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3D-2D ASYMPTOTIC OBSERVATION FOR MINIMIZATION PROBLEMS ASSOCIATED WITH DEGENERATIVE ENERGY-COEFFICIENTS

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ABSTRACT. In this paper, a class of minimization problems, labeled by an index $0 < h < 1$, is considered. Each minimization problem is for a *free-energy*, motivated by the magnetics in 3D-ferromagnetic thin film, and in the context, the index h denotes the thickness of the observing film. The Main Theorem consists of two themes, which are concerned with the study of the solvability (existence of minimizers) and the 3D-2D asymptotic analysis for our minimization problems. These themes will be discussed under degenerative setting of the material coefficients, and such degenerative situation makes the energy-domain be variable with respect to h . In conclusion, assuming some restrictive conditions for the domain-variation, a definite association between our 3D-minimization problems, for very thin h , and a 2D-limiting problem, as $h \searrow 0$, will be demonstrated with helps from the theory of Γ -convergence.

1. Introduction. Let $S \subset \mathbb{R}^2$ be a two-dimensional bounded domain with a smooth boundary, and let $\Omega \subset \mathbb{R}^3$ be a three-dimensional cylindrical domain, given by $\Omega := S \times (0, 1)$. Let $\alpha : \overline{\Omega} \rightarrow [0, \infty)$ be a given nonnegative and continuous function.

In this paper, let us imagine the situation that a ferromagnetic thin film is applied on a thin region $\Omega^{(h)} := S \times (0, h)$ with a (small) thickness $0 < h < 1$. As a possible free-energy for the magnetic study in such situation, the following functional, denoted by $\mathcal{E}_\alpha^{(h)}$:

$$\mathcal{E}_\alpha^{(h)}(m) := \Psi_\alpha^{(h)}(m) + \int_{\Omega^{(h)}} \left(\varphi(m) + \frac{1}{2} \nabla \zeta \cdot m \right) d\mathcal{L}^3, \quad (1)$$

for any $m = (m_1, m_2, m_3) \in L^2(\Omega^{(h)}; \mathbb{R}^3)$;

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subject to:

$$\operatorname{div}(-\nabla\zeta + \overline{0m}) = 0, \quad \text{in } \mathbb{R}^3, \quad (2)$$

$$|m| = m_s, \quad \mathcal{L}^3\text{-a.e. in } \Omega; \quad (3)$$

was proposed by Brown [7] (1963), where Ψ_α is the lower semi-continuous envelopment of a functional:

$$\psi \in W^{1,2}(\Omega^{(h)}; \mathbb{R}^3) \cap L^2(\Omega^{(h)}; \mathbb{S}^2) \mapsto \int_{\Omega^{(h)}} \alpha |\nabla\psi|^2 d\mathcal{L}^3;$$

onto the space $L^2(\Omega^{(h)}; \mathbb{R}^3)$.

In (1), the value of $\mathcal{E}_\alpha^{(h)}$ denotes an energy quantity, per unit volume in $\Omega^{(h)}$, and the variable $m = (m_1, m_2, m_3)$ denotes the magnetization in the region $\Omega^{(h)}$ of magnetic thin film. In this light, the minimizers of $\mathcal{E}_\alpha^{(h)}$ are supposed to represent the most probable profile of the magnetization distribution applied on $\Omega^{(h)}$. Here, the given function α is the so-called material coefficient, and this coefficient is supposed to be degenerative somewhere in Ω . $\varphi : \mathbb{R}^3 \rightarrow [0, \infty)$ is a given continuous function, which is involved in the magnetization anisotropy.

The function $\zeta : \mathbb{R}^3 \rightarrow \mathbb{R}$ as in (1)-(2) denotes the magnetic field potential, and hence, it is prescribed as the solution of the simplified Maxwell equation (2). Here, the notation “ $\overline{0}$ ” denotes the zero-extension of functions. In addition to the above, let us note that the free-energy $\mathcal{E}_\alpha^{(h)}$ is considered under the constrained condition (3), by a positive constant m_s of the magnetization saturation.

In this paper, we set:

$$\mathcal{L}^2(S) = 1 \quad (\text{and hence } \mathcal{L}^3(\Omega) = 1), \quad \text{and } m_s = 1;$$

for simplicity. On that basis, let us denote by $T^{(h)}$ the scale transform, defined as:

$$T^{(h)} : x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mapsto (x_1, x_2, hx_3) \in \mathbb{R}^3;$$

to consider a rescaled minimization problem, denoted by (MP)^(h).

(MP)^(h) Find a vectorial function $m^{(h)} = (m_1^{(h)}, m_2^{(h)}, m_3^{(h)}) \in L^2(\Omega; \mathbb{R}^3)$ of three variables, which minimizes the following functional on $L^2(\Omega; \mathbb{R}^3)$:

$$\begin{aligned} \mathcal{F}_\alpha^{(h)}(m) &:= \Phi_\alpha^{(h)}(m) + \int_{\Omega} \varphi(m) d\mathcal{L}^3 \\ &\quad + \frac{1}{2} \int_{\Omega} \left(\nabla_{\mathbb{P}} \zeta \cdot m_{\mathbb{P}} + \frac{1}{h} \partial_3 \zeta m_3 \right) d\mathcal{L}^3, \end{aligned} \quad (4)$$

$$\text{for any } m = (m_1, m_2, m_3) \in L^2(\Omega; \mathbb{R}^3);$$

subject to:

$$\nabla_{\mathbb{P}} \cdot (-\nabla_{\mathbb{P}} \zeta + \overline{0m_{\mathbb{P}}}) + \frac{1}{h} \partial_3 \left(-\frac{1}{h} \partial_3 \zeta + \overline{0m_3} \right) = 0, \quad \text{in } \mathbb{R}^3; \quad (5)$$

where the subscript “ \mathbb{P} ” denotes the restriction of the situation onto the two-dimensional plane \mathbb{R}^2 , e.g.:

$$y_{\mathbb{P}} := (y_1, y_2), \quad \text{for } y = (y_1, y_2, y_3) \in \mathbb{R}^3,$$

$$\mu_{\mathbb{P}} := (\mu_1, \mu_2) \in L^2(\Omega; \mathbb{R}^2), \quad \text{for } \mu = (\mu_1, \mu_2, \mu_3) \in L^2(\Omega; \mathbb{R}^3),$$

