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Abstract. In this paper we study the validity of the so-called Oberbeck-Boussinesq approximation for compressible viscous perfect gases in the whole three-dimensional space. Both the cases of fluids with positive heat conductivity and zero conductivity are considered. For small perturbations of a constant equilibrium, we establish the global existence of unique strong solutions in a critical regularity functional framework. Next, taking advantage of Strichartz estimates for the associated system of acoustic waves, and of uniform estimates with respect to the Mach number, we obtain all-time convergence to the Boussinesq system with a explicit decay rate.

1. Introduction

This work aims at giving a mathematical justification of the *Oberbeck-Boussinesq approximation* that is commonly used to model stratified fluids such as e.g. atmosphere or oceans. One of the characteristics of the this approximation is that, although the primitive system is the full compressible Navier-Stokes system, the limit equations are incompressible, and the density is a constant. In fact, the velocity field just convects an active scalar creating buoyancy force, proportional to the discrepancy between the temperature and its equilibrium.

1.1. Formal derivation.

The starting point of our analysis is the full Navier-Stokes system for compressible viscous fluids, namely

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u\otimes u) - \operatorname{div}\tau + \dfrac{1}{\mathrm{Ma}^2}\nabla P = \dfrac{1}{\mathrm{Fr}^2}\,\rho\nabla V, \\ \partial_t(\rho s) + \operatorname{div}(\rho s) + \operatorname{div}(q/\mathcal{T}) = \sigma. \end{cases}$$

Above $\rho = \rho(t,x) \in \mathbb{R}^+$, $u = u(t,x) \in \mathbb{R}^3$ and $\mathcal{T} = \mathcal{T}(t,x) \in \mathbb{R}^+$ stand for the density, velocity field and temperature, respectively. The scalar function V stands for some (given) external potential (e.g. the gravity potential). We concentrate on the study of the evolution toward the future in the whole space $\mathbb{R}^3$ (hence the time variable t belongs to $\mathbb{R}^+$ and the space variable x, to $\mathbb{R}^3$).

In the Newtonian case that we shall consider, the stress tensor τ is given by

$$\tau = \mu(\nabla u + Du) + \lambda\operatorname{div} u\,\mathrm{Id}.$$

For simplicity, the viscosity coefficients λ and μ are assumed to be constant. As we only consider *viscous* fluids, those two coefficients satisfy

$$\mu > 0 \quad\text{and}\quad \nu := \lambda + 2\mu > 0.$$

This ensures ellipticity for the second order operator $\mathcal{A} := \mu\Delta + (\lambda+\mu)\nabla\operatorname{div}$.

The heat flux q is equal to $-\kappa\nabla\mathcal{T}$ for some constant conductivity coefficient $\kappa \geq 0$. The pressure P, the internal energy e and the specific entropy s are related to ρ and $\mathcal{T}$ through the Gibbs relation

$$\mathcal{T}\,ds = de + P\,d(1/\rho).$$

We focus on perfect gases, namely we assume that for some $a > 0$ and $b > 0$, $P = a\rho\mathcal{T}$ and $e = b\mathcal{T}$. After rescaling, it is non restrictive to take $a = b = 1$.

Finally, in the velocity equation, the Mach number Ma and the Froude number Fr are two dimensionless small parameters accounting for the compressibility and the stratification of the fluid. Formally, Oberbeck-Boussinesq approximation is obtained in the asymptotics $\varepsilon \to 0$ if

$$\mathrm{Ma} = \varepsilon \quad \text{and} \quad \mathrm{Fr} = \sqrt{\varepsilon},$$

an assumption that we shall make from now on.

Gathering all the above assumptions over the coefficients and state laws, we end up with the following system (with exponents ε emphasizing the dependency with respect to ε):

$$\left\{\begin{array}{l} \partial_t\rho^\varepsilon + \mathrm{div}\,(\rho^\varepsilon u^\varepsilon) = 0, \\ \partial_t(\rho^\varepsilon u^\varepsilon) + \mathrm{div}\,(\rho^\varepsilon u^\varepsilon\otimes u^\varepsilon) - \mu\Delta u^\varepsilon - (\lambda+\mu)\nabla\mathrm{div}\,u^\varepsilon + \dfrac{\nabla P^\varepsilon}{\varepsilon^2} = \dfrac{1}{\varepsilon}\rho^\varepsilon\nabla V^\varepsilon, \\ \partial_t(\rho^\varepsilon\mathcal{T}^\varepsilon) + \mathrm{div}\,(u^\varepsilon\rho^\varepsilon\mathcal{T}^\varepsilon) - \kappa\Delta\mathcal{T}^\varepsilon = \varepsilon^2[2\mu|Du^\varepsilon|^2 + \lambda(\mathrm{div}\,u^\varepsilon)^2]. \end{array}\right. \tag{1.1}$$

Let us first provide a formal derivation of the Oberbeck-Boussinesq approximation in the case where the heat conductivity κ is positive. We want to consider so-called *ill-prepared data* of the form $\rho_0^\varepsilon = 1+\varepsilon a_0^\varepsilon$, u_0^ε and $\mathcal{T}_0^\varepsilon = 1+\varepsilon\theta_0^\varepsilon$ where $(a_0^\varepsilon, u_0^\varepsilon, \theta_0^\varepsilon)$ are bounded in a sense that will be specified later on. Setting $\rho^\varepsilon = 1+\varepsilon a^\varepsilon$ and $\mathcal{T}^\varepsilon = 1+\varepsilon\theta^\varepsilon$, we get the following governing equations for $(a^\varepsilon, u^\varepsilon, \theta^\varepsilon)$:

$$\left\{\begin{array}{l} \partial_t a^\varepsilon + \dfrac{\mathrm{div}\,u^\varepsilon}{\varepsilon} = -\mathrm{div}\,(a^\varepsilon u^\varepsilon), \\ \partial_t u^\varepsilon + u^\varepsilon\cdot\nabla u^\varepsilon - \dfrac{\mathcal{A}u^\varepsilon}{1+\varepsilon a^\varepsilon} + \dfrac{\nabla(a^\varepsilon+\theta^\varepsilon+\varepsilon a^\varepsilon\theta^\varepsilon)}{\varepsilon(1+\varepsilon a^\varepsilon)} = \dfrac{1}{\varepsilon}\nabla V^\varepsilon, \\ \partial_t\theta^\varepsilon + \dfrac{\mathrm{div}\,u^\varepsilon}{\varepsilon} + \mathrm{div}\,(\theta^\varepsilon u^\varepsilon) - \dfrac{\kappa\Delta\theta^\varepsilon}{1+\varepsilon a^\varepsilon} = \dfrac{\varepsilon}{1+\varepsilon a^\varepsilon}[2\mu|Du^\varepsilon|^2 + \lambda(\mathrm{div}\,u^\varepsilon)^2]. \end{array}\right. \tag{1.2}$$

In order to handle the singular potential term in the r.h.s. of the velocity equation, it is usual to work with the *modified* deviation of density $b^\varepsilon := a^\varepsilon - V^\varepsilon$. We get

$$\left\{\begin{array}{l} \partial_t b^\varepsilon + u^\varepsilon\cdot\nabla b^\varepsilon + \dfrac{\mathrm{div}\,u^\varepsilon}{\varepsilon} = -\partial_t V^\varepsilon - \mathrm{div}\,(V^\varepsilon u^\varepsilon) - b^\varepsilon\mathrm{div}\,u^\varepsilon, \\ \partial_t u^\varepsilon + u^\varepsilon\cdot\nabla u^\varepsilon - \mathcal{A}u^\varepsilon + \dfrac{\nabla(b^\varepsilon+\theta^\varepsilon)}{\varepsilon} = \left(\dfrac{a^\varepsilon-\theta^\varepsilon}{1+\varepsilon a^\varepsilon}\right)\nabla a^\varepsilon - \dfrac{\varepsilon a^\varepsilon}{1+\varepsilon a^\varepsilon}\mathcal{A}u^\varepsilon, \\ \partial_t\theta^\varepsilon + u^\varepsilon\cdot\nabla\theta^\varepsilon + \dfrac{\mathrm{div}\,u^\varepsilon}{\varepsilon} - \kappa\Delta\theta^\varepsilon = \dfrac{\varepsilon}{1+\varepsilon a^\varepsilon}[2\mu|Du^\varepsilon|^2 + \lambda(\mathrm{div}\,u^\varepsilon)^2] \\ \qquad\qquad - \kappa\dfrac{\varepsilon a^\varepsilon}{1+\varepsilon a^\varepsilon}\Delta\theta^\varepsilon - \theta^\varepsilon\mathrm{div}\,u^\varepsilon, \end{array}\right. \tag{1.3}$$

which may formally written as follows:

$$\frac{\partial}{\partial t}\begin{pmatrix} b^\varepsilon \\ u^\varepsilon \\ \theta^\varepsilon \end{pmatrix} + \frac{1}{\varepsilon}\begin{pmatrix} 0 & \mathrm{div} & 0 \\ \nabla & 0 & \nabla \\ 0 & \mathrm{div} & 0 \end{pmatrix}\begin{pmatrix} b^\varepsilon \\ u^\varepsilon \\ \theta^\varepsilon \end{pmatrix} = \mathcal{O}(1).$$

The notation $\mathcal{O}(1)$ designates terms that are expected to be bounded uniformly with respect to ε.

As a consequence of our considering ill-prepared data, the first order time derivatives are likely to blow-up like $1/\varepsilon$ for ε going to 0. At the 'physical' level, this means that highly oscillating acoustic waves may propagate in the fluid.

In order to better understand the action of those singular terms, we may first look at the kernel $\mathrm{Ker}\, L$ of the 5×5 first order antisymmetric differential matrix operator L above. The basic idea is that modes that are in $\mathrm{Ker}\, L$ will not be affected, while modes that are in $(\mathrm{Ker}\, L)^\perp$ may experience wild oscillations. A straightforward computation shows that

$$\begin{aligned}\mathrm{Ker}\, L &= \Big\{(b,u,\theta) : \mathrm{div}\, u = 0 \quad \text{and} \quad \nabla(b+\theta)=0\Big\},\\ (\mathrm{Ker}\, L)^\perp &= \Big\{(b,u,\theta) : \mathrm{curl}\, u = 0 \quad \text{and} \quad \nabla(b-\theta)=0\Big\}.\end{aligned}$$

Hence it is natural to look more closely at the equations satisfied by $(q^\varepsilon,\mathcal{Q}u^\varepsilon)$ and $(\Theta^\varepsilon,\mathcal{P}u^\varepsilon)$ where $\mathcal{P}$ and $\mathcal{Q}$ stand for the orthogonal projectors over divergence-free and curl-free vector fields, respectively, and

$$q^\varepsilon := \frac{\theta^\varepsilon + b^\varepsilon}{\sqrt{2}}, \quad \Theta^\varepsilon := \frac{\theta^\varepsilon - b^\varepsilon}{\sqrt{2}}\cdot$$

As L is antisymmetric, we expect the oscillating components of the solution, namely $\mathcal{Q}u^\varepsilon$ and q^ε to be dispersed whereas L will have no effect on $\mathcal{P}u^\varepsilon$ and Θ^ε. Let us be more accurate: we see that $(q^\varepsilon,\mathcal{Q}u^\varepsilon)$ satisfies

$$\left\{\begin{aligned}&\partial_t q^\varepsilon + \frac{\sqrt{2}}{\varepsilon}\mathrm{div}\,\mathcal{Q}u^\varepsilon = -\mathrm{div}\,(q^\varepsilon u^\varepsilon) - \frac{\sqrt{2}}{2}\biggl(\partial_t V^\varepsilon + \mathrm{div}\,(V^\varepsilon u^\varepsilon) + \kappa\frac{\Delta\theta^\varepsilon}{1+\varepsilon a^\varepsilon}\biggr)\\ &\qquad\qquad + \frac{\sqrt{2}}{2}\frac{\varepsilon}{1+\varepsilon a^\varepsilon}[2\mu|Du^\varepsilon|^2 + \lambda(\mathrm{div}\,u^\varepsilon)^2],\\ &\partial_t\mathcal{Q}u^\varepsilon + \frac{\sqrt{2}}{\varepsilon}\nabla q^\varepsilon = \mathcal{Q}\biggl(\Bigl(\frac{a^\varepsilon-\theta^\varepsilon}{1+\varepsilon a^\varepsilon}\Bigr)\nabla a^\varepsilon - \frac{\mathcal{A}u^\varepsilon}{1+\varepsilon a^\varepsilon} - u^\varepsilon\cdot\nabla u^\varepsilon\biggr)\end{aligned}\right. \tag{1.4}$$

whereas $(\Theta^\varepsilon,\mathcal{P}u^\varepsilon)$ fulfills

$$\left\{\begin{aligned}&\partial_t\Theta^\varepsilon + \mathcal{P}u^\varepsilon\cdot\nabla\Theta^\varepsilon - \frac{\kappa}{2}\Delta\Theta^\varepsilon = -\mathrm{div}\,(\Theta^\varepsilon\mathcal{Q}u^\varepsilon) + \frac{\sqrt{2}}{2}\bigl(\partial_t V^\varepsilon + \mathcal{P}u^\varepsilon\cdot\nabla V^\varepsilon + \mathrm{div}\,(V^\varepsilon\mathcal{Q}u^\varepsilon)\bigr)\\ &\qquad\qquad + \frac{\kappa}{2}\Delta q^\varepsilon - \frac{\sqrt{2}\kappa}{2}\frac{\varepsilon a^\varepsilon}{1+\varepsilon a^\varepsilon}\Delta\theta^\varepsilon + \frac{\sqrt{2}}{2}\frac{\varepsilon}{1+\varepsilon a^\varepsilon}[2\mu|Du^\varepsilon|^2 + \lambda(\mathrm{div}\,u^\varepsilon)^2],\\ &\partial_t\mathcal{P}u^\varepsilon - \mu\Delta\mathcal{P}u^\varepsilon + \mathcal{P}(\mathcal{P}u^\varepsilon\cdot\nabla\mathcal{P}u^\varepsilon) + \mathcal{P}(\theta^\varepsilon\nabla a^\varepsilon) = -\mathcal{P}\bigl(u^\varepsilon\cdot\nabla\mathcal{Q}u^\varepsilon + \mathcal{Q}u^\varepsilon\cdot\nabla\mathcal{P}u^\varepsilon\bigr)\\ &\qquad\qquad - \mathcal{P}\biggl(\frac{\varepsilon a^\varepsilon}{1+\varepsilon a^\varepsilon}\mathcal{A}u^\varepsilon\biggr) + \mathcal{P}\biggl(\frac{\varepsilon a^\varepsilon(\theta^\varepsilon-a^\varepsilon)}{1+\varepsilon a^\varepsilon}\nabla a^\varepsilon\biggr)\cdot\end{aligned}\right. \tag{1.5}$$

If we assume the solution $(b^\varepsilon,u^\varepsilon,\theta^\varepsilon)$ and the data to be bounded independently of ε then the right-hand side of (1.4) is bounded, too. Hence, owing to the antisymmetric (and nondegenerate) structure of the left-hand side of (1.4), one may expect $(q^\varepsilon,\mathcal{Q}u^\varepsilon)$ to tend weakly to 0. We shall see later on that in the whole space setting that is here considered, it is possible to get strong convergence (for suitable negative Besov norms), with an explicit rate.

In order to find out what the limit system for (1.5) is, let us observe that

$$\begin{aligned}\sqrt{2}\mathcal{P}(\theta^\varepsilon\nabla a^\varepsilon) &= \mathcal{P}(\Theta^\varepsilon\nabla V^\varepsilon)+\mathcal{P}(q^\varepsilon\nabla V^\varepsilon)+\sqrt{2}\mathcal{P}(\theta^\varepsilon\nabla b^\varepsilon)\\ &= \mathcal{P}(\Theta^\varepsilon\nabla V^\varepsilon)+\mathcal{P}(q^\varepsilon\nabla V^\varepsilon)+\sqrt{2}\mathcal{P}((\theta^\varepsilon+b^\varepsilon)\nabla b^\varepsilon)\\ &= \mathcal{P}(\Theta^\varepsilon\nabla V^\varepsilon)+\mathcal{P}(q^\varepsilon\nabla V^\varepsilon)+2\mathcal{P}(q^\varepsilon\nabla b^\varepsilon).\end{aligned}\tag{1.6}$$

Because q^ε tends to 0, we expect that

$$\sqrt{2}\mathcal{P}(\theta^\varepsilon\nabla a^\varepsilon)-\mathcal{P}(\Theta^\varepsilon\nabla V^\varepsilon)\to 0 \quad\text{for}\quad \varepsilon \quad\text{going to}\quad 0.$$

Hence, if we assume in addition that $V^\varepsilon\to V$, $\mathcal{P}u_0^\varepsilon\to v_0$ and $\Theta_0^\varepsilon\to\Theta_0$, then $(\Theta^\varepsilon,\mathcal{P}u^\varepsilon)$ should tend to the solution (Θ,v) to the following *Boussinesq system:*

$$\begin{cases}\partial_t\Theta+v\cdot\nabla\Theta-\dfrac{\kappa}{2}\Delta\Theta=\dfrac{\sqrt{2}}{2}(\partial_t+v\cdot\nabla)V,\\[2mm] \partial_t v+v\cdot\nabla v-\mu\Delta v+\nabla\Pi=-\dfrac{\sqrt{2}}{2}\Theta\nabla V,\qquad \operatorname{div}v=0,\\[2mm] (\Theta,v)|_{t=0}=(\Theta_0,v_0).\end{cases}\tag{1.7}$$

Setting $\tilde\Theta=\Theta-\sqrt{2}/2\,V$, and changing $\nabla\Pi$ accordingly, we see that this system is equivalent to the following one, which is commonly used:

$$\begin{cases}\partial_t\tilde\Theta+v\cdot\nabla\tilde\Theta-\dfrac{\kappa}{2}\Delta\tilde\Theta=\dfrac{\sqrt{2}}{4}\kappa\Delta V,\\[2mm] \partial_t v+v\cdot\nabla v-\mu\Delta v+\nabla\tilde\Pi=-\dfrac{\sqrt{2}}{2}\tilde\Theta\nabla V,\qquad \operatorname{div}v=0.\end{cases}\tag{1.8}$$

Note that although the density is constant in the limit system, it comes into play in the buoyancy force where it is related to the temperature and the potential.

We end this paragraph with a formal derivation *in the case* $\kappa=0$. It turns out to be easier to work with the pressure rather than with the temperature. We thus set $\rho^\varepsilon=1+\varepsilon a^\varepsilon$ and $P^\varepsilon=\rho^\varepsilon\mathcal{T}^\varepsilon=1+\varepsilon(\mathcal{R}^\varepsilon+V^\varepsilon)$, and obtain that

$$\begin{cases}\partial_t a^\varepsilon+\dfrac{\operatorname{div}u^\varepsilon}{\varepsilon}=-\operatorname{div}(a^\varepsilon u^\varepsilon),\\[2mm] \partial_t u^\varepsilon+u^\varepsilon\cdot\nabla u^\varepsilon-\dfrac{\mathcal{A}u^\varepsilon}{1+\varepsilon a^\varepsilon}+\dfrac{\nabla\mathcal{R}^\varepsilon}{\varepsilon(1+\varepsilon a^\varepsilon)}=\dfrac{a^\varepsilon}{1+\varepsilon a^\varepsilon}\nabla V^\varepsilon,\\[2mm] \partial_t\mathcal{R}^\varepsilon+\dfrac{\operatorname{div}u^\varepsilon}{\varepsilon}+\operatorname{div}(\mathcal{R}^\varepsilon u^\varepsilon)=\varepsilon[2\mu|Du^\varepsilon|^2+\lambda(\operatorname{div}u^\varepsilon)^2]-\partial_t V^\varepsilon-\operatorname{div}(V^\varepsilon u^\varepsilon).\end{cases}\tag{1.9}$$

Setting $\Theta^\varepsilon:=a^\varepsilon-\mathcal{R}^\varepsilon-V^\varepsilon$, we thus get

$$\begin{cases}\partial_t\Theta^\varepsilon+\operatorname{div}(\Theta^\varepsilon u^\varepsilon)=-\varepsilon[2\mu|Du^\varepsilon|^2+\lambda(\operatorname{div}u^\varepsilon)^2],\\[2mm] \partial_t\mathcal{P}u^\varepsilon+\mathcal{P}(u^\varepsilon\cdot\nabla u^\varepsilon)-\mu\Delta\mathcal{P}u^\varepsilon=-\mathcal{P}\Big(\dfrac{\varepsilon a^\varepsilon}{1+\varepsilon a^\varepsilon}\mathcal{A}u^\varepsilon\Big)+\mathcal{P}\Big(\dfrac{a^\varepsilon}{1+\varepsilon a^\varepsilon}\nabla(V^\varepsilon+\mathcal{R}^\varepsilon)\Big),\\[2mm] \partial_t\mathcal{Q}u^\varepsilon+\mathcal{Q}(u^\varepsilon\cdot\nabla u^\varepsilon)-\nu\Delta\mathcal{Q}u^\varepsilon+\dfrac{\nabla\mathcal{R}^\varepsilon}{\varepsilon}=-\mathcal{Q}\Big(\dfrac{\varepsilon a^\varepsilon}{1+\varepsilon a^\varepsilon}\mathcal{A}u^\varepsilon\Big)+\mathcal{Q}\Big(\dfrac{a^\varepsilon}{1+\varepsilon a^\varepsilon}\nabla(V^\varepsilon+\mathcal{R}^\varepsilon)\Big),\\[2mm] \partial_t\mathcal{R}^\varepsilon+\dfrac{\operatorname{div}u^\varepsilon}{\varepsilon}+\operatorname{div}(\mathcal{R}^\varepsilon u^\varepsilon)=\varepsilon[2\mu|Du^\varepsilon|^2+\lambda(\operatorname{div}u^\varepsilon)^2]-\partial_t V^\varepsilon-\operatorname{div}(V^\varepsilon u^\varepsilon).\end{cases}\tag{1.10}$$

As before, owing to the first order antisymmetric terms, we expect $(\mathcal{Q}u^\varepsilon,\mathcal{R}^\varepsilon)$ to go to 0. Concerning $(\Theta^\varepsilon,\mathcal{P}u^\varepsilon)$, we notice that

$$\mathcal{P}(a^\varepsilon\nabla(V^\varepsilon+\mathcal{R}^\varepsilon))=\mathcal{P}(\Theta^\varepsilon\nabla V^\varepsilon)+\mathcal{P}(\Theta^\varepsilon\nabla\mathcal{R}^\varepsilon).$$

Therefore the limit system for $(\Theta^\varepsilon, \mathcal{P}u^\varepsilon)$ reads

$$\begin{cases} \partial_t \Theta + v \cdot \nabla \Theta = 0, \\ \partial_t v + v \cdot \nabla v - \mu \Delta v + \nabla \Pi = \Theta \nabla V, \\ \operatorname{div} v = 0. \end{cases} \tag{1.11}$$

Note that in contrast with (1.7), this system is not fully parabolic.

1.2. **Some related works.** There is an important literature dedicated to the limit system, that is the Oberbeck-Boussinesq equations (1.7), (1.8) and (1.11), under various hypotheses over the coefficients κ and μ, and the potential V (although the most common assumption is that $V = x_3$). Loosely speaking the classical results concerning the existence issue are (see e.g. [8, 9, 16] and the references therein):

- Dimension 2: Global existence of strong solutions if $(\mu, \kappa) \neq (0, 0)$.
- Dimension 3 with $\mu \neq 0$: Global weak solutions and local strong solutions (which become global if the data are small).
- Dimension 3 with $\mu = 0$: only local-in-time strong solutions are available.

In contrast, although the Oberbeck-Boussinesq approximation is commonly used in geophysics (see e.g. the books by J. Pedlosky [20] or R. K. Zeytounian [21]) there are few results concerning the rigorous justification of the derivation that we presented in the previous subsection. To our knowledge, the first mathematical justification of Oberbeck-Boussinesq approximation in this context has been given only rather recently in the framework of the so-called *variational weak solutions* (see [11] for a complete presentation of such solutions for the full Navier-Stokes equations). The case of bounded domains with potential $V = x_3$ (or more generally, in $W^{1,\infty}(\Omega)$) has been treated by E. Feireisl and A. Novotny in [12, 13], while the exterior domain case has been studied by E. Feireisl and M. Schonbek in [14] (still under the assumption $V \in W^{1,\infty}(\Omega)$, thus ruling out the common but not so physical assumption that $V = x_3$). For passing to the limit, all those works borrow some seminal ideas that have been introduced by P.-L. Lions in his book [19] and B. Desjardins et al in [10] in the related context of low Mach number limit for the isentropic Navier-Stokes equations[1].

