HAL
open science

The influence of boundary conditions on the contact problem in a 3D Navier-Stokes Flow

David Gérard-Varet, Matthieu Hillairet, Chao Wang

To cite this version:

David Gérard-Varet, Matthieu Hillairet, Chao Wang. The influence of boundary conditions on the contact problem in a 3D Navier-Stokes Flow. Journal de Mathématiques Pures et Appliquées, 2015, 103 (1), pp.1–38. hal-00795366

HAL Id: hal-00795366

https://hal.science/hal-00795366v1

Submitted on 27 Feb 2013

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

The influence of boundary conditions on the contact problem in a 3D Navier-Stokes Flow

David Gerard-Varet[*], Matthieu Hillairet[†] and Chao Wang[‡]

February 28, 2013

Abstract

We consider the free fall of a sphere above a wall in a viscous incompressible fluid. We investigate the influence of boundary conditions on the finite-time occurrence of contact between the sphere and the wall. We prove that slip boundary conditions enable to circumvent the "no-collision" paradox associated with no-slip boundary conditions. We also examine the case of mixed boundary conditions.

1 Introduction

Understanding the close-range dynamics of particles inside a fluid is crucial to several fields of application, such as rheology, sedimentation or slurry erosion. A related problem is to compute the drag induced by nearby solid bodies: it is important to lubrication theory, or to the control of microswimmers, among others. In the case of a viscous incompressible fluid, a standard way to model the fluid-solid interaction is through the coupling of the Navier-Stokes equations for the fluid part and the conservation of momentum for the bodies. However, such model reveals unrealistic features: in particular, it can overvalue the drag force induced on bodies that are close to collision. Consider for instance the free fall of a rigid sphere over a wall, and assume standard no-slip conditions at the solid surface of the sphere and the wall. Then, the model predicts that no collision is possible between the sphere and the wall, no matter their relative density and the viscosity of the fluid.This is a "no-collision paradox": it has been recognized at first for reduced models (Stokes approximation, see [5, 6]), and more recently for the full constant-density incompressible Navier Stokes system [13, 19, 21]. These results show the need for a more accurate model of the fluid behavior in the "small-distance" regime. Several refinements were proposed in the past [1, 8, 27], see the introduction of [26] for a review. In this paper, we investigate the influence of boundary conditions at the fluid-solid interfaces.

[*]Institut de Mathématiques de Jussieu, CNRS UMR 7586, Université Paris 7, 175 rue du Chevaleret, 75013 Paris. Mail: `gerard-varet@math.jussieu.fr`

[†]Ceremade, CNRS UMR 7534, Univerisité Paris Dauphine, Place du Maréchal de Lattre de Tassigny, 75775 Paris Cedex 16 and Institut Camille Jordan, CNRS UMR 5208, Université Lyon 1, 43 boulevard du 11 novembre 1918, 69622 Villeurbane Cedex, Mail: `hillairet@ceremade.dauphine.fr`

[‡]Institut de Mathématiques de Jussieu, CNRS UMR 7586, Université Paris 7, 175 rue du Chevaleret, 75013 Paris. Mail: `wangc@math.jussieu.fr`

Boundary conditions are constitutive equations that can be derived theoretically or experimentally. A popular general type of boundary conditions was proposed by Navier in the early time of analytical fluid dynamics. This Navier law states that, on the one hand, the normal component of the velocity is continuous at the fluid/solid interface (impermeability condition) and that, on the other hand, the amount of slip in the tangential part of the velocity is proportional to the tangential part of the normal stress exerted by the fluid on the boundary. The proportionality coefficient is called "slip length". Assuming the slip length vanishes (*i.e.*, assuming no-slip boundary conditions) suffices to make theoretical and experimental results to coincide in numerous cases. However, developments such as the "no-collision paradox" lead to question this assumption [23]. We refer the reader to [2, 28] for a further description of experimental and theoretical validations of the Navier boundary conditions with slip. *We propose here new insights on the contact-problem, by considering the effect of Navier boundary conditions. We shall study this effect in the context of the full Navier-Stokes system, which is, up to our knowledge, new.*

We consider a simplified case where only one homogeneous solid is moving in the fluid. We denote $S(t) \subseteq \mathbb{R}^3, F(t) \subseteq \mathbb{R}^3$ the solid and the fluid domains at time t, and $\Omega \triangleq \overline{S(t)} \cup F(t)$ the total domain. We assume that the fluid is governed by the constant-density incompressible Navier-Stokes equations :

$$\rho_F(\partial_t u_F + u_F \cdot \nabla u_F) - \mu_F \Delta u_F = -\nabla p_F - \rho_F g e_3, \quad t > 0, \quad x \in F(t), \tag{1.1}$$

$$\text{div } u_F = 0, \quad t > 0, \quad x \in F(t), \tag{1.2}$$

where u_F and p_F stand respectively for the velocity and internal pressure, ρ_F the density, μ_F the viscosity in the fluid. We emphasize that we consider here the influence of gravity whose direction is the third vector of the basis (e_1, e_2, e_3), *i.e.*, the "vertical" unit vector. The solid motion is governed by Newton laws. Denoting $x_S(t) \in \mathbb{R}^3$ the position of the center of mass of the solid at time t, $U_S(t) \in \mathbb{R}^3$ its velocity and $\omega_S(t) \in \mathbb{R}^3$ the angular velocity, the Newton laws read

$$m_S \frac{d}{dt} U_S(t) = -\int_{\partial S(t)} (2\mu_F D(u_F) - p_F I) n d\sigma - m_S g e_3, \tag{1.3}$$

$$\frac{d}{dt}(J_S \omega_S(t)) = -\int_{\partial S(t)} (x - x_S(t)) \times (2\mu_F D(u_F) - p_F I) n d\sigma. \tag{1.4}$$

Here ρ_S and $m_S \triangleq \rho_S |S(0)|$ are the density and the mass of the solid respectively. Symbol n stands for the unit normal vector to $\partial S(t)$ pointing outside the fluid domain. The 3x3 inertia matrix $J_S(t)$ is defined by

$$J_S(t) \triangleq \rho_S \int_{S(t)} (|x - x_S(t)|^2 I_d - (x - x_S(t)) \otimes (x - x_S(t))) dx.$$

We emphasize that the solid is homogeneous, so that no angular momentum is induced by gravity in (1.4). Given (U_S, ω_S, x_S), the solid velocity-field u_S at each point $x \in S(t)$ reads

$$u_S(t, x) \triangleq U_S(t) + \omega_S \times (x - x_S(t)). \tag{1.5}$$

In this paper, we complete the system with Navier boundary conditions :

$$(u_F - u_S) \cdot n|_{\partial S(t)} = 0, \qquad (u_F - u_S) \times n|_{\partial S(t)} = -2\beta_S (D(u_F) n) \times n, \tag{1.6}$$

$$u_F \cdot n|_{\partial \Omega} = 0, \qquad u_F \times n|_{\partial \Omega} = -2\beta_\Omega (D(u_F) n) \times n, \tag{1.7}$$

where $\beta_S, \beta_\Omega \geq 0$ are the slip lengths at the fluid-solid and fluid-container interface respectively. We remind that assuming that β_S or β_Ω vanishes means assuming no-slip boundary conditions at the corresponding interface.

System (1.1)–(1.7) is completed with initial conditions :

$$S(0) = S_0, \qquad u_F(0,x) = u_{F,0}(x), \quad x \in \Omega \setminus \bar{S}_0 \tag{1.8}$$

$$U(0) = U_0, \qquad \omega(0) = \omega_0 \,. \tag{1.9}$$

Equation (1.9) may also be written :

$$u_S(0,x) = u_{S,0}(x), \quad x \in S_0 \,. \tag{1.9}$$

where $u_{S,0}$ is computed with respect to U_0, ω_0 and S_0 *via* (1.5). When $S_0 \subset \Omega$ and $u_{F,0}$ has finite energy, namely $u_{F,0} \in L^2(\Omega \setminus \bar{S}_0)$, standard computations show that a reasonable solution to (1.1)–(1.9) should satisfy the energy estimate :

$$\begin{aligned}
&\frac{1}{2}\left[\int_{F(t)} \rho_F |u_F(t)|^2 + \int_{S(t)} \rho_S |u_S(t)|^2\right] \\
&+ \mu_F \left(2\int_0^t \int_{F(t)} |D(u_F(t))|^2 + \frac{1}{\beta_\Omega}\int_0^t \int_{\partial\Omega} |u_F \times n|^2 \mathrm{d}\sigma + \frac{1}{\beta_S}\int_0^t \int_{\partial S(t)} |(u_F - u_S)\times n|^2 \mathrm{d}\sigma\right) \\
&\leq -\int_0^t \int_{F(t)} \rho_F g e_3 \cdot u_F - \int_0^t \int_{S(t)} \rho_S g e_3 \cdot u_S + \frac{1}{2}\int_{F_0} \rho_F |u_{F,0}|^2 + \frac{1}{2}\int_{S_0} \rho_S |u_S(0)|^2 \,.
\end{aligned} \tag{1.10}$$

where $F_0 \triangleq \Omega \setminus \bar{S}_0$. When $\beta_S = 0$ or $\beta_\Omega = 0$, the corresponding boundary term vanishes.