and the distributional gradient

$$\nabla_{\mathbb{P}} \mu := \begin{pmatrix} \partial_1 \mu_1 & \partial_2 \mu_1 \\ \partial_1 \mu_2 & \partial_2 \mu_2 \\ \partial_1 \mu_3 & \partial_2 \mu_3 \end{pmatrix}, \quad \text{for } \mu = (\mu_1, \mu_2, \mu_3) \in L^2(\Omega; \mathbb{R}^3);$$

and $\Phi_\alpha^{(h)}$ is the rescaled version of the lower semi-continuous envelopment $\Psi_\alpha^{(h)}$ by $T^{(h)}$, and it is rigorously defined as:

$$\Phi_\alpha^{(h)}(m) := \begin{cases} \inf_{\{\psi^{(i)}\} \in \mathcal{Q}_\Omega(m)} \liminf_{i \rightarrow \infty} \int_\Omega \alpha^{(h)} \left(|\nabla_P \psi^{(i)}|^2 + \frac{1}{h^2} |\partial_3 \psi^{(i)}|^2 \right) d\mathcal{L}^3, \\ \text{if } |m| = 1, \mathcal{L}^3\text{-a.e. in } \Omega, \\ \infty, \text{ otherwise, for any } m \in L^2(\Omega; \mathbb{R}^3); \end{cases} \quad (6)$$

by using a composition:

$$\alpha^{(h)}(x) := (\alpha \circ T^{(h)})(x) = \alpha(x_1, x_2, hx_3), \quad \text{for all } x = (x_1, x_2, x_3) \in \bar{\Omega};$$

and a class of approximating functions:

$$\mathcal{Q}_\Omega(m) := \left\{ \{\psi^{(i)}\} \mid \begin{array}{l} \psi^{(i)} \in W^{1,2}(\Omega; \mathbb{R}^3) \cap L^2(\Omega; \mathbb{S}^2), \quad i = 1, 2, 3, \dots, \\ \text{and } \psi^{(i)} \rightarrow m \text{ in } L^2(\Omega; \mathbb{R}^3) \text{ as } i \rightarrow \infty \end{array} \right\}; \quad (7)$$

for any $m \in L^2(\Omega; \mathbb{S}^2)$.

As is easily seen, the inverse transform $(T^{(h)})^{-1}$ provides a bijective correspondence between the minimizers $m^{(h)}$ of $(\text{MP})^{(h)}$ and the minimizers $m_{\text{org}}^{(h)} := m^{(h)} \circ (T^{(h)})^{-1}$ of the original free-energy $\mathcal{E}_\alpha^{(h)}$. Besides, let us note that the domains $\text{Dom}(\mathcal{F}_\alpha^{(h)})$ of free-energies are not uniform, but variable with respect to $0 < h < 1$, and the variation is directly governed by the degenerating part of the coefficient:

$$A_0^{(h)} := (\alpha^{(h)})^{-1}(0) \subset \bar{\Omega}, \quad \text{for } 0 < h < 1.$$

Under very thin situation of the thickness h , it is naturally expected that the minimization problem $(\text{MP})^{(h)}$ can be reduced to a simpler problem, considered in two-dimensional domain S . Such reduction will be realized through the limiting observation for $(\text{MP})^{(h)}$ as $h \searrow 0$, and then, the binary function:

$$\begin{aligned} \alpha^\circ(x_1, x_2) &:= \alpha(x_1, x_2, 0) \text{ for any } (x_1, x_2) \in \bar{S}, \\ &\text{with the degenerating part } A_0^\circ := (\alpha^\circ)^{-1}(0); \end{aligned}$$

will be the material coefficient in the limiting problem. Actually, in the h -independent case of $A_0^{(h)}$, a number of like-minded study results, such as [1, 2, 3, 4, 5, 6, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18], were reported, from various viewpoints, and some of them concluded a definite association between the limiting profile of $(\text{MP})^{(h)}$, as $h \searrow 0$, and the following minimization problem, denoted by $(\text{MP})^\circ$.

$(\text{MP})^\circ$ Find a vectorial function $m^\circ = (m_1^\circ, m_2^\circ, m_3^\circ) \in L^2(S; \mathbb{R}^3)$ of two variables, which minimizes the following functional:

$$\begin{aligned} \mathcal{F}_\alpha^\circ(m) &:= \Phi_\alpha^\circ(m) + \int_S \varphi(m) d\mathcal{L}^2 + \frac{1}{2} \int_S |m_3|^2 d\mathcal{L}^2, \\ &\text{for any } m = (m_1, m_2, m_3) \in L^2(S; \mathbb{R}^3); \end{aligned} \quad (8)$$

where Φ_α° is a convex function on $L^2(S; \mathbb{R}^3)$, defined as:

$$\Phi_\alpha^\circ(m) := \begin{cases} \inf_{\{\psi^{(i)}\} \in \mathcal{Q}_S(m)} \liminf_{i \rightarrow \infty} \int_S \alpha^\circ |\nabla \psi^{(i)}|^2 d\mathcal{L}^2, \\ \text{if } |m| = 1, \mathcal{L}^2\text{-a.e. in } S, \\ \infty, \text{ otherwise, for any } m \in L^2(S; \mathbb{R}^3); \end{cases} \quad (9)$$

by using a class of approximating sequences:

$$\mathcal{Q}_S(m) := \left\{ \{\psi^{(i)}\} \mid \begin{array}{l} \psi^{(i)} \in W^{1,2}(S; \mathbb{R}^3) \cap L^2(S; \mathbb{S}^2), \quad i = 1, 2, 3, \dots, \\ \text{and } \psi^{(i)} \rightarrow m \text{ in } L^2(S; \mathbb{R}^3) \text{ as } i \rightarrow \infty \end{array} \right\}.$$

Now, the main theme of this paper is to verify whether analogous observation is available even under h -variable situation of $A_0^{(h)}$ (or energy domains), or not. To this end, we here impose the following two conditions for the material coefficient α :

- (a1) $\mathcal{L}^3(A_0^{(h)}) = 0$, for $0 < h < 1$;
- (a2) $\mathcal{L}^2(A_0^\circ) = 0$, and $\alpha^\circ(x_P) \leq \alpha(x)$, for all $x = (x_1, x_2, x_3) \in \bar{\Omega}$.

Consequently, a certain positive answer for our theme will be demonstrated in the main theorem, stated as follows.