On the one hand, those results are very general for one may consider any finite energy data. On the other hand, the convergence results are not very accurate for they strongly rely on compactness methods : in particular convergence holds up to extraction only, and no rate may be given.

1.3. **Aim of the paper.** Getting stronger results of convergence that is in particular convergence of the *whole* sequence with an explicit rate, is the main purpose of the present work. Considering general variational solutions is hopeless. We shall focus on strong solutions with the so-called critical regularity, a framework which is nowadays classical for the study of viscous compressible fluids (see e.g. [2, 4, 5]). Of course, this will enforce us to restrict considerably the set of admissible data, but we will get much more accurate results of convergence.

Working in a functional framework that has the same scaling invariance as (1.2), if any, is the basic idea. Here we see that (if $V^\varepsilon \equiv 0$ to simplify the presentation), the system is "almost" invariant for all $\ell > 0$ by the rescaling

$$a^\varepsilon(t,x) \to a^\varepsilon(\lambda^2 t, \lambda x), \quad u^\varepsilon(t,x) \to \lambda u^\varepsilon(\lambda^2 t, \lambda x), \quad \theta^\varepsilon(t,x) \to \lambda^2 \theta^\varepsilon(\lambda^2 t, \lambda x).$$

[1]For other recent results concerning the low Mach number asymptotics for the *full* Navier-Stokes equations, the reader may refer to [1, 15, 17, 18].

If we believe in an energy type method then a good candidate for initial data is thus the homogeneous Sobolev space

$$\dot{H}^{\frac{3}{2}}(\mathbb{R}^3)\times\bigl(\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)\bigr)^3\times\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3),$$

or rather the slightly smaller following homogeneous Besov space:

$$\dot{B}^{\frac{3}{2}}_{2,1}(\mathbb{R}^3)\times\bigl(\dot{B}^{\frac{1}{2}}_{2,1}(\mathbb{R}^3)\bigr)^3\times\dot{B}^{-\frac{1}{2}}_{2,1}(\mathbb{R}^3)$$

which has nicer embedding properties ($\dot{B}^{\frac{3}{2}}_{2,1}$ is embedded in bounded functions for instance) and better behaves with respect to maximal parabolic estimates.

However, owing to the lower order pressure term, the above scaling invariance is not quite respected. Consequently, we have to work at a different level of regularity for the low frequencies of a^ε and θ^ε, to compensate this scaling defect. All this is now well understood and already occurs in the isentropic case [4].

Finally, in the case $\kappa=0$ that we shall also consider (and that cannot be studied in the framework of variational solutions), only the velocity is smoothed out during the evolution, and it is no longer possible to use a critical regularity framework: we will have to assume much more regularity.

We end this introductory part with a short description of the rest of the paper. After an unavoidable introduction of some notations and functional spaces, the next section is devoted to the presentation of the main results of the paper. The analysis of the heat conducting case is carried out in Section 3 while $\kappa=0$ is considered in Section 4. Some technical estimates are postponed in the Appendix.

2. Results

Before presenting the main statements of the paper, we briefly introduce some notations and function spaces. We are given an homogeneous Littlewood-Paley decomposition $(\dot{\Delta}_j)_{j\in\mathbb{Z}}$ that is a dyadic decomposition in the Fourier space for $\mathbb{R}^3$. One may for instance set $\dot{\Delta}_j:=\varphi(2^{-j}D)$ with $\varphi(\xi):=\chi(\xi/2)-\chi(\xi)$, and χ a non-increasing nonnegative smooth function supported in $B(0,4/3)$, and value 1 on $B(0,3/4)$ (see [2], Chap. 2 for more details).

We then define, for $p\in[1,+\infty]$ and $s\in\mathbb{R}$, the semi-norms

$$\|z\|_{\dot{B}^s_{p,1}}:=\sum_{j\in\mathbb{Z}}2^{js}\|\dot{\Delta}_jz\|_{L^p}.$$

In order to avoid complications due to polynomials, we adopt the following definition of homogeneous Besov spaces:

$$\dot{B}^s_{p,1}=\Bigl\{z\in\mathcal{S}'(\mathbb{R}^3):\|z\|_{\dot{B}^s_{p,1}}<\infty\ \text{ and }\ \lim_{j\to-\infty}\dot{S}_jz=0\Bigr\}\quad\text{with}\quad\dot{S}_j:=\chi(2^{-j}D).$$

To compensate the lack of strict scaling invariance of the system under consideration (as pointed out in the previous section), we also need to introduce the following *hybrid Besov spaces* with different regularity exponent in low and high frequencies:

Definition 2.1. *For $s\in\mathbb{R}$, $p\in[1,\infty]$ and $\alpha>0$, we set*

$$\|z\|_{\tilde{B}^{s,\pm}_{p,\alpha}}:=\sum_{j\in\mathbb{Z}}2^{js}\bigl(\min(\alpha^{-1},2^j)\bigr)^{\pm1}\|\dot{\Delta}_jz\|_{L^p}$$

and define

$$\tilde{B}^{s,\pm}_{p,\alpha} := \Big\{ z \in \mathcal{S}'(\mathbb{R}^3) : \|z\|_{\tilde{B}^{s,\pm}_{p,\alpha}} < \infty \} \quad \textit{and} \quad \lim_{j\to-\infty} \dot{S}_j z = 0 \Big\}.$$

We shall mainly use the above definition with $p = 2$, in which case, the corresponding hybrid Besov space will be simply denoted by $\tilde{B}^{s,\pm}_{\alpha}$, if the fact that $p = 2$ is clear from the context.

We agree that[2]:

$$z^\ell := \sum_{2^j\alpha\leq 1} \dot{\Delta}_j z \quad \text{and} \quad z^h := \sum_{2^j\alpha>1} \dot{\Delta}_j z. \tag{2.12}$$

With this notation, we have

$$\|z\|_{\tilde{B}^{s,\pm}_{p,\alpha}} = \|z^\ell\|_{\dot{B}^{s\pm1}_{p,1}} + \alpha^{\mp1}\|z^h\|_{\dot{B}^{s}_{p,1}}.$$

Therefore $\dot{B}^s_{p,1}$ is the bulk regularity of a function in $\tilde{B}^{s,\pm}_{p,\alpha}$ while the behavior at infinity is given by the low frequency part which is in $\dot{B}^{s\pm1}_{p,1}$. Of course, changing the value of α does not affect the space, and the corresponding norms are equivalent. However a suitable choice of α will enable us to get uniform estimates with respect to ε.

As we shall work with *time-dependent functions* with values in Besov spaces, we introduce the norms:

$$\|u\|_{L^q_T(\dot{B}^s_{p,1})} := \Big\| \|u(t,\cdot)\|_{\dot{B}^s_{p,1}} \Big\|_{L^q(0,T)} \quad \text{and} \quad \|u\|_{L^q_T(\tilde{B}^{s,\pm1}_{p,\alpha})} := \Big\| \|u(t,\cdot)\|_{\tilde{B}^{s,\pm1}_{p,\alpha}} \Big\|_{L^q(0,T)}.$$

As in many works using parabolic estimates in Besov spaces, it is somehow natural to take the time-Lebesgue norm *before* performing the summation for computing the Besov norm. This motivates us to introduce the following quantities:

$$\|u\|_{\tilde{L}^q_T(\dot{B}^s_{p,1})} := \sum_{j\in\mathbb{Z}} 2^{js}\|\dot{\Delta}_j u\|_{L^q_T(L^p)} \quad \text{and} \quad \|u\|_{\tilde{L}^q_T(\tilde{B}^{s,\pm}_{p,\alpha})} := \sum_{j\in\mathbb{Z}} 2^{js}\big(\min(\alpha^{-1},2^j)\big)^{\pm1}\|\dot{\Delta}_j u\|_{L^q_T(L^p)}.$$

The index T will be omitted if $T = +\infty$ and we shall denote by $\tilde{C}(\dot{B}^s_{p,1})$ (resp. $\tilde{C}(\tilde{B}^{s,\pm}_{p,\alpha})$) the subset of $\tilde{L}^\infty(\dot{B}^s_{p,1})$ (resp. $\tilde{L}^\infty(\tilde{B}^{s,\pm}_{p,\alpha})$) constituted by continuous functions over $\mathbb{R}^+$ with values in $\dot{B}^s_{p,1}$ (resp. $\tilde{B}^{s,\pm}_{p,\alpha}$).

Let us emphasize that, owing to Minkowski inequality, we have

$$\|u\|_{L^q_T(\dot{B}^s_{p,1})} \leq \|u\|_{\tilde{L}^q_T(\dot{B}^s_{p,1})}$$

with equality if and only if $q = 1$. Similar properties hold for hybrid Besov spaces.

Throughout, we shall denote

$$\tilde{\kappa} := \kappa/\nu, \quad \tilde{\lambda} := \lambda/\nu, \quad \tilde{\mu} := \mu/\nu \ \text{ with } \ \nu := \lambda + 2\mu. \tag{2.13}$$

One can state our first main result : the global existence of solutions corresponding to small (critical) data with *estimates independent of* ε in the case $\kappa > 0$.

Theorem 2.1. *Assume that the initial data* $(b^\varepsilon_0, u^\varepsilon_0, \theta^\varepsilon_0)$ *and that the potential term* V^ε *satisfy, for a small enough constant* η depending only on $\tilde{\kappa}$ and $\tilde{\mu}$*:*

$$\|b^\varepsilon_0\|_{\tilde{B}^{\frac{3}{2},-}_{\varepsilon\nu}} + \|u^\varepsilon_0\|_{\dot{B}^{\frac12}_{2,1}} + \|\theta^\varepsilon_0\|_{\tilde{B}^{-\frac12,+}_{\varepsilon\nu}} \leq \eta\nu, \tag{2.14}$$

$$\nu^{\frac12}\|\nabla V^\varepsilon\|_{L^2(\tilde{B}^{\frac{3}{2},-}_{\varepsilon\nu})} + \|V^\varepsilon\|_{\tilde{L}^\infty(\tilde{B}^{\frac{3}{2},-}_{\varepsilon\nu})} + \|\partial_t V^\varepsilon\|_{L^1(\tilde{B}^{\frac{3}{2},-}_{\varepsilon\nu})} \leq \eta\nu. \tag{2.15}$$

[2]We omit the dependency with respect to the threshold α in the above notation because the value of α will be always clear from the context.

Let $a_0^\varepsilon := b_0^\varepsilon + V^\varepsilon(0)$. Then System (1.2) with initial data $(1+\varepsilon a_0^\varepsilon, u_0^\varepsilon, 1+\varepsilon\theta_0^\varepsilon)$ has a unique global solution $(a^\varepsilon, u^\varepsilon, \theta^\varepsilon)$ (with $a^\varepsilon = b^\varepsilon + V^\varepsilon$) which satisfies

$$b^\varepsilon \in \tilde{C}(\tilde{B}^{\frac{3}{2},-}_{\varepsilon\nu}) \cap L^1(\tilde{B}^{\frac{3}{2},+}_{\varepsilon\nu}), \quad u^\varepsilon \in \tilde{C}(\dot{B}^{\frac{1}{2}}_{2,1}) \cap L^1(\dot{B}^{\frac{5}{2}}_{2,1}), \quad \theta^\varepsilon \in \tilde{C}(\tilde{B}^{-\frac{1}{2},+}_{\varepsilon\nu}) \cap L^1(\tilde{B}^{\frac{3}{2},+}_{\varepsilon\nu})$$

and, for a constant K depending only on $\tilde{\kappa}$ and $\tilde{\mu}$,

$$\begin{aligned}\|b^\varepsilon\|_{\tilde{L}^\infty(\tilde{B}^{\frac{3}{2},-}_{\varepsilon\nu})} + \nu\|b^\varepsilon\|_{L^1(\tilde{B}^{\frac{3}{2},+}_{\varepsilon\nu})} + \|u^\varepsilon\|_{\tilde{L}^\infty(\dot{B}^{\frac{1}{2}}_{2,1})} + \nu\|u^\varepsilon\|_{L^1(\dot{B}^{\frac{5}{2}}_{2,1})} + \|\theta^\varepsilon\|_{\tilde{L}^\infty(\tilde{B}^{-\frac{1}{2},+}_{\varepsilon\nu})}\\ +\nu\|\theta^\varepsilon\|_{L^1(\tilde{B}^{\frac{3}{2},+}_{\varepsilon\nu})} \leq K\big(\|b_0^\varepsilon\|_{\tilde{B}^{\frac{3}{2},-}_{\varepsilon\nu}} + \|u_0^\varepsilon\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \|\theta_0^\varepsilon\|_{\tilde{B}^{-\frac{1}{2},+}_{\varepsilon\nu}} + \|\partial_t V^\varepsilon\|_{L^1(\tilde{B}^{\frac{3}{2},-}_{\varepsilon\nu})}\big).\end{aligned}$$

Remark 2.1. *Smoother data give rise to smoother solutions. For example if in addition to the above hypotheses, we have*

$$\varepsilon\|b_0^\varepsilon\|_{\tilde{B}^{\frac{5}{2},-}_{\varepsilon\nu}} + \nu^{-1}\|(\theta_0^\varepsilon, u_0^\varepsilon)\|_{\tilde{B}^{\frac{3}{2},-}_{\varepsilon\nu}} + \varepsilon\|\partial_t V^\varepsilon\|_{L^1_t(\tilde{B}^{\frac{5}{2},-}_{\varepsilon\nu})} + \varepsilon\|\nabla V^\varepsilon\|_{L^2(\tilde{B}^{\frac{5}{2},-}_{\varepsilon\nu})} \leq \eta,$$

then the above solution also satisfies

$$\varepsilon\|b^\varepsilon\|_{\tilde{L}^\infty(\tilde{B}^{\frac{5}{2},-}_{\varepsilon\nu})} + \nu^{-1}\|(\theta^\varepsilon, u^\varepsilon)\|_{\tilde{L}^\infty(\tilde{B}^{\frac{3}{2},-}_{\varepsilon\nu})} + \varepsilon\nu\|b^\varepsilon\|_{L^1(\tilde{B}^{\frac{5}{2},+}_{\varepsilon\nu})} + \varepsilon\nu\|(u^\varepsilon, \theta^\varepsilon)\|_{L^1(\tilde{B}^{\frac{7}{2},-}_{\varepsilon\nu})} \leq K\eta.$$

Next, combining this result with Strichartz estimates, we shall prove the following result of convergence to the Boussinesq system.

Theorem 2.2. *Consider a family of data $(b_0^\varepsilon, u_0^\varepsilon, \theta_0^\varepsilon, V^\varepsilon)_{\varepsilon>0}$ satisfying the conditions of Theorem 2.1 with in addition*

$$\begin{aligned}M_0 := \sup_{\varepsilon>0}\big(&\|b_0^\varepsilon\|_{\tilde{B}^{\frac{3}{2},-}_{\varepsilon\nu}} + \|u_0^\varepsilon\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \|\theta_0^\varepsilon\|_{\tilde{B}^{-\frac{1}{2},+}_{\varepsilon\nu}}\\ &+ \nu^{\frac{1}{2}}\|\nabla V^\varepsilon\|_{L^2(\tilde{B}^{\frac{3}{2},-}_{\varepsilon\nu})} + \|V^\varepsilon\|_{\tilde{L}^\infty(\tilde{B}^{\frac{3}{2},-}_{\varepsilon\nu})} + \|\partial_t V^\varepsilon\|_{L^1(\tilde{B}^{\frac{3}{2},-}_{\varepsilon\nu})}\big) \leq \eta\nu.\end{aligned} \tag{2.16}$$

Let $q^\varepsilon := (\theta^\varepsilon + b^\varepsilon)/\sqrt{2}$ and $\Theta^\varepsilon := (\theta^\varepsilon - b^\varepsilon)/\sqrt{2}$. Assume that $(\mathcal{P}u_0^\varepsilon, \Theta_0^\varepsilon, V^\varepsilon)$ converges (in the sense of distributions) to some triplet (v_0, Θ_0, V) such that

$$v_0 \in \dot{B}^{\frac{1}{2}}_{2,1}, \quad \Theta_0 \in \dot{B}^{\frac{1}{2}}_{2,1}, \quad \nabla V \in L^2(\dot{B}^{\frac{1}{2}}_{2,1}), \quad \partial_t V \in L^1(\dot{B}^{\frac{1}{2}}_{2,1}).$$

Then the following properties hold true :

1. *System (1.2) with initial data $(1+\varepsilon a_0^\varepsilon, u_0^\varepsilon, 1+\varepsilon\theta_0^\varepsilon)$ has a unique global solution with the properties described in Theorem 2.1;*
2. *Boussinesq system (1.7) admits a unique global solution (v, Θ) in $\tilde{\mathcal{C}}(\dot{B}^{\frac{1}{2}}_{2,1}) \cap L^1(\dot{B}^{\frac{5}{2}}_{2,1})$ satisfying for some constant $K = K(\tilde{\kappa}, \tilde{\mu})$:*
$$\|(v,\Theta)\|_{\tilde{L}^\infty(\dot{B}^{\frac{1}{2}}_{2,1})} + \nu\|(v,\Theta)\|_{L^1(\dot{B}^{\frac{5}{2}}_{2,1})} \leq K\big(\|(v_0,\Theta_0)\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \|\partial_t V\|_{L^1(\dot{B}^{\frac{1}{2}}_{2,1})}\big).$$
3. *The functions q^ε and $\mathcal{Q}u^\varepsilon$ go to 0 in the following meaning for all $p \in [2,\infty]$ and $s \in [-1/2+4/p, 3/p]$:*
$$\nu^{\frac{1}{2}}\|q^\varepsilon\|_{\tilde{L}^2(\tilde{B}^{s-1,+}_{p,\varepsilon\nu})} + \nu^{\frac{1}{2}}\|\mathcal{Q}u^\varepsilon\|_{\tilde{L}^2(\dot{B}^s_{p,1})} \leq K(\varepsilon\nu)^{\frac{3}{p}-s}M_0.$$
4. *The couple $(\mathcal{P}u^\varepsilon, \Theta^\varepsilon)$ tends to (v,Θ) in the following meaning for all $p \in [2,\infty]$ and $s \in [-1/2+4/p, 3/p]$ with $s > 1/2$:*

$$\begin{aligned}&\nu^{1/2}\|\delta\Theta^\varepsilon\|_{\tilde{L}^2(\tilde{B}^{s-1,+}_{p,\varepsilon\nu})} + \|\delta\Theta^\varepsilon\|_{\tilde{L}^\infty(\tilde{B}^{s-2,+}_{p,\varepsilon\nu})} + \nu\|\delta v^\varepsilon\|_{L^1(\tilde{B}^{s,+}_{p,\varepsilon\nu})} + \|\delta v^\varepsilon\|_{\tilde{L}^\infty(\tilde{B}^{s-2,+}_{p,\varepsilon\nu})}\\ &\quad\leq C\big(\|(\delta\Theta_0^\varepsilon, \delta v_0^\varepsilon)\|_{\tilde{B}^{s-2,+}_{p,\varepsilon\nu}} + \|\partial_t\delta V^\varepsilon\|_{L^1(\tilde{B}^{s-2,+}_{p,\varepsilon\nu})+L^2(\tilde{B}^{s-3,+}_{p,\varepsilon\nu})} + M_0^2\varepsilon^{\frac{3}{p}-s} + M_0\|\nabla\delta V^\varepsilon\|_{L^2(\dot{B}^{s-1}_{p,1})}\big)\\ &\quad\text{with } \delta\Theta^\varepsilon := \Theta^\varepsilon - \Theta,\ \delta v^\varepsilon := \mathcal{P}u^\varepsilon - v,\ \delta V^\varepsilon := V^\varepsilon - V \text{ and } C = C(\tilde{\mu},\tilde{\kappa},s,p).\end{aligned}$$

Remark 2.2. *If the data are smoother, e.g. as in Remark 2.1 then the results of convergence hold for stronger norms. For instance, it may be shown that* $(\mathcal{Q}u^\varepsilon, q^\varepsilon) \to 0$ *in* $\tilde{L}^2(\dot{B}^{\frac{4}{p}-\frac{1}{2}}_{p,1})$, *that* $\mathcal{P}u^\varepsilon \to v$ *in* $L^1(\dot{B}^{\frac{4}{p}+\frac{1}{2}}_{p,1}) \cap \tilde{L}^\infty(\dot{B}^{\frac{4}{p}-\frac{3}{2}}_{p,1})$, *and that* $\Theta^\varepsilon \to \Theta$ *in* $\tilde{L}^2(\dot{B}^{\frac{4}{p}-\frac{1}{2}}_{p,1}) \cap \tilde{L}^\infty(\dot{B}^{\frac{4}{p}-\frac{3}{2}}_{p,1})$, *with the decay rate* $\varepsilon^{\frac{1}{2}-\frac{1}{p}}$.

Let us finally state our main global existence and convergence result for nonconducting fluids.

Theorem 2.3. *Assume that the initial data* $(a_0^\varepsilon, u_0^\varepsilon, \mathcal{R}_0^\varepsilon)$ *and the force term* V^ε *verify that*

$$C_0^\varepsilon := \|(a_0^\varepsilon, u_0^\varepsilon, \mathcal{R}_0^\varepsilon)\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + (\varepsilon\nu)^3 \|(a_0^\varepsilon, u_0^\varepsilon, \mathcal{R}_0^\varepsilon)\|_{\dot{B}^{\frac{7}{2}}_{2,1}} + (\varepsilon\nu)^2 \|u_0^\varepsilon\|_{\dot{B}^{\frac{5}{2}}_{2,1}} + \|\partial_t V^\varepsilon\|_{L^1(\dot{B}^{\frac{1}{2}}_{2,1})} + (\varepsilon\nu)^3 \|\partial_t V^\varepsilon\|_{L^1(\dot{B}^{\frac{7}{2}}_{2,1})} \leq \eta\nu,$$

$$\|V^\varepsilon\|_{\tilde{L}^\infty(\dot{B}^{\frac{1}{2}}_{2,1})} + (\varepsilon\nu)^3 \|V^\varepsilon\|_{\tilde{L}^\infty(\dot{B}^{\frac{7}{2}}_{2,1})} + \nu \|V^\varepsilon\|_{L^1(\dot{B}^{\frac{5}{2}}_{2,1})} + \nu(\varepsilon\nu)^2 \|V^\varepsilon\|_{L^1(\dot{B}^{\frac{9}{2}}_{2,1})} \leq \eta\nu$$

where the constant η *is sufficiently small and depends only on* $\tilde{\mu}$.