Roughly, with regards to the classical theory of the Navier-Stokes equation in a fixed domain, one can expect two kinds of solutions for (1.1)–(1.9):

- Strong solutions, locally in time: they should be unique, with possible blow-up prior to collision in 3D.
- Weak solutions: they should exist at least up to collision, and be possibly non-unique.

In the no-slip case ($\beta_S = \beta_\Omega = 0$), both kinds of solutions have been built in various contexts: 2D/3D problems, bounded/unbounded container, smooth/singular shape of the solid bodies. Existence of strong solutions locally in time (prior to collision) is now well-known, see [9, 12, 13, 17, 32, 33]. The existence of weak solutions up to collision has been studied in [4, 7, 10, 18, 24, 25, 30, 34]. Uniqueness of weak solutions up to collision in 2D is established in the recent paper [16]. Finally , extension of weak solutions after contact is given in [29] for 2D problems and [11] for 3D problems.

In the case of slip conditions, when $\beta_S > 0$ and $\beta_\Omega > 0$, much less in known. It is partly because genuine new mathematical difficulties arise: for instance, at the level of weak solutions, discontinuity of the tangential velocity at the fluid/solid interface forbids global H^1 bound on the extended velocity-field u as defined by:

$$u = u_S \text{ in } S(t)\,, \qquad u = u_F \text{ in } F(t). \tag{1.11}$$

Such global bound is a key point in the treatment of the Dirichlet case. Hence, to investigate the contact problem for system (1.1)–(1.9), we shall rely on the recent paper [15], which shows the existence of weak solutions up to collision. Both the definition of weak solutions and the existence result given in [15] will be recalled in the next section. Let us also mention the work [31], related to a single body without container.

Considering a special class of weak solutions, we will show that slip conditions allow collision in finite time. Hence, taking into account slight slip at the solid boundaries allows to clear the no-collision paradox. More precisely, we shall consider configurations obeying the following assumptions:

A1. The solid $S(t) \sim S_0$ is a ball of radius 1.

A2. Ω is a smooth convex domain of $\{x_3 > 0\}$, flat near:

$$\partial\Omega \supset \{x_3 = 0, |x| < 2\delta\} \text{ for some } \delta \in (0, \frac{1}{4}).$$

A3. $S(t)$ is axisymmetric with respect to $\{x_1 = x_2 = 0\}$.

A4. The only possible contact between the solid and the container is at $x = 0$. More precisely,

$$dist\,(S(t), \partial\Omega \setminus \{x_3 = 0, |x| < \delta\}) \geq d_\delta > 0.$$

Parameters δ and d_δ will be involved in the proof. Let us comment on these assumptions. *Their only goal is to ensure that the geometry of the potential contact zone does not change with time.* Namely, it is an aperture between a sphere and a plane, only varying with the distance $h(t)$ between the south pole of $S(t)$ and $x = 0$. The whole point is then to determine if $h(t)$ can vanish in finite time or not. This simplifies greatly our computations. However, as will be clear from our proof, our result is genuinely local in space time: it is independent from the global characteristics of the solid/container geometry.

Note that A1-A2 are just assumptions on S_0 and Ω. If $S_0 \subset\subset \Omega$, and if the initial data $(u_{F,0}, u_{S,0})$ has finite energy, article [15] ensures the existence of a weak solution up to contact between $S(t)$ and $\partial\Omega$ (see next section). Assumption A3 states that the axisymmetry of the solid is preserved with time. Actually, it will be automatically satisfied if S_0, Ω and the initial velocities are axisymmetric. More precisely, in this case, one can adapt the construction of weak solutions from [15] by including the symmetry constraint, and obtain directly an axisymmetric weak solution. Finally, assumption A4 guarantees that the sphere remains away from the top and lateral boundaries. Again, this can be ensured by a proper choice of the container and the initial data. As regards lateral boundaries, it is enough by assumption A3 that they are at distance > 1 to the vertical axis $\{x_1 = x_2 = 0\}$. Also, there are several ways to avoid merging of the solid body and the top boundary. One can for instance assume that the solid is heavier than the fluid ($\rho_S > \rho_F$), and that the initial kinetic energy of the fluid and the solid is small enough. As the total energy of weak solutions is non-increasing, this means that the increase of the potential energy through time can only be small: hence, the solid will not rise too much, and so will remain away from the top boundary. Again, we insist that our method of proof could extend to more general settings.

We can now state our main results:

Theorem 1.1 (Slip Case)**.** *Assume $\beta_S, \beta_\Omega > 0$ and $\rho_S > \rho_F$. For any weak solution (S, u) (see Definition 2.1) satisfying A1-A4, the solid body S touches $\partial\Omega$ in finite time.*

Theorem 1.2 (Mixed Case)**.** *Let β_S or β_Ω vanish. Any weak solution (S, u) (see Definition 2.3) satisfying A1-A4 is global. In particular, the solid body S never touches $\partial\Omega$.*

We recall that the full no-slip case (when $\beta_S = \beta_\Omega = 0$) was already treated in [21]. Theorems 1.1 and 1.2 show that one possible way to obtain realistic contact, and to circumvent the no-collision paradox, is to introduce slip in the boundary conditions *on all solid boundaries of the fluid domain.*

We conclude with a few hints on the proof of Theorem 1.1 (ideas for Theorem 1.2 are similar). The proof elaborates on the paper [14], devoted to a simplified linear system: one takes there $\Omega = \mathbf{R}^3_+$, while the fluid outside the sphere $S(t)$ is governed by a steady Stokes flow (with $\mu_F = 1$ for simplicity):

$$-\Delta u_F = -\nabla p_F - \rho_F g e_3, \quad \nabla \cdot u_F = 0, \quad x \in F(t).$$

In this simplified setting, the fluid and solid domains $F(t) = F_{h(t)}$ and $S(t) = S_h(t)$ are characterized by the distance $h(t)$ beween the sphere and the plane wall. Then, it is easily seen that $u_F(t) = h'(t) w_{h(t)}$ where w_h is the solution of a normalized Stokes equation, set in the domain F_h "frozen" at distance h:

$$\begin{cases} -\Delta w_h + \nabla p_h = 0, \quad \nabla \cdot w_h = 0, \quad x \in F_h, \quad (w_h - e_3) \cdot n|_{\partial S_h} = 0, \quad w_h \cdot n|_{\partial\Omega} = 0, \\ (w_h - e_3) \times n|_{\partial S_h} = -2\beta_S (D(w_h) n) \times n, \\ \qquad\qquad w_h \times n|_{\partial\Omega} = -2\beta_\Omega (D(w_h) n) \times n. \end{cases} \tag{1.12}$$

We refer to [14] for all details. Eventually, the dynamics reduces to an ODE of the type

$$h''(t) = -h'(t)\mathcal{D}(h(t)) + \frac{\rho_S - \rho_F}{\rho_S} g$$

where the *drag term* $\mathcal{D}(h)$ is the energy of w_h:

$$\mathcal{D}(h) \;=\; \mathcal{E}_h(w_h), \quad \mathcal{E}_h(u) \;:=\; \int_{F_h} |\nabla u|^2 + \left(\frac{1}{\beta_S} + 1\right) \int_{\partial S_h} |(u - e_3) \times n|^2 + \frac{1}{\beta_\Omega} \int_{\partial\Omega} |u \times n|^2.$$

For this reduced model, the contact problem resumes to the computation of the drag $\mathcal{D}_h$ in the limit $h \to 0$. The difficulty is that there is no simple formula for the solution w_h of the Stokes system in F_h.

A general approach to this problem was recently proposed in [14]. It works for no-slip conditions, slip ones, and for more general geometries. The starting idea is to use a variational characterization of the drag: one can identify $\mathcal{D}(h)$ as the minimizer of an energy $\mathcal{E}_h$, over an admissible set of fields $\mathcal{A}_h$:

$$\mathcal{D}(h) \;=\; \inf_{u \in \mathcal{A}_h} \mathcal{E}_h(u) = \mathcal{E}_h(w_h).$$

Then, instead of computing the true minimizer, the point is to find an appropriate relaxed minimization problem, for which the minimizer can be easily computed, and such that the corresponding minimum is close to the exact one asymptotically in h. The choice of such

relaxed problem is developed in various situations in [14]. Roughly, in the case of slip conditions, one obtains that $c|\ln(h)| \leq D(h) \leq C|\ln(h)|$, in agreement with older computations of Hocking [23]. Hence, the drag is weak enough to allow for collision, within this linear Stokes approximation. See also [20] for numerical insight.