Main Theorem. (I) *Let us assume the condition (a1). Then, for any $0 < h < 1$, the problem $(MP)^{(h)}$ admits at least one solution (minimizer) $m^{(h)}$.*
 (II) *Under the conditions (a1)-(a2), there exist a sequence $\{h_i \mid i = 1, 2, 3, \dots\} \subset (0, 1)$ and a function $m^\circ \in L^2(S; \mathbb{R}^3)$ of two variables, such that:*
 (i) $h_i \searrow 0$, $m^{(h_i)} \rightarrow m^\circ$ in $L^2(\Omega; \mathbb{R}^3)$, $\mathcal{F}_\alpha^{(h_i)}(m^{(h_i)}) \rightarrow \mathcal{F}_\alpha^\circ(m^\circ)$, as $i \rightarrow \infty$;
 (ii) *the limit m° solves the problem $(MP)^\circ$;*

where $\{m^{(h)} \mid 0 < h < 1\}$ is the sequence of minimizers $m^{(h)}$, $0 < h < 1$, obtained in (I).

The content of this paper is as follows. In the next Section 2, some key-properties for the minimization problems $(MP)^{(h)}$, $0 < h < 1$, and $(MP)^\circ$ are briefly mentioned as preliminaries. In subsequent Section 3, the continuous dependence between the energy sequence $\{\mathcal{F}_\alpha^{(h)} \mid 0 < h < 1\}$ and the energy \mathcal{F}_α° , as $h \searrow 0$, will be shown by means of the notion of Γ -convergence (cf. [9]). On that basis, the final Section 4 will be devoted to the proof of Main Theorem.

Notation. Throughout this paper, the Lebesgue measure is denoted by \mathcal{L}^n , for any observing dimension $n \in \mathbb{N}$.

For any abstract Banach space, the norm of X is denoted by $|\cdot|_X$. However, when X is an Euclidean space, the norm is simply denoted by $|\cdot|$. Besides, for any functional $F : X \rightarrow (-\infty, \infty]$, we denote by $\text{Dom}(F)$ the domain of F , and for any $r > 0$, we denote by $L(r; F)$ the sublevel set of F , more precisely:

$$\text{Dom}(F) := \{ \xi \in X \mid F(\xi) < \infty \} \quad \text{and} \quad L(r; F) := \{ \xi \in X \mid F(\xi) \leq r \}.$$

For any abstract Hilbert space H , the inner product of H is denoted by $(\cdot, \cdot)_H$. However, when H is an Euclidean space, the inner product between two vectors $\xi, \eta \in H$ is simply denoted by $\xi \cdot \eta$.

2. Preliminaries. Let us start with summarizing the known-facts, concerned with the coupled Maxwell equation (5).

(Fact 1) (Summary of [18, Lemma 3.1]) Let us fix any $0 < h < 1$. Then, for any function $m = (m_1, m_2, m_3) \in L^2(\Omega; \mathbb{R}^3)$, the solution $\zeta^{(h)}$ of the Maxwell equation (5) is prescribed in the scope of a Hilbert space:

$$V^{(h)} := \left\{ v \in H_{\text{loc}}^1(\mathbb{R}^3) \mid \nabla v \in L^2(\mathbb{R}^3; \mathbb{R}^3) \quad \text{and} \quad \int_{B_\Omega} v \, d\mathcal{L}^3 = 0 \right\};$$

endowed with a h -dependent inner product:

$$(u, v)_{V^{(h)}} := (\nabla_P u, \nabla_P v)_{L^2(\mathbb{R}^3; \mathbb{R}^{3 \times 2})} + \frac{1}{h^2} (\partial_3 u, \partial_3 v)_{L^2(\mathbb{R}^3; \mathbb{R}^3)}, \quad \text{for } u, v \in V^{(h)};$$

where B_Ω is an (fixed) open ball containing Ω . Then, the solution $\zeta^{(h)} \in V^{(h)}$ is supposed to fulfill a weak formulation by the following variational identity:

$$\int_{\Omega} \left((\nabla_{\mathbb{P}} \zeta^{(h)} - m_{\mathbb{P}}) \cdot \nabla_{\mathbb{P}} v + \frac{1}{h} \left(\frac{1}{h} \partial_3 \zeta^{(h)} - m_3 \right) \partial_3 v \right) d\mathcal{L}^3 = 0, \quad (10)$$

for any $v \in V^{(h)}$;

Moreover, taking more one function $\tilde{m} \in L^2(\Omega; \mathbb{R}^3)$, arbitrarily, and taking another solution $\tilde{\zeta}^{(h)}$ of (10) when $m = \tilde{m}$, it follows that:

$$|\zeta^{(h)} - \tilde{\zeta}^{(h)}|_{V^{(h)}} \leq |m - \tilde{m}|_{L^2(\Omega; \mathbb{R}^3)}. \quad (11)$$

Hence, the variational problem (10) is well-posed.

(Fact 2) (Summary of [15, Proposition 4.1]) Let us set:

$$\begin{cases} F_{\text{mag}}^{(h)}(m) := \frac{1}{2} \int_{\Omega} \left(\nabla_{\mathbb{P}} \zeta^{(h)} \cdot m_{\mathbb{P}} + \frac{1}{h} \partial_3 \zeta^{(h)} m_3 \right) d\mathcal{L}^3, \\ F_{\text{mag}}^{\circ}(m) := \frac{1}{2} \int_{\Omega} |m_3|^2 d\mathcal{L}^3, \end{cases} \quad (12)$$

for any $m = (m_1, m_2, m_3) \in L^2(\Omega; \mathbb{R}^3)$;

by using the solution $\zeta^{(h)}$ of the variational identity (10). On that basis, let us assume that $\{\tilde{m}^{(h)} \mid 0 < h < 1\} \subset L^2(\Omega; \mathbb{R}^3)$, and $\tilde{m}^{(h)} \rightarrow \tilde{m}$ in $L^2(\Omega; \mathbb{R}^3)$ as $h \searrow 0$, for some $\tilde{m} = (\tilde{m}_1, \tilde{m}_2, \tilde{m}_3) \in L^2(\Omega; \mathbb{R}^3)$. Then:

$$F_{\text{mag}}^{(h)}(\tilde{m}^{(h)}) \rightarrow F_{\text{mag}}^{\circ}(\tilde{m}), \quad \text{as } h \searrow 0.$$

Next, let us look toward the key-properties of the lower semi-continuous envelopments $\Phi_{\alpha}^{(h)}$, $0 < h < 1$, and Φ_{α}° .