Then System (1.9) *admits a unique global solution* $(a^\varepsilon, u^\varepsilon, \mathcal{R}^\varepsilon)$ *which satisfies*

$$a^\varepsilon \in \tilde{C}(\dot{B}^{\frac{1}{2}}_{2,1} \cap \dot{B}^{\frac{7}{2}}_{2,1}), \quad u^\varepsilon \in \tilde{C}(\dot{B}^{\frac{1}{2}}_{2,1} \cap \dot{B}^{\frac{5}{2}}_{2,1}) \cap L^1(\dot{B}^{\frac{5}{2}}_{2,1} \cap \dot{B}^{\frac{9}{2}}_{2,1}), \quad \mathcal{R}^\varepsilon \in \tilde{C}(\dot{B}^{\frac{1}{2}}_{2,1} \cap \dot{B}^{\frac{7}{2}}_{2,1}) \cap L^1(\tilde{B}^{\frac{3}{2},+}_{\varepsilon\nu} \cap \tilde{B}^{\frac{7}{2},+}_{\varepsilon\nu})$$

and, for some constant K *depending only on* $\tilde{\mu}$,

$$\|(a^\varepsilon, \mathcal{R}^\varepsilon)\|_{\tilde{L}^\infty(\dot{B}^{\frac{1}{2}}_{2,1})} + (\varepsilon\nu)^3 \|(a^\varepsilon, \mathcal{R}^\varepsilon)\|_{\tilde{L}^\infty(\dot{B}^{\frac{7}{2}}_{2,1})} + \|u^\varepsilon\|_{\tilde{L}^\infty(\dot{B}^{\frac{1}{2}}_{2,1})} + (\varepsilon\nu)^2 \|u^\varepsilon\|_{\tilde{L}^\infty(\dot{B}^{\frac{5}{2}}_{2,1})}$$
$$+ \nu \|\mathcal{R}^\varepsilon\|_{L^1(\tilde{B}^{\frac{3}{2},+}_{\varepsilon\nu})} + \nu(\varepsilon\nu)^2 \|\mathcal{R}^\varepsilon\|_{L^1(\tilde{B}^{\frac{7}{2},+}_{\varepsilon\nu})} + \nu \|u^\varepsilon\|_{L^1(\dot{B}^{\frac{5}{2}}_{2,1})} + \nu(\varepsilon\nu)^2 \|u^\varepsilon\|_{L^1(\dot{B}^{\frac{9}{2}}_{2,1})} \leq K C_0^\varepsilon.$$

Suppose in addition that $\Theta_0^\varepsilon \to \Theta_0$, *that* $\mathcal{P}u_0^\varepsilon \to v_0$ *and that* $V^\varepsilon \to V$ *with*

$$\|v_0\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \|\nabla V\|_{L^1(\dot{B}^{\frac{3}{2}}_{2,1})} + \|\Theta_0\|_{\dot{B}^{\frac{1}{2}}_{2,1}} \leq \eta\mu. \tag{2.17}$$

Then the corresponding limit Boussinesq system (1.11) *admits a unique global solution* (Θ, v) *in* $\tilde{C}(\dot{B}^{\frac{1}{2}}_{2,1}) \times \big(\tilde{C}(\dot{B}^{\frac{1}{2}}_{2,1}) \cap L^1(\dot{B}^{\frac{5}{2}}_{2,1})\big)$. *Furthermore we have*

$$\|(\Theta, v)\|_{\tilde{L}^\infty(\dot{B}^{\frac{1}{2}}_{2,1})} + \mu \|v\|_{L^1(\dot{B}^{\frac{5}{2}}_{2,1})} \leq K \|(\Theta_0, v_0)\|_{\dot{B}^{\frac{1}{2}}_{2,1}}. \tag{2.18}$$

In addition, if C_0^ε *is bounded by some constant* C_0 *when* ε *goes to* 0 *then* $(\mathcal{Q}u^\varepsilon, \mathcal{R}^\varepsilon)$ *goes to zero with the following rates of convergence for all* $p \in [2, \infty)$:

$$\|(\mathcal{Q}u^\varepsilon, \mathcal{R}^\varepsilon)\|_{\tilde{L}^{\frac{2p}{p-2}}(\dot{B}^{\frac{2}{p}-\frac{1}{2}}_{p,1})} \leq K C_0 \varepsilon^{\frac{1}{2}-\frac{1}{p}} \tag{2.19}$$

$$\nu^{\frac{1}{2}} \|(\mathcal{Q}u^\varepsilon, \mathcal{R}^\varepsilon)\|_{\tilde{L}^2(\dot{B}^s_{p,1})} \leq K C_0 (\varepsilon\nu)^{\frac{3}{p}-s} \quad \text{if } s \in [-1/2 + 4/p, 3/p]. \tag{2.20}$$

Finally, if Θ_0^ε *and* $\mathcal{P}u_0^\varepsilon$ *are independent*[3] *of* ε *then for all* p *and* s *as above (with in addition* $s > 1/2$*), and* $T > 0$,

$$\Theta^\varepsilon - \Theta \to 0 \text{ in } \tilde{C}_T(\dot{B}^{s-2}_{p,1}) \quad \text{and} \quad \mathcal{P}u^\varepsilon - v \to 0 \text{ in } \tilde{C}_T\big(\dot{B}^{s-1}_{p,1} + \dot{B}^{s-2}_{p,1}\big) \cap \big(\tilde{L}^2_T(\dot{B}^s_{p,1}) + L^1_T(\dot{B}^s_{p,1})\big),$$

and the rate of convergence is $\varepsilon^{\frac{3}{p}-s}$.

The above statements deserve some comments:

[3]The reader may refer to Inequalities (4.116) and (4.117) for the general case.

(1) In this paper, for simplicity, we focussed on the physical dimension 3. However similar statements may be established in any dimension $d \geq 2$.
(2) In the case of large data, we expect, as for the isentropic Navier-Stokes equations studied in [6], the lifespan of the solutions to (1.2) to tend to that of the limit Oberbeck-Boussinesq equations. Global existence for the limit equations should entail global existence for (1.2) with small ε, if $\kappa > 0$. This is of particular interest in dimension two, as the limit equations are globally well-posed for any data with the above smoothness. We reserve this study to future works.
(3) We also reserve the case of other boundary conditions, in particular the periodic ones, to future works. We want to point out that the global existence statements (that is Theorem 2.1 as well as the first part of Theorem 2.3) extend to that case. At the same time, no dispersive inequalities are available, hence the approach for proving convergence is expected to be completely different, provided based on the filtering method, as in the isentropic case [7].

We end this section by explaining the general strategy for the proof of convergence. The first step consists in proving uniform global a priori estimates. This in fact corresponds to the statement of Theorem 2.1 and to the first part of Theorem 2.3. We shall see that the proof reduces to the case $\varepsilon = 1$ after suitable rescaling of the equations. Then, proving convergence requires two steps : first we establish that the oscillating part of the solution converges to 0 (this relies on Strichartz estimates), and next establish strong convergence to Oberbeck-Boussinesq for the incompressible modes. Note that, owing to the fact that only small solutions are considered, we do not need to resort to bootstrap arguments.

3. Global existence and convergence in the case $\kappa > 0$

Let us first notice that performing the change of unknown[4]:

$$(b,u,\theta)(t,x) := \varepsilon(b^\varepsilon,u^\varepsilon,\theta^\varepsilon)(\varepsilon^2\nu t,\varepsilon\nu x) \tag{3.21}$$

and the change of data

$$(b_0,u_0,\theta_0)(x) := \varepsilon(b_0^\varepsilon,u_0^\varepsilon,\theta_0^\varepsilon)(\varepsilon\nu x) \quad\text{and}\quad \widetilde{V}(t,x) := \varepsilon V^\varepsilon(\varepsilon^2\nu t,\varepsilon\nu x) \tag{3.22}$$

reduces the study to the case $\nu = 1$ and $\varepsilon = 1$. Indeed it is obvious that $(b^\varepsilon,u^\varepsilon,\theta^\varepsilon)$ satisfies (1.2) if and only if (b,u,θ) satisfies the same system with $\varepsilon = 1$, Lamé coefficients $(\tilde\lambda,\tilde\mu) := \nu^{-1}(\lambda,\mu)$ and heat conductivity $\tilde\kappa := \nu^{-1}\kappa$, provided the data have been changed according to (3.22). This change of variables has the desired effect on the norms that are used in Theorem 2.1. For example, we have, up to a constant independent of ε and ν,

$$\|b_0\|_{\tilde B_1^{\frac32,-}} = \nu^{-1}\|b_0^\varepsilon\|_{\tilde B_{\varepsilon\nu}^{\frac32,-}}, \quad \|u_0\|_{\dot B_{2,1}^{\frac12}} = \nu^{-1}\|u_0^\varepsilon\|_{\dot B_{2,1}^{\frac12}}, \quad \|\theta_0\|_{\tilde B_1^{-\frac12,+}} = \nu^{-1}\|\theta_0^\varepsilon\|_{\tilde B_{\varepsilon\nu}^{-\frac12,+}},$$
$$\|\nabla\widetilde{V}(t,\cdot)\|_{\tilde B_1^{\frac32,-}} = \varepsilon\|\nabla V^\varepsilon(\varepsilon^2\nu t,\cdot)\|_{\tilde B_{\varepsilon\nu}^{\frac32,-}} \quad\text{and}\quad \|\partial_t\widetilde{V}(t,\cdot)\|_{\tilde B_1^{\frac32,-}} = \varepsilon^2\|\partial_t V^\varepsilon(\varepsilon^2\nu t,\cdot)\|_{\tilde B_{\varepsilon\nu}^{\frac32,-}},$$

hence

$$\|\nabla\widetilde{V}\|_{L^2(\tilde B_1^{\frac32,-})} = \nu^{-\frac12}\|\nabla V^\varepsilon\|_{L^2(\tilde B_{\varepsilon\nu}^{\frac32,-})} \quad\text{and}\quad \|\partial_t\widetilde{V}\|_{L^1(\tilde B_1^{\frac32,-})} = \nu^{-1}\|\partial_t V^\varepsilon\|_{L^1(\tilde B_{\varepsilon\nu}^{\frac32,-})}.$$

Consequently, in order to prove Theorem 2.1, it is suffices to consider the case $\nu = 1$ and $\varepsilon = 1$. We shall resume to the original variables only at the end of this section, for getting the convergence results of Theorem 2.2.

[4]Recall that $\nu = \lambda + 2\mu$

3.1. **The linearized system.** In the case $\varepsilon=\nu=1$, the linearized equations about $(0,0,0)$ read

$$\begin{cases}\partial_t b+\operatorname{div} u=0,\\ \partial_t u-\tilde{\mu}\Delta u-(\tilde{\lambda}+\tilde{\mu})\nabla\operatorname{div} u+\nabla(b+\theta)=0,\\ \partial_t\theta+\operatorname{div} u-\tilde{\kappa}\Delta\theta=0.\end{cases}\tag{3.23}$$

We aim at proving energy type estimates for (b,u,θ). Roughly speaking, we shall exhibit a low frequency parabolic type smoothing for all the components of the solution whereas, in high frequency, only (u,θ) will experience a parabolic smoothing. As for b, it will be damped with no gain of regularity whatsoever. Throughout our proof (which will require several steps) we shall also pinpoint where one has to work in different level of regularities to get the aforementioned features of the system.

Let us first notice that the gradient terms in the velocity equation involve only the potential part of the velocity. More precisely, setting $d:=\Lambda^{-1}\operatorname{div} u$ (with $\Lambda^s:=|D|^s$) and $w:=\mathcal{P}u=u+\nabla(-\Delta^{-1})\operatorname{div} u$, we get

$$\begin{cases}\partial_t b+\Lambda d=0,\\ \partial_t d-\Delta d-\Lambda(b+\theta)=0,\\ \partial_t\theta+\Lambda d-\tilde{\kappa}\Delta\theta=0,\\ \partial_t w-\tilde{\mu}\Delta w=0.\end{cases}\tag{3.24}$$

As the last equation is the standard heat equation with constant diffusion, we focus on the proof of estimates for the first three equations. After localization by means of the homogeneous Littlewood-Paley decomposition $(\dot{\Delta}_j)_{j\in\mathbb{Z}}$, the obtained system reads

$$\begin{cases}\partial_t b_j+\Lambda d_j=0,\\ \partial_t d_j-\Delta d_j-\Lambda(b_j+\theta_j)=0,\\ \partial_t\theta_j+\Lambda d_j-\tilde{\kappa}\Delta\theta_j=0\end{cases}\tag{3.25}$$

with $b_j:=\dot{\Delta}_j b$, $d_j:=\dot{\Delta}_j d$ and $\theta_j:=\dot{\Delta}_j\theta$.

Step 1: Basic Energy Estimate for (b,d,θ). Owing to the antisymmetric structure of the first order terms in (3.25), we readily get

$$\frac{1}{2}\frac{d}{dt}\bigl[\|b_j\|_{L^2}^2+\|d_j\|_{L^2}^2+\|\theta_j\|_{L^2}^2\bigr]+\|\Lambda d_j\|_{L^2}^2+\tilde{\kappa}\|\Lambda\theta_j\|_{L^2}^2=0.\tag{3.26}$$

Step 2: Improved Energy Estimate for (b,d,θ). We want to track the decay properties of b. For that we notice that the auxiliary function $\Lambda b-d$ satisfies:

$$\partial_t[\Lambda b_j-d_j]+\Lambda(b_j+\theta_j)=0.$$

Hence taking the L^2 inner product with Λb_j-d_j yields

$$\frac{1}{2}\frac{d}{dt}\|\Lambda b_j-d_j\|_{L^2}^2+\|\Lambda b_j\|_{L^2}^2+\bigl(\Lambda\theta_j|\Lambda b_j\bigr)_{L^2}-\bigl((b_j+\theta_j)|\Lambda d_j\bigr)_{L^2}=0,$$

from which we deduce that

$$\frac{1}{2}\frac{d}{dt}\Bigl[\|\Lambda b_j-d_j\|_{L^2}^2+\|b_j\|_{L^2}^2+\|\theta_j\|_{L^2}^2\Bigr]+\|\Lambda b_j\|_{L^2}^2+\bigl(\Lambda\theta_j|\Lambda b_j\bigr)_{L^2}+\tilde{\kappa}\|\Lambda\theta_j\|_{L^2}^2=0.\tag{3.27}$$

Putting (3.26) and (3.27) together, we thus get for any $\alpha \geq 0$,

$$\frac{1}{2}\frac{d}{dt}\Big[\alpha\|d_j\|_{L^2}^2 + \|\Lambda b_j - d_j]\|_{L^2}^2 + (1+\alpha)\|b_j\|_{L^2}^2 + (1+\alpha)\|\theta_j\|_{L^2}^2\Big] + \|\Lambda b_j\|_{L^2}^2 + \big(\Lambda\theta_j|\Lambda b_j\big)_{L^2} + \tilde{\kappa}(1+\alpha)\|\Lambda\theta_j\|_{L^2}^2 + \alpha\|\Lambda d_j\|_{L^2}^2 = 0. \tag{3.28}$$

Let us denote

$$f_j^2 := \alpha\|d_j\|_{L^2}^2 + (1+\alpha)\|b_j\|_{L^2}^2 + \|\Lambda b_j - d_j\|_{L^2}^2 + (1+\alpha)\|\theta_j\|_{L^2}^2, \tag{3.29}$$

$$H_j^2 := \frac{1}{2}\|\Lambda b_j\|_{L^2}^2 + \alpha\|\Lambda d_j\|_{L^2}^2 + \Big(\tilde{\kappa}(1+\alpha) - \frac{1}{2}\Big)\|\Lambda\theta_j\|_{L^2}^2. \tag{3.30}$$

Then combining (3.28) with the following Young inequality:

$$\big|\big(\Lambda\theta_j|\Lambda b_j\big)_{L^2}\big| \leq \frac{1}{2}\|\Lambda\theta_j\|_{L^2}^2 + \frac{1}{2}\|\Lambda b_j\|_{L^2}^2,$$

implies that

$$\frac{1}{2}\frac{d}{dt}f_j^2 + H_j^2 \leq 0. \tag{3.31}$$

Let us notice that

$$f_j^2 = (\alpha+1)\|(b_j, d_j, \theta_j)\|_{L^2}^2 + \|\Lambda b_j\|_{L^2}^2 - 2(\Lambda b_j|d_j)_{L^2}.$$

Therefore, because

$$2|(\Lambda b_j|d_j)_{L^2}| \leq \frac{2}{3}\|\Lambda b_j\|_{L^2}^2 + \frac{3}{2}\|d_j\|_{L^2}^2,$$

we have

$$\Big(\alpha - \frac{1}{2}\Big)\|d_j\|_{L^2}^2 + \frac{1}{3}\|\Lambda b_j\|_{L^2}^2 \leq f_j^2 - (\alpha+1)\|(b_j,\theta_j)\|_{L^2}^2 \leq \Big(\alpha + \frac{5}{2}\Big)\|d_j\|_{L^2}^2 + \frac{5}{3}\|\Lambda b_j\|_{L^2}^2. \tag{3.32}$$

Let us first assume that $\tilde{\kappa} \leq 1$. Then we take $\alpha = 2/\tilde{\kappa} - 1$ and (3.32) thus implies that

$$f_j^2 \approx \begin{cases} \tilde{\kappa}^{-1}\|(b_j, d_j, \theta_j)\|_{L^2}^2 & \text{if} \quad \tilde{\kappa}2^{2j} \leq 1, \\ \tilde{\kappa}^{-1}\|(d_j,\theta_j)\|_{L^2}^2 + \|\Lambda b_j\|_{L^2}^2 & \text{if} \quad \tilde{\kappa}2^{2j} \geq 1. \end{cases}$$

At the same time, we have

$$H_j^2 \gtrsim \begin{cases} 2^{2j}\|(b_j, d_j, \theta_j)\|_{L^2}^2 & \text{if} \quad \tilde{\kappa}2^{2j} \leq 1, \\ \tilde{\kappa}^{-1}\|(d_j,\theta_j)\|_{L^2}^2 + \|\Lambda b_j\|_{L^2}^2 & \text{if} \quad \tilde{\kappa}2^{2j} \geq 1. \end{cases}$$

Therefore, one may easily conclude that for some (universal) constant $c \in (0,1]$,

$$\|(b_j, d_j, \theta_j)(t)\|_{L^2} \lesssim e^{-c\tilde{\kappa}2^{2j}t}\|(b_j, d_j, \theta_j)(0)\|_{L^2} \quad \text{if} \quad 2^{2j}\tilde{\kappa} \leq 1, \tag{3.33}$$

$$\|(\tilde{\kappa}\Lambda b_j, d_j, \theta_j)(t)\|_{L^2} \lesssim e^{-ct}\|(\tilde{\kappa}\Lambda b_j, d_j, \theta_j)(0)\|_{L^2} \quad \text{if} \quad 2^{2j}\tilde{\kappa} \geq 1. \tag{3.34}$$

Let us now assume that $\tilde{\kappa} \geq 1$. Then we take $\alpha = 1$ so that following the above computations after replacing everywhere $\tilde{\kappa}$ by 1, it is easy to conclude that

$$\|(b_j, d_j, \theta_j)(t)\|_{L^2} \lesssim e^{-c2^{2j}t}\|(b_j, d_j, \theta_j)(0)\|_{L^2} \quad \text{if} \quad j \leq 0, \tag{3.35}$$

$$\|(\Lambda b_j, d_j, \theta_j)(t)\|_{L^2} \lesssim e^{-ct}\|(\Lambda b_j, d_j, \theta_j)(0)\|_{L^2} \quad \text{if} \quad j \geq 0. \tag{3.36}$$

Therefore, denoting $\check{\kappa} = \min(1, \tilde{\kappa})$ and putting together (3.33), (3.34), (3.35) and (3.36), we end up with

$$\begin{aligned} &\|(b_j, d_j, \theta_j)(t)\|_{L^2} \lesssim e^{-c\check{\kappa}2^{2j}t}\|(b_j, d_j, \theta_j)(0)\|_{L^2} && \text{if} \quad 2^{2j}\check{\kappa} \leq 1, \\ &\|\Lambda b_j(t)\|_{L^2} + \check{\kappa}^{-1}\|(d_j,\theta_j)(t)\|_{L^2} \lesssim e^{-ct}\big(\|\Lambda b_j(0)\|_{L^2} + \|\check{\kappa}^{-1}(d_j,\theta_j)(0)\|_{L^2}\big) && \text{if} \quad 2^{2j}\check{\kappa} \geq 1. \end{aligned} \tag{3.37}$$

For reasons that will appear more clearly in the following steps, it is suitable to work *with one less derivative* in the high frequency regime. Now from the second inequality of (3.37) and Bernstein inequality, we get for $2^{2j}\check{\kappa} \geq 1$,

$$\|\check{\kappa} b_j(t)\|_{L^2} + \|\Lambda^{-1}(d_j,\theta_j)(t)\|_{L^2} \lesssim e^{-ct}\big(\|\check{\kappa} b_j(0)\|_{L^2} + \|\Lambda^{-1}(d_j,\theta_j)(0)\|_{L^2}\big). \tag{3.38}$$

Step 3: Parabolic smoothing for θ. We here aim at tracking the high-frequency parabolic smoothing for θ. For that, we rewrite the last two equations of (3.25) as follows

$$\begin{cases} \partial_t \Lambda^{-1} d_j - \Delta(\Lambda^{-1} d_j) - \theta_j = b_j, \\ \partial_t \Lambda^{-1}\theta_j - \tilde{\kappa}\Delta(\Lambda^{-1}\theta_j) + d_j = 0. \end{cases}$$

Then applying a direct energy method, we readily get

$$\frac{1}{2}\frac{d}{dt}\Big(\|\Lambda^{-1} d_j\|_{L^2}^2 + \|\Lambda^{-1}\theta_j\|_{L^2}^2\Big) + \|d_j\|_{L^2}^2 + \tilde{\kappa}\|\theta_j\|_{L^2}^2 = (b_j|\Lambda^{-1} d_j).$$

Therefore, performing a time integration yields

$$\|\Lambda^{-1}(d_j,\theta_j)(t)\|_{L^2} + c\check{\kappa}\int_0^t \|\Lambda(d_j,\theta_j)\|_{L^2}\,d\tau \leq \|\Lambda^{-1}(d_j,\theta_j)(0)\|_{L^2} + \int_0^t \|b_j\|_{L^2}\,d\tau,$$

and taking advantage of the second inequality of (3.37) eventually leads to

$$\|\Lambda^{-1}(d_j,\theta_j)(t)\|_{L^2} + \check{\kappa}\int_0^t \|\Lambda(d_j,\theta_j)\|_{L^2}\,d\tau \lesssim \|b_j(0)\|_{L^2} + \check{\kappa}^{-1}\|\Lambda^{-1}(d_j,\theta_j)(0)\|_{L^2} \tag{3.39}$$

in the high frequency regime, that is whenever $2^j\sqrt{\check{\kappa}} \geq 1$.

Step 4: Parabolic smoothing for d. Given that

$$\partial_t d_j - \Delta d_j = \Lambda(b_j + \theta_j),$$

one may write that

$$\|d_j(t)\|_{L^2} + c2^{2j}\int_0^t \|d_j\|_{L^2}\,d\tau \leq \|d_j(0)\|_{L^2} + \int_0^t \|\Lambda(b_j,\theta_j)\|_{L^2}\,d\tau.$$

The previous steps ensure that, for $2^j\sqrt{\check{\kappa}} \geq 1$,

$$\begin{aligned} \int_0^t \|\Lambda b_j\|_{L^2}\,d\tau &\lesssim \|\Lambda b_j(0)\|_{L^2} + \check{\kappa}^{-1}\|(d_j,\theta_j)(0)\|_{L^2}, \\ \int_0^t \|\Lambda\theta_j\|_{L^2}\,d\tau &\lesssim \check{\kappa}^{-1}\|b_j(0)\|_{L^2} + \check{\kappa}^{-2}\|\Lambda^{-1}(d_j,\theta_j)(0)\|_{L^2}. \end{aligned}$$

Therefore we have

$$\begin{aligned} 2^{2j}\int_0^t \|d_j\|_{L^2}\,d\tau \lesssim \|\Lambda b_j(0)\|_{L^2} + \check{\kappa}^{-1}\|b_j(0)\|_{L^2} \\ + \check{\kappa}^{-2}\|\Lambda^{-1}(d_j,\theta_j)(0)\|_{L^2} + \check{\kappa}^{-1}\|(d_j,\theta_j)(0)\|_{L^2}. \end{aligned} \tag{3.40}$$

Step 5: Final a priori estimate for (b,u,θ). Putting together inequalities (3.37), (3.39) and (3.40) and using the standard properties of the heat equation (as regards w), we get if $j\leq 0$:

$$\|(b_j,u_j,\theta_j)(t)\|_{L^2}+2^{2j}\int_0^t\|(b_j,u_j,\theta_j)\|_{L^2}\,d\tau\leq C\|(b_j,u_j,\theta_j)(0)\|_{L^2}, \tag{3.41}$$

and, if $j\geq 0$:

$$\|(2^jb_j,u_j,2^{-j}\theta_j)(t)\|_{L^2}+\int_0^t\|(2^jb_j,2^{2j}u_j,2^j\theta_j)\|_{L^2}\,d\tau\leq C\|(2^jb_j,u_j,2^{-j}\theta_j)(0)\|_{L^2}. \tag{3.42}$$

The above constant C depends only on $\check{\mu}$ and $\check{\kappa}$.

3.2. **A priori estimates for the paralinearized system.** As pointed out in the previous subsection (see in particular (3.42)), there is no gain of regularity for b throughout the evolution (only damping in fact). Therefore, the convection term $v\cdot\nabla b$ cannot just be considered as a source term, tractable by Duhamel formula, for the presence of ∇b will induce a loss of one derivative in the estimates.