In the present paper, we manage to use ideas of the linear study, in the context of the nonlinear Navier-Stokes flow. At a formal level, the first idea is to multiply the momentum equation by $w(t,x) = w_{h(t)}(x)$, w_h solution of (1.12) (where this time Ω is the container). After formal manipulations, one obtains

$$h''(t) \;=\; -h'\mathcal{D}(h) \;+\; \frac{\rho_S - \rho_F}{\rho_S} g \;+\; \text{remainder} \tag{1.13}$$

where the remainder term involves the test function w and the convective derivative of u. However, the Stokes solution w_h is not known explicitly. *Hence, the main point in our proof is to use another test function, based on the relaxed minimizer introduced in [14]. Then, one must show that for such test function, the remainder term is indeed small, that is controlled by the gravity when the solid is heavier than the fluid.* Let us stress that several difficulties arise in this process. For instance, the relaxed minimizer introduced in the linear study is not accurate enough, so that we must improve it. Moreover, due to the slip conditions, the natural quantity is the symmetric gradient of the weak solution, not the full gradient. This makes the derivation of the various bounds more difficult than in the no-slip case.

The outline of the paper is as follows. In the next section, we introduce a suitable notion of weak solutions for system (1.1)–(1.9), in the slip case and in the mixed case. We recall the existence theorems up to collision available for such weak solutions. in the slip case and in the mixed case. The two last sections are devoted to the proofs of the two theorems.

2 On the definition of weak solutions

We introduce here a definition of weak solutions for (1.1)–(1.8) in two cases: slip boundary conditions at all boundaries, mixed slip/no-slip boundary conditions. We mostly follow [15]. Given Ω a Lipschitz domain, we set:

$$\begin{aligned}
\mathcal{D}_\sigma(\Omega) &\triangleq \{\varphi \in C_c^\infty(\Omega), \quad \operatorname{div}\varphi = 0\}, && \mathcal{D}_\sigma(\overline{\Omega}) \triangleq \{\varphi|_\Omega,\ \varphi \in \mathcal{D}_\sigma(\mathbb{R}^3)\},\\
L^2_\sigma(\Omega) &\triangleq \text{the closure of } \mathcal{D}_\sigma(\Omega) \text{ in } L^2(\Omega), && H^1_\sigma(\Omega) \triangleq H^1(\Omega) \cap L^2_\sigma(\Omega),\\
H^1_\sigma(\overline{\Omega}) &\triangleq \text{the closure of } \mathcal{D}_\sigma(\overline{\Omega}) \text{ in } H^1(\Omega),\\
\mathcal{R} &\triangleq \{\varphi_s, \quad \varphi_s = V + \omega \times x, \quad \text{for some} \quad V \in \mathbb{R}^3, \omega \in \mathbb{R}^3\},
\end{aligned}$$

2.1 The case of slip boundary conditions: $\beta_S > 0$ and $\beta_\Omega > 0$.

We consider here that the slip lengths β_S and β_Ω do not vanish *i.e.*, that the fluid slips on both boundaries. The corresponding Cauchy problem is studied in [15], where the following definition of weak solution is given:

Definition 2.1. *Let Ω and $S_0 \Subset \Omega$ be two Lipschitz bounded domains of $\mathbb{R}^3$. Let $u_{F,0} \in L^2_\sigma(\Omega), u_{S,0} \in \mathcal{R}$ such that $u_{F,0} \cdot n = u_{S,0} \cdot n$ on ∂S_0. Let $T > 0$, or $T = \infty$.*

A weak solution of (1.1)–(1.9) on $[0,T)$ is a pair (S,u) satisfying
1) $S(t) \Subset \Omega$ is a bounded domain of $\mathbb{R}^3$ for all $t \in [0,T)$ such that

$$\chi_S(t,x) \triangleq 1_{S(t)}(x) \in L^\infty((0,T)\times\Omega).$$

2) u belongs to the space

$$\begin{aligned}\mathcal{S}_T \triangleq \{&u \in L^\infty(0,T;L^2_\sigma(\Omega)),\ \text{there exists } u_F \in L^2_{loc}([0,T);H^1_\sigma(\Omega)),\ u_S \in L^\infty(0,T;\mathcal{R})\\ &\text{such that } u(t,\cdot)=u_F(t,\cdot) \text{ on } F(t),\quad u(t,\cdot)=u_S(t,\cdot) \text{ on } S(t), \text{ for all } t\in[0,T)\}\end{aligned}$$

where $F(t) \triangleq \Omega \setminus \overline{S(t)}$.
3) For all φ belonging to the space

$$\begin{aligned}\mathcal{T}^s_T \triangleq \{&\varphi \in C([0,T];L^2_\sigma(\Omega)),\ \text{there exists } \varphi_F \in \mathcal{D}([0,T);\mathcal{D}_\sigma(\overline{\Omega})), \varphi_S \in \mathcal{D}([0,T);\mathcal{R})\\ &\text{such that } \varphi(t,\cdot)=\varphi_F(t,\cdot) \text{ on } F(t),\quad \varphi(t,\cdot)=\varphi_S(t,\cdot) \text{ on } S(t), \text{ for all } t\in[0,T]\}\end{aligned}$$

there holds

$$\begin{aligned}&-\int_0^T\int_{F(t)}\rho_F u_F\cdot\partial_t\varphi_F-\int_0^T\int_{S(t)}\rho_S u_S\cdot\partial_t\varphi_S-\int_0^T\int_{F(t)}\rho_F u_F\otimes u_F:\nabla\varphi_F\\ &+\frac{\mu_F}{\beta_\Omega}\int_0^T\int_{\partial\Omega}(u_F\times n)\cdot(\varphi_F\times n)+\frac{\mu_F}{\beta_S}\int_0^T\int_{\partial S(t)}((u_F-u_S)\times n)\cdot((\varphi_F-\varphi_S)\times n)\\ &+\int_0^T\int_{F(t)}2\mu_F D(u_F):D(\varphi_F)\quad=\quad-\int_0^T\int_{F(t)}\rho_F g e_3\varphi_F-\int_0^T\int_{S(t)}\rho_S g e_3\varphi_S\\ &+\int_{F_0}\rho_F u_{F,0}\varphi_F(0,\cdot)+\int_{S_0}\rho_S u_{S,0}\varphi_S(0,\cdot),\end{aligned}\tag{2.14}$$

4) For all $\psi \in \mathcal{D}([0,T);\mathcal{D}(\overline{\Omega}))$, we have

$$-\int_0^T\int_{S(t)}\partial_t\psi-\int_0^T\int_{S(t)}u_S\cdot\nabla\psi=\int_{S_0}\psi(0,\cdot),$$

5) We have the following energy inequality for a.a. $t\in(0,T)$:

$$\begin{aligned}&\frac{1}{2}\left[\int_{F(t)}\rho_F|u_F(t)|^2+\int_{S(t)}\rho_S|u_S(t)|^2\right]\\ &+\mu_F\left(2\int_0^t\int_{F(t)}|D(u_F(t))|^2+\frac{1}{\beta_\Omega}\int_0^t\int_{\partial\Omega}|u_F\times n|^2\mathrm{d}\sigma+\frac{1}{\beta_S}\int_0^t\int_{\partial S(t)}|(u_F-u_S)\times n|^2\mathrm{d}\sigma\right)\\ &\quad\le-\int_0^t\int_{F(t)}\rho_F g e_3\cdot u_F-\int_0^t\int_{S(t)}\rho_S g e_3\cdot u_S+\frac{1}{2}\int_{F(0)}\rho_F|u_{F,0}|^2+\frac{1}{2}\int_{S(0)}\rho_S|u_S(0)|^2.\end{aligned}\tag{2.15}$$

We refer to [15] for a detailed discussion of this definition. Note that it is restricted to solutions up to collision, through the condition $S(t) \Subset \Omega$. We recall that item *3)* is the identity one gets by multiplying (1.1) by a divergence-free test-function φ_F and then combining with

(1.3)–(1.4). Item *4)* is the weak form of the transport equation satisfied by χ_S, the indicator function of $S(t)$ as defined in *1)*. Finally, *2)* summarizes the regularity that is implied by the energy estimate (2.15). Note that u_F and u_S share the same normal component, so that the $L^\infty L^2$ bound on u_F and u_S yields a global $L^\infty L^2_\sigma(\Omega)$ on the extended velocity field u (see (1.11) for the definition of u). Moreover, the $L^\infty L^2$ bound on u_F and $L^2 L^2$ bound on $D(u_F)$ imply an $L^2_{loc} H^1$ bound, through Korn inequality (that is valid as long as the solid is away from the boundary of the container).

From [15], there is a weak solution for the system up to a contact between $S(t)$ and $\partial\Omega$:

Theorem 2.2. *Let Ω and $S_0 \Subset \Omega$ two $C^{1,1}$ bounded domain of $\mathbb{R}^3$. Given $(\beta_S, \beta_\Omega) \in (0,\infty)^2$, and initial data $u_{F,0} \in L^2_\sigma(\Omega), u_{S,0} \in \mathcal{R}$ satisfying:*

$$u_{F,0} \cdot n = u_{S,0} \cdot n, \quad on \quad \partial S_0,$$

there exists $T \in \mathbf{R}^*_+ \cup \{\infty\}$ and a weak solution of (1.1)-(1.9) on $[0,T)$. Moreover, such weak solution exists up to collision, that is either we can take $T = \infty$, or we can take $T \in \mathbf{R}^*_+$ in such a way that

$$S(t) \Subset \Omega, \quad for\ all \quad t \in [0,T), \quad and \quad \lim_{t \to T^-} dist(\partial S(t), \partial\Omega) = 0.$$

The aim of Theorem 1.1 is to prove that $T < \infty$ under assumption A1-A4.