Lemma 2.1. (I) For any $0 < h < 1$, the functional $\Phi_{\alpha}^{(h)}$ is a maximal functional in the class of l.s.c. functionals on $L^2(\Omega; \mathbb{R}^3)$, supporting the functional:

$$\psi \in W^{1,2}(\Omega; \mathbb{R}^3) \cap L^2(\Omega; \mathbb{S}^2) \mapsto \int_{\Omega} \alpha^{(h)} \left(|\nabla_{\mathbb{P}} \psi|^2 + \frac{1}{h^2} |\partial_3 \psi|^2 \right) d\mathcal{L}^3.$$

Moreover:

$$(i-1) \quad \Phi_{0,\alpha}^{(h)}(m) := \int_{\Omega \setminus A_0^{(h)}} \alpha^{(h)} \left(|\nabla_{\mathbb{P}} m|^2 + \frac{1}{h^2} |\partial_3 m|^2 \right) d\mathcal{L}^3 \leq \Phi_{\alpha}^{(h)}(m),$$

for any $m \in W_{\text{loc}}^{1,2}(\Omega \setminus A_0^{(h)}; \mathbb{R}^3) \cap L^2(\Omega; \mathbb{S}^2)$;

$$(i-2) \quad W^{1,2}(\Omega; \mathbb{R}^3) \cap L^2(\Omega; \mathbb{S}^2) \subset \text{Dom}(\Phi_{\alpha}^{(h)}) \subset \text{Dom}(\Phi_{0,\alpha}^{(h)}) \subset W_{\text{loc}}^{1,2}(\Omega \setminus A_0^{(h)}; \mathbb{R}^3),$$

and $\Phi_{0,\alpha}^{(h)} = \Phi_{\alpha}^{(h)}$ on $W^{1,2}(\Omega; \mathbb{R}^3) \cap L^2(\Omega; \mathbb{S}^2)$;

(i-3) for any $m \in \text{Dom}(\Phi_{\alpha}^{(h)})$, there exists a sequence $\{\mu^{(i)} \mid i = 1, 2, 3, \dots\} \subset W^{1,2}(\Omega; \mathbb{R}^3) \cap L^2(\Omega; \mathbb{S}^2)$ such that

$$\mu^{(i)} \rightarrow m \text{ in } L^2(\Omega; \mathbb{R}^3) \text{ and } \Phi_{0,\alpha}^{(h)}(\mu^{(i)}) \rightarrow \Phi_{\alpha}^{(h)}(m), \text{ as } i \rightarrow \infty.$$

(II) The functional Φ_{α}° is a maximal functional in the class of l.s.c. functionals on $L^2(S; \mathbb{R}^3)$, supporting the functional:

$$\psi \in W^{1,2}(S; \mathbb{R}^3) \cap L^2(S; \mathbb{S}^2) \mapsto \int_S \alpha^{\circ} |\nabla \psi|^2 d\mathcal{L}^2.$$

Moreover:

- (ii-1) $\Phi_{0,\alpha}^\circ(m) := \int_{S \setminus A_0^\circ} \alpha^\circ |\nabla m|^2 d\mathcal{L}^2 \leq \Phi_\alpha^{(h)}(m)$, for any $m \in W_{\text{loc}}^{1,2}(S \setminus A_0^\circ; \mathbb{R}^3) \cap L^2(S; \mathbb{S}^2)$;
- (ii-2) $W^{1,2}(S; \mathbb{R}^3) \cap L^2(S; \mathbb{S}^2) \subset \text{Dom}(\Phi_\alpha^\circ) \subset \text{Dom}(\Phi_{0,\alpha}^\circ) \subset W_{\text{loc}}^{1,2}(S \setminus A_0^\circ; \mathbb{R}^3)$, and $\Phi_{0,\alpha}^\circ = \Phi_\alpha^\circ$ on $W^{1,2}(S; \mathbb{R}^3)$
- (ii-3) for any $m \in \text{Dom}(\Phi_\alpha^{(h)})$, there exists a sequence $\{\mu^{(i)} \mid i = 1, 2, 3, \dots\} \subset W^{1,2}(S; \mathbb{R}^3) \cap L^2(S; \mathbb{S}^2)$ such that
- $$\mu^{(i)} \rightarrow m \text{ in } L^2(S; \mathbb{R}^3) \text{ and } \Phi_{0,\alpha}^\circ(\mu^{(i)}) \rightarrow \Phi_\alpha^\circ(m), \text{ as } i \rightarrow \infty.$$

Proof. This lemma follows, directly, from the definition formulas (6) and (9). \square

Remark 1. (Key-properties for free-energies) By virtue of (4), (8) and Lemma 2.1, the functional $\mathcal{F}_\alpha^{(h)}$ (resp. \mathcal{F}_α°) turns out to be l.s.c. in $L^2(\Omega; \mathbb{R}^3)$ (resp. in $L^2(S; \mathbb{R}^3)$), and $\text{Dom}(\mathcal{F}_\alpha^{(h)}) = \text{Dom}(\Phi_\alpha^{(h)})$, for $0 < h < 1$ (resp. $\text{Dom}(\mathcal{F}_\alpha^\circ) = \text{Dom}(\Phi_\alpha^\circ)$). Furthermore, under the conditions (a1)-(a2), as in introduction, the variation of energy-domains $\text{Dom}(\mathcal{F}_\alpha^{(h)})$, with respect to h , will be restrictive in the sense that $\text{Dom}(\mathcal{F}_\alpha^{(h)})$ will be included in $W_{\text{loc}}^{1,2}(\Omega \setminus (A_0^\circ \times (0, 1)); \mathbb{R}^3)$, uniformly, for all $0 < h < 1$.

Taking into account of Lemma 2.1, Remark 1 and [16, Corollary 2], we can derive the following corollary.