At the same time, at the level of L^2 estimates, this convection term is rather harmless provided $\operatorname{div}v$ is in $L^1(\mathbb{R}^+;L^\infty)$ (it is only a matter of integrating by parts). The natural idea is thus to keep the convection terms in the linearized equations[5] and to resume to the method of the previous paragraph. As however the Littlewood-Paley localization operator $\dot{\Delta}_j$ *does not* commute with the material derivative $(\partial_t+v\cdot\nabla)$, it is convenient to keep only the 'bad' part of the convection term, that is the one which does induce a loss of one derivative. In order to better explain what we mean, we have to give a short presentation of Bony's decomposition (first introduced in [3]) and paraproduct calculus. The paraproduct is the bilinear operator defined on the set of couples of tempered distributions, by

$$T_fg:=\sum_j\dot{S}_{j-1}f\,\dot{\Delta}_jg\quad\text{with}\quad\dot{S}_{j-1}:=\chi(2^{-(j-1)}D).$$

The (formal) Bony decomposition of the product fg reads

$$fg=T_fg+T'_gf.$$

The basic idea is that the term T_fg is always defined but cannot be more regular than g, and that under suitable assumptions the other term T'_gf is more regular. If we look at the convection term, the 'bad' part that may cause a loss of one derivative and has to be included in the linear analysis is thus (with the summation convention over repeated indices) $T_{u^k}\partial_kb$. This motivates us to extend the analysis of the previous subsection to the following 'paralinearized' system:

$$\begin{cases}\partial_tb+\Lambda d+T_{v^k}\partial_kb=B,\\ \partial_td+T_{v^k}\partial_kd-\Delta d-\Lambda(b+\theta)=D,\\ \partial_t\theta+\Lambda d+T_{v^k}\partial_k\theta-\tilde{\kappa}\Delta\theta=G,\\ \partial_tw+T_{v^k}\partial_kw-\tilde{\mu}\Delta w=W,\end{cases} \tag{3.43}$$

where the source terms B, D, G, W and the vector field v are given.

[5]We keep *all* the terms just for questions of symmetry, but only $v\cdot\nabla b$ may cause a loss of derivative.

Proposition 3.1. *Let $\mathcal{V}(t) := \int_0^t \|\nabla v\|_{L^\infty}\,d\tau$. For all $s \in \mathbb{R}$, there exists a constant K depending only on $\tilde{\mu}$, $\check{\kappa}$, and a universal constant C such that the following inequality holds true:*

$$\begin{aligned}\|b\|_{\tilde{L}^\infty_t(\tilde{B}^{s+1,-}_1)}+\|(d,w)\|_{\tilde{L}^\infty_t(\dot{B}^s_{2,1})}+\|\theta\|_{\tilde{L}^\infty_t(\tilde{B}^{s-1,+}_1)}+\int_0^t\big(\|b\|_{\tilde{B}^{s+1,+}_1}+\|(d,w)\|_{\dot{B}^{s+2}_{2,1}}+\|\theta\|_{\tilde{B}^{s+1,+}_1}\big)\,d\tau\\ \leq Ke^{C\mathcal{V}(t)}\bigg(\|b_0\|_{\tilde{B}^{s+1,-}_1}+\|(d_0,w_0)\|_{\dot{B}^s_{2,1}}+\|\theta_0\|_{\tilde{B}^{s-1,+}_1}\\ +\int_0^t e^{-C\mathcal{V}(\tau)}\big(\|B\|_{\tilde{B}^{s+1,-}_1}+\|(D,W)\|_{\dot{B}^s_{2,1}}+\|G\|_{\tilde{B}^{s-1,+}_1}\big)\,d\tau\bigg)\cdot\end{aligned}$$

Proof. Compared to the study of the previous subsection, the main additional difficulty lies in the paraconvection terms. Indeed, the source terms may be easily dealt with by means of the Duhamel formula.

The paraconvection terms may be handled thanks to the following inequality:

$$\big|\big(\phi(2^{-j}D)(T_{v^k}\partial_k z)|\phi(2^{-j}D)z\big)_{L^2}\big|\leq C\|\nabla v\|_{L^\infty}\|\phi(2^{-j}D)z\|_{L^2}\sum_{|j'-j|\leq N}\|\phi(2^{-j'}D)z\|_{L^2} \tag{3.44}$$

which holds true for any smooth function ϕ with compact support away from the origin and large enough integer N depending only on $\operatorname{Supp}\phi$ and $\operatorname{Supp}\varphi$.

Let us justify (3.44). We fix some integer N so that

$$\operatorname{Supp}\phi(2^{-j}\cdot)\cap\operatorname{Supp}\big(\chi(2^{-j'}\cdot)*\varphi(2^{-j'}\cdot)\big)=\emptyset\quad\text{whenever}\quad|j-j'|>N.$$

Then we use the following algebraic identity:

$$\begin{aligned}\big(\phi(2^{-j}D)(T_{v^k}\partial_k z)|\phi(2^{-j}D)z\big)_{L^2}&=\sum_{|j'-j|\leq N}\big(\phi(2^{-j}D)(\dot{S}_{j'-1}v^k\partial_k\dot{\Delta}_{j'}z)|\phi(2^{-j}D)z\big)_{L^2}\\ &=\sum_{|j'-j|\leq N}\big(\phi(2^{-j}D)((\dot{S}_{j'-1}-\dot{S}_{j-1})v^k\partial_k\dot{\Delta}_{j'}z)|\phi(2^{-j}D)z\big)_{L^2}\\ &\quad+\sum_{|j'-j|\leq N}\big([\phi(2^{-j}D),\dot{S}_{j-1}v^k]\partial_k\dot{\Delta}_{j'}z|\phi(2^{-j}D)z\big)_{L^2}\\ &\quad+(\dot{S}_{j-1}v^k\partial_k\phi(2^{-j}D)z|\phi(2^{-j}D)z)_{L^2}.\end{aligned}$$

The first term may be bounded thanks to spectral localization and Bernstein inequality, and the second, to a standard commutator estimate (see e.g. [2], Lemma 2.97). The last term may be dealt with according to the following integration by parts:

$$\int\dot{S}_{j-1}v^k\partial_k\phi(2^{-j}D)z\,\phi(2^{-j}D)z\,dx=-\frac{1}{2}\int\operatorname{div}\dot{S}_{j-1}v\,(\phi(2^{-j}D)z)^2\,dx.$$

Let us now resume to the proof of Proposition 3.1. As an example, we show how the first two steps of the previous subsection have to be adapted for (3.43). So we apply $\dot{\Delta}_j$ to the first three equations and get:

$$\begin{cases}\partial_t b_j+\Lambda d_j+\dot{\Delta}_j(T_{v^k}\partial_k b)=B_j,\\ \partial_t d_j+\dot{\Delta}_j(T_{v^k}\partial_k d)-\Delta d_j-\Lambda(b_j+\theta_j)=D_j,\\ \partial_t\theta_j+\Lambda d_j+\dot{\Delta}_j(T_{v^k}\partial_k\theta)-\tilde{\kappa}\Delta\theta_j=G_j.\end{cases}$$

Taking the L^2-inner product of the first, second and third equations with b_j, d_j and θ_j, respectively, we find that

$$\frac12\frac{d}{dt}\bigl[\|b_j\|_{L^2}^2+\|d_j\|_{L^2}^2+\|\theta_j\|_{L^2}^2\bigr]+\|\Lambda d_j\|_{L^2}^2+\tilde\kappa\|\Lambda\theta_j\|_{L^2}^2+\bigl(\dot\Delta_j(T_{v^k}\partial_k b)|\dot\Delta_j b\bigr)_{L^2}$$
$$+\bigl(\dot\Delta_j(T_{v^k}\partial_k d)|\dot\Delta_j d\bigr)_{L^2}+\bigl(\dot\Delta_j(T_{v^k}\partial_k\theta)|\dot\Delta_j\theta\bigr)_{L^2}=(B_j|b_j)_{L^2}+(D_j|d_j)_{L^2}+(G_j|\theta_j)_{L^2}.$$

Therefore using Inequality (3.44) we readily get

$$\frac12\frac{d}{dt}\|(b_j,d_j,\theta_j)\|_{L^2}^2+\|\Lambda d_j\|_{L^2}^2+\tilde\kappa\|\Lambda\theta_j\|_{L^2}^2\leq\|(b_j,d_j,\theta_j)\|_{L^2}$$
$$\times\Bigl(\|(B_j,D_j,G_j)\|_{L^2}+C\|\nabla v\|_{L^\infty}\sum_{|j'-j|\leq N}\|(b_{j'},d_{j'},\theta_{j'})\|_{L^2}\Bigr).$$

Next, we use the fact that Λb_j-d_j satisfies

$$\partial_t(\Lambda b_j-d_j)+\Lambda(b_j+d_j)+\Lambda\dot\Delta_j(T_{v^k}\partial_k b)-\dot\Delta_j(T_{v^k}\partial_k d)=\Lambda B_j-D_j.$$

Therefore arguing as in the second step of the previous section, we get

$$\frac12\frac{d}{dt}f_j^2+H_j^2+\bigl((\Lambda\dot\Delta_j(T_{v^k}\partial_k b)-\dot\Delta_j(T_{v^k}\partial_k d))|(\Lambda b_j-d_j)\bigr)_{L^2}$$
$$+(1+\alpha)\bigl(\dot\Delta_j(T_{v^k}\partial_k b)|\dot\Delta_j b\bigr)_{L^2}+\alpha\bigl(\dot\Delta_j(T_{v^k}\partial_k d)|\dot\Delta_j d\bigr)_{L^2}+(1+\alpha)\bigl(\dot\Delta_j(T_{v^k}\partial_k\theta)|\dot\Delta_j\theta\bigr)_{L^2}$$
$$=(1+\alpha)(B_j|b_j)_{L^2}+\alpha(D_j|d_j)_{L^2}+(1+\alpha)(G_j|\theta_j)_{L^2}+\bigl((\Lambda B_j-D_j)|(\Lambda b_j-d_j)\bigr)_{L^2}$$

where f^j and H^j have been defined in (3.29) and (3.30), and $\alpha=2/\check\kappa-1$.

Note that all the paraconvection terms except the first one may be directly dealt with according to (3.44). As for the first one, we may use the decomposition:

$$\Lambda\dot\Delta_j(T_{v^k}\partial_k b)-\dot\Delta_j(T_{v^k}\partial_k d)=\dot\Delta_j T_{v^k}\partial_k(\Lambda b-d)+2^j[\phi(2^{-j}D),T_{v^k}]\partial_k b$$

with $\phi(\xi):=|\xi|\varphi(\xi)$. Therefore, applying again (3.44) and Lemma 2.97 in [2], we end up with

$$\bigl|\bigl((\Lambda\dot\Delta_j(T_{v^k}\partial_k b)-\dot\Delta_j(T_{v^k}\partial_k d))|(\Lambda b_j-d_j)\bigr)_{L^2}\bigr|$$
$$\lesssim\|\nabla v\|_{L^\infty}\|\Lambda b_j-d_j\|_{L^2}\sum_{|j'-j|\leq N}\bigl(\|\Lambda b_{j'}-d_{j'}\|_{L^2}+\|\Lambda b_{j'}\|_{L^2}\bigr).$$

The following steps may be done similarly, once noticed that operators such as $\Lambda^{\pm1}\dot\Delta_j$ may be written $2^{\pm j}\phi(2^{-j}D)$ for some suitable function ϕ with the same support as φ. The final inequality may be obtained after multiplying by 2^{js}, performing a summation over j and applying Gronwall's lemma. The details are left to the reader. □

3.3. **The proof of global existence.** This paragraph is devoted to proving Theorem 2.1 in the case $\varepsilon=\nu=1$. As explained at the incipit of this section, this will imply the global existence for general positive ε and ν. The proof of existence and uniqueness is similar to that for the full Navier-Stokes system in [5]. The only difference here is that the source term ∇V^ε is not in $L^1(\mathbb{R}^+;\dot B^{\frac12}_{2,1})$. However it still belongs to $L^1_{loc}(\mathbb{R}^+;\dot B^{\frac12}_{2,1})$ which suffices to establish local-in-time results, global results being a consequence of the following a priori estimates. Note that a direct proof based on Friedrichs spectral truncation method may also be easily implemented as we are interested in L^2 type estimates.

So let us now derive global a priori estimates under the smallness assumptions (2.14) and (2.15). Such estimates rely on Proposition 3.1 with $s=1/2$, once noticed that

$$u=\mathcal{P}u+(\mathrm{Id}-\mathcal{P})u=w-\nabla\Lambda^{-1}d,$$

that (b,d,θ,w) satisfies (3.43) with $v=u$ and, using the summation convention over repeated indices,

$$\begin{aligned}
B &:= T_{u^k}\partial_k b-u\cdot\nabla b-b\mathrm{div}\,u-\partial_t\widetilde{V}-\mathrm{div}\,(u\widetilde{V}),\\
D &:= T_{u^k}\partial_k d-\Lambda^{-1}\mathrm{div}\,(u\cdot\nabla u)-\Lambda^{-1}\mathrm{div}\left[\frac{a}{1+a}(\tilde{\mu}\Delta u+(\tilde{\lambda}+\tilde{\mu})\nabla\mathrm{div}\,u)+\frac{(\theta-a)\nabla a}{(1+a)}\right],\\
G &:= T_{u^k}\partial_k\theta-u\cdot\nabla\theta-\theta\mathrm{div}\,u-\frac{a}{1+a}\tilde{\kappa}\Delta\theta+\frac{1}{1+a}[2\tilde{\mu}|Du|^2+\tilde{\lambda}(\mathrm{div}\,u)^2],\\
W &:= T_{u^k}\partial_k w-\mathcal{P}(u\cdot\nabla u)-\mathcal{P}\left[\frac{a}{1+a}(\tilde{\mu}\Delta u+(\tilde{\lambda}+\tilde{\mu})\nabla\mathrm{div}\,u)+\frac{(\theta-a)\nabla a}{(1+a)}\right].
\end{aligned}$$

Setting $U(t):=\int_0^t\|\nabla u\|_{L^\infty}\,d\tau$ and

$$X(t):=\|b\|_{\tilde{L}^\infty_t(\tilde{B}^{\frac{3}{2},-}_1)}+\|u\|_{\tilde{L}^\infty_t(\dot{B}^{\frac12}_{2,1})}+\|\theta\|_{\tilde{L}^\infty_t(\tilde{B}_1^{-\frac12,+})}+\int_0^t\big(\|b\|_{\tilde{B}^{\frac{3}{2},+}_1}+\|u\|_{\dot{B}^{\frac52}_{2,1}}+\|\theta\|_{\tilde{B}^{\frac{3}{2},+}_1}\big)\,d\tau,$$

we may write

$$X(t)\leq Ke^{CU(t)}\bigg(X(0)+\int_0^t e^{-CU(\tau)}\big(\|B\|_{\tilde{B}^{\frac{3}{2},-}_1}+\|(D,W)\|_{\dot{B}^{\frac12}_{2,1}}+\|G\|_{\tilde{B}_1^{-\frac12,+}}\big)\,d\tau\bigg)\cdot\tag{3.45}$$

Throughout we suppose that $1+a$ is bounded and bounded away from 0, an assumption that is satisfied provided $\|a\|_{L^\infty(\dot{B}^{\frac{3}{2}}_{2,1})}$ is small enough.

Bounding $\|B\|_{\tilde{B}^{\frac12,\frac32}_1}$. According to Bony's decomposition, we have

$$u\cdot\nabla b-T_{u^k}\partial_k b=T'_{\partial_k b}u^k.$$

Hence standard results for the paraproduct imply (just decompose b into low and high frequencies):

$$\|T_{u^k}\partial_k b-u\cdot\nabla b\|_{\tilde{B}^{\frac{3}{2},-}_1}\lesssim\|\nabla b\|_{\tilde{B}^{\frac12,-}_1}\|u\|_{\dot{B}^{\frac52}_{2,1}}\cdot\tag{3.46}$$

Likewise, according to Lemma 5.1, we have

$$\|b\,\mathrm{div}\,u\|_{\tilde{B}^{\frac{3}{2},-}_1}\lesssim\|b\|_{\tilde{B}^{\frac{3}{2},-}_1}\|\mathrm{div}\,u\|_{\dot{B}^{\frac32}_{2,1}}\cdot\tag{3.47}$$

Finally, because $\mathrm{div}\,(\widetilde{V}u)=\widetilde{V}\mathrm{div}\,u+u\cdot\nabla\widetilde{V}$, we have

$$\|\mathrm{div}\,(u\widetilde{V})\|_{\tilde{B}^{\frac{3}{2},-}_1}\lesssim\|\widetilde{V}\|_{\tilde{B}^{\frac{3}{2},-}_1}\|\mathrm{div}\,u\|_{\dot{B}^{\frac32}_{2,1}}+\|\nabla\widetilde{V}\|_{\tilde{B}^{\frac{3}{2},-}_1}\|u\|_{\dot{B}^{\frac32}_{2,1}}\cdot\tag{3.48}$$

Bounding $\|(D,W)\|_{\dot B^{\frac12}_{2,1}}$. We concentrate on D, proving estimates for W being similar. We have

$$T_{u^k}\partial_k d-\Lambda^{-1}\mathrm{div}\,(u\cdot\nabla u)=[T_{u^k},\Lambda^{-1}\partial_i]\partial_k u^i-\Lambda^{-1}\partial_i T'_{\partial_k u^i}u^k.$$

Therefore, resorting to standard commutator estimates and continuity results for the paraproduct (see e.g. [2]), we get

$$\|T_{u^k}\partial_k d-\Lambda^{-1}\mathrm{div}\,(u\cdot\nabla u)\|_{\dot B^{\frac12}_{2,1}}\lesssim\|\nabla u\|_{L^\infty}\|u\|_{\dot B^{\frac12}_{2,1}}. \tag{3.49}$$

Next, combining composition and product estimates yields

$$\|\frac{a}{1+a}\nabla^2 u\|_{\dot B^{\frac12}_{2,1}}\lesssim\|a\|_{\dot B^{\frac32}_{2,1}}\|u\|_{\dot B^{\frac52}_{2,1}}, \tag{3.50}$$

and also

$$\Big\|\frac{(\theta-a)\nabla a}{1+a}\Big\|_{\dot B^{\frac12}_{2,1}}\lesssim\big(1+\|a\|_{\dot B^{\frac32}_{2,1}}\big)\|\nabla a\|_{\dot B^{\frac12}_{2,1}}\big(\|\theta^\ell\|_{\dot B^{\frac32}_{2,1}}+\|\theta^h\|_{\dot B^{\frac32}_{2,1}}+\|a\|_{\dot B^{\frac32}_{2,1}}\big).$$

Note that we expect θ^ℓ and θ^h to belong to $L^2(\mathbb{R}^+;\dot B^{\frac32}_{2,1})$ and $L^1(\mathbb{R}^+;\dot B^{\frac32}_{2,1})$, respectively, and that, applying Hölder inequality yields

$$\Big\|\frac{(\theta-a)\nabla a}{1+a}\Big\|_{L^1(\dot B^{\frac12}_{2,1})}\lesssim\big(1+\|a\|_{L^\infty(\dot B^{\frac32}_{2,1})}\big)\big(\|a\|_{L^2(\dot B^{\frac32}_{2,1})}\|(a,\theta^\ell)\|_{L^2(\dot B^{\frac32}_{2,1})}+\|a\|_{L^\infty(\dot B^{\frac32}_{2,1})}\|\theta^h\|_{L^1(\dot B^{\frac32}_{2,1})}\big).$$

So finally, because $\tilde B^{\frac32,-}_1\hookrightarrow\dot B^{\frac32}_{2,1}$,

$$\begin{aligned}\Big\|\frac{(\theta-a)\nabla a}{1+a}\Big\|_{L^1(\dot B^{\frac12}_{2,1})}&\lesssim\big(1+\|a\|_{L^\infty(\tilde B^{\frac32,-}_1)}\big)\\&\quad\times\big(\|a\|^2_{L^2(\dot B^{\frac32}_{2,1})}+\|a\|_{L^2(\dot B^{\frac32}_{2,1})}\|\theta\|_{L^2(\tilde B^{\frac12,+}_1)}+\|a\|_{L^\infty(\tilde B^{\frac32,-}_1)}\|\theta\|_{L^1(\tilde B^{\frac32,+}_1)}\big).\end{aligned} \tag{3.51}$$

Bounding $\|G\|_{\tilde B^{\frac12,+}_1}$. We first use the fact that

$$u\cdot\nabla\theta-T_{u^k}\partial_k\theta=T'_{\partial_k\theta}u^k.$$

Therefore

$$\|T_{u^k}\partial_k\theta-u\cdot\nabla\theta\|_{\tilde B^{-\frac12,+}_1}\lesssim\|\nabla\theta\|_{\tilde B^{-\frac32,+}_1}\|u\|_{\dot B^{\frac52}_{2,1}}. \tag{3.52}$$

Next, Lemma 5.1 implies that

$$\|\theta\,\mathrm{div}\,u\|_{\tilde B^{-\frac12,+}_1}\lesssim\|\theta\|_{\tilde B^{-\frac12,+}_1}\|\mathrm{div}\,u\|_{\dot B^{\frac32}_{2,1}}, \tag{3.53}$$

$$\|\frac{a}{1+a}\Delta\theta\|_{\tilde B^{-\frac12,+}_1}\lesssim\|a\|_{\dot B^{\frac32}_{2,1}}\|\theta\|_{\tilde B^{\frac32,+}_1}. \tag{3.54}$$

Finally, since $\dot B^{-\frac12}_{2,1}\hookrightarrow\tilde B_1^{-\frac12,+}$, standard product laws enable us to write that

$$\Big\|\frac{1}{1+a}\nabla u\otimes\nabla u\Big\|_{\tilde B^{-\frac12,+}_1}\lesssim\big(1+\|a\|_{\dot B^{\frac32}_{2,1}}\big)\|\nabla u\|^2_{\dot B^{1/2}_{2,1}}. \tag{3.55}$$

Plugging inequalities (3.46) to (3.55) in (3.45) and making the assumption that

$$\|\nabla u\|_{L^1(L^\infty)}\ll1\quad\text{and}\quad\|\widetilde V\|_{L^\infty(\tilde B^{\frac32,-}_1)}+\|\nabla\widetilde V\|_{L^2(\tilde B^{\frac32,-}_1)}\ll1, \tag{3.56}$$

we thus get

$$X(t) \leq C\big(X(0) + \|\partial_t \tilde{V}\|_{L^1(\tilde{B}_1^{\frac{3}{2},-})} + X^2(t) + X^4(t)\big).$$

It is now clear that the solution may be bounded for all time if $X(0)$ and $\tilde{V}$ are small enough: we get for some constant K depending only on $\tilde{\kappa}$, $\tilde{\mu}$ and $\tilde{\lambda}$,

$$X(t) \leq KC_0 \tag{3.57}$$

with

$$C_0 := \|b_0\|_{\tilde{B}_1^{\frac{3}{2},-}} + \|u_0\|_{\dot{B}_{2,1}^{\frac{1}{2}}} + \|\theta_0\|_{\tilde{B}_1^{-\frac{1}{2},+}} + \|\partial_t \widetilde{V}\|_{L^1(\tilde{B}_1^{\frac{3}{2},-})}.$$

3.4. **Convergence to the viscous and diffusive Boussinesq system.** The key observation is that in the asymptotics ε going to 0, the leading order part of the system for $(q^\varepsilon, \mathcal{Q}u^\varepsilon)$ is the acoustic wave equation, which has dispersive properties. This will enable us to show (first step) that $(q^\varepsilon, \mathcal{Q}u^\varepsilon)$ tends strongly to 0 in some negative Besov space. Next, we shall check that the limit Boussinesq system (1.7) supplemented with small data $v_0 \in \dot{B}_{2,1}^{\frac{1}{2}}$, $\Theta_0 \in \dot{B}_{2,1}^{-\frac{1}{2}}$ and potential V with $\partial_t V \in L^1(\dot{B}_{2,1}^{\frac{1}{2}})$ and $\nabla V \in L^2(\dot{B}_{2,1}^{\frac{1}{2}})$ has a unique global solution. Finally, resorting to maximal regularity estimates for the heat equation, we will conclude that $(\mathcal{P}u^\varepsilon, \Theta^\varepsilon) \to (v, \Theta)$.