2.2 The mixed case: $\beta_S = 0$ and $\beta_\Omega > 0$.

We proceed with the case where slip is imposed on one of the boundaries only. For simplicity, we consider that no-slip boundary conditions are imposed on $\partial S(t)$ *i.e.* $\beta_S = 0$ and $\beta_\Omega > 0$. To our knowledge, this particular system has not been treated explicitly in previous studies. Nevertheless, as slip is allowed on the exterior boundary of $F(t)$ only, it is straightforward to adapt the method of [10] or [25], for instance, to tackle a Cauchy theory for weak solutions before contact. In particular, the extension trick (1.11) yields a sufficiently smooth velocity-field in this case so that we can reduce the full system to a global weak formulation on this extended velocity field. This reads as follows:

Definition 2.3. *Let Ω and $S_0 \subset \Omega$ be two Lipschitz bounded domains of $\mathbb{R}^3$. Let $u_{F,0} \in L^2_\sigma(\Omega), u_{S,0} \in \mathcal{R}$ such that $u_{F,0} \cdot n = u_{S,0} \cdot n$ on ∂S_0. A weak solution of (1.1)–(1.9) is a pair (S,u) satisfying*

1) $S(t) \Subset \Omega$ is a bounded domain of $\mathbb{R}^3$ for all $t \in [0,T)$ such that

$$\chi_S(t,x) \triangleq 1_{S(t)}(x) \in L^\infty((0,T) \times \Omega).$$

2) $u \in L^\infty(0,T; L^2_\sigma(\Omega)) \cap L^2(0,T; H^1_\sigma(\Omega))$ satisfies

$$u(t,\cdot)_{|_{S(t)}} \in \mathcal{R} \quad for\ a.a.\ t \in (0,T)\,.$$

3) For all φ belonging to

$$\mathcal{T}^m_T \triangleq \{\varphi \in \mathcal{D}([0,T) \times \bar{\Omega}), \quad \varphi(t,\cdot)_{|_{S(t)}} \in \mathcal{R}\ for\ all\ t \in [0,T)\}$$

there holds

$$-\int_0^T\int_\Omega \rho\,(u\partial_t\varphi + u\otimes u : D(\varphi)) + \int_0^T\int_\Omega 2\mu_F D(u) : D(\varphi)$$
$$+\frac{1}{\beta_\Omega}\int_0^T\int_{\partial\Omega}(u\times n)\cdot(\varphi\times n) = -\int_0^T\int_\Omega \rho g e_3\varphi - \int_\Omega \rho_0 u_0\varphi(0,\cdot), \quad (2.16)$$

where

$$\begin{aligned}
\rho(t,x) &= \rho_S\, 1_{S(t)}(x) + \rho_F 1_{F(t)}(x), && \text{for } (t,x)\in(0,T)\times\Omega,\\
\rho_0(x) &= \rho_S\, 1_{S_0}(x) + \rho_F 1_{F_0}(x), && \text{for } x\in\Omega,\\
u_0(x) &= u_{S,0}\, 1_{S_0}(x) + u_{F,0} 1_{F_0}(x), && \text{for } x\in\Omega,
\end{aligned}$$

4) For all $\psi\in\mathcal{D}([0,T);\mathcal{D}(\overline{\Omega}))$, we have

$$-\int_0^T\int_{S(t)}\partial_t\psi - \int_0^T\int_{S(t)} u\cdot\nabla\psi = \int_{S_0}\psi(0,\cdot),$$

5) We have the following energy inequality for a.a. $t\in(0,T)$:

$$\frac{1}{2}\int_\Omega \rho|u(t)|^2 + 2\mu_F\int_0^t\int_\Omega |D(u)|^2 + \frac{1}{\beta_\Omega}\int_0^t\int_{\partial\Omega}|u\times n|^2 \mathrm{d}\sigma$$
$$\leq -\int_0^t\int_{F(t)}\rho g e_3 u + \frac{1}{2}\int_\Omega \rho(0,\cdot)|u_0|^2\,. \quad (2.17)$$

Following the method of proof of [10], one obtains:

Theorem 2.4. *Let Ω and $S_0 \Subset \Omega$ two $C^{1,1}$ bounded domain of $\mathbb{R}^3$. Given $\beta_\Omega\in(0,\infty)$ and initial data $u_{F,0}\in L^2_\sigma(\Omega), u_{S,0}\in\mathcal{R}$ satisfying*

$$u_{F,0}\cdot n = u_{S,0}\cdot n, \quad \text{on} \quad \partial S_0,$$

the same conclusion as in Theorem 2.2 holds.

3 Proof of collision with slip conditions

This section is devoted to the proof of Theorem 1.1. We assume β_S and β_Ω are fixed non-negative slip lengths, and $\rho_S > \rho_F$. Let (S,u) be a weak solution over $(0,T)$, as given by Theorem 2.2. We consider a solution up to its "maximal time of existence": that is $T=\infty$ if there is no collision, or $T<\infty$ the time of collision. We assume that this weak solution satisfies A1-A4 for $t\in[0,T)$. We denote by $h=h(t)$ the distance between the south pole of $S(t)$ and $x=0$. By assumption A4, when $h(t)$ is small enough, it is equal to the distance between $S(t)$ and the boundary $\partial\Omega$. Our goal is to prove that $h(t)$ goes to zero in finite time. In other words, $T<\infty$, that is collision occurs in finite time.

To build up our proof, we first need some general considerations on the connection between the weak formulation (2.14) and the distance h, for an arbitrary weak solution satisfying A4. We then design a special test-function in order to prove contact in finite time.

3.1 Some features of the weak formulation (2.14)

Let (S,u) a weak solution defined on $(0,T)$ and $\varphi \in \mathcal{T}_T^s$ of the form

$$\varphi = \zeta(t)\varphi_{h(t)} \text{ with } \qquad \varphi_S = \zeta(t)e_3\,. \tag{3.18}$$

for some functions $\zeta \in \mathcal{D}([0,T))$ and $h \mapsto \varphi_h$ to be made precise. Integrating by parts, we get that for any $q \in \mathcal{D}([0,T)\times\bar{\Omega})$, for a.a. $t \in (0,T)$

$$\begin{aligned}
& 2\int_{F(t)} D(\varphi_F) : D(u_F) \\
= & \int_{F(t)} (2D(\varphi_F) - qI) : \nabla u_F \\
= & -\int_{F(t)} (\Delta\varphi_F - \nabla q)\cdot u_F + \int_{\partial F(t)} (2D(\varphi_F) - qI)n\cdot u_F d\sigma \\
= & -\int_{F(t)} (\Delta\varphi_F - \nabla q)\cdot u_F + \int_{\partial\Omega} (2D(\varphi_F) - qI)n\cdot u_F d\sigma + \int_{\partial S(t)} (2D(\varphi_F) - qI)n\cdot u_F d\sigma \\
= & -\int_{F(t)} (\Delta\varphi_F - \nabla q)\cdot u_F + \int_{\partial\Omega} (2D(\varphi_F) - qI)n\cdot u_F d\sigma \\
& + \int_{\partial S(t)} (2D(\varphi_F) - qI)n\cdot (u_F - u_S) d\sigma + \int_{\partial S(t)} (2D(\varphi_F) - qI)n\cdot u_S d\sigma,
\end{aligned}$$

Chosing also q of the form $q = \zeta(t)q_{h(t)}$, the last term can be calculated by

$$\int_{\partial S(t)} (2D(\varphi_F) - qI)n\cdot u_S d\sigma = h'(t)\zeta(t)\int_{\partial S(t)} (2D(\varphi_{h(t)}) - q_{h(t)}I)n\cdot e_3 d\sigma \triangleq \zeta(t)h'(t)n(h(t)).$$

We emphasize that the domain $S(t)$ is completely fixed by the value of $h(t)$ (we shall denote $S_{h(t)}$ in what follows) so that the last integral does depend on $h(t)$ only. Combining all the equations with (2.14), we find

$$\begin{aligned}
& \mu_F \int_0^T \zeta(t)h'(t)n(h(t))\mathrm{d}t \\
= & \int_0^T\int_{F(t)} \rho_F u_F\cdot\partial_t\varphi_F + \int_0^T\int_{S(t)} \rho_S u_S\cdot\partial_t\varphi_S + \int_0^T\int_{F(t)} \rho_F u_F\otimes u_F : \nabla\varphi_F \\
- & \int_0^T\int_{F(t)} \rho_F g e_3\varphi_F - \int_0^T\int_{S(t)} \rho_S g e_3\varphi_S + \int_{F_0} \rho_F u_{F,0}\varphi_F(0,\cdot) + \int_{S_0} \rho_S u_{S,0}\varphi_S(0,\cdot) \\
- & \frac{\mu_F}{\beta_\Omega}\int_0^T\int_P (u_F\times n)\cdot(\varphi_F\times n) - \frac{\mu_F}{\beta_S}\int_0^T\int_{\partial S(t)} ((u_F - u_S)\times n)\cdot((\varphi_F - \varphi_S)\times n) \\
+ & \mu_F\Bigg(\int_0^T\int_{F(t)} (\Delta\varphi_F - \nabla q)\cdot u_F - \int_0^T\int_{\partial\Omega} (2D(\varphi_F) - qI)n\cdot u_F d\sigma \\
- & \int_0^T\int_{\partial S(t)} (2D(\varphi_F) - qI)n\cdot(u_F - u_S) d\sigma\Bigg)
\end{aligned} \tag{3.19}$$