Corollary 1. (Compactness) *Let us assume the condition (a1), as in introduction. Then, for any $0 < h < 1$ and any $r > 0$, the sublevel sets $L(r; \Phi_\alpha^{(h)})$ and $L(r; \mathcal{F}_\alpha^{(h)})$ are compact in $L^2(\Omega; \mathbb{R}^3)$. Additionally, if we assume the conditions (a1)-(a2), as in introduction, then for any $r > 0$, the sublevel sets $L(r; \Phi_\alpha^\circ)$ and $L(r; \mathcal{F}_\alpha^\circ)$ are compact in $L^2(S; \mathbb{R}^3)$, and the unions*

$$\mathcal{U}_\Phi(r) := \bigcup_{0 < h < 1} L(r; \Phi_\alpha^{(h)}) \text{ and } \mathcal{U}_\mathcal{F}(r) := \bigcup_{0 < h < 1} L(r; \mathcal{F}_\alpha^{(h)});$$

are relatively compact in $L^2(\Omega; \mathbb{R}^3)$.

Proof. Let us assume (a1), and let us fix any $0 < h < 1$ and any $r > 0$. Here, taking the solution $\zeta^{(h)}$ as the test function of (10), we have:

$$F_{\text{mag}}^{(h)}(m) = \frac{1}{2} |\zeta^{(h)}|_{V^{(h)}}^2 \geq 0, \text{ for any } m \in L^2(\Omega; \mathbb{R}^3). \quad (13)$$

Subsequently, we see from (4), (13) and (i-1) of Lemma 2.1 that:

$$L(r; \mathcal{F}_\alpha^{(h)}) \subset L(r; \Phi_\alpha^{(h)}) \subset L(r; \Phi_{0,\alpha}^{(h)}).$$

Since the compactness of $L(r; \Phi_{0,\alpha}^{(h)})$ in $L^2(\Omega; \mathbb{R}^3)$ is already concluded in [16, Corollary 2], we can say that its closed subsets $L(r; \Phi_\alpha^{(h)})$ and $L(r; \mathcal{F}_\alpha^{(h)})$ are also so.

Just as in the above, the assertion for the sublevel sets $L(r; \Phi_\alpha^\circ)$ and $L(r; \mathcal{F}_\alpha^\circ)$ (resp. for the unions $\mathcal{U}_\Phi(r)$ and $\mathcal{U}_\mathcal{F}(r)$) can be concluded, with the helps from the condition (a2) and [16, Corollary 2] (resp. [16, Theorem 3.4]). \square

3. Continuous dependence of energies. The objective in this section is summarized in the following theorem, concerned with continuous dependence (Γ -convergence) of lower semi-continuous envelopments, as $h \searrow 0$.

Theorem 3.1. (*Γ -convergence from $\Phi_\alpha^{(h)}$ to Φ_α° as $h \searrow 0$)* Let us assume the conditions (a1)-(a2), as in introduction. Then, the sequence $\{\Phi_\alpha^{(h)} \mid 0 < h < 1\}$ of the lower semi-continuous envelopments converges to Φ_α° , in the sense of Γ -convergence, as $h \searrow 0$. More precisely, referring to [9] (or [1, Lemma 2.3]), the above assertion is equivalent to:

- ($\gamma 1$) $\liminf_{h \searrow 0} \Phi_\alpha^{(h)}(\tilde{m}^{(h)}) \geq \Phi_\alpha^\circ(\tilde{m}^\circ)$, if $\{\tilde{m}^{(h)} \mid 0 < h < 1\} \subset L^2(\Omega; \mathbb{R}^3)$, $\tilde{m}^\circ \in L^2(\Omega; \mathbb{R}^3)$, and $\tilde{m}^{(h)} \rightarrow \tilde{m}^\circ$ in $L^2(\Omega; \mathbb{R}^3)$ as $h \searrow 0$;
 ($\gamma 2$) for any $\hat{m}^\circ \in \text{Dom}(\Phi_\alpha^\circ)$, there exists a sequence $\{\hat{m}^{(h)} \mid 0 < h < 1\} \subset L^2(\Omega; \mathbb{R}^3)$, such that $\hat{m}^{(h)} \rightarrow \hat{m}^\circ$ in $L^2(\Omega; \mathbb{R}^3)$ and $\Phi_\alpha^{(h)}(\hat{m}^{(h)}) \rightarrow \Phi_\alpha^\circ(\hat{m}^\circ)$, as $h \searrow 0$.

This theorem is proved by relying on some classes of open sets, mentioned in the following lemma.

Lemma 3.2. (*Open coverings for non-degenerate parts*) There exists a sequence $\{\bar{\alpha}_\ell^\circ \mid \ell = 1, 2, 3, \dots\} \subset (0, 1)$ and a covering $\{S_\ell \mid \ell = 1, 2, 3, \dots\} \subset S \setminus A_0^\circ$ of $S \setminus A_0^\circ$ with smooth boundaries ∂S_ℓ ($\ell = 1, 2, 3, \dots$), such that:

$$\begin{cases} \emptyset \neq S_1 \subset \subset \dots \subset S_\ell \subset \subset \dots \subset S \setminus A_0^\circ = \bigcup_{\ell=0}^{\infty} S_\ell, \\ \bar{\alpha}_1^\circ > \dots > \bar{\alpha}_\ell^\circ > \dots > 0 = \lim_{\ell \rightarrow \infty} \bar{\alpha}_\ell^\circ, \quad \text{and } \alpha^\circ \geq \bar{\alpha}_\ell^\circ \text{ on } S_\ell, \text{ for } \ell = 1, 2, 3, \dots \end{cases}$$

Hence, if we assume the conditions (a1)-(a2) as in introduction, then a sequence $\{\Omega_\ell^\dagger\} := \{S_\ell \times (0, 1) \mid \ell = 1, 2, 3, \dots\}$ turns out to be a covering of an open set $\Omega^\dagger := \Omega \setminus (A_0^\circ \times (0, 1))$, with Lipschitz boundaries $\partial \Omega_\ell^\dagger$ ($\ell = 1, 2, 3, \dots$), such that:

$$\begin{cases} \emptyset \neq \Omega_1^\dagger \subset \subset \dots \subset \Omega_\ell^\dagger \subset \subset \dots \subset \Omega^\dagger = \bigcup_{\ell=0}^{\infty} \Omega_\ell^\dagger \subset \Omega \setminus A_0^\circ, \quad \mathcal{L}^3(\Omega \setminus \Omega^\dagger) = 0, \\ \alpha^{(h)} \geq \alpha^\circ \geq \bar{\alpha}_\ell^\circ > 0 \text{ on } \Omega_\ell^\dagger, \text{ for } \ell = 1, 2, 3, \dots \text{ and } 0 < h < 1. \end{cases} \quad (14)$$

Proof of Lemma 3.2. This lemma is a direct consequence of the line of arguments, discussed in [16, Lemma 4.1, Remark 4-5]. \square

Proof of Theorem 3.1. First, we verify the assertion ($\gamma 1$). Then, it is enough to consider only the case when $\liminf_{h \searrow 0} \Phi_\alpha^{(h)}(\tilde{m}^{(h)}) < \infty$, since another case is trivial.