3.4.1. *Convergence to zero for the oscillating modes* $(q^\varepsilon, \mathcal{Q}u^\varepsilon)$. In order to exhibit the decay properties of $(q^\varepsilon, \mathcal{Q}u^\varepsilon)$, we only have to consider the case $\varepsilon = 1$ and $\nu = 1$ thanks to the rescaling (3.21), which implies in particular that

$$(q, \mathcal{Q}u)(t,x) = \varepsilon(q^\varepsilon, \mathcal{Q}u^\varepsilon)(\varepsilon^2\nu t, \varepsilon\nu x).$$

Then using Strichartz estimates for the acoustic wave equation (see Proposition 5.1 in the appendix) will enable us to bound some suitable norm of $(q, \mathcal{Q}u)$. Resuming to the original variables, we then get for free the convergence to 0 for $(q^\varepsilon, \mathcal{Q}u^\varepsilon)$, with an explicit rate.

Let us give more details : $(q, \mathcal{Q}u)$ satisfies

$$\begin{cases} \partial_t q + \sqrt{2}\operatorname{div} \mathcal{Q}u = -\operatorname{div}(qu) - \dfrac{\sqrt{2}}{2}\Big(\partial_t \widetilde{V} + \operatorname{div}(\widetilde{V}u) + \tilde{\kappa}\dfrac{\Delta\theta}{1+a}\Big) \\ \qquad\qquad + \dfrac{\sqrt{2}}{2}\dfrac{1}{1+a}[2\mu|Du|^2 + \lambda(\operatorname{div} u)^2], \\ \partial_t \mathcal{Q}u + \sqrt{2}\nabla q = \mathcal{Q}\Big(\Big(\dfrac{a-\theta}{1+a}\Big)\nabla a - \dfrac{\mathcal{A}u}{1+a} - u\cdot\nabla u\Big). \end{cases} \tag{3.58}$$

Therefore Strichartz estimates (first inequality of Proposition 5.1 with $s = 1/2$) enable us to bound the norm of $(q, \mathcal{Q}u)$ in $\tilde{L}^{\frac{2p}{p-2}}(\dot{B}_{p,1}^{\frac{2}{p}-\frac{1}{2}})$ for all $p \in [2, \infty)$ in terms of the norm of the data in $\dot{B}_{2,1}^{\frac{1}{2}}$ and of the right-hand side in $L^1(\dot{B}_{2,1}^{\frac{1}{2}})$. Under our present assumptions however, the last term in the r.h.s. of the first equation belongs only to the larger space $L^1(\tilde{B}_1^{-\frac{1}{2},+})$. So one has to use the *second* inequality of Proposition 5.1 and just get estimates in the wider space $\tilde{L}^{\frac{2p}{p-2}}(\tilde{B}_{p,1}^{\frac{2}{p}-\frac{3}{2},+})$.

Let us bound the r.h.s. of (3.58) in $L^1(\tilde{B}_1^{-\frac{1}{2},+})$. All the terms may be dealt with by taking advantage of standard product laws and Lemma 5.1. More precisely we have, keeping in

mind the smallness of a in $L^\infty(\tilde B^{\frac32,-}_1)$ (and thus also in $L^\infty(\dot B^{\frac32}_{2,1})$ and $L^\infty(\mathbb{R}^+\times\mathbb{R}^3)$):

$$\begin{aligned}
&\|\mathrm{div}\,(\widetilde V u)\|_{L^1(\dot B^{\frac12}_{2,1})} &&\lesssim \|\widetilde V\|_{L^2(\dot B^{\frac32}_{2,1})}\|u\|_{L^2(\dot B^{\frac32}_{2,1})},\\
&\|(1+a)^{-1}\Delta\theta\|_{L^1(\tilde B_1^{-\frac12,+})} &&\lesssim \|\theta\|_{L^1(\tilde B_1^{\frac32,+})},\\
&\|(1+a)^{-1}\nabla u\otimes\nabla u\|_{L^1(\dot B^{-\frac12}_{2,1})} &&\lesssim \|u\|^2_{L^2(\dot B^{\frac32}_{2,1})},\\
&\|(1+a)^{-1}(a-\theta)\nabla a\|_{L^1(\dot B^{\frac12}_{2,1})} &&\lesssim \|a\|_{L^2(\dot B^{\frac32}_{2,1})}(\|a\|_{L^2(\dot B^{\frac32}_{2,1})}+\|\theta\|_{L^2(\tilde B_1^{\frac12,+})})+\|a\|_{L^\infty(\tilde B_1^{\frac32,-})}\|\theta\|_{L^1(\tilde B_1^{\frac32,+})}\\
&\|(1+a)^{-1}\mathcal{A}u\|_{L^1(\dot B^{\frac12}_{2,1})} &&\lesssim \|u\|_{L^1(\dot B^{\frac52}_{2,1})},\\
&\|u\cdot\nabla u\|_{L^1(\dot B^{\frac12}_{2,1})} &&\lesssim \|u\|^2_{L^2(\dot B^{\frac32}_{2,1})}.
\end{aligned}$$

Given that $\mathcal{Q}$ is a 0-th order multiplier (hence maps all Besov spaces involved here into themselves), that $\dot B^{-\frac12}_{2,1}$ and $\dot B^{\frac12}_{2,1}$ are continuously embedded in $\tilde B_1^{-\frac12,+}$, and that $\tilde B_1^{\frac32,-}$ is continuously embedded in $\tilde B_1^{-\frac12,+}$, we eventually conclude that (with the notation of (3.45) and (3.57)):

$$\|(q,\mathcal{Q}u)\|_{\tilde L^{\frac{2p}{p-2}}(\tilde B^{\frac2p-\frac32,+}_{p,1})}\lesssim \|(q_0,\mathcal{Q}u_0)\|_{\tilde B_1^{-\frac12,+}}+X+X^2+\|\partial_t\tilde V\|_{L^1(\tilde B_1^{\frac32,-})}+\|\tilde V\|_{L^2(\dot B^{\frac32}_{2,1})}X.$$

Therefore, given that $X(t)\leq KC_0$ and that C_0 is small,

$$\|(q,\mathcal{Q}u)\|_{\tilde L^{\frac{2p}{p-2}}(\tilde B^{\frac2p-\frac32,+}_{p,1})}\leq KC_0\quad\text{for all}\quad p\in[2,\infty) \tag{3.59}$$

with K depending only on p, $\tilde\kappa$ and $\tilde\mu$.

On the other hand, Inequality (3.57) implies that

$$\|(q,\mathcal{Q}u)\|_{L^1(\tilde B_1^{\frac32,+})}\leq KC_0.$$

Therefore, using the fact that

$$[L^1(\tilde B_1^{\frac32,+}),\tilde L^{\frac{2p}{p-2}}(\tilde B^{\frac2p-\frac32,+}_{p,1})]_{p/(p+2)}\subset\widetilde L^2(\tilde B^{\frac4q-\frac32}_{q,1})\quad\text{with } q=(p+2)/2,$$

we get also

$$\|(q,\mathcal{Q}u)\|_{\widetilde L^2(\tilde B^{\frac4q-\frac32,+}_{q,1})}\leq KC_0\quad\text{for all}\quad q\in[2,\infty).$$

Given that $(q,\mathcal{Q}u)$ is in $\tilde L^2(\tilde B_1^{\frac12,+})$ hence in $\tilde L^2(\tilde B^{-1+\frac3q,+}_{q,1})$, an ultimate interpolation ensures that

$$\|(q,\mathcal{Q}u)\|_{\widetilde L^2(\tilde B^{s-1,+}_{p,1})}\leq KC_0\quad\text{for all}\quad s\in[-1/2+4/p,3/p]\ \text{ and }\ p\in[2,\infty). \tag{3.60}$$

Of course, we also have $\mathcal{Q}u$ in $\widetilde L^2(\dot B^{\frac32}_{2,1})$ whence in $\widetilde L^2(\dot B^{\frac3p}_{p,1})$ for all $p\geq2$. Therefore, interpolating with (3.60), we deduce that

$$\|\mathcal{Q}u\|_{\widetilde L^2(\dot B^{s}_{p,1})}\leq KC_0\quad\text{for all}\quad s\in[-1/2+4/p,3/p]. \tag{3.61}$$

Now coming back to the initial variables, (3.59), (3.60) and (3.61) translate into

$$\nu^{\frac12-\frac1p}\|(q^\varepsilon,\mathcal{Q}u^\varepsilon)\|_{\tilde L^{\frac{2p}{p-2}}(\tilde B^{\frac2p-\frac32,+}_{p,\varepsilon\nu})}\leq K(\varepsilon\nu)^{\frac12-\frac1p}C_0^\varepsilon\quad\text{for all}\quad p\in[2,\infty), \tag{3.62}$$

$$\nu^{\frac12}\|q^\varepsilon\|_{\tilde L^2(\tilde B^{s-1,+}_{p,\varepsilon\nu})}\leq K(\varepsilon\nu)^{\frac3p-s}C_0^\varepsilon\quad\text{for all}\quad s\in[-1/2+4/p,3/p], \tag{3.63}$$

$$\nu^{\frac12}\|\mathcal{Q}u^\varepsilon\|_{\tilde L^2(\dot B^s_{p,1})}\leq K(\varepsilon\nu)^{\frac3p-s}C_0^\varepsilon\quad\text{for all}\quad s\in[-1/2+4/p,3/p]. \tag{3.64}$$

3.4.2. *Global existence for the Boussinesq system* (1.7). Let us first briefly justify that, under our assumptions, the limit data (Θ_0,v_0,V) give rise to a global solution to (1.7). Establishing this is an obvious modification of the proof for the standard incompressible Navier-Stokes equation. It is only a matter of rewriting the system as

$$\Theta(t)=e^{t\frac\kappa2\Delta}\Theta_0+\int_0^t e^{(t-\tau)\frac\kappa2\Delta}\Big(\frac{\sqrt2}{2}(\partial_tV+v\cdot\nabla V)-v\cdot\nabla\Theta\Big)\,d\tau,$$
$$v(t)=e^{t\mu\Delta}v_0-\int_0^t e^{\mu(t-\tau)\Delta}\mathcal{P}\Big(v\cdot\nabla v+\frac{\sqrt2}{2}\Theta\nabla V\Big)\,d\tau,$$

and the global-in-time solvability for small data may be achieved as a consequence of the Banach fixed point theorem. Let us just check that global a priori estimates are available in the case of small data. Applying Proposition 5.2 and using that the product is continuous from $\dot B^{\frac12}_{2,1}\times\dot B^{\frac32}_{2,1}$ to $\dot B^{\frac12}_{2,1}$ implies that

$$\|\Theta\|_{\tilde L^\infty(\dot B^{\frac12}_{2,1})}+\kappa\|\Theta\|_{L^1(\dot B^{\frac52}_{2,1})}\lesssim\|\Theta_0\|_{\dot B^{\frac12}_{2,1}}+\|\partial_tV\|_{L^1(\dot B^{\frac12}_{2,1})}$$
$$+\|v\|_{L^2(\dot B^{\frac32}_{2,1})}\big(\|\nabla\Theta\|_{L^2(\dot B^{\frac12}_{2,1})}+\|\nabla V\|_{L^2(\dot B^{\frac12}_{2,1})}\big)$$

and that

$$\|v\|_{\tilde L^\infty(\dot B^{\frac12}_{2,1})}+\mu\|v\|_{L^1(\dot B^{\frac52}_{2,1})}\lesssim\|v_0\|_{\dot B^{\frac12}_{2,1}}+\|v\|_{L^2(\dot B^{\frac32}_{2,1})}\|\nabla v\|_{L^2(\dot B^{\frac12}_{2,1})}+\|\nabla V\|_{L^2(\dot B^{\frac12}_{2,1})}\|\Theta\|_{L^2(\dot B^{\frac32}_{2,1})}.$$

Hence, setting

$$Y:=\|(\Theta,v)\|_{\tilde L^\infty(\dot B^{\frac12}_{2,1})}+\nu\|(\Theta,v)\|_{L^1(\dot B^{\frac52}_{2,1})},$$

we get for some constant $K=K(\tilde\mu,\tilde\kappa)$,

$$Y\leq K\big(Y_0+\|\partial_tV\|_{L^1(\dot B^{\frac12}_{2,1})}+Y(Y+\nu^{-\frac12}\|\nabla V\|_{L^2(\dot B^{\frac12}_{2,1})})\big),$$

and it thus easy to close the estimates globally if Y_0, $\|\partial_tV\|_{L^1(\dot B^{\frac12}_{2,1})}$ and $\nu^{\frac12}\|\nabla V\|_{L^2(\dot B^{\frac12}_{2,1})}$ are small compared to ν.

3.4.3. *Convergence for the "incompressible" modes* $(\Theta^\varepsilon,\mathcal{P}u^\varepsilon)$. In this paragraph, we prove the convergence of $(\Theta^\varepsilon,\mathcal{P}u^\varepsilon)$ to the solution (Θ,v) to the Boussinesq equation (1.7). We claim that for any $p\in[2,\infty)$ and $s\in[-1/2+4/p,3/p]$ with $s>1/2$:

- $\delta\Theta^\varepsilon:=\Theta^\varepsilon-\Theta$ tends to 0 in $\tilde L^2(\tilde B^{s-1,+}_{p,\varepsilon\nu})\cap\tilde L^\infty(\tilde B^{s-2,+}_{p,\varepsilon\nu})$,
- $\delta v^\varepsilon:=\mathcal{P}u^\varepsilon-v$ tends to 0 in $L^1(\tilde B^{s,+}_{p,\varepsilon\nu})\cap\tilde L^\infty(\tilde B^{s-2,+}_{p,\varepsilon\nu})$.

For proving that, we shall use the parabolic estimates of Proposition 5.2 for the system satisfied by $(\delta\Theta^\varepsilon, \delta v^\varepsilon)$. Let us first focus on $\delta\Theta^\varepsilon$. By performing the difference between (1.5) and (1.7), we see that

$$\partial_t\delta\Theta^\varepsilon - \frac{\kappa}{2}\Delta\delta\Theta^\varepsilon = -\mathcal{P}u^\varepsilon\cdot\nabla\delta\Theta^\varepsilon - \delta v^\varepsilon\cdot\nabla\Theta + \frac{\sqrt{2}}{2}\big(\partial_t\delta V^\varepsilon + \mathcal{P}u^\varepsilon\cdot\nabla\delta V^\varepsilon + \delta v^\varepsilon\cdot\nabla V\big)$$
$$+\mathrm{div}\,((V^\varepsilon-\Theta^\varepsilon)\mathcal{Q}u^\varepsilon) + \frac{\kappa}{2}\Delta q^\varepsilon - \frac{\sqrt{2}}{2}\kappa\frac{\varepsilon a^\varepsilon}{1+\varepsilon a^\varepsilon}\Delta\theta^\varepsilon + \frac{\sqrt{2}}{2}\frac{\varepsilon}{1+\varepsilon a^\varepsilon}[2\mu|Du^\varepsilon|^2+\lambda(\mathrm{div}\,u^\varepsilon)^2].$$

Hence, according to Proposition 5.2, it suffices to get suitable estimates for the right-hand side in $L^1(\tilde{B}^{s-2,+}_{p,\varepsilon\nu})+\tilde{L}^2(\tilde{B}^{s-3,+}_{p,\varepsilon\nu})$. From product estimates (see Lemma 5.1) we easily get under the assumption that $s>1/2$ (in fact here we just need $s>-1/2$ owing to $\mathrm{div}\,\delta v^\varepsilon = \mathrm{div}\,\mathcal{P}u^\varepsilon = 0$):

$$\|\mathcal{P}u^\varepsilon\cdot\nabla\delta\Theta^\varepsilon\|_{L^1(\tilde{B}^{s-2,+}_{p,\varepsilon\nu})} \lesssim \|\mathcal{P}u^\varepsilon\|_{L^2(\dot{B}^{\frac32}_{2,1})}\|\nabla\delta\Theta^\varepsilon\|_{L^2(\tilde{B}^{s-2,+}_{p,\varepsilon\nu})}, \tag{3.65}$$

$$\|\delta v^\varepsilon\cdot\nabla\Theta\|_{L^1(\tilde{B}^{s-2,+}_{p,\varepsilon\nu})} \lesssim \|\nabla\Theta\|_{L^2(\dot{B}^{\frac12}_{2,1})}\|\delta v^\varepsilon\|_{L^2(\tilde{B}^{s-1,+}_{p,\varepsilon\nu})}, \tag{3.66}$$

$$\|\mathcal{P}u^\varepsilon\cdot\nabla\delta V^\varepsilon\|_{L^1(\tilde{B}^{s-2,+}_{p,\varepsilon\nu})} \lesssim \|\mathcal{P}u^\varepsilon\|_{L^2(\dot{B}^{\frac32}_{2,1})}\|\nabla\delta V^\varepsilon\|_{L^2(\tilde{B}^{s-2,+}_{p,\varepsilon\nu})}, \tag{3.67}$$

$$\|\delta v^\varepsilon\cdot\nabla V\|_{L^1(\tilde{B}^{s-2,+}_{p,\varepsilon\nu})} \lesssim \|\nabla V\|_{L^2(\dot{B}^{\frac12}_{2,1})}\|\delta v^\varepsilon\|_{L^2(\tilde{B}^{s-1,+}_{p,\varepsilon\nu})}. \tag{3.68}$$

We split the next term into (referring to the notation introduced in (2.12) with $\alpha=\varepsilon\nu$)

$$\mathrm{div}\,((V^\varepsilon-\Theta^\varepsilon)\mathcal{Q}u^\varepsilon) = \mathrm{div}\,((V^\varepsilon-\Theta^{\varepsilon,\ell})\mathcal{Q}u^\varepsilon) - \mathrm{div}\,(\Theta^{\varepsilon,h}\mathcal{Q}u^\varepsilon).$$

First we have

$$\begin{aligned}\|\mathrm{div}\,((V^\varepsilon-\Theta^{\varepsilon,\ell})\mathcal{Q}u^\varepsilon)\|_{L^1(\tilde{B}^{s-2,+}_{p,\varepsilon\nu})} &\lesssim \|(V^\varepsilon-\Theta^{\varepsilon,\ell})\mathcal{Q}u^\varepsilon\|_{L^1(\tilde{B}^{s-1,+}_{p,\varepsilon\nu})}\\ &\lesssim (\|V^\varepsilon\|_{L^2(\dot{B}^{\frac32}_{2,1})} + \|\Theta^{\varepsilon,\ell}\|_{L^2(\dot{B}^{\frac32}_{2,1})})\|\mathcal{Q}u^\varepsilon\|_{L^2(\tilde{B}^{s-1,+}_{p,\varepsilon\nu})},\end{aligned} \tag{3.69}$$

and, second

$$\|\mathrm{div}\,(\Theta^{\varepsilon,h}\mathcal{Q}u^\varepsilon)\|_{L^1(\tilde{B}^{s-2,+}_{p,\varepsilon\nu})} \lesssim \frac{1}{\epsilon\nu}\|\Theta^{\varepsilon,h}\mathcal{Q}u^\varepsilon\|_{L^1(\dot{B}^{s-1}_{p,1})} \lesssim \|\mathcal{Q}u^\varepsilon\|_{L^2(\dot{B}^{s}_{p,1})}\|\Theta^{\varepsilon,h}\|_{L^2(\tilde{B}^{\frac12,+}_{\varepsilon\nu})}. \tag{3.70}$$

Next, we see that, for all $\alpha\in[0,1)$,

$$\|\frac{\varepsilon a^\varepsilon}{1+\varepsilon a^\varepsilon}\Delta\theta^\varepsilon\|_{L^1(\tilde{B}^{-\frac12-\alpha,+}_{\varepsilon\nu})} \lesssim \|\varepsilon a^\varepsilon\|_{L^\infty(\dot{B}^{\frac32-\alpha}_{2,1})}\|\Delta\theta^\varepsilon\|_{L^1(\tilde{B}^{-\frac12,+}_{\varepsilon\nu})}.$$

Now, by interpolation

$$\|a^\varepsilon\|_{\dot{B}^{\frac32-\alpha}_{2,1}} \lesssim \|a^\varepsilon\|^{1-\alpha}_{\dot{B}^{\frac32}_{2,1}}\|a^\varepsilon\|^{\alpha}_{\dot{B}^{\frac12}_{2,1}}$$

and the definition of the norm in $\tilde{B}^{\frac32,-}_{\varepsilon\nu}$ implies that

$$\|a^\varepsilon\|_{\dot{B}^{\frac12}_{2,1}} + \varepsilon\nu\|a^\varepsilon\|_{\dot{B}^{\frac32}_{2,1}} \lesssim \|a^\varepsilon\|_{\tilde{B}^{\frac32,-}_{2,\varepsilon\nu}}.$$

Therefore

$$\|\varepsilon\nu a^\varepsilon\|_{\dot{B}^{\frac32-\alpha}_{2,1}} \lesssim (\varepsilon\nu)^\alpha\|a^\varepsilon\|_{\tilde{B}^{\frac32,-}_{2,\varepsilon\nu}}. \tag{3.71}$$

We also notice that $\tilde{B}^{-\frac12-\alpha,+}_{2,\varepsilon\nu} \hookrightarrow \tilde{B}^{\frac3p-2-\alpha,+}_{p,\varepsilon\nu}$ for $p\geq 2$. Therefore if we take

$$\alpha := 3/p - s,$$

then we get, keeping in mind that $\|a^\varepsilon\|_{L^\infty(\tilde B^{\frac32,-}_{2,\varepsilon\nu})}$ is small,

$$\|\frac{\varepsilon a^\varepsilon}{1+\varepsilon a^\varepsilon}\Delta\theta^\varepsilon\|_{L^1(\tilde B^{s-2,+}_{p,\varepsilon\nu})}\lesssim\nu^{-1}(\varepsilon\nu)^\alpha\|a^\varepsilon\|_{L^\infty(\tilde B^{\frac32,-}_{2,\varepsilon\nu})}\|\theta^\varepsilon\|_{L^1(\tilde B^{\frac32,+}_{2,\varepsilon\nu})}. \tag{3.72}$$

Finally,

$$\begin{aligned}\|\frac{\varepsilon}{1+\varepsilon a^\varepsilon}[2\mu|Du^\varepsilon|^2+\lambda(\operatorname{div}u^\varepsilon)^2]\|_{L^1(\dot B^{-\frac12}_{2,1})}&\lesssim\varepsilon(1+\|\varepsilon a^\varepsilon\|_{L^\infty(\dot B^{\frac32}_{2,1})})\|\nabla u^\varepsilon\|^2_{L^2(\dot B^{\frac12}_{2,1})},\\&\lesssim\varepsilon(1+\nu^{-1}\|a^\varepsilon\|_{L^\infty(\tilde B^{\frac32,-}_{2,\varepsilon\nu})})\|u^\varepsilon\|^2_{L^2(\dot B^{\frac32}_{2,1})}.\end{aligned}$$

At this point, let us notice that for all $z\in\dot B^{-\frac12}_{2,1}$ and $\alpha\in[0,1]$,

$$\begin{aligned}\|z\|_{\dot B^{-\frac12-\alpha,+}_{2,\varepsilon\nu}}&=\|z^\ell\|_{\dot B^{\frac12-\alpha}_{2,1}}+(\varepsilon\nu)^{-1}\|z^h\|_{\dot B^{-\frac12-\alpha}_{2,1}}\\&\lesssim(\varepsilon\nu)^{\alpha-1}\|z\|_{\dot B^{-\frac12}_{2,1}}.\end{aligned}$$

Since $\tilde B^{-\frac12-\alpha,+}_{2,\varepsilon\nu}\hookrightarrow\tilde B^{s-2,+}_{p,\varepsilon\nu}$ (with $\alpha=3/p-s$), we thus end up with

$$\|\frac{\varepsilon}{1+\varepsilon a^\varepsilon}[2\mu|Du^\varepsilon|^2+\lambda(\operatorname{div}u^\varepsilon)^2]\|_{L^1(\tilde B^{s-2,+}_{p,\varepsilon\nu})}\lesssim\nu^{-1}(\varepsilon\nu)^\alpha\|u^\varepsilon\|^2_{L^2(\dot B^{\frac32}_{2,1})}. \tag{3.73}$$