This identity enables to describe the time evolution of h through the function n that we have to choose now. To compute $n(h)$, we shall take advantage of the following identity:

$$\begin{aligned} n(h) &= \int_{\partial S_h} (2D(\varphi_h) - q_h I)n \cdot \varphi_h d\sigma + \int_{\partial S_h} (D(\varphi_h) - q_h I)n \cdot (e_3 - \varphi_h) d\sigma \\ &= \int_{F_h} (\Delta\varphi_h - \nabla q_h) \cdot \varphi_h dx + 2\int_{F_h} D(\varphi_h) : D(\varphi_h) dx \\ &- \int_{\partial\Omega} (2D(\varphi_h) - q_h I)n \cdot \varphi_h d\sigma + \int_{\partial S_h} (D(\varphi_h) - q_h I)n \cdot (e_3 - \varphi_h) d\sigma. \end{aligned} \tag{3.20}$$

where $F_h \triangleq \Omega \setminus \bar{S}_h$. This identity holds true for all h such that $S_h \subset\subset \Omega$.

3.2 Construction of test-functions

We construct now the test function φ_h and the pressure q_h associated with the solid particle S_h frozen at distance h. It will be defined for $h \in (0, h_M)$, with $h_M \triangleq \sup_{t\in[0,T)} h(t)$. When $h \to 0$, a cusp arises in $F_h = \Omega \setminus \overline{S_h}$. This cusp is contained in the domain :

$$\Omega_{h,r} := \{(x_1, x_2, x_3) \in F_h,\ x_3 < \frac{1}{2} + h \text{ and } |(x_1, x_2)| < r\},$$

for arbitrary $r \in (0, \frac{1}{2})$. In particular, in all forthcoming estimates, we shall pay special attention to the region $\Omega_{h,2\delta}$, where δ was introduced in assumption A2.

To compute test-functions and associated pressures, we introduce cylindrical coordinates (r, θ, z) associated with cartesian coordinates (x_1, x_2, x_3) in such a way that $x_3 = z$. The local basis reads (e_r, e_θ, e_z). First, we introduce an approximate solution of the steady Stokes equation in the domain $\Omega_{h,2\delta}$. It will satisfy locally the slip boundary conditions (1.6)-(1.7), with u_S replaced by e_z:

$$\widetilde{\varphi}_h = \frac{1}{2}\left(-\partial_z\Phi\left(r, \frac{z}{h+\gamma_s(r)}\right)x_1, -\partial_z\Phi\left(r, \frac{z}{h+\gamma_s(r)}\right)x_2, \frac{1}{r}\partial_r\left(\Phi\left(r, \frac{z}{h+\gamma_s(r)}\right)r^2\right)\right)$$

where

$$\gamma_s(r) = 1 - \sqrt{1-r^2}, \quad \Phi(r,t) = P_1(r)t + P_2(r)t^2 + P_3 t^3. \tag{3.21}$$

Here

$$P_1(r) = \frac{6(2+\alpha_S)}{12 + 4(\alpha_S + \alpha_P) + \alpha_S\alpha_P}, \tag{3.22}$$

$$P_2(r) = \frac{3(2+\alpha_S)\alpha_P}{12 + 4(\alpha_S + \alpha_P) + \alpha_S\alpha_P}, \tag{3.23}$$

$$P_3(r) = -\frac{2(\alpha_S + \alpha_S\alpha_P + \alpha_P)}{12 + 4(\alpha_S + \alpha_P) + \alpha_S\alpha_P}, \tag{3.24}$$

where

$$\alpha_P = \frac{h + \gamma_s(r)}{\beta_\Omega}, \quad \alpha_S = \left(\frac{1}{\beta_S} + 2\right)(h + \gamma_s(r)). \tag{3.25}$$

This approximation is inspired by the computations in [14]. We emphasize that the value of α_S does not match exactly the one in [14]. This modification is required so that further boundary estimates hold true (see (3.39)). *We stress that notation* $\partial_z^k \Phi(r, \frac{z}{h+\gamma_s(r)})$, *resp.* $\partial_r^k \Phi(r, \frac{z}{h+\gamma_s(r)})$, *stands for* $\partial_z^k \Psi(r,z)$, *resp.* $\partial_r^k \Psi(r,z)$, $\Psi(r,z) \triangleq \Phi(r, \dfrac{z}{h+\gamma_s(r)})$. With this structure, $\tilde{\varphi}_h$ is divergence-free. Moreover, the function Φ is constructed so that it satisfies the following boundary conditions:

- for $z = 0$

$$\Phi(r,0) = 0, \tag{3.26}$$

$$\partial_{zz}\Phi\left(r, \frac{z}{h+\gamma_s(r)}\right) - \frac{1}{\beta_\Omega}\partial_z\Phi\left(r, \frac{z}{h+\gamma_s(r)}\right) = 0 \tag{3.27}$$

- for $z = h + \gamma_S(r)$

$$\Phi(r,1) = 1 \tag{3.28}$$

$$\partial_{zz}\Phi\left(r, \frac{z}{h+\gamma_s(r)}\right) + \left(\frac{1}{\beta_S} + 2\right)\partial_z\Phi\left(r, \frac{z}{h+\gamma_s(r)}\right) = 0 \tag{3.29}$$

In the local cylindrical basis, the approximate solution $\widetilde{\varphi}_h$ reads:

$$\widetilde{\varphi}_h = \frac{1}{2}\left[-\partial_z\Phi\left(r, \frac{z}{h+\gamma_s(r)}\right) re_r + \frac{1}{r}\partial_r\left(\Phi\left(r, \frac{z}{h+\gamma_s(r)}\right) r^2\right) e_z\right]. \tag{3.30}$$

In particular, if we remark that n satisfies $n = -re_r + (1+h-z)e_z$ on $\partial S_h \cap \Omega_{h,1/2}$ and $n = e_3$ on $\partial\Omega \cap \Omega_{h,1/2}$, we obtain (see the introduction of Appendix A for further details) :

$$\left\{\begin{array}{rcl} (\widetilde{\varphi}_h - e_3)\cdot n|_{\partial S_h \cap \partial\Omega_{h,2\delta}} & = & 0, \\ \widetilde{\varphi}_h \cdot n|_{\partial\Omega\cap\partial\Omega_{h,2\delta}} & = & 0, \\ \widetilde{\varphi}_h \times n|_{\partial\Omega\cap\partial\Omega_{h,2\delta}} & = & -2\beta_\Omega(D(\widetilde{\varphi}_h)n)\times n|_{\partial\Omega\cap\partial\Omega_{h,2\delta}}. \end{array}\right. \tag{3.31}$$

Thus, up to its extension to the whole of Ω, the function $\widetilde{\varphi}_h$ is a good candidate for applying the computations of the previous section. Such extension φ_h should satisfy $\varphi_h|_{S_h} = e_z$. It remains to define it into the whole of F_h.

To this end, we introduce a smooth function $\phi = \phi(x)$ with compact support satisfying

$$\begin{array}{ll} \phi = 1 & \text{on a } \frac{d_\delta}{2}\text{-neighborhood of the unit ball } B(0,1)\ , \\ \phi = 0 & \text{outside a } d_\delta\text{-neighborhood of } B(0,1). \end{array}$$

We recall that d_δ was introduced in A4. We also define a cut-off function $\chi : \mathbb{R}^3 \to [0,1]$ s.t.

$$\chi = 1 \text{ in } (-\delta,\delta)^3, \qquad \chi = 0 \text{ outside } (-2\delta, 2\delta)^3. \tag{3.32}$$

With these cut-off functions, we set (in cartesian coordinates):

$$\varphi_h = \frac{1}{2}\left(\begin{array}{c} -x_1\partial_z\left[(1-\chi(x))\phi\left(x-(1+h)e_3\right) + \chi(x)\Phi(r, \frac{z}{h+\gamma_s(r)})\right] \\ -x_2\partial_z\left[(1-\chi(x))\phi\left(x-(1+h)e_3\right) + \chi(x)\Phi(r, \frac{z}{h+\gamma_s(r)})\right] \\ \frac{1}{r}\partial_r\left[r^2((1-\chi(x))\phi\left(x-(1+h)e_3\right) + \chi(x)\Phi(r, \frac{z}{h+\gamma_s(r)}))\right] \end{array}\right) \text{ in } F_h,$$

$$\varphi_h = e_3 \text{ in } S_h. \tag{3.33}$$

By the definition of φ_h, we get

Lemma 3.1. *The map $(h,x) \mapsto \varphi_h(x)$ is smooth on all domains:*

$$\{(h,x), h \in (0,h_M), x \in F_h\}\,,\ \{(h,x) \in (0,h_M), x \in S_h\}\,,\ \{(h,x), h \in [0,h_M), x \in F_h \setminus \Omega_{h,\delta}\}.$$

Furthermore, there holds:

$$\varphi_h \cdot n = 0 \quad \text{on } \partial\Omega \tag{3.34}$$

and

$$\varphi_h|_{F_h} \cdot n = e_3 \cdot n \quad \text{on } \partial S_h. \tag{3.35}$$

This comes from direct calculations, that we leave to the reader for brevity. We now state some refined estimates, to be used in the proof of Theorem 1.1. The whole point is to have an accurate bound on the blow-up of the test function as h goes to zero.