On account of (i-3) in Lemma 2.1, we find a sequence $\{\check{h}_i \mid i = 1, 2, 3, \dots\} \subset (0, 1)$ and a sequence $\{\check{\mu}^{(i)} \mid i = 1, 2, 3, \dots\} \subset W^{1,2}(\Omega; \mathbb{R}^3) \cap L^2(\Omega; \mathbb{S}^2)$, such that:

$$\begin{cases} \check{h}_i \searrow 0 \text{ and } \check{\mu}^{(i)} \rightarrow \tilde{m}^\circ \text{ in } L^2(\Omega; \mathbb{R}^3), \text{ as } i \rightarrow \infty, \\ \lim_{i \rightarrow \infty} \Phi_{0,\alpha}^{(\check{h}_i)}(\check{\mu}^{(i)}) = \liminf_{h \searrow 0} \Phi_\alpha^{(h)}(\tilde{m}^{(h)}). \end{cases}$$

Here, in the light of (14) and (i-1) in Lemma 2.1,

$$\begin{aligned} |\partial_3 \tilde{m}^\circ|_{L^2(\Omega_\ell^\dagger; \mathbb{R}^3)}^2 &\leq \liminf_{i \rightarrow \infty} |\partial_3 \check{\mu}^{(i)}|_{L^2(\Omega_\ell^\dagger; \mathbb{R}^3)}^2 \\ &\leq \frac{1}{\bar{\alpha}_\ell^\circ} \sup_{i \geq 1} \Phi_{0,\alpha}^{(\check{h}_i)}(\check{\mu}^{(i)}) \lim_{i \rightarrow \infty} \check{h}_i^2 = 0, \quad \ell = 1, 2, 3, \dots, \end{aligned} \quad (15)$$

which implies $\partial_3 \check{m}^\circ = 0$, \mathcal{L}^3 -a.e. in Ω^\dagger . Hence, the limit \check{m}° can be regarded to belong to the class $L^2(S \setminus A_0^\circ; \mathbb{R}^3)$ ($= L^2(S; \mathbb{R}^3)$) of binary functions. Subsequently, let us set:

$$\check{\mu}^{(i)}(x_1, x_2) := \check{\mu}^{(i)}(x_1, x_2, c_i), \text{ for } \mathcal{L}^2\text{-a.e. } (x_1, x_2) \in S \text{ and } i = 1, 2, 3, \dots;$$

by using a collection $\{c_i \mid i = 1, 2, 3, \dots\} \subset (0, 1)$ of constants, such that:

$$\int_S \alpha^{(\check{h}_i)}(x_1, x_2, c_i) |\nabla_{\mathbb{P}} \check{\mu}^{(i)}(x_1, x_2, c_i)|^2 d\mathcal{L}^2 \leq \int_\Omega \alpha^{(\check{h}_i)} |\nabla_{\mathbb{P}} \check{\mu}^{(i)}|^2 d\mathcal{L}^3, \quad i = 1, 2, 3, \dots.$$

Then, with the helps from (15) and Fubini's theorem, it is computed that:

$$\begin{aligned} & \lim_{i \rightarrow \infty} |\check{\psi}^{(i)} - \check{\mu}^{(i)}|_{L^2(\Omega; \mathbb{R}^3)}^2 \\ &= \lim_{i \rightarrow \infty} \left(|\check{\psi}^{(i)} - \check{\mu}^{(i)}|_{L^2(\Omega_\ell^\dagger; \mathbb{R}^3)}^2 + |\check{\psi}^{(i)} - \check{\mu}^{(i)}|_{L^2(\Omega \setminus \Omega_\ell^\dagger; \mathbb{R}^3)}^2 \right) \\ &\leq \lim_{i \rightarrow \infty} \int_{\Omega_\ell^\dagger} \int_0^1 |\partial_3 \check{\mu}^{(i)}|^2 d\mathcal{L}^1 d\mathcal{L}^3 + 4\mathcal{L}^3(\Omega \setminus \Omega_\ell^\dagger) \\ &\leq \lim_{i \rightarrow \infty} |\partial_3 \check{\mu}^{(i)}|_{L^2(\Omega_\ell^\dagger)}^2 + 4\mathcal{L}^3(\Omega \setminus \Omega_\ell^\dagger) = 4\mathcal{L}^3(\Omega \setminus \Omega_\ell^\dagger), \quad \text{for } \ell = 1, 2, 3, \dots \end{aligned}$$

It implies that:

$$\check{\psi}^{(i)} \rightarrow \check{m}^\circ \text{ in } L^2(\Omega; \mathbb{R}^3), \text{ as } i \rightarrow \infty; \quad (16)$$

since $\mathcal{L}^3(\Omega \setminus \Omega_\ell^\dagger) \rightarrow 0$ as $\ell \rightarrow \infty$. Taking into account of (a2), (16), (ii-2) in Lemma 2.1 and the lower semi-continuity of Φ_α° , we deduce that:

$$\begin{aligned} \liminf_{h \searrow 0} \Phi_\alpha^{(h)}(\check{m}^{(h)}) &= \lim_{i \rightarrow \infty} \Phi_\alpha^{(\check{h}_i)}(\check{\mu}^{(i)}) \geq \liminf_{i \rightarrow \infty} \int_\Omega \alpha^{(\check{h}_i)} |\nabla_{\mathbb{P}} \check{\mu}^{(i)}|^2 d\mathcal{L}^3 \\ &\geq \liminf_{i \rightarrow \infty} \int_S \alpha^{(\check{h}_i)}(x_1, x_2, c_i) |\nabla \check{\psi}^{(i)}(x_1, x_2)|^2 d\mathcal{L}^2 \geq \liminf_{i \rightarrow \infty} \Phi_\alpha^\circ(\check{\psi}^{(i)}) \geq \Phi_\alpha^\circ(\check{m}^\circ). \end{aligned}$$

Thus, the assertion ($\gamma 1$) is concluded.