So putting (3.65) to (3.73) together and using (2.16), we conclude that

$$\begin{aligned}\nu^{\frac12}\|\delta\Theta^\varepsilon\|_{\tilde L^2(\tilde B^{s-1,+}_{p,\varepsilon\nu})}+\|\delta\Theta^\varepsilon\|_{\tilde L^\infty(\tilde B^{s-2,+}_{p,\varepsilon\nu})}&\lesssim\|\delta\Theta^\varepsilon_0\|_{\tilde B^{s-2,+}_{p,\varepsilon\nu}}+M_0\|(\delta v^\varepsilon,\delta\Theta^\varepsilon)\|\|_{\tilde L^2(\tilde B^{s-1,+}_{p,\varepsilon\nu})}\\&+M_0(\varepsilon\nu)^\alpha(M_0+1)+M_0\|\nabla\delta V^\varepsilon\|_{L^2(\tilde B^{s-2,+}_{p,\varepsilon\nu})}+\|\partial_t\delta V^\varepsilon\|_{L^1(\tilde B^{s-2,+}_{p,\varepsilon\nu})+\tilde L^2(\tilde B^{s-3,+}_{p,\varepsilon\nu})}.\end{aligned} \tag{3.74}$$

Let us now concentrate on the proof of estimates for δv^ε. We have, subtracting (1.7) from (1.5) and using (1.6),

$$\begin{aligned}\partial_t\delta v^\varepsilon-\mu\Delta\delta v^\varepsilon+\mathcal P(\mathcal Pu^\varepsilon\cdot\nabla\delta v^\varepsilon+\delta v^\varepsilon\cdot\nabla v)=&-\frac{\sqrt2}{2}\mathcal P\big(\Theta^\varepsilon\nabla\delta V^\varepsilon+\delta\Theta^\varepsilon\nabla V+q^\varepsilon\nabla V^\varepsilon-2q^\varepsilon\nabla b^\varepsilon\big)\\&-\mathcal P\bigg(u^\varepsilon\cdot\nabla\mathcal Qu^\varepsilon+\mathcal Qu^\varepsilon\cdot\nabla\mathcal Pu^\varepsilon+\frac{\varepsilon a^\varepsilon}{1+\varepsilon a^\varepsilon}\mathcal Au^\varepsilon-\frac{\varepsilon a^\varepsilon(\theta^\varepsilon-a^\varepsilon)}{1+\varepsilon a^\varepsilon}\nabla a^\varepsilon\bigg).\end{aligned}$$

Therefore, according to Proposition 5.2 and to the fact that $\mathcal P$ is a self-map on any homogeneous Besov space, we have

$$\begin{aligned}\|\delta v^\varepsilon\|_{\tilde L^\infty(\tilde B^{s-2,+}_{p,\varepsilon\nu})}+\nu\|\delta v^\varepsilon\|_{L^1(\tilde B^{s,+}_{p,\varepsilon\nu})}&\lesssim\|\delta v^\varepsilon_0\|_{\tilde B^{s-2,+}_{p,\varepsilon\nu}}+\|\mathcal Pu^\varepsilon\cdot\nabla\delta v^\varepsilon\|_{L^1(\tilde B^{s-2,+}_{p,\varepsilon\nu})}\\&+\|\delta v^\varepsilon\cdot\nabla v\|\|_{L^1(\tilde B^{s-2,+}_{p,\varepsilon\nu})}+\|\Theta^\varepsilon\nabla\delta V^\varepsilon\|_{L^1(\tilde B^{s-2,+}_{p,\varepsilon\nu})}+\|\delta\Theta^\varepsilon\nabla V\|_{L^1(\tilde B^{s-2,+}_{p,\varepsilon\nu})}\\&+\|q^\varepsilon\nabla V^\varepsilon\|_{L^1(\tilde B^{s-2,+}_{p,\varepsilon\nu})}+\|q^\varepsilon\nabla b^\varepsilon\|_{L^1(\tilde B^{s-2,+}_{p,\varepsilon\nu})}+\|u^\varepsilon\cdot\nabla\mathcal Qu^\varepsilon\|_{L^1(\tilde B^{s-2,+}_{p,\varepsilon\nu})}\\&+\|\mathcal Qu^\varepsilon\cdot\nabla\mathcal Pu^\varepsilon\|_{L^1(\tilde B^{s-2,+}_{p,\varepsilon\nu})}+\|\frac{\varepsilon a^\varepsilon}{1+\varepsilon a^\varepsilon}\mathcal Au^\varepsilon\|_{L^1(\tilde B^{s-2,+}_{p,\varepsilon\nu})}+\|\frac{\varepsilon a^\varepsilon(\theta^\varepsilon-a^\varepsilon)}{1+\varepsilon a^\varepsilon}\nabla a^\varepsilon\|_{L^1(\tilde B^{s-2,+}_{p,\varepsilon\nu})}.\end{aligned}$$

The following inequalities stem from product laws (see Lemma 5.1), under the assumption that $s>-1/2$:

$$\|\mathcal Pu^\varepsilon\cdot\nabla\delta v^\varepsilon\|_{L^1(\tilde B^{s-2,+}_{p,\varepsilon\nu})}\lesssim\|\mathcal Pu^\varepsilon\|_{L^2(\dot B^{\frac32}_{2,1})}\|\nabla\delta v^\varepsilon\|_{L^2(\tilde B^{s-2,+}_{p,\varepsilon\nu})}, \tag{3.75}$$

$$\|\delta v^\varepsilon\cdot\nabla v\|\|_{L^1(\tilde B^{s-2,+}_{p,\varepsilon\nu})}\lesssim\|\nabla v\|_{L^2(\dot B^{\frac12}_{2,1})}\|\delta v^\varepsilon\|_{L^2(\tilde B^{s-1,+}_{p,\varepsilon\nu})}. \tag{3.76}$$

Next we have, if $s>1/2$,

$$\|u^\varepsilon\cdot\nabla\mathcal{Q}u^\varepsilon\|_{L^1(\tilde B^{s-2,+}_{p,\varepsilon\nu})}\lesssim\|u^\varepsilon\|_{L^2(\dot B^{\frac32}_{2,1})}\|\nabla\mathcal{Q}u^\varepsilon\|_{L^2(\tilde B^{s-2,+}_{p,\varepsilon\nu})}, \tag{3.77}$$

$$\|\mathcal{Q}u^\varepsilon\cdot\nabla\mathcal{P}u^\varepsilon\|_{L^1(\dot B^{s-1}_{p,1})}\lesssim\|\nabla\mathcal{P}u^\varepsilon\|_{L^2(\dot B^{\frac12}_{2,1})}\|\mathcal{Q}u^\varepsilon\|_{L^2(\dot B^{s}_{p,1})}, \tag{3.78}$$

$$\|\Theta^\varepsilon\nabla\delta V^\varepsilon\|_{L^1(\tilde B^{s-2,+}_{p,\varepsilon\nu})}\lesssim\|\Theta^\varepsilon\|_{L^2(\tilde B^{\frac12,+}_{\varepsilon\nu})}\|\nabla\delta V^\varepsilon\|_{L^2(\dot B^{s-1}_{p,1})}, \tag{3.79}$$

$$\|\delta\Theta^\varepsilon\nabla V\|_{L^1(\tilde B^{s-2,+}_{p,\varepsilon\nu})}\lesssim\|\nabla V\|_{L^2(\dot B^{\frac12}_{2,1})}\|\delta\Theta^\varepsilon\|_{L^2(\tilde B^{s-1,+}_{p,\varepsilon\nu})}, \tag{3.80}$$

$$\|q^\varepsilon\nabla V^\varepsilon\|_{L^1(\tilde B^{s-2,+}_{p,\varepsilon\nu})}\lesssim\|\nabla V^\varepsilon\|_{L^2(\dot B^{\frac12}_{2,1})}\|q^\varepsilon\|_{L^2(\tilde B^{s-1,+}_{p,\varepsilon\nu})}, \tag{3.81}$$

$$\|q^\varepsilon\nabla b^\varepsilon\|_{L^1(\tilde B^{s-2,+}_{p,\varepsilon\nu})}\lesssim\|\nabla b^\varepsilon\|_{L^2(\dot B^{\frac12}_{2,1})}\|q^\varepsilon\|_{L^2(\tilde B^{s-1,+}_{p,\varepsilon\nu})}. \tag{3.82}$$

So arguing as in the proof of (3.72), we get

$$\|\frac{\varepsilon a^\varepsilon}{1+\varepsilon a^\varepsilon}\mathcal{A}u^\varepsilon\|_{L^1(\tilde B^{s-2,+}_{p,\varepsilon\nu})}\lesssim\nu^{-1}(\varepsilon\nu)^\alpha\|a^\varepsilon\|_{L^\infty(\tilde B^{\frac32,-}_{2,\varepsilon\nu})}\|u^\varepsilon\|_{L^1(\dot B^{\frac52}_{2,1})}. \tag{3.83}$$

Finally,

$$\|\frac{\varepsilon a^\varepsilon(\theta^\varepsilon-a^\varepsilon)}{1+\varepsilon a^\varepsilon}\nabla a^\varepsilon\|_{L^1(\tilde B^{-\frac12-\alpha,+}_{2,\varepsilon\nu})}\lesssim\|\nabla a^\varepsilon\|_{L^2(\dot B^{\frac12}_{2,1})}\|\theta^\varepsilon-a^\varepsilon\|_{L^2(\tilde B^{\frac12,+}_{2,\varepsilon\nu})}\|\varepsilon a^\varepsilon\|_{L^\infty(\dot B^{\frac32-\alpha}_{2,1})}.$$

Hence using again that $\tilde B^{-\frac12,+}_{2,\varepsilon\nu}\hookrightarrow\tilde B^{s-2,+}_{p,\varepsilon\nu}$ and (3.71), we conclude that

$$\begin{aligned}\|\frac{\varepsilon a^\varepsilon(\theta^\varepsilon-a^\varepsilon)}{1+\varepsilon a^\varepsilon}\nabla a^\varepsilon\|_{L^1(\tilde B^{s-2,+}_{p,\varepsilon\nu})}&\\ \lesssim\nu^{-1}(\varepsilon\nu)^\alpha\|a^\varepsilon\|_{L^\infty(\tilde B^{\frac32,-}_{\varepsilon\nu})}\|a^\varepsilon\|_{L^2(\dot B^{\frac32}_{2,1})}&(\|\theta^\varepsilon\|_{L^2(\tilde B^{\frac12,+}_{2,\varepsilon\nu})}+\|a^\varepsilon\|_{L^2(\dot B^{\frac32}_{2,1})}).\end{aligned} \tag{3.84}$$

So putting together inequalities (3.75) to (3.84), we end up with

$$\begin{aligned}\nu\|\delta v^\varepsilon\|_{L^1(\tilde B^{s,+}_{p,\varepsilon\nu})}+\|\delta v^\varepsilon\|_{\tilde L^\infty(\tilde B^{s-2,+}_{p,\varepsilon\nu})}&\lesssim\|\delta v^\varepsilon_0\|_{\tilde B^{s-2,+}_{p,\varepsilon\nu}}\\ &+M_0(\|(\delta v^\varepsilon,\delta\Theta^\varepsilon)\|_{L^2(\tilde B^{s-1,+}_{p,\varepsilon\nu})}+\|\nabla\delta V^\varepsilon\|_{L^2(\dot B^{s-1}_{p,1})})+(\varepsilon\nu)^\alpha M_0^2(1+\nu^{-1}M_0).\end{aligned}$$

Bearing in mind (3.65), we thus see that if M_0 is small enough with respect to ν,

$$\begin{aligned}\nu^{\frac12}\|\delta\Theta^\varepsilon\|_{\tilde L^2(\tilde B^{s-1,+}_{p,\varepsilon\nu})}+\|\delta\Theta^\varepsilon\|_{\tilde L^\infty(\tilde B^{s-2,+}_{p,\varepsilon\nu})}+\nu\|\delta v^\varepsilon\|_{L^1(\tilde B^{s,+}_{p,\varepsilon\nu})}+\|\delta v^\varepsilon\|_{\tilde L^\infty(\tilde B^{s-2,+}_{p,\varepsilon\nu})}&\lesssim\|(\delta\Theta^\varepsilon_0,\delta v^\varepsilon_0)\|_{\tilde B^{s-2,+}_{p,\varepsilon\nu}}\\ +M_0^2(\varepsilon\nu)^\alpha+M_0\|\nabla\delta V^\varepsilon\|_{L^2(\dot B^{s-1,+}_{p,1})}+\|\partial_t\delta V^\varepsilon\|_{L^1(\tilde B^{s-2,+}_{p,\varepsilon\nu})+L^2(\tilde B^{s-3,+}_{p,\varepsilon\nu})}&\end{aligned} \tag{3.85}$$

whenever $s>1/2$, $4/p-1/2\leq s\leq 3/p$ and $2\leq p<\infty$. This completes the proof of the theorem.

3.5. **The case of smoother data.** In order to improve the results of convergence (see Remark 2.2), we need to have higher order a priori estimates for the linear system (3.43). In effect, if we want to have convergence in (3.62) for the norm $\tilde L^{\frac{2p}{p-2}}(\dot B^{\frac2p-\frac12}_{p,1})$ rather than $\tilde L^{\frac{2p}{p-2}}(\tilde B^{\frac2p-\frac32,+}_{p,\varepsilon\nu})$ then we need θ to have the same regularity as b, namely $\dot B^{\frac32}_{2,1}$. So we need in addition that $\theta_0\in\tilde B^{\frac32,-}_1$ and, owing to linear coupling, this will enforce us to take $u_0\in\tilde B^{\frac32,-}_1$.

Here we just point out what has to be modified to our previous arguments so as to handle such data. Let us start with (3.25). We concentrate on the high frequency regime. First we notice that

$$\partial_t\theta - \tilde\kappa\Delta\theta = -\Lambda d.$$

Hence standard energy estimates ensure that

$$\|\Lambda\theta_j(t)\|_{L^2} + \tilde\kappa 2^{2j}\|\Lambda\theta_j\|_{L^1_t(L^2)} \leq \|\Lambda\theta_j(0)\|_{L^2} + \|\Lambda^2 d_j\|_{L^1_t(L^2)}.$$

Taking advantage of (3.42), we thus get

$$2^j\|\theta_j(t)\|_{L^2} + 2^{3j}\|\theta_j\|_{L^1_t(L^2)} \leq C\|(2^j b_j, d_j, 2^j\theta_j)(0)\|_{L^2}. \tag{3.86}$$

We also need more regularity for (b,d). This is given by (3.42) after multiplying by 2^j:

$$\|(2^{2j}b_j, 2^j d_j, \theta_j)(t)\|_{L^2} + \int_0^t \|(2^{2j}b_j, 2^{3j}d_j, 2^{2j}\theta_j)\|_{L^2}\,d\tau \leq C\|(2^{2j}b_j, 2^j d_j, \theta_j)(0)\|_{L^2}. \tag{3.87}$$

Arguing as in the proof of Proposition 3.1, we thus deduce that

$$\begin{aligned}\|b\|_{\tilde L^\infty_t(\tilde B^{s+1,-}_1\cap\tilde B^{s+2,-}_1)} + \|(d,w,\theta)\|_{\tilde L^\infty_t(\tilde B^{s+1,-}_1)} + \int_0^t \Big(\|b\|_{\tilde B^{s+1,+}_1\cap\tilde B^{s+2,+}_1} + \|(d,w,\theta)\|_{\tilde B^{s+3,-}_1}\Big)\,d\tau \\ \leq Ke^{CV(t)}\bigg(\|b_0\|_{\tilde B^{s+1,-}_1\cap\tilde B^{s+2,-}_1} + \|(d_0,w_0,\theta_0)\|_{\tilde B^{s+1,-}_1} \\ + \int_0^t e^{-CV(\tau)}\Big(\|B\|_{\tilde B^{s+1,-}_1\cap B^{s+2,-}_1} + \|(D,W,G)\|_{\tilde B^{s+1,-}_1}\Big)\,d\tau\bigg).\end{aligned}$$

Starting from this inequality and following the computations of Subsection 3.3, it is easy to get the result of Remark 2.1. Next, resorting to the first inequality of Proposition 5.1 with $s = 1/2$ and to nonlinear estimates, we get Remark 2.2.

4. The nonconducting case

As pointed out in the introduction, in the case $\kappa = 0$, it is easier to work with $\mathcal{R}^\varepsilon$. The reason why is that the linearized equations for $(u^\varepsilon, \mathcal{R}^\varepsilon)$ are the same as those of the classical barotropic Navier-Stokes equations (see next paragraph). Apart from this purely technical point and the fact that one has to work with smoother data, the overall approach for investigating the global existence and low Mach number issues is the same : first we perform the change of variables

$$(a,u,\mathcal{R})(t,x) = \varepsilon(a^\varepsilon, u^\varepsilon, \mathcal{R}^\varepsilon)(\varepsilon^2\nu t, \varepsilon\nu x) \quad\text{and}\quad \widetilde V(t,x) = \varepsilon V^\varepsilon(\varepsilon^2\nu t, \varepsilon\nu x), \tag{4.88}$$

so as to reduce the proof of existence to the case $\varepsilon = \nu = 1$, and next we take advantage of dispersive properties of the acoustic wave equation, and of parabolic estimates to establish the convergence to some suitable solution of the Boussinesq system with no heat conduction (namely (1.11)).

4.1. Linear and paralinear estimates.

If we decompose, as in the heat-conducting case, the velocity field u into its (reduced) potential part d, and its divergence-free part w, then the linearized system about 0 reads

$$\begin{cases}\partial_t a + \Lambda d = 0,\\ \partial_t d - \Delta d - \Lambda\mathcal{R} = 0,\\ \partial_t\mathcal{R} + \Lambda d = 0,\\ \partial_t w - \tilde\mu\Delta w = 0.\end{cases} \tag{4.89}$$

As in the heat-conducting case, w just fulfills the heat equation. Next, we notice that $(\mathcal{R},d)$ satisfies the linearized equation for the compressible modes of the barotropic Navier-Stokes equations. Hence, following the method of [4], we gather that for some universal constant C,

$$\|(\mathcal{R}_j,d_j)(t)\|_{L^2}+2^{2j}\int_0^t\|(\mathcal{R}_j,d_j)\|_{L^2}\,d\tau\leq C\|(\mathcal{R}_j,d_j)(0)\|_{L^2}\qquad\text{if}\quad j\leq 0,$$
$$\|(2^j\mathcal{R}_j,d_j)(t)\|_{L^2}+\int_0^t\|(2^j\mathcal{R}_j,2^{2j}d_j)\|_{L^2}\,d\tau\leq C\|(2^j\mathcal{R}_j,d_j)(0)\|_{L^2}\quad\text{if}\quad j>0.$$

Now, from the first and last equations of (4.89), we see that

$$a_j(t)-\mathcal{R}_j(t)=a_j(0)-\mathcal{R}_j(0)\quad\text{for all}\quad t\in\mathbb{R}^+.$$

Hence, taking advantage of the above estimate for $\mathcal{R}_j$, we get

$$\max(1,2^j)\|a_j(t)\|_{L^2}\leq C\big(\max(1,2^j)\|(a_j(0),\mathcal{R}_j(0))\|_{L^2}+\|d_j(0)\|_{L^2}\big).$$

From those inequalities, arguing as in the case $\kappa>0$, one may deduce a priori estimates for the following paralinearized equations:

$$\begin{cases}\partial_t a+\Lambda d+T_{v^k}\partial_k a=A,\\ \partial_t d+T_{v^k}\partial_k d-\Delta d-\Lambda\mathcal{R}=D,\\ \partial_t\mathcal{R}+\Lambda d+T_{v^k}\partial_k\mathcal{R}=R,\\ \partial_t w+T_{v^k}\partial_k w-\tilde{\mu}\Delta w=W,\end{cases}\tag{4.90}$$

where the source terms A, D, R, W and the vector field v are given.

More precisely, we have

Proposition 4.1. *Let $\mathcal{V}(t):=\int_0^t\|\nabla v\|_{L^\infty}\,d\tau$. There exists a constant K depending only on $\tilde{\mu}$ and a universal constant C such that for all $s\in\mathbb{R}$, the following inequality holds true:*

$$\|(a,\mathcal{R})\|_{\tilde{L}^\infty_t(\tilde{B}^{s+1,-}_1)}+\|(d,w)\|_{\tilde{L}^\infty_t(\dot{B}^s_{2,1})}+\int_0^t\big(\|(d,w)\|_{\dot{B}^{s+2}_{2,1}}+\|\mathcal{R}\|_{\tilde{B}^{s+1,+}_1}\big)\,d\tau$$
$$\leq Ke^{C\mathcal{V}(t)}\bigg(\|(a_0,\mathcal{R}_0)\|_{\tilde{B}^{s+1,-}_1}+\|(d_0,w_0)\|_{\dot{B}^s_{2,1}}$$
$$+\int_0^t e^{-C\mathcal{V}(\tau)}\big(\|(A,R)\|_{\tilde{B}^{s+1,-}_1}+\|(D,W)\|_{\dot{B}^s_{2,1}}\big)\,d\tau\bigg).$$

4.2. **The proof of global existence.** Here, in the case $\varepsilon=\nu=1$, we want to prove the existence of a global solution $(a,u,\mathcal{R})$ to (1.9) with

$$a\in\widetilde{C}(\dot{B}^{\frac12}_{2,1}\cap\dot{B}^{\frac72}_{2,1}),\quad u\in\widetilde{C}(\dot{B}^{\frac12}_{2,1}\cap\dot{B}^{\frac52}_{2,1})\cap L^1(\dot{B}^{\frac52}_{2,1}\cap\dot{B}^{\frac92}_{2,1}),\quad\mathcal{R}\in\widetilde{C}(\dot{B}^{\frac12}_{2,1}\cap\dot{B}^{\frac72}_{2,1})\cap L^1(\tilde{B}^{\frac32,+}_1\cap\tilde{B}^{\frac72,+}_1).$$

For that, this is mainly a matter of proving a priori estimates in this space, taking for granted the existence of a solution. Indeed, the a priori estimates that we are going to prove below would be the same for the system truncated by means of the Friedrichs method (see e.g. [2], Chap. 10 for the related case of the barotropic Navier-Stokes equation).

More precisely, we have to bound:

$$X:=\|(a,\mathcal{R})\|_{\tilde{L}^\infty(\dot{B}^{\frac12}_{2,1}\cap\dot{B}^{\frac72}_{2,1})}+\|u\|_{\tilde{L}^\infty(\dot{B}^{\frac12}_{2,1}\cap\dot{B}^{\frac52}_{2,1})}+\|u\|_{L^1(\dot{B}^{\frac52}_{2,1}\cap\dot{B}^{\frac92}_{2,1})}+\|\mathcal{R}\|_{L^1(\tilde{B}^{\frac32,+}_1\cap\tilde{B}^{\frac72,+}_1)}.\tag{4.91}$$

As we have in mind to apply Proposition 4.1 (twice: once with $s=3/2$ and once with $s=7/2$), we rewrite (1.9) as follows:

$$\left\{\begin{array}{l}\partial_t a + T_{u^k}\partial_k a + \Lambda d = A,\\ \partial_t d + T_{u^k}\partial_k d - \Delta d - \Lambda\mathcal{R} = D,\\ \partial_t \mathcal{R} + \Lambda d + T_{u^k}\partial_k \mathcal{R} = R,\\ \partial_t w + T_{u^k}\partial_k w - \mu\Delta w = W,\end{array}\right. \tag{4.92}$$

where

$$\begin{aligned}
A &:= T_{u^k}\partial_k a - u\cdot\nabla a - a\mathrm{div}\, u,\\
D &:= T_{u_k}\partial_k d - \Lambda^{-1}\mathrm{div}\,(u\cdot\nabla u) - \Lambda^{-1}\mathrm{div}\left[\frac{a}{1+a}\big(\tilde\mu\Delta u + (\tilde\lambda+\tilde\mu)\nabla\mathrm{div}\, u\big) - \frac{a\nabla(\mathcal{R}+\widetilde{V})}{(1+a)}\right],\\
R &:= T_{u^k}\partial_k\mathcal{R} - u\cdot\nabla\mathcal{R} - \mathcal{R}\mathrm{div}\, u - \partial_t\widetilde{V} - \mathrm{div}\,(\widetilde{V}u) + [2\tilde\mu|Du|^2 + \tilde\lambda(\mathrm{div}\, u)^2],\\
W &:= T_{u^k}\partial_k w - \mathcal{P}(u\cdot\nabla u) - \mathcal{P}\left[\frac{a}{1+a}\big(\tilde\mu\Delta u + (\tilde\lambda+\tilde\mu)\nabla\mathrm{div}\, u\big) - \frac{a\nabla(\mathcal{R}+\widetilde{V})}{(1+a)}\right].
\end{aligned}$$

According to Proposition 4.1, we thus have to bound A, R in $L^1(\dot B^{\frac12}_{2,1}\cap\dot B^{\frac72}_{2,1})$ and D, W in $L^1(\dot B^{\frac12}_{2,1}\cap\dot B^{\frac52}_{2,1})$. We shall assume throughout that $\|a\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^3)}$ is small.