Lemma 3.2. *There exist constants $0 < c < C$ so that, when $h < h_M$:*

$$\|\varphi_h\|_{L^2(F_h)} \leq C, \tag{3.36}$$

$$\|\nabla\varphi_h\|^2_{L^2(F_h)} \leq C|\ln h|, \tag{3.37}$$

$$\|D(\varphi_h)\|^2_{L^2(F_h)} \geq c|\ln(h)|, \tag{3.38}$$

and

$$\|2\beta_S(D(\varphi_h)n) \times n + (\varphi_h - e_3) \times n\|_{L^2(\partial S_h)} \leq C, \tag{3.39}$$

Furthermore, for $i = 1,2,3$, denoting by exponent i cartesian coordinates, there holds:

$$\|\int_0^{h+\gamma_s(r)} \partial_h\varphi_h^i dz\|^2_{L^2(\partial\Omega\cap\partial\Omega_{h,\delta})} \leq C|\ln h|, \tag{3.40}$$

$$\|\int_z^{h+\gamma_s(r)} \partial_h\varphi_h^i ds\|_{L^2(\Omega_{h,\delta})} \leq C, \tag{3.41}$$

$$\|(h+\gamma_s(r))\partial_i \int_z^{h+\gamma_s(r)} \partial_h\varphi_h^i(x_1,x_2,s)ds\|_{L^2(\Omega_{h,\delta})} \leq C, \tag{3.42}$$

and for $(i,j) \in \{1,2,3\}^2$ and $x \in \Omega_{h,\delta}$:

$$\left|\int_z^{h+\gamma_S(r)} (\partial_i\varphi_h^j(x_1,x_2,s) + \partial_j\varphi_h^i(x_1,x_2,s))ds\right| \leq C, \tag{3.43}$$

$$\left|(h+\gamma_S(r))\partial_i \int_z^{h+\gamma_S(r)} (\partial_i\varphi_h^j(x_1,x_2,s) + \partial_j\varphi_h^i(x_1,x_2,s))ds\right| \leq C. \tag{3.44}$$

The proof of this lemma is postponed to Appendix A.

When computing (3.19), we introduced a pressure q. The aim was to smoothen singularities in $\Delta\varphi_h$ which must arise when $h \to 0$. The construction of this pressure is the content of the next lemma:

Lemma 3.3. *There exists a map $(h,x) \mapsto q_h$ which is smooth on the domains*

$$\{(h,x), h \in (0,h_M), x \in F_h\} \text{ and } \{(h,x), h \in [0,h_M), x \in F_h \setminus \Omega_{h,\delta}\}$$

and which satisfies :

- *there exist constants* $0 < c < C < \infty$ *such that :*

$$c|\ln h| \leq \int_{\partial S_h} (2D(\varphi_h) - q_h I)n \cdot (e_3 - \varphi_h) d\sigma - \int_{\partial\Omega} (2D(\varphi_h) - q_h I)n \cdot \varphi_h d\sigma \leq C|\ln h|;$$

- *there exists a positive constant* C *such that, for any* $h \in (0, h_M)$ *and* $v \in H^1(F_h)$ *satisfying* $v \cdot n = 0$ *on* $\partial\Omega$*, we have:*

$$\left| \int_{F_h} (\Delta\varphi_h - \nabla q_h) v \right| \leq C \left(\|D(v)\|_{L^2(F_h)} + \|v\|_{L^2(\partial\Omega)} \right). \tag{3.45}$$

The proof of this lemma is also postponed to Appendix A.

3.3 Conclusion of the proof of Theorem 1.1

We are now in a position to prove Theorem 1.1. Combining the energy inequality (2.15) with the classical identity

$$-\int_0^t \int_{F(t)} \rho_F g e_z \cdot u_F - \int_0^t \int_{S(t)} \rho_S g e_z \cdot u_S = \frac{4(\rho_F - \rho_S) g\pi}{3} (h(t) - h(0))$$

we obtain easily (remember that $\rho_S > \rho_F$):

$$\sup_{(0,T)} \int_\Omega |u(t)|^2 + \int_0^T \int_{F(t)} |D(u_F(t))|^2 + \int_0^T \int_{\partial\Omega} |u_F \times n|^2 \mathrm{d}\sigma + \int_0^T \int_{\partial S(t)} |(u_F - u_S) \times n|^2 \mathrm{d}\sigma \leq C_0. \tag{3.46}$$

This energy bound will be of constant use.

As mentioned before, we take a special test function in the variational formulation (2.14). Namely, we take $\varphi(t,x) = \zeta(t)\, \varphi_{h(t)}(x)$, with the field φ_h built in paragraph 3.2. From Lemma 3.1, we have that $\varphi \in \mathcal{T}_T$ so that identity (3.19) holds true. The whole point is then to bound properly the r.h.s. of (3.19):

$$\begin{aligned}
RHS \;\triangleq\; & \int_0^T \int_{F(t)} \rho_F u_F \cdot \partial_t \varphi_F + \int_0^T \int_{S(t)} \rho_S u_S \cdot \partial_t \varphi_S + \int_0^T \int_{F(t)} \rho_F u_F \otimes u_F : \nabla \varphi_F \\
- \; & \int_0^T \left(\int_{F(t)} \rho_F g e_z \varphi_F + \int_{S(t)} \rho_S g e_z \varphi_S \right) + \left(\int_{F_0} \rho_F u_{F,0} \varphi_F(0,\cdot) + \int_{S_0} \rho_S u_{S,0} \varphi_S(0,\cdot) \right) \\
- \; & \mu_F \int_0^T \int_{\partial\Omega} (u_F \times n) \cdot \left(\frac{1}{\beta_\Omega} (\varphi_F \times n) + (2D(\varphi_F) - qI)n \right) d\sigma \\
- \; & \mu_F \int_0^T \int_{\partial S(t)} ((u_F - u_S) \times n) \cdot \left(\frac{1}{\beta_S} (\varphi_F - \varphi_S) \times n) + (2D(\varphi_F) - qI)n \right) d\sigma \\
+ \; & \mu_F \int_0^T \int_{F(t)} (\Delta\varphi_F - \nabla q) \cdot u_F \\
= \; & I_1 + I_2 + I_3 + I_4 + I_5 - \mu_F (I_6 + I_7 + I_8),
\end{aligned}$$

where we have applied the continuity of normal traces on $\partial S(t)$ and $\partial\Omega$ in the boundary terms.

- We first deal with

$$I_3 = \int_0^T \int_{F(t)} \rho_F u_F \otimes u_F : \nabla\varphi_F \triangleq \int_0^T \rho_F \zeta(t) I_3(t).$$

We introduce the functions $x \mapsto P_z\varphi_h^{i,j}(x)$ for $(i,j) \in \{1,2,3\}^2$ defined by:

$$\begin{aligned} P_z\varphi_h^{i,j}(x) &= \int_z^{h+\gamma_S(r)} (\partial_i\varphi_h^j + \partial_j\varphi_h^i)(x_1,x_2,s)ds\,, \quad \forall\, r<1, \quad z < h+\gamma_s(r), \\ P_z\varphi_h^{i,j}(x) &= \int_z^{h+1} (\partial_i\varphi_h^j + \partial_j\varphi_h^i)(x_1,x_2,s)ds \quad \text{otherwise.} \end{aligned}$$

Let us stress that $P_z\varphi_h^{i,j}$ is smooth in F_h, notably across $r=1$: it follows from the cancellation of the symmetric gradient $\partial_i\varphi_h^j + \partial_j\varphi_h^i$ near $(r=1, z=1+h)$ that both expressions coincide there.