Next, we verify the assertion ($\gamma 2$). Let us take any $\hat{m}^\circ \in \text{Dom}(\Phi_\alpha^\circ)$. Then, construction of the required sequence $\{\hat{m}^{(h)} \mid 0 < h < 1\}$ will be on the basis of a sequence $\{\hat{\mu}^{(i)} \mid i = 1, 2, 3, \dots\} \subset W^{1,2}(S; \mathbb{R}^3) \cap L^2(S; \mathbb{R}^3)$, which will be obtained as the approximating sequence, as in (ii-3) of Lemma 2.1, when $m = \hat{m}^\circ$. Here, noting that $\alpha^{(h)} \rightarrow \alpha^\circ$ in $C(\bar{\Omega})$ as $h \searrow 0$, there exists a sequence $\{\hat{h}_i \mid i = 1, 2, 3, \dots\} \subset (0, 1)$, such that:

$$\begin{cases} \bullet \hat{h}_1 > \dots > \hat{h}_i > \dots > 0 = \lim_{i \rightarrow \infty} \hat{h}_i, \\ \bullet 0 \leq \Phi_{0,\alpha}^{(h)}(\hat{\mu}^{(i)}) - \Phi_{0,\alpha}^\circ(\hat{\mu}^{(i)}) = \int_\Omega (\alpha^{(h)} - \alpha^\circ) |\nabla_{\mathbb{P}} \hat{\mu}^{(i)}|^2 d\mathcal{L}^3 < \frac{1}{2^i}, \\ \bullet \text{ for any } 0 < h < \hat{h}_i, \quad i = 1, 2, 3, \dots \end{cases}$$

On that basis, the finding sequence $\{\hat{m}^{(h)}\}$ will be constructed by putting:

$$\hat{m}^{(h)} := \hat{\mu}^{(i)} \text{ in } L^2(\Omega; \mathbb{R}^3), \text{ if } \hat{h}_{i+1} \leq h < \hat{h}_i, \quad i = 1, 2, 3, \dots;$$

with an optional setting $\hat{m}^{(h)} := \hat{\mu}^{(1)}$ in $L^2(\Omega; \mathbb{R}^3)$, for $\hat{h}_1 \leq h < 1$. \square

The above Theorem 3.1 actually implies the Γ -convergence of free-energies, stated as follows.

Corollary 2. (*Γ -convergence from $\mathcal{F}_\alpha^{(h)}$ to \mathcal{F}_α° as $h \searrow 0$) Let us assume the conditions (a1)-(a2). Then, the sequence $\{\mathcal{F}_\alpha^{(h)} \mid 0 < h < 1\}$ of free-energies converges to the limiting one \mathcal{F}_α° , in the sense of Γ -convergence, as $h \searrow 0$.*

Proof of Corollary 2. This corollary is immediately concluded, by taking into account of Theorem 3.1 and (Fact 2) in Section 2. \square

Remark 2. On account of (13) and (Fact 2), we will see that the sequence $\{\hat{m}^{(h)} \mid 0 < h < 1\}$ as in ($\gamma 2$) of Theorem 3.1 will realize the convergence:

$$\mathcal{F}_\alpha^{(h)}(\hat{m}^{(h)}) \rightarrow \mathcal{F}_\alpha^\circ(\hat{m}^\circ) \text{ as } h \searrow 0.$$

4. Proof of Main Theorem. We divide this section into two subsections, for the respective assertions (I) and (II) of Main Theorem.

4.1. Proof of (I) of Main Theorem. The proof of this assertion will be a slight modification of the argument, discussed in [16, Section 5.1]. In fact, under (a1), and under the fixed setting of $0 < h < 1$, we can take the so-called *minimizing sequence* $\{m_*^{(i)} \mid i = 1, 2, 3, \dots\} \subset \text{Dom}(\mathcal{F}_\alpha^{(h)})$ that is supposed to satisfy:

$$\mathcal{F}_\alpha^{(h)}(m_*^{(i)}) \searrow F_*^{(h)} := \inf_{m \in L^2(\Omega; \mathbb{R}^3)} \mathcal{F}_\alpha^{(h)}(m) \text{ as } i \rightarrow \infty.$$

Here, on account of (13), (Fact 1) and Corollary 1, a convergence subsequence $\{m_*^{(i_k)} \mid k = 1, 2, 3, \dots\} \subset \{m_*^{(i)}\}$ will be found with the limit $m_* \in L^2(\Omega; \mathbb{R}^3)$, and it will be seen that:

$$\begin{cases} m_*^{(i_k)} \rightarrow m_* \text{ in } L^2(\Omega; \mathbb{R}^3), & \varphi(m_*^{(i_k)}) \rightarrow \varphi(m_*) \text{ in } L^1(\Omega), \\ F_{\text{mag}}^{(h)}(m_*^{(i_k)}) \rightarrow F_{\text{mag}}^{(h)}(m_*), \end{cases} \text{ as } k \rightarrow \infty.$$

In response to the above, we infer from the lower semi-continuity of $\mathcal{F}_\alpha^{(h)}$ that the limit m_* is one of minimizers of (MP)^(h). \square

4.2. Proof of (II) of Main Theorem. Let us assume the conditions (a1)-(a2), and let us take a sequence $\{m^{(h)} \mid 0 < h < 1\}$ of minimizers of $\mathcal{F}_\alpha^{(h)}$, $0 < h < 1$.