Bounds for $\|A\|_{L^1(\dot B^{\frac12}_{2,1}\cap\dot B^{\frac72}_{2,1})}$. Recall that

$$A = -T'_{\partial_k a}u^k - a\mathrm{div}\, u.$$

Using standard product laws for the paraproduct and remainder (see e.g. [2]), we get

$$\|T'_{\partial_k a}u^k\|_{L^1(\dot B^{\frac12}_{2,1})} \lesssim \|\nabla a\|_{L^\infty(\dot B^{-\frac12}_{2,1})}\|u\|_{L^1(\dot B^{\frac52}_{2,1})}, \tag{4.93}$$

$$\|T'_{\partial_k a}u^k\|_{L^1(\dot B^{\frac72}_{2,1})} \lesssim \|\nabla a\|_{L^\infty(\dot B^{\frac12}_{2,1})}\|u\|_{L^1(\dot B^{\frac92}_{2,1})}, \tag{4.94}$$

$$\|a\mathrm{div}\, u\|_{L^1(\dot B^{\frac12}_{2,1})} \lesssim \|a\|_{L^\infty(\dot B^{\frac12}_{2,1})}\|\mathrm{div}\, u\|_{L^1(\dot B^{\frac32}_{2,1})}, \tag{4.95}$$

$$\|a\mathrm{div}\, u\|_{L^1(\dot B^{\frac72}_{2,1})} \lesssim \|a\|_{L^\infty(\dot B^{\frac32}_{2,1})}\|\mathrm{div}\, u\|_{L^1(\dot B^{\frac72}_{2,1})} + \|a\|_{L^\infty(\dot B^{\frac72}_{2,1})}\|\mathrm{div}\, u\|_{L^1(\dot B^{\frac32}_{2,1})}. \tag{4.96}$$

Hence

$$\|A\|_{L^1(\dot B^{\frac12}_{2,1}\cap\dot B^{\frac72}_{2,1})} \lesssim \|a\|_{L^\infty(\dot B^{\frac12}_{2,1}\cap\dot B^{\frac72}_{2,1})}\|u\|_{L^1(\dot B^{\frac52}_{2,1}\cap\dot B^{\frac92}_{2,1})}. \tag{4.97}$$

Bounds for $\|D\|_{L^1(\dot B^{\frac12}_{2,1}\cap\dot B^{\frac52}_{2,1})}$ *and* $\|W\|_{L^1(\dot B^{\frac12}_{2,1}\cap\dot B^{\frac52}_{2,1})}$. We may rewrite D as follows:

$$D = [T_{u_k},\Lambda^{-1}\partial_i]\partial_k u^i - \Lambda^{-1}\partial_i T'_{\partial_k u^i}u^k - \Lambda^{-1}\mathrm{div}\left[\frac{a}{1+a}\big(\tilde\mu\Delta u + (\tilde\lambda+\tilde\mu)\nabla\mathrm{div}\, u - \nabla(\mathcal{R}+V)\big)\right].$$

The first two terms of D may be treated as in (3.49): we get for any $s>0$,

$$\|[T_{u_k},\Lambda^{-1}\partial_i]\partial_k u^i - \Lambda^{-1}\partial_i T'_{\partial_k u^i}u^k\|_{\dot B^s_{2,1}} \lesssim \|\nabla u\|_{L^\infty}\|u\|_{\dot B^s_{2,1}}. \tag{4.98}$$

Next, classical composition and tame estimates yield for $s>0$,

$$\|\frac{a}{1+a}\mathcal{A}u\|_{\dot B^s_{2,1}} \lesssim \|a\|_{L^\infty}\|\mathcal{A}u\|_{\dot B^s_{2,1}} + \|\mathcal{A}u\|_{L^\infty}\|a\|_{\dot B^s_{2,1}}.$$

Hence, using the embedding $\dot B^{\frac32}_{2,1}\hookrightarrow L^\infty$, we easily get

$$\|\frac{a}{1+a}\mathcal{A}u\|_{L^1(\dot B^{\frac12}_{2,1}\cap\dot B^{\frac52}_{2,1})}\lesssim\|a\|_{L^\infty(\dot B^{\frac12}_{2,1}\cap\dot B^{\frac52}_{2,1})}\|u\|_{L^1(\dot B^{\frac52}_{2,1}\cap\dot B^{\frac92}_{2,1})}. \tag{4.99}$$

Finally, we have

$$\|\frac{a}{1+a}\nabla(\mathcal{R}+\widetilde V)\|_{L^1(\dot B^{\frac12}_{2,1})}\lesssim\|a\|_{L^\infty(\dot B^{\frac12}_{2,1})}\|\nabla(\mathcal{R}+\widetilde V)\|_{L^1(\dot B^{\frac32}_{2,1})}, \tag{4.100}$$

$$\begin{aligned}\|\frac{a}{1+a}\nabla(\mathcal{R}+\widetilde V)\|_{L^1(\dot B^{\frac52}_{2,1})}&\lesssim\|a\|_{L^\infty(\dot B^{\frac32}_{2,1})}\|\nabla(\mathcal{R}+\widetilde V)\|_{L^1(\dot B^{\frac52}_{2,1})}\\&\quad+\|a\|_{L^\infty(\dot B^{\frac52}_{2,1})}\|\nabla(\mathcal{R}+\widetilde V)\|_{L^1(\dot B^{\frac32}_{2,1})}.\end{aligned} \tag{4.101}$$

So putting (4.98) to (4.101) together, we get

$$\begin{aligned}\|D\|_{L^1(\dot B^{\frac12}_{2,1}\cap\dot B^{\frac52}_{2,1})}&\lesssim\|u\|_{L^1(\dot B^{\frac52}_{2,1})}\|u\|_{L^\infty(\dot B^{\frac12}_{2,1}\cap\dot B^{\frac52}_{2,1})}\\&\quad+\|a\|_{L^\infty(\dot B^{\frac12}_{2,1}\cap\dot B^{\frac52}_{2,1})}\Big(\|u\|_{L^1(\dot B^{\frac52}_{2,1}\cap\dot B^{\frac92}_{2,1})}+\|\mathcal{R}\|_{L^1(\dot B^{\frac52}_{2,1}\cap\dot B^{\frac72}_{2,1})}+\|\nabla\widetilde V\|_{L^1(\dot B^{\frac32}_{2,1}\cap\dot B^{\frac52}_{2,1})}\Big).\end{aligned} \tag{4.102}$$

It is clear that W satisfies exactly the same inequality.

Bounds for $\|R\|_{L^1(\dot B^{\frac12}_{2,1}\cap\dot B^{\frac72}_{2,1})}$. Recall that

$$R=-T'_{\partial_k\mathcal{R}}u^k-\mathcal{R}\mathrm{div}\,u-\partial_t\widetilde V-\mathrm{div}\,(\widetilde V u)+[2\tilde\mu|Du|^2+\tilde\lambda(\mathrm{div}\,u)^2].$$

First we have for any $s>0$,

$$\|T'_{\partial_k\mathcal{R}}u^k\|_{\dot B^s_{2,1}}\lesssim\|\nabla\mathcal{R}\|_{L^\infty}\|u\|_{\dot B^s_{2,1}}.$$

Hence

$$\begin{aligned}\|T'_{\partial_k\mathcal{R}}u^k\|_{L^1(\dot B^{\frac12}_{2,1})}&\lesssim\|\mathcal{R}\|_{L^1(\dot B^{\frac52}_{2,1})}\|u\|_{L^\infty(\dot B^{\frac12}_{2,1})},\\\|T'_{\partial_k\mathcal{R}}u^k\|_{L^1(\dot B^{\frac72}_{2,1})}&\lesssim\|\mathcal{R}\|_{L^\infty(\dot B^{\frac52}_{2,1})}\|u\|_{L^1(\dot B^{\frac72}_{2,1})}.\end{aligned} \tag{4.103}$$

Next, product estimates imply that

$$\begin{aligned}\|\mathcal{R}\mathrm{div}\,u\|_{L^1(\dot B^{\frac12}_{2,1})}&\lesssim\|\mathcal{R}\|_{L^2(\dot B^{\frac32}_{2,1})}\|u\|_{L^2(\dot B^{\frac32}_{2,1})},\\\|\mathcal{R}\mathrm{div}\,u\|_{L^1(\dot B^{\frac72}_{2,1})}&\lesssim\|\mathcal{R}\|_{L^\infty(\dot B^{\frac32}_{2,1})}\|u\|_{L^1(\dot B^{\frac92}_{2,1})}+\|u\|_{L^1(\dot B^{\frac52}_{2,1})}\|\mathcal{R}\|_{L^\infty(\dot B^{\frac72}_{2,1})}.\end{aligned} \tag{4.104}$$

We also have

$$\|\mathrm{div}\,(\widetilde V u)\|_{L^1(\dot B^{\frac12}_{2,1})}\lesssim\|\widetilde V\|_{L^\infty(\dot B^{\frac12}_{2,1})}\|u\|_{L^1(\dot B^{\frac52}_{2,1})}+\|\widetilde V\|_{L^1(\dot B^{\frac52}_{2,1})}\|u\|_{L^\infty(\dot B^{\frac12}_{2,1})} \tag{4.105}$$

$$\|\mathrm{div}\,(\widetilde V u)\|_{L^1(\dot B^{\frac72}_{2,1})}\lesssim\|\widetilde V\|_{L^\infty(\dot B^{\frac32}_{2,1})}\|u\|_{L^1(\dot B^{\frac92}_{2,1})}+\|\widetilde V\|_{L^1(\dot B^{\frac92}_{2,1})}\|u\|_{L^\infty(\dot B^{\frac32}_{2,1})}. \tag{4.106}$$

And finally,

$$\|\nabla u\otimes\nabla u\|_{L^1(\dot B^{\frac12}_{2,1})}\lesssim\|\nabla u\|_{L^1(L^\infty)}\|\nabla u\|_{L^\infty(\dot B^{\frac12}_{2,1})}, \tag{4.107}$$

$$\|\nabla u\otimes\nabla u\|_{L^1(\dot B^{\frac72}_{2,1})}\lesssim\|\nabla u\|_{L^\infty(L^\infty)}\|\nabla u\|_{L^1(\dot B^{\frac72}_{2,1})}. \tag{4.108}$$

Therefore, combining inequalities (4.103) to (4.108), and using embedding, we end up with

$$\begin{aligned}\|R\|_{L^1(\dot B^{\frac12}_{2,1}\cap\dot B^{\frac72}_{2,1})} &\lesssim \|u\|_{L^1(\dot B^{\frac52}_{2,1}\cap\dot B^{\frac92}_{2,1})}\|u\|_{L^\infty(\dot B^{\frac12}_{2,1}\cap\dot B^{\frac52}_{2,1})} + \|\partial_t\widetilde V\|_{L^1(\dot B^{\frac12}_{2,1}\cap\dot B^{\frac72}_{2,1})}\\ &+\|\widetilde V\|_{L^1(\dot B^{\frac52}_{2,1}\cap\dot B^{\frac92}_{2,1})}\|u\|_{L^\infty(\dot B^{\frac12}_{2,1}\cap\dot B^{\frac32}_{2,1})} + \|u\|_{L^1(\dot B^{\frac52}_{2,1}\cap\dot B^{\frac92}_{2,1})}\|\widetilde V\|_{L^\infty(\dot B^{\frac12}_{2,1}\cap\dot B^{\frac32}_{2,1})}\\ &+\|\mathcal R\|_{L^\infty(\dot B^{\frac12}_{2,1}\cap\dot B^{\frac72}_{2,1})}\|u\|_{L^1(\dot B^{\frac52}_{2,1}\cap\dot B^{\frac92}_{2,1})}+\|\mathcal R\|_{L^2(\dot B^{\frac32}_{2,1})}\|u\|_{L^2(\dot B^{\frac32}_{2,1})}+\|\mathcal R\|_{L^1(\dot B^{\frac52}_{2,1})}\|u\|_{L^\infty(\dot B^{\frac12}_{2,1})}.\end{aligned}\tag{4.109}$$

Putting (4.97), (4.102) and (4.109) together, one may finally conclude that for some constant K depending only on $\tilde\lambda$ and $\tilde\mu$, we have

$$X\le K\Big(X(0)+X^2+\big(\|\widetilde V\|_{L^\infty(\dot B^{\frac12}_{2,1}\cap\dot B^{\frac32}_{2,1})}+\|\widetilde V\|_{L^1(\dot B^{\frac52}_{2,1}\cap\dot B^{\frac92}_{2,1})}\big)X+\|\partial_t\widetilde V\|_{L^1(\dot B^{\frac12}_{2,1}\cap\dot B^{\frac72}_{2,1})}\Big).$$

From this, we see that if $X(0)$ and the terms pertaining to $\widetilde V$ are small enough, then

$$X\le 2K\Big(X(0)+\|\partial_t\widetilde V\|_{L^1(\dot B^{\frac12}_{2,1}\cap\dot B^{\frac72}_{2,1})}\Big).\tag{4.110}$$

Going back to the original variables according to (4.88), we then get the global existence part of Theorem 2.3, for any $\varepsilon>0$.

4.3. **The proof of convergence.** As in the case where $\kappa>0$, we first show that $(\mathcal Qu^\varepsilon,\mathcal R^\varepsilon)$ goes to 0, a consequence of Strichartz estimates, then establish that $(\mathcal Pu^\varepsilon,\Theta^\varepsilon)$ goes to the solution (v,Θ) of the Boussinesq system (1.11).

4.3.1. *Convergence to* 0 *for* $(\mathcal Qu^\varepsilon,\mathcal R^\varepsilon)$. It suffices to prove dispersion estimates in the case $\varepsilon=1$. The change of variable (4.88) will provide us with decay estimates in the general case. Now, the system for $(\mathcal Qu,\mathcal R)$ reads

$$\begin{cases}\partial_t\mathcal Qu+\nabla\mathcal R=-\mathcal Q(u\cdot\nabla u)-\mathcal Q\Big(\dfrac{\mathcal Au}{1+a}\Big)+\mathcal Q\Big(\dfrac{a}{1+a}\nabla(f+\mathcal R)\Big)=:\mathbb H_1,\\ \partial_t\mathcal R+\operatorname{div}\mathcal Qu=-\partial_tV-\operatorname{div}\big((V+\mathcal R)u\big)-2\mu|Du|^2-(\lambda+\mu)(\operatorname{div}u)^2=:\mathbb H_2.\end{cases}$$

Therefore, Strichartz estimates imply that for all $p\in[2,\infty)$,

$$\|(\mathcal Qu,\mathcal R)\|_{\widetilde L^{\frac{2p}{p-2}}(\dot B^{\frac2p-\frac12}_{p,1})}\lesssim\|(\mathcal Qu_0,\mathcal R_0)\|_{\dot B^{\frac12}_{2,1}}+\|(\mathbb H_1,\mathbb H_2)\|_{L^1(\dot B^{\frac12}_{2,1})}.\tag{4.111}$$

So it is only a matter of bounding $\mathbb H_1$ and $\mathbb H_2$ in $L^1(\dot B^{\frac12}_{2,1})$, which may be done by using standard results of continuity in Besov spaces and the fact that $\mathcal Q$ is an homogeneous multiplier of degree 0. More precisely, we have

$$\begin{aligned}\|\mathcal Q(u\cdot\nabla u)\|_{L^1(\dot B^{\frac12}_{2,1})}&\lesssim\|u\|_{L^\infty(\dot B^{\frac12}_{2,1})}\|u\|_{L^1(\dot B^{\frac52}_{2,1})},\\ \|\mathcal Q\Big(\frac{\mathcal Au}{1+a}\Big)\|_{L^1(\dot B^{\frac12}_{2,1})}&\lesssim(1+\|a\|_{L^\infty(\dot B^{\frac32}_{2,1})})\|u\|_{L^1(\dot B^{\frac52}_{2,1})},\\ \|\mathcal Q\Big(\frac{a}{1+a}\nabla(V+\mathcal R)\Big)\|_{L^1(\dot B^{\frac12}_{2,1})}&\lesssim\|a\|_{L^\infty(\dot B^{\frac12}_{2,1})}(\|\nabla V\|_{L^1(\dot B^{\frac32}_{2,1})}+\|\nabla\mathcal R\|_{L^1(\dot B^{\frac32}_{2,1})}),\\ \|\operatorname{div}\big((V+\mathcal R)u\big)\|_{L^1(\dot B^{\frac12}_{2,1})}&\lesssim\|u\|_{L^1(\dot B^{\frac52}_{2,1})}\|V+\mathcal R\|_{L^\infty(\dot B^{\frac12}_{2,1})}+\|u\|_{L^\infty(\dot B^{\frac12}_{2,1})}\|V+\mathcal R\|_{L^1(\dot B^{\frac52}_{2,1})},\\ \|\nabla u\otimes\nabla u\|_{L^1(\dot B^{\frac12}_{2,1})}&\lesssim\|u\|_{L^1(\dot B^{\frac52}_{2,1})}\|u\|_{L^\infty(\dot B^{\frac32}_{2,1})}.\end{aligned}$$

Therefore, if we set

$$C_0 := \|(a_0,\mathcal{R}_0)\|_{\dot B^{\frac12}_{2,1}\cap\dot B^{\frac72}_{2,1}} + \|u_0\|_{B^{\frac12}_{2,1}\cap\dot B^{\frac52}_{2,1}} + \|\partial_t V\|_{L^1(\dot B^{\frac12}_{2,1}\cap\dot B^{\frac72}_{2,1})},$$

then plugging the above inequalities in (4.111) and using (4.110) leads to

$$\|(\mathcal{Q}u,\mathcal{R})\|_{\tilde L^{\frac{2p}{p-2}}(\dot B^{\frac2p-\frac12}_{p,1})} \leq KC_0 \quad \text{for all} \quad p\in[2,\infty).$$

From (4.110), we also know that $(\mathcal{Q}u,\mathcal{R})$ is bounded by KC_0 in $L^1(\dot B^{\frac52}_{2,1})$. Hence using interpolation exactly as in the case $\kappa>0$ leads to

$$\|(\mathcal{Q}u,\mathcal{R})\|_{\tilde L^2(\dot B^s_{p,1})} \leq KC_0 \quad \text{for all} \quad p\geq 2 \quad \text{and} \quad s\in[-1/2+4/p,3/p].$$

Now, going back to the original variables, we gather that for $\varepsilon>0$, we have

$$\|(\mathcal{Q}u^\varepsilon,\mathcal{R}^\varepsilon)\|_{\tilde L^{\frac{2p}{p-2}}(\dot B^{\frac2p-\frac12}_{p,1})} \leq KC_0^\varepsilon\varepsilon^{\frac12-\frac1p} \quad \text{if} \quad 2\leq p<\infty, \tag{4.112}$$

$$\nu^{\frac12}\|(\mathcal{Q}u^\varepsilon,\mathcal{R}^\varepsilon)\|_{\tilde L^2(\dot B^s_{p,1})} \leq KC_0^\varepsilon(\varepsilon\nu)^{\frac3p-s} \quad \text{for all} \quad p\geq 2 \quad \text{and} \quad s\in[-1/2+4/p,3/p]. \tag{4.113}$$

with C_0^ε defined in the statement of Theorem 2.3.

4.3.2. *Global existence of a solution to* (1.11). Under the assumption that (2.17), the existence of a global solution (Θ,v) to (1.11) satisfying (2.18) is an easy modification of the corresponding proof for the standard incompressible Navier-Stokes equations, combined with the following a priori estimate for the transport equation (see e.g. [2], Chap. 3):

$$\|\Theta\|_{\tilde L^\infty_T(\dot B^{\frac12}_{2,1})} \leq \|\Theta_0\|_{\dot B^{\frac12}_{2,1}} \exp\biggl(\int_0^T \|v\|_{\dot B^{\frac52}_{2,1}}\,dt\biggr).$$

Indeed, using once again Proposition 5.2 and product estimates, we see that

$$\|u\|_{\tilde L^\infty_T(\dot B^{\frac12}_{2,1})} + \mu\|u\|_{L^1_T(\dot B^{\frac52}_{2,1})} \lesssim \|u_0\|_{\dot B^{\frac12}_{2,1}} + \|u\|_{L^\infty_T(\dot B^{\frac12}_{2,1})}\|u\|_{L^1_T(\dot B^{\frac52}_{2,1})} + \|\Theta\|_{L^\infty_T(\dot B^{\frac12}_{2,1})}\|\nabla V\|_{L^1_T(\dot B^{\frac32}_{2,1})}.$$

Hence if (2.17) is fulfilled then one may close the a priori estimates globally in time.

4.3.3. *Convergence of* $(\mathcal{P}u^\varepsilon,\Theta^\varepsilon)$. Let us first notice that (recall that $\Theta^\varepsilon=a^\varepsilon-\mathcal{R}^\varepsilon-V^\varepsilon$)

$$\begin{aligned}\mathcal{P}\biggl(\tfrac{a^\varepsilon}{1+\varepsilon a^\varepsilon}\nabla(V^\varepsilon+\mathcal{R}^\varepsilon)\biggr) &= \mathcal{P}\Bigl(a^\varepsilon\nabla(V^\varepsilon+\mathcal{R}^\varepsilon)\Bigr) - \mathcal{P}\biggl(a^\varepsilon\tfrac{\varepsilon a^\varepsilon}{1+\varepsilon a^\varepsilon}\nabla(V^\varepsilon+\mathcal{R}^\varepsilon)\biggr)\\ &= \mathcal{P}\bigl(\Theta^\varepsilon\nabla(V^\varepsilon+\mathcal{R}^\varepsilon)\bigr) - \mathcal{P}\biggl(a^\varepsilon\tfrac{\varepsilon a^\varepsilon}{1+\varepsilon a^\varepsilon}\nabla(V^\varepsilon+\mathcal{R}^\varepsilon)\biggr).\end{aligned}$$

Therefore the system for $(\delta\Theta^\varepsilon,\delta v^\varepsilon):=(\Theta^\varepsilon-\Theta,\mathcal{P}u^\varepsilon-v)$ writes

$$\begin{cases}\partial_t\delta\Theta^\varepsilon + \mathcal{P}u^\varepsilon\cdot\nabla\delta\Theta^\varepsilon = -\delta v^\varepsilon\cdot\nabla\Theta - \mathcal{Q}u^\varepsilon\cdot\nabla\Theta^\varepsilon - \Theta^\varepsilon\mathrm{div}\,\mathcal{Q}u^\varepsilon - \varepsilon\bigl(2\mu|Du^\varepsilon|^2+\lambda(\mathrm{div}\,u^\varepsilon)^2\bigr),\\ \partial_t\delta v^\varepsilon - \mu\Delta\delta v^\varepsilon + \mathcal{P}(v\cdot\nabla\delta v^\varepsilon) + \mathcal{P}(\delta v^\varepsilon\cdot\nabla\mathcal{P}u^\varepsilon) = \mathcal{P}(\delta\Theta^\varepsilon\nabla V + \Theta^\varepsilon\nabla\delta V^\varepsilon + \Theta^\varepsilon\nabla\mathcal{R}^\varepsilon)\\ \qquad\qquad -\mathcal{P}\biggl(\mathcal{Q}u^\varepsilon\cdot\nabla\mathcal{P}u^\varepsilon + u^\varepsilon\cdot\nabla\mathcal{Q}u^\varepsilon + \tfrac{\varepsilon a^\varepsilon}{1+\varepsilon a^\varepsilon}\bigl(\mathcal{A}u^\varepsilon + a^\varepsilon\nabla(V^\varepsilon+\mathcal{R}^\varepsilon)\bigr)\biggr).\end{cases}$$

In contrast with the heat-conducting case, we do not know how to prove convergence *globally in time*. This is due to the fact that some terms in the right-hand side of the equations for $(\delta\Theta^\varepsilon,\delta v^\varepsilon)$ decay to 0 *only in* L^2*-in time spaces* and that $\delta\Theta^\varepsilon$ satisfies a mere transport equation (hence the r.h.s. should be bounded in L^1*-in-time space* if we want to get a time independent bound for $\delta\Theta^\varepsilon$).