Moreover, by (3.43), there exists a constant C independent of t such that, for all $(i,j) \in \{1,2,3\}^2$, there holds:

$$|P_z\varphi_h^{i,j}(x_1,x_2,x_3)| + |(h+\gamma_S(r))\partial_i P_z\varphi_h^{i,j}(x_1,x_2,x_3)| \le C, \quad \forall\, x \in \Omega_{h(t),\delta}, \tag{3.47}$$

$$|P_z\varphi_h^{i,j}(x_1,x_2,x_3)| + |\partial_i P_z\varphi_h^{i,j}(x_1,x_2,x_3)| \le C, \quad \forall\, x \in F(t)\setminus\Omega_{h(t),\delta}. \tag{3.48}$$

Then, by performing integration by parts, we obtain (summation on indices i and j is implicit):

$$\begin{aligned} I_3(t) &= -\int_{F(t)} u_F^i u_F^j\, \partial_z[P_z\varphi_h^{i,j}] = \int_{F(t)} \partial_3(u_F^i u_F^j) P_z\varphi_h^{i,j} - \int_{\partial\Omega} u_F^i u_F^j P_z\varphi_h^{i,j} n^3 \mathrm{d}\sigma \\ &= \int_{F(t)} ((\partial_3 u_F^i + \partial_i u_F^3)u_F^j + (\partial_3 u_F^j + \partial_j u_F^3)u_F^i) P_z\varphi_h^{i,j} \\ &\quad - \int_{F(t)} (\partial_i u_F^3 u_F^j + \partial_j u_F^3 u_F^i) P_z\varphi_h^{i,j} - \int_{\partial\Omega} (u_F^i u_F^j) P_z\varphi_h^{i,j} n^3 \mathrm{d}\sigma \\ &= \int_{F(t)} ((\partial_3 u_F^i + \partial_i u_F^3)u_F^j + (\partial_3 u_F^j + \partial_j u_F^3)u_F^i) P_z\varphi_h^{i,j} \\ &\quad + \int_{F(t)} \Big((\partial_i u_F^j + \partial_j u_F^i) u_F^3 P_z\varphi_h^{i,j} + u_F^3 u_F^j \partial_i P_z\varphi_h^{i,j} + u_F^3 u_F^i \partial_j P_z\varphi_h^{i,j} \Big) \\ &\quad - \int_{\partial\Omega} (u_F^i u_F^j n^3 + u_F^j u_F^3 n^i + u_F^i u_F^3 n^j) P_z\varphi_h^{i,j} \mathrm{d}\sigma. \end{aligned}$$

In this last identity, we apply (3.47)-(3.48) to obtain :

$$\left| \int_{\partial\Omega} (u_F^i u_F^j n^3 + u_F^j u_F^3 n^i + u_F^i u_F^3 n^j) P_z\varphi_h^{i,j} \mathrm{d}\sigma \right| \le C \|u_F(t)\|^2_{L^2(\partial\Omega)}\,.$$

Concerning the other terms, we have for instance :

$$\begin{aligned}\left|\int_{F(t)} u_F^3 u_F^j \partial_i P_z \varphi_h^{i,j}\right| &\leq \left|\int_{F(t)\cap\Omega_{h(t),\delta}} \frac{u_F^3}{h+\gamma_S(r)} u_F^j (h+\gamma_S(r))\partial_i P_z \varphi_h^{i,j}\right| \\ &\quad + \left|\int_{F(t)\setminus\Omega_{h(t),\delta}} u_F^3 u_F^j \partial_i P_z \varphi_h^{i,j}\right| \\ \leq \ & \|\frac{u_F^3}{h+\gamma_S(r)}\|_{L^2(F(t)\cap\Omega_{h(t),\delta})} \|u_F\|_{L^2(F(t))} \|(h+\gamma_S(r))\partial_i P_z \varphi_h^{i,j}\|_{L^\infty(F(t)\cap\Omega_{h(t),\delta}))} \\ & + \|u_F^3\|_{L^2(F(t)\cap\{|(x_1,x_2)|<(1+d_\delta)\})} \|u_F\|_{L^2(F(t))} \|\partial_i P_z \varphi_h^{i,j}\|_{L^\infty(F(t)\setminus\Omega_{h(t),\delta}))}.\end{aligned}$$

As $u_F^3(x) = 0$ for $x \in \{|(x_1,x_2)| < 1+d_\delta, x_3 = 0\}$, a straightforward Poincaré inequality yields that there exists a constant $C > 0$ independent of time so that:

$$\begin{aligned}\|\frac{u_F^3}{h+\gamma_S(r)}\|_{L^2(F(t)\cap\Omega_{h(t),\delta})} &\leq C\|\partial_3 u_F^3\|_{L^2(F(t))} \\ \|u_F^3\|_{L^2(F(t)\cap\{|(x_1,x_2)|<(1+d_\delta)\})} &\leq C\left(\|\partial_3 u_F^3\|_{L^2(F(t))} + \|u_F\times n\|_{L^2(\partial\Omega)}\right).\end{aligned}$$

Introducing (3.47)–(3.48) then yields :

$$|I_3(t)| \leq C\left[\|D(u_F)\|^2_{L^2(F(t))} + \|D(u_F(t))\|_{L^2(F(t))}\|u_F(t)\|_{L^2(F(t))} + \|u_F(t)\|^2_{L^2(\partial\Omega)}\right],$$

so that (3.46) implies there exists a constant C depending only on initial data for which

$$|I_3| \leq C(1+\sqrt{T})\|\zeta\|_{L^\infty(0,T)}.$$

- To compute I_1, we split:

$$I_1 = \int_0^T\int_{F(t)} \rho_F u_F\cdot\partial_t\varphi_F = \int_0^T\int_{F(t)} \rho_F u_F \zeta'(t)\varphi_{h(t)} + \int_0^T\int_{F(t)} \rho_F u_F \zeta(t) h'(t)\partial_h\varphi_F \triangleq I_1^a + I_1^b.$$

Applying (3.36) yields :

$$|I_1^a| \leq C\|\zeta'\|_{L^1(0,T)}\|u\|_{L^\infty(0,T;L^2(\Omega))} \leq C_0\|\zeta'\|_{L^1(0,T)}.$$

Then, we have $I_1^b = \rho_F\int_0^t \zeta(t)h'(t)I_1^b(t)$ where, for all $t\in(0,T)$:

$$I_1^b(t) = \sum_{i=1}^3 \int_{F(t)} u_F^i\cdot\partial_h\varphi_{h(t)}^i.$$

To bound this term, we define as previously $P_z\varphi_h^i$ which satisfies, for all $i = 1,2,3$:

$$\begin{aligned}P_z\varphi_h^i(x) &= \int_z^{h+\gamma_S(r)} \partial_h\varphi_h^i(x_1,x_2,s)ds, \quad \forall\, r<1, \quad z<h+\gamma_s(r), \\ P_z\varphi_h^i(x) &= \int_z^{h+1} \partial_h\varphi_h^i(x_1,x_2,s)ds \quad \text{otherwise.}\end{aligned}$$

Again, $P_z\varphi_h^i$ is continuous in F_h, notably near $r=1$, as $\partial_h\varphi_h$ vanishes near $(r=1, z=1+h)$.

With (3.41) together with (3.42), there is a constant C such that:

$$\|P_z\varphi_h^i\|_{L^2(F(t))} + \|(h+\gamma_S(r))\partial_i P_z\varphi_h^i\|_{L^2(F(t)\cap\Omega_{h(t),2\delta})} + \|\partial_i P_z\varphi_h^i\|_{L^2(F(t)\setminus\Omega_{h(t),2\delta})} \leq C, \quad (3.49)$$

$$\|P_z\varphi_h^i\|_{L^2(\partial\Omega)} \leq C|\ln(h)|^{1/2}. \quad (3.50)$$

From now on, we restrict implicitly the integral $I_1^b(t)$ to $|(x_1,x_2)| < (1+d_\delta)$ as $\partial_h\varphi_h$ vanishes elsewhere. Performing integration by parts (summation over index i is implicit), we obtain :

$$\begin{aligned} I_1^b(t) &= -\int_{F(t)} u_F^i \partial_z P_z\varphi_h^i \\ &= \int_{F(t)} \partial_3 u_F^i P_z\varphi_h^i - \int_{\partial\Omega} u_F^i P_z\varphi_h^i n^3 \mathrm{d}\sigma \\ &\leq \int_{F(t)} (\partial_3 u_F^i + \partial_i u_F^3) P_z\varphi_h^i + \int_{F(t)} u_F^3 \partial_i P_z\varphi_h^i - \int_{\partial\Omega} (u_F^3 n^i + u_F^i n^3) P_z\varphi_h^i \end{aligned}$$

Relying on (3.49)-(3.50) and (3.46), we might then reproduce the arguments we used to compute I_3, resulting in:

$$|I_1^b| \leq C\|\zeta\|_{L^\infty}\left(\int_0^T |h'(t)||\ln h|^{\frac{1}{2}}\|u_F\times n\|_{L^2(\partial\Omega)} + \sqrt{T}\right).$$

At this point, we remark that, given the computations of [14, Section 3.2], $|\ln(h(t))|^{1/2}$ is the minimum dissipation energy of functions $v \in H^1(F(t))$ satisfying $v\cdot n = e_3\cdot n$ on $\partial S(t)$ and $v\cdot n = 0$ on $\partial\Omega$. Consequently, there exists a constant C independent of time such that:

$$|\dot{h}||\ln(h)|^{\frac{1}{2}} \leq C\left(\int_{F(t)} |D(u_F(t))|^2 + \int_{\partial\Omega} |u_F\times n|^2 \mathrm{d}\sigma + \int_{\partial S(t)} |(u_F - u_S)\times n|^2 \mathrm{d}\sigma\right)^{\frac{1}{2}},$$

and (3.46) implies:

$$\int_0^T |h'(t)||\ln h|^{\frac{1}{2}}\|u_F\times n\|_{L^2(\partial\Omega)} \leq C\,.$$

