*) \leq 1$ , for any  $0 < h < 1$  (see [16, Section 5.2], for details). In view of this,

$$\begin{aligned} \Phi_\alpha^{(h)}(m^{(h)}) &\leq \mathcal{F}_\alpha^{(h)}(m^{(h)}) \leq \mathcal{F}_\alpha^{(h)}(\nu^*) = \Phi_\alpha^{(h)}(\nu^*) + |\varphi(\nu^*)|_{L^1(\Omega)} + F_{\text{mag}}^{(h)}(\nu^*) \\ &\leq \varphi(\nu^*) + 1, \text{ for all } 0 < h < 1. \end{aligned} \quad (17)$$

Since, the above (17) implies that  $\{m^{(h)}\} \subset \mathcal{U}_{\mathcal{F}}(\varphi(\nu^*) + 1)$ , we can apply Corollary 1, to find a sequence  $\{h_i \mid i = 1, 2, 3, \dots\} \subset (0, 1)$  and a limiting function  $m^\circ \in L^2(\Omega; \mathbb{R}^3)$ , such that:

$$\begin{cases} h_i \searrow 0, & m^{(h_i)} \rightarrow m^\circ \text{ in } L^2(\Omega; \mathbb{R}^3), \\ \varphi(m^{(h_i)}) \rightarrow \varphi(m^\circ) \text{ in } L^1(\Omega), \end{cases} \text{ as } i \rightarrow \infty.$$

Here, taking into account of Theorem 3.1, Corollary 2 and (17),

$$\mathcal{F}_\alpha^\circ(m^\circ) \leq \liminf_{i \rightarrow \infty} \mathcal{F}_\alpha^{(h_i)}(m^{(h_i)}) \leq \varphi(\nu^*) + 1;$$

and hence  $m^\circ \in \text{Dom}(\mathcal{F}_\alpha^\circ) \subset L^2(S; \mathbb{R}^3)$ .

Next, taking any  $\hat{m}^\circ \in \text{Dom}(\mathcal{F}_\alpha^\circ)$  ( $= \text{Dom}(\Phi_\alpha^\circ)$ ), and taking the sequence  $\{\hat{m}^{(h)} \mid 0 < h < 1\} \subset L^2(\Omega; \mathbb{R}^3)$ , obtained in ( $\gamma 2$ ) of Theorem 3.1, it will be seen from Theorem 3.1 Corollary 2 and Remark 2 that:

$$\begin{aligned} \mathcal{F}_\alpha^\circ(m^\circ) &\leq \liminf_{i \rightarrow \infty} \mathcal{F}_\alpha^{(h_i)}(m^{(h_i)}) \leq \limsup_{i \rightarrow \infty} \mathcal{F}_\alpha^{(h_i)}(m^{(h_i)}) \\ &\leq \lim_{i \rightarrow \infty} \mathcal{F}_\alpha^{(h_i)}(\hat{m}^{(h_i)}) = \mathcal{F}_\alpha^\circ(\hat{m}^\circ). \end{aligned} \quad (18)$$

It implies that  $m^\circ$  solves the limiting problem  $(\text{MP})^\circ$ . Furthermore, putting  $\hat{m}^\circ = m^\circ$  in (18), it is deduced that:

$$\mathcal{F}_\alpha^{(h)}(m^{(h)}) \rightarrow \mathcal{F}_\alpha^\circ(m^\circ) \text{ as } h \searrow 0.$$

Thus, we conclude the assertion (II).  $\square$

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