We claim nevertheless that $\Theta^\varepsilon \to \Theta$ in $\tilde{L}^\infty_{loc}(\dot{B}^{s-2}_{p,1})$ with s as in the previous step, and that $\mathcal{P}u^\varepsilon \to v$ in

$$\big(\tilde{L}^\infty_{loc}(\dot{B}^{s-1}_{p,1}) \cap \tilde{L}^2_{loc}(\dot{B}^{s}_{p,1})\big) + \big(\tilde{L}^\infty_{loc}(\dot{B}^{s-2}_{p,1}) \cap L^1_{loc}(\dot{B}^{s}_{p,1})\big).$$

Let us first examine $\delta\Theta^\varepsilon$. Denoting by $\mathbb{K}_1$ the r.h.s. of the equation for $\delta\Theta^\varepsilon$, standard estimates for the transport equation ensure that, if $s > -1/2$ then we have for all $T \geq 0$,

$$\|\delta\Theta^\varepsilon\|_{\tilde{L}^\infty_T(\dot{B}^{s-2}_{p,1})} \leq \exp\biggl(\int_0^T \|\nabla\mathcal{P}u^\varepsilon\|_{\dot{B}^{\frac{3}{2}}_{2,1}}\,dt\biggr)\biggl(\|\delta\Theta^\varepsilon_0\|_{\dot{B}^{s-2}_{p,1}} + \int_0^T \|\mathbb{K}_1\|_{\dot{B}^{s-2}_{p,1}}\,dt\biggr)\cdot \tag{4.114}$$

Product laws give if, in addition, $s > 1/2$,

$$\begin{aligned}
\|\delta v^\varepsilon \cdot \nabla\Theta\|_{\dot{B}^{s-2}_{p,1}} &\lesssim \|\delta v^\varepsilon\|_{\dot{B}^{s}_{p,1}} \|\nabla\Theta\|_{\dot{B}^{-\frac{1}{2}}_{2,1}},\\
\|\mathcal{Q}u^\varepsilon \cdot \nabla\Theta^\varepsilon\|_{\dot{B}^{s-2}_{p,1}} &\lesssim \|\mathcal{Q}u^\varepsilon\|_{\dot{B}^{s}_{p,1}} \|\nabla\Theta^\varepsilon\|_{\dot{B}^{-\frac{1}{2}}_{2,1}},\\
\|\Theta^\varepsilon \mathrm{div}\,\mathcal{Q}u^\varepsilon\|_{\dot{B}^{s-2}_{p,1}} &\lesssim \|\mathrm{div}\,\mathcal{Q}u^\varepsilon\|_{\dot{B}^{s-1}_{p,1}} \|\Theta^\varepsilon\|_{\dot{B}^{\frac{1}{2}}_{2,1}}.
\end{aligned}$$

For the last term of $\mathbb{K}_1$, we use the fact that the product maps $\dot{B}^{-\frac{1}{2}}_{2,1} \times \dot{B}^{\frac{3}{2}-\alpha}_{2,1}$ in $\dot{B}^{-\frac{1}{2}-\alpha}_{2,1}$ if $0 \leq \alpha < 1$. Hence using the embedding $\dot{B}^{-\frac{1}{2}-\alpha}_{2,1} \hookrightarrow \dot{B}^{s-2}_{p,1}$ with $\alpha = 3/p - s$, we get

$$\|2\mu|Du^\varepsilon|^2 + \lambda(\mathrm{div}\,u^\varepsilon)^2\|_{\dot{B}^{s-2}_{p,1}} \lesssim \|u^\varepsilon\|_{\dot{B}^{\frac{1}{2}}_{2,1}} \|u^\varepsilon\|_{\dot{B}^{\frac{5}{2}-\alpha}_{2,1}}.$$

Inserting those inequalities in (4.114) and keeping in mind that $\nabla\mathcal{P}u^\varepsilon$ is uniformly bounded in $L^1(\dot{B}^{\frac{3}{2}}_{2,1})$, we get for any $s \in [-\frac{1}{2}+\frac{4}{p}, \frac{3}{p}] \cap (\frac{1}{2}, \infty)$:

$$\begin{aligned}
\|\delta\Theta^\varepsilon\|_{\tilde{L}^\infty_T(\dot{B}^{s-2}_{p,1})} \lesssim \|\delta\Theta^\varepsilon_0\|_{\dot{B}^{s-2}_{p,1}} &+ \int_0^T \|\delta v^\varepsilon\|_{\dot{B}^{s}_{p,1}} \|\Theta\|_{\dot{B}^{\frac{1}{2}}_{2,1}}\,dt\\
&+ \int_0^T \Bigl(\|\mathcal{Q}u^\varepsilon\|_{\dot{B}^{s}_{p,1}} \|\Theta^\varepsilon\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \varepsilon\|u^\varepsilon\|_{\dot{B}^{\frac{1}{2}}_{2,1}} \|u^\varepsilon\|_{\dot{B}^{\frac{5}{2}-\alpha}_{2,1}}\Bigr)dt,
\end{aligned}$$

whence

$$\begin{aligned}
\|\delta\Theta^\varepsilon\|_{\tilde{L}^\infty_T(\dot{B}^{s-2}_{p,1})} \lesssim \|\delta\Theta^\varepsilon_0\|_{\dot{B}^{s-2}_{p,1}}&\\
+ (1+T^{\frac{1}{2}})\|\Theta_0\|_{\dot{B}^{\frac{1}{2}}_{2,1}} \|\delta v^\varepsilon\|_{\tilde{L}^2_T(\dot{B}^{s}_{p,1})+L^1_T(\dot{B}^{s}_{p,1})} &+ \nu^{-1}(C^\varepsilon_0)^2((\nu T)^{\frac{1}{2}}(\varepsilon\nu)^\alpha + \varepsilon\nu(\nu T)^{\frac{\alpha}{2}}).
\end{aligned} \tag{4.115}$$

In order to bound δv^ε, we shall make use once again of the parabolic estimates given by Proposition 5.2. The main difficulty here is that some terms of the r.h.s. $\mathbb{K}_2$ of the equation for δv^ε cannot be bounded in global L^1-in-time spaces. Hence we shall use the following inequality which may be easily deduced from Proposition 5.2 (we do not track the dependency with respect to μ):

$$\|\delta v^\varepsilon\|_{\tilde{L}^\infty_T(\dot{B}^{s-1}_{p,1}+\dot{B}^{s-2}_{p,1})} + \|\delta v^\varepsilon\|_{\tilde{L}^2_T(\dot{B}^{s}_{p,1})+L^1_T(\dot{B}^{s}_{p,1})} \lesssim \|\delta v^\varepsilon_0\|_{\dot{B}^{s-1}_{p,1}+\dot{B}^{s-2}_{p,1}} + \|\mathbb{K}_2\|_{\tilde{L}^2_T(\dot{B}^{s-2}_{p,1})+L^1_T(\dot{B}^{s-2}_{p,1}+\dot{B}^{s-1}_{p,1})}.$$

Now, from product estimates in Besov spaces, we get

$$\begin{aligned}
\|v\cdot\nabla\delta v^\varepsilon\|_{L^1_T(\dot B^{s-1}_{p,1}+\dot B^{s-2}_{p,1})} &\lesssim (\|v\|_{L^2_T(\dot B^{\frac32}_{2,1})}+\|v\|_{L^\infty_T(\dot B^{\frac12}_{2,1})})\|\delta v^\varepsilon\|_{L^2_T(\dot B^s_{p,1})+L^1_T(\dot B^s_{p,1})},\\
\|\delta v^\varepsilon\cdot\nabla\mathcal{P}u^\varepsilon\|_{L^1_T(\dot B^{s-1}_{p,1}+\dot B^{s-2}_{p,1})} &\lesssim \|\nabla\mathcal{P}u^\varepsilon\|_{L^1_T(\dot B^{\frac32}_{2,1})}\|\delta v^\varepsilon\|_{L^\infty_T(\dot B^{s-1}_{p,1}+\dot B^{s-2}_{p,1})},\\
\|\delta\Theta^\varepsilon\nabla V\|_{\dot B^{s-2}_{p,1}} &\lesssim \|\nabla V\|_{\dot B^{\frac32}_{2,1}}\|\delta\Theta^\varepsilon\|_{\dot B^{s-2}_{p,1}},\\
\|\Theta^\varepsilon\nabla\delta V^\varepsilon\|_{\tilde L^2_T(\dot B^{s-2}_{p,1})} &\lesssim \|\Theta^\varepsilon\|_{\tilde L^\infty_T(\dot B^{\frac12}_{2,1})}\|\nabla\delta V^\varepsilon\|_{\tilde L^2_T(\dot B^{s-1}_{p,1})},\\
\|\Theta^\varepsilon\nabla\mathcal{R}^\varepsilon\|_{\tilde L^2_T(\dot B^{s-2}_{p,1})} &\lesssim \|\Theta^\varepsilon\|_{\tilde L^\infty_T(\dot B^{\frac12}_{2,1})}\|\nabla\mathcal{R}^\varepsilon\|_{\tilde L^2_T(\dot B^{s-1}_{p,1})},\\
\|\mathcal{Q}u^\varepsilon\cdot\nabla\mathcal{P}u^\varepsilon\|_{L^1_T(\dot B^{s-1}_{p,1})} &\lesssim \|\nabla\mathcal{P}u^\varepsilon\|_{L^2_T(B^{\frac12}_{2,1})}\|\mathcal{Q}u^\varepsilon\|_{L^2_T(\dot B^s_{p,1})},\\
\|u^\varepsilon\cdot\nabla\mathcal{Q}u^\varepsilon\|_{L^1_T(\dot B^{s-1}_{p,1})} &\lesssim \|u^\varepsilon\|_{L^2_T(B^{\frac32}_{2,1})}\|\nabla\mathcal{Q}u^\varepsilon\|_{L^2_T(\dot B^{s-1}_{p,1})},
\end{aligned}$$

and arguing as in the proof of (3.72),

$$\begin{aligned}
\|\tfrac{\varepsilon a^\varepsilon}{1+\varepsilon a^\varepsilon}\mathcal{A}u^\varepsilon\|_{L^1_T(\dot B^{s-1}_{p,1})} &\lesssim \|\varepsilon a^\varepsilon\|_{L^\infty(\dot B^{\frac32-\alpha}_{2,1})}\|\nabla^2u^\varepsilon\|_{L^1_T(\dot B^{\frac12}_{2,1})}\\
&\lesssim \nu^{-1}(\varepsilon\nu)^\alpha\|a^\varepsilon\|_{L^\infty_T(\tilde B^{\frac32,-}_{\varepsilon\nu})}\|u^\varepsilon\|_{L^1_T(\dot B^{\frac52}_{2,1})}.
\end{aligned}$$

Finally, because $\dot B^{-\frac12-\alpha}_{2,1}\hookrightarrow\dot B^{s-2}_{p,1}$,

$$\begin{aligned}
\|\tfrac{\varepsilon a^\varepsilon}{1+\varepsilon a^\varepsilon}a^\varepsilon\nabla(V^\varepsilon+\mathcal{R}^\varepsilon)\|_{\tilde L^2_T(\dot B^{s-2}_{p,1})} &\lesssim \|\varepsilon a^\varepsilon\|_{\tilde L^\infty_T(\dot B^{\frac32-\alpha}_{2,1})}\|a^\varepsilon\|_{\tilde L^\infty_T(\dot B^{\frac12}_{2,1})}\|\nabla(V^\varepsilon+\mathcal{R}^\varepsilon)\|_{\tilde L^2_T(\dot B^{\frac12}_{2,1})}\\
&\lesssim \nu^{-1}(\varepsilon\nu)^\alpha\|a^\varepsilon\|_{\tilde L^\infty_T(\tilde B^{\frac32,-}_{\varepsilon\nu})}\|a^\varepsilon\|_{\tilde L^\infty_T(\dot B^{\frac12}_{2,1})}\|V^\varepsilon+\mathcal{R}^\varepsilon\|_{\tilde L^2_T(\dot B^{\frac32}_{2,1})}.
\end{aligned}$$

Therefore, putting together all those inequalities and using the estimates provided by the previous steps we conclude that

$$\begin{aligned}
&\|\delta v^\varepsilon\|_{\tilde L^\infty_T(\dot B^{s-1}_{p,1}+\dot B^{s-2}_{p,1})}+\|\delta v^\varepsilon\|_{\tilde L^2_T(\dot B^s_{p,1})+L^1_T(\dot B^s_{p,1})}\lesssim\|\delta v^\varepsilon_0\|_{\dot B^{s-1}_{p,1}+\dot B^{s-2}_{p,1}}+\int_0^T\|\nabla V\|_{\dot B^{\frac32}_{2,1}}\|\delta\Theta^\varepsilon\|_{\dot B^{s-2}_{p,1}}\,dt\\
&+C_0^\varepsilon\big(\|\delta v^\varepsilon\|_{L^2_T(\dot B^s_{p,1})+L^1_T(\dot B^s_{p,1})}+\|\delta v^\varepsilon\|_{\tilde L^\infty_T(\dot B^{s-1}_{p,1}+\dot B^{s-2}_{p,1})}+\|\nabla\delta V^\varepsilon\|_{\tilde L^2_T(\dot B^{s-1}_{p,1})}\big)+(C_0^\varepsilon)^2(1+\nu^{-1}C_0^\varepsilon)(\varepsilon\nu)^\alpha.
\end{aligned}$$

If $\nu^{-1}C_0^\varepsilon$ is suitably small, we thus deduce that

$$\|\delta v^\varepsilon\|_{\tilde L^\infty_T(\dot B^{s-1}_{p,1}+\dot B^{s-2}_{p,1})}+\|\delta v^\varepsilon\|_{\tilde L^2_T(\dot B^s_{p,1})+L^1_T(\dot B^s_{p,1})}\le\beta(\varepsilon)+K\int_0^T\|\nabla V\|_{\dot B^{\frac32}_{2,1}}\|\delta\Theta^\varepsilon\|_{\dot B^{s-2}_{p,1}}\,dt$$

with $\beta(\varepsilon):=\|\delta v^\varepsilon_0\|_{\dot B^{s-1}_{p,1}+\dot B^{s-2}_{p,1}}+C_0^\varepsilon(\|\nabla\delta V^\varepsilon\|_{\tilde L^2_T(\dot B^{s-1}_{p,1})}+C_0^\varepsilon\varepsilon^\alpha)$.

Therefore, plugging (4.115) in the above integral, and using Gronwall lemma, we get

$$\begin{aligned}
\|\delta\Theta^\varepsilon\|_{\tilde L^\infty_T(\dot B^{s-2}_{p,1})}\le\Big(&\|\delta\Theta^\varepsilon_0\|_{\dot B^{s-2}_{p,1}}+(1+T^{\frac12})\|\Theta_0\|_{\dot B^{\frac12}_{2,1}}\big(\|\delta v^\varepsilon_0\|_{\dot B^{s-1}_{p,1}+\dot B^{s-2}_{p,1}}\\
&+C_0^\varepsilon(\|\nabla\delta V^\varepsilon\|_{\tilde L^2_T(\dot B^{s-1}_{p,1})}+C_0^\varepsilon\varepsilon^{\frac3p-s})\big)+(C_0^\varepsilon)^2(T^{\frac12}\varepsilon^{\frac3p-s}+\varepsilon T^{\frac{3}{2p}-\frac{s}{2}})\Big)\\
&\times\exp\Big(K\|\Theta_0\|_{\dot B^{\frac12}_{2,1}}(1+T^{\frac12})\|\nabla V\|_{L^1_T(\dot B^{\frac32}_{2,1})}\Big),
\end{aligned}\tag{4.116}$$

and

$$\begin{aligned}(4.117)\qquad \|\delta v^\varepsilon\|_{\tilde L^\infty_T(\dot B^{s-1}_{p,1}+\dot B^{s-2}_{p,1})} + \|\delta v^\varepsilon\|_{\tilde L^2_T(\dot B^s_{p,1})+L^1_T(\dot B^s_{p,1})}&\\ \leq \|\delta v_0^\varepsilon\|_{\dot B^{s-1}_{p,1}+\dot B^{s-2}_{p,1}} + C_0^\varepsilon(\|\nabla\delta V^\varepsilon\|_{\tilde L^2_T(\dot B^{s-1}_{p,1})} + C_0^\varepsilon\varepsilon^{\frac{3}{p}-s})&\\ +K\bigg(\|\delta\Theta_0^\varepsilon\|_{\dot B^{s-2}_{p,1}} + (1+T^{\frac12})\|\Theta_0\|_{\dot B^{\frac12}_{2,1}}\big(\|\delta v_0^\varepsilon\|_{\dot B^{s-1}_{p,1}+\dot B^{s-2}_{p,1}}&\\ +C_0^\varepsilon(\|\nabla\delta V^\varepsilon\|_{\tilde L^2_T(\dot B^{s-1}_{p,1})} + C_0^\varepsilon\varepsilon^{\frac{3}{p}-s})\big) + (C_0^\varepsilon)^2(T^{\frac12}\varepsilon^{\frac{3}{p}-s} + \varepsilon T^{\frac{3}{2p}-\frac{s}{2}})\bigg)\|\nabla V\|_{L^1_T(\dot B^{\frac32}_{2,1})}&\\ \times\exp\bigg(K\|\Theta_0\|_{\dot B^{\frac12}_{2,1}}(1+T^{\frac12})\|\nabla V\|_{L^1_T(\dot B^{\frac32}_{2,1})}\bigg)&\end{aligned}$$

whenever $s\in[-1/2+4/p,3/p]$ and $s>1/2$. This ensures the convergence of $(\Theta^\varepsilon,\mathcal{P}u^\varepsilon)$ to (Θ,v) with an explicit rate. ∎

5. Appendix

In this Appendix, we give some a priori estimates involving hybrid Besov spaces. Let us start with product estimates.

Lemma 5.1. *Suppose that $p\in[2,\infty]$ and $\beta\geq0$. There exists a constant C such that for all $\alpha>0$, we have*

$$\begin{aligned}\|fg\|_{\tilde B^{s-\beta,-}_{p,\alpha}} &\lesssim \|f\|_{\tilde B^{s,-}_{p,\alpha}}\|g\|_{\dot B^{\frac32-\beta}_{2,1}} \quad \text{if } \beta-1/2<s\leq3/p,\\ \|fg\|_{\tilde B^{s-\beta,+}_{p,\alpha}} &\lesssim \|f\|_{\tilde B^{s,+}_{p,\alpha}}\|g\|_{\dot B^{\frac32-\beta}_{2,1}} \quad \text{if } \beta-3/2<s\leq3/p-1.\end{aligned}$$

Proof. We may assume that $\alpha=1$ making a change of variables if the case may be. In order to prove the first inequality, it suffices to notice that for all $\sigma\in\mathbb{R}$ we have

$$(5.118)\qquad \|\cdot\|_{\tilde B^{\sigma,-}_{p,1}}\approx\|\cdot\|_{\dot B^{\sigma-1}_{p,1}\cap\dot B^{\sigma}_{p,1}}.$$

Now, it is well known (see e.g. [2]) that the usual product maps $\dot B^\sigma_{p,1}\times\dot B^{\frac32-\beta}_{2,1}$ in $\dot B^{\sigma-\beta}_{p,1}$ whenever $\beta-3/2<\sigma\leq3/p$ and $\beta\geq0$. Therefore

$$\|fg\|_{\dot B^{s-\beta-1}_{p,1}}\lesssim\|f\|_{\dot B^{s-1}_{p,1}}\|g\|_{\dot B^{\frac32-\beta}_{2,1}} \quad\text{and}\quad \|fg\|_{\dot B^{s-\beta}_{p,1}}\lesssim\|f\|_{\dot B^{s}_{p,1}}\|g\|_{\dot B^{\frac32-\beta}_{2,1}}.$$

This implies the first inequality.

Proving the second inequality is rather similar: now we use the fact that

$$(5.119)\qquad \|\cdot\|_{\tilde B^{s,+}_{p,1}}\approx\|\cdot\|_{\dot B^{s}_{p,1}+\dot B^{s+1}_{p,1}}.$$

Decomposing f into low and high frequencies according to (2.12), we have

$$fg=f^\ell g+f^h g.$$

Now, the aforementioned product law ensures that

$$\|f^\ell g\|_{\dot B^{s+1-\beta}_{p,1}}\lesssim\|f^\ell\|_{\dot B^{s+1}_{p,1}}\|g\|_{\dot B^{\frac32-\beta}_{2,1}} \quad\text{and}\quad \|f^h g\|_{\dot B^{s-\beta}_{p,1}}\lesssim\|f^h\|_{\dot B^{s}_{p,1}}\|g\|_{\dot B^{\frac32-\beta}_{2,1}}.$$

So taking advantage of (5.119) completes the proof of the second inequality. ∎

The following Strichartz estimates for the acoustic wave equation are the key to the proof of convergence.

Proposition 5.1. *Let $(q,\mathcal{Q}u)$ (with $\operatorname{curl}\mathcal{Q}u=0$) satisfy the 3D acoustic wave equation*

$$\begin{cases}\partial_t q+\sqrt{2}\operatorname{div}\mathcal{Q}u=F,\\ \partial_t\mathcal{Q}u+\sqrt{2}\,\nabla q=G.\end{cases}$$

Then for any $\alpha>0$, $s\in\mathbb{R}$ and $p\in[2,\infty)$ the following estimates hold true

$$\begin{aligned}\|(q,\mathcal{Q}u)\|_{\tilde L^{\frac{2p}{p-2}}(\dot B^{s+\frac{2}{p}-1}_{p,1})}&\leq C\Big(\|(q_0,\mathcal{Q}u_0)\|_{\dot B^s_{2,1}}+\|(F,G)\|_{L^1(\dot B^s_{2,1})}\Big),\\ \|(q,\mathcal{Q}u)\|_{\tilde L^{\frac{2p}{p-2}}(\tilde B^{s+\frac{2}{p}-1,\pm}_{p,\alpha})}&\leq C\Big(\|(q_0,\mathcal{Q}u_0)\|_{\tilde B^{s,\pm}_{2,\alpha}}+\|(F,G)\|_{L^1(\tilde B^{s,\pm}_{2,\alpha})}\Big).\end{aligned}$$

Proof. The first inequality has been proved in [6]. In order to prove the second one, one just has to decompose $(q,\mathcal{Q}u)$ into low and high frequencies, that is $(q,\mathcal{Q}u)=(q^\ell,\mathcal{Q}u^\ell)+(q^h,\mathcal{Q}u^h)$ and apply the first inequality with $s\pm1$ (resp. s) to $(q^\ell,\mathcal{Q}u^\ell)$ (resp. $(q^h,\mathcal{Q}u^h)$). ■

Let us finally state maximal regularity estimates for the heat equation, in hybrid Besov spaces.

Proposition 5.2. *Let u be a solution to the heat equation*

$$\begin{cases}\partial_t u-\Delta u=f,\\ u|_{t=0}=u_0.\end{cases}$$

Then we have the following estimates for any $\sigma\in\mathbb{R}$, $\alpha>0$, $p\in[1,\infty]$ and $q\geq r$:

$$\begin{aligned}\|u\|_{\widetilde L^q_T(\dot B^{\sigma+\frac{2}{q}}_{p,1})}&\lesssim\|u_0\|_{\dot B^\sigma_{p,1}}+\|f\|_{\widetilde L^r_T(\dot B^{\sigma+\frac{2}{r}-2}_{p,1})},\\ \|u\|_{\widetilde L^q_T(\widetilde B^{\sigma+\frac{2}{q},\pm}_{p,\alpha})}&\lesssim\|u_0\|_{\widetilde B^{\sigma,\pm}_{p,\alpha}}+\|f\|_{\widetilde L^r_T(\widetilde B^{\sigma+\frac{2}{r}-2,\pm}_{p,\alpha})}.\end{aligned}$$

Proof. The first inequality is classical (see e.g. [2], Chap. 3). The second inequality may be obtained from the first one after decomposing u, u_0 and f into low and high frequencies. ■

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