- Next, we recall that $\int_{F(t)} \rho_F g e_3 \varphi_{h(t)} = -\dfrac{4\pi\rho_F g}{3}$, which implies:

$$I_4 = -\int_0^T\int_{F(t)} \rho_F g e_3\varphi_F - \int_0^T\int_{S(t)} \rho_S g e_3\varphi_S = \frac{4\pi(\rho_F-\rho_S)g}{3}\int_0^T \zeta(t)\mathrm{d}t. \quad (3.51)$$

- Then, we apply (3.31) together with (3.46). It leads to

$$\begin{aligned} |I_6| &= \left|\int_0^T \zeta(t)\int_{\partial\Omega} (u_F\times n)\cdot\left(\frac{1}{\beta_\Omega}(\varphi_h\times n) + (2D(\varphi_h) - qI)n\right) d\sigma\right|, \\ &= \left|\int_0^T \zeta(t)\int_{\partial\Omega} (u_F\times n)\cdot\left(\frac{1}{\beta_\Omega}(\varphi_h\times n) + 2D(\varphi_h)n\right) d\sigma\right|, \\ &\leq C\int_0^T |\zeta(t)|\|u_F\times n\|_{L^2(\partial\Omega)} \leq C'\|\zeta\|_{L^\infty(0,T)}\sqrt{T}, \end{aligned}$$

- By (3.39) together with (3.46):

$$
\begin{aligned}
|I_7| &= \left| \int_0^T \int_{\partial S(t)} ((u_F - u_S) \times n) \cdot \left(\frac{1}{\beta_S} (\varphi_h - e_3) \times n) + 2D(\varphi_h) n \right) d\sigma \right| \\
&\leq C \int_0^T |\zeta(t)| \|(u_F - u_S) \times n\|_{L^2(\partial S(t))} \|(\varphi_h - e_3) \times n) + 2\beta_S D(\varphi_h) n\|_{L^2(\partial S(t))} \mathrm{d}t \\
&\leq C' \|\zeta\|_{L^\infty(0,T)} \sqrt{T}.
\end{aligned}
$$

- As for I_8, applying (3.45), we have that

$$
\begin{aligned}
I_8 = \int_0^T \int_{F(t)} (\Delta \varphi_F - \nabla q) u_F &\leq C \int_0^T |\zeta(t)| \left(\|D(u_F)\|_{L^2(F(t))} + \|u_F\|_{L^2(\partial\Omega)} \right) dt \\
&\leq C' \|\zeta\|_{L^\infty(0,T)} \sqrt{T}.
\end{aligned}
$$

- Eventually, we get easily the following controls:

$$
|I_2| \leq C \, \|\zeta'\|_{L^1(0,T)}, \quad I_5 \leq C.
$$

Combining all these estimates to bound RHS, we end up with:

$$
\begin{aligned}
&\mu_F \int_0^T h'(t) \zeta(t) n(h(t)) \mathrm{d}t \\
&\qquad = \frac{4\pi(\rho_F - \rho_S) g}{3} \int_0^T \zeta(t) \mathrm{d}t + O(C_0(\|\zeta\|_{L^\infty(0,T)} + \|\zeta'\|_{L^1(0,T)})(1 + \sqrt{T})) \quad (3.52)
\end{aligned}
$$

We conclude by introducing a sequence ζ_n in $\mathcal{D}([0,T))$ such that

$$
\zeta_n \to 1_{[0,T)} \text{ pointwise}, \quad \text{with} \quad \|\zeta_n\|_{L^\infty} \leq C, \quad \|\zeta_n'\|_{L^1(0,T)} \leq C.
$$

A possible choice is $\zeta_n(t) = 1_{[0,T)}(t) - \chi(n(T-t))$, for some smooth χ compactly supported with $\chi = 1$ near 0. We take $\zeta = \zeta_n$ in (3.52), and let n go to infinity, to get

$$
\mu_F \int_{h(0)}^{h(T)} n(s) \mathrm{d}s + \frac{4\pi(\rho_S - \rho_F) g}{3} T \leq C_0(1 + \sqrt{T}). \quad (3.53)
$$

Introducing the bounds of Lemma 3.2 and Lemma 3.3 into (3.20), we show there exists a constant C such that:

$$
n(h) \leq C |\ln h| \quad \forall \, h < h_M,
$$

which implies that there exists a constant C for which:

$$
\int_{h(0)}^{h} n(s) ds \geq -C \, , \quad \forall \, h < h_M \, .
$$

Hence, we obtain a contradiction in (3.53) if T is large enough compared to C and C_0 which are fixed by h_M and the initial data only.

4 Proof of Theorem 1.2

The proof is very close to the one performed above for slip boundary conditions. Therefore, we only sketch the proof, insisting on its few specific features.

Computation of the drag. Let (S, u) be a weak solution given by Theorem 2.4 on $[0, T)$. As before, we consider a solution up to its "maximal time of existence": that is $T = \infty$ if there is no collision, or $T < \infty$ the time of collision. We assume that (S, u) satisfies A1-A4 over $[0, T)$. This time, we want to show that there is no collision, which means that h does not go to zero in finite time (and thus $T = \infty$). We shall consider a test function $\varphi \in \mathcal{T}_T^m$ of the form

$$\varphi = \zeta(t)\varphi_{h(t)} \qquad \text{with} \qquad \varphi_S = \zeta(t)e_z\,. \tag{4.54}$$

associated with a pressure $q \in \mathcal{D}([0, T) \times \bar{\Omega})$, of the form

$$q = \zeta(t)q_{h(t)}.$$

The values of φ_h and q_h are given below. As in the previous section, we shall use functions $\zeta = \zeta_n$ approximating $1_{(0,T)}$. Here and in what follows, we keep the convention that $\varphi_F = \varphi 1_{F(t)}$ and $\varphi_S = \varphi 1_{S(t)}$. With computations similar to those of the previous section, we obtain:

$$\begin{aligned}
& \mu_F \int_0^T \zeta(t)h'(t)n(h(t))\mathrm{d}t \\
= \; & \int_0^T \int_{F(t)} \rho_F u_F \cdot \partial_t \varphi_F + \int_0^T \int_{F(t)} \rho_F u_F \otimes u_F : \nabla \varphi_F \\
- \; & \int_0^T \int_{F(t)} \rho_F g e_z \varphi_F - \int_0^T \int_{S(t)} \rho_S g e_z \varphi_S + \int_{F_0} \rho_F u_{F,0} \varphi_F(0,\cdot) + \int_{S_0} \rho_S u_{S,0} \varphi_S(0,\cdot) \\
- \; & \frac{\mu_F}{\beta_\Omega} \int_0^T \int_P (u_F \times n) \cdot (\varphi_F \times n) \\
+ \; & \mu_F \left(\int_0^T \int_{F(t)} (\Delta \varphi_F - \nabla q) \cdot u_F - \int_0^T \int_{\partial\Omega} (2D(\varphi_F) - qI)n \cdot u_F d\sigma \right)
\end{aligned} \tag{4.55}$$

Again, we compute $n(h)$ applying the identity:

$$\begin{aligned}
n(h) \; = \; & \int_{F_h} (\Delta \varphi_h - \nabla q_h) \cdot \varphi_h dx + 2 \int_{F_h} D(\varphi_h) : D(\varphi_h) dx \\
- \; & \int_{\partial\Omega} (2D(\varphi_h) - q_h I)n \cdot \varphi_h d\sigma.
\end{aligned} \tag{4.56}$$

Construction of the test-function. For the function $(x, h) \mapsto \varphi_h(x)$, we keep the structure of (3.33) with a test-function $\widetilde{\varphi}_h$ still of the form (3.30). The only point that differs is the choice of the coefficients in the polynomial Φ. We set now:

$$P_1(r) \triangleq \frac{6}{4 + \alpha_P} \quad P_2(r) \triangleq \frac{3\alpha_P}{4 + \alpha_P} \qquad P_3(r) \triangleq -2\frac{1 + \alpha_P}{4 + \alpha_P}\,,$$

with α_P given by (3.25). We remark that this is equivalent to taking the limit $\alpha_S \to \infty$ in our choice of test-function in the previous section. With this choice, the function Φ satisfies boundary conditions:

- for $z = 0$

$$\Phi(r,0) = 0, \tag{4.57}$$

$$\partial_{zz}\Phi\left(r, \frac{z}{h+\gamma_s(r)}\right) - \frac{1}{\beta_\Omega}\partial_z\Phi\left(r, \frac{z}{h+\gamma_s(r)}\right) = 0 \tag{4.58}$$

- for $z = h + \gamma_S(r)$

$$\Phi(r,1) = 1 \tag{4.59}$$

$$\partial_z\Phi(r,1) = 0. \tag{4.60}$$

Similarly to the previous case, we provide also the following technical lemmas :

Lemma 4.1. *The map $(h,x) \mapsto \varphi_h(x)$ is smooth on all domains :*