Let us set $\nu^* := [1, 0, 0] \in \mathbb{S}^2$. Then, for the variational identity (10) when $m \equiv \nu^*$, taking the solution itself as the test function v in (10) yields that $F_{\text{mag}}^{(h)}(\nu^*) \leq 1$, for any $0 < h < 1$ (see [16, Section 5.2], for details). In view of this,

$$\begin{aligned} \Phi_\alpha^{(h)}(m^{(h)}) &\leq \mathcal{F}_\alpha^{(h)}(m^{(h)}) \leq \mathcal{F}_\alpha^{(h)}(\nu^*) = \Phi_\alpha^{(h)}(\nu^*) + |\varphi(\nu^*)|_{L^1(\Omega)} + F_{\text{mag}}^{(h)}(\nu^*) \\ &\leq \varphi(\nu^*) + 1, \text{ for all } 0 < h < 1. \end{aligned} \quad (17)$$

Since, the above (17) implies that $\{m^{(h)}\} \subset \mathcal{U}_{\mathcal{F}}(\varphi(\nu^*) + 1)$, we can apply Corollary 1, to find a sequence $\{h_i \mid i = 1, 2, 3, \dots\} \subset (0, 1)$ and a limiting function $m^\circ \in L^2(\Omega; \mathbb{R}^3)$, such that:

$$\begin{cases} h_i \searrow 0, & m^{(h_i)} \rightarrow m^\circ \text{ in } L^2(\Omega; \mathbb{R}^3), \\ \varphi(m^{(h_i)}) \rightarrow \varphi(m^\circ) \text{ in } L^1(\Omega), \end{cases} \text{ as } i \rightarrow \infty.$$

Here, taking into account of Theorem 3.1, Corollary 2 and (17),

$$\mathcal{F}_\alpha^\circ(m^\circ) \leq \liminf_{i \rightarrow \infty} \mathcal{F}_\alpha^{(h_i)}(m^{(h_i)}) \leq \varphi(\nu^*) + 1;$$

and hence $m^\circ \in \text{Dom}(\mathcal{F}_\alpha^\circ) \subset L^2(S; \mathbb{R}^3)$.

Next, taking any $\hat{m}^\circ \in \text{Dom}(\mathcal{F}_\alpha^\circ)$ ($= \text{Dom}(\Phi_\alpha^\circ)$), and taking the sequence $\{\hat{m}^{(h)} \mid 0 < h < 1\} \subset L^2(\Omega; \mathbb{R}^3)$, obtained in ($\gamma 2$) of Theorem 3.1, it will be seen from Theorem 3.1 Corollary 2 and Remark 2 that:

$$\begin{aligned} \mathcal{F}_\alpha^\circ(m^\circ) &\leq \liminf_{i \rightarrow \infty} \mathcal{F}_\alpha^{(h_i)}(m^{(h_i)}) \leq \limsup_{i \rightarrow \infty} \mathcal{F}_\alpha^{(h_i)}(m^{(h_i)}) \\ &\leq \lim_{i \rightarrow \infty} \mathcal{F}_\alpha^{(h_i)}(\hat{m}^{(h_i)}) = \mathcal{F}_\alpha^\circ(\hat{m}^\circ). \end{aligned} \quad (18)$$

It implies that m° solves the limiting problem $(\text{MP})^\circ$. Furthermore, putting $\hat{m}^\circ = m^\circ$ in (18), it is deduced that:

$$\mathcal{F}_\alpha^{(h)}(m^{(h)}) \rightarrow \mathcal{F}_\alpha^\circ(m^\circ) \text{ as } h \searrow 0.$$

Thus, we conclude the assertion (II). \square

REFERENCES

- [1] F. Acanfora, G. Cardone and S. Mortola, *On the variational convergence of non-coercive quadratic integral functionals and semicontinuity problems*, NoDEA Nonlinear Differential Equations Appl., **10** (2003), 347–373.
- [2] R. Alicandro and C. Leone, *3D-2D asymptotic analysis for micromagnetic thin films*, ESAIM Control Optim. Calc. Var., **6** (2001), 489–498.
- [3] H. Ammari, L. Halpern and K. Hamdache, *Asymptotic behaviour of thin ferromagnetic films*, Asymptot. Anal., **24** (2000), 277–294.
- [4] J.-F. Babadjian and M. Baía, *3D-2D analysis of a thin film with periodic microstructure*, Proc. Roy. Soc. Edinburgh Sect. A, **136** (2006), 223–243.
- [5] M. Baía and E. Zappale, *A note on the 3D-2D dimensional reduction of a micromagnetic thin film with nonhomogeneous profile*, Appl. Anal., **86** (2007), 555–575.
- [6] A. Braides, I. Fonseca and G. Francfort, *3D-2D asymptotic analysis for inhomogeneous thin films*, Indiana Univ. Math. J., **49** (2000), 1367–1404.
- [7] W. F. Brown, “Micromagnetics,” Interscience Tracts on Physics and Astronomy, no. 18, John Wiley & Sons, New York, 1963.
- [8] G. Carbou, *Thin layers in micromagnetism*, Math. Models Methods Appl. Sci., **11** (2001), 1529–1546.
- [9] G. Dal Maso, “An introduction to Γ -convergence,” Birkhäuser, 1993.
- [10] A. Desimone, R. Kohn, S. Müller and F. Otto, *A reduced theory for thin-film micromagnetics*, Comm. Pure Appl. Math., **55** (2002), 1408–1460.
- [11] I. Fonseca and G. Francfort, *3D-2D asymptotic analysis of an optimal design problem for thin films*, J. Reine Angew. Math., **505** (1998), 173–202.
- [12] A. Gaudiello and R. Hadiji, *Junction of one-dimensional minimization problems involving S^2 valued maps*, Adv. Differential Equations, **13** (2008), 935–958.
- [13] A. Gaudiello and R. Hadiji, *Asymptotic analysis, in a thin multidomain, of minimizing maps with values in S^2* , Ann. Inst. H. Poincaré Anal. Non Linéaire, **26** (2009), 59–80.
- [14] A. Gaudiello and R. Hadiji, *Junction of ferromagnetic thin films*, to appear in Calc. Var. Partial Differential Equations (2010).
- [15] G. Gioia and R. D. James, *Micromagnetics of very thin film*, Proc. R. Soc. Lond. A, **453** (1997), 213–223.
- [16] R. Hadiji and K. Shirakawa, *Asymptotic analysis for micromagnetics of thin films governed by indefinite material coefficients*, Commun. Pure Appl. Anal., **9** (to appear).
- [17] O. A. Hafsá, *Variational formulations on thin elastic plates with constraints*, J. Convex Anal., **12** (2005), 365–382.
- [18] R. D. James and D. Kinderlehrer, *Frustration in ferromagnetic materials*, Contin. Mech. Thermodyn., **2** (1990), 215–239.
- [19] R. Kohn and V. Slastikov, *Another thin-film limit of micromagnetics*, Arch. Ration. Mech. Anal., **178** (2005), 227–245.
- [20] A. Visintin, *On Landau-Lifshitz’ equations for ferromagnetism*, Japan J. Appl. Math., **2** (1985), 69–84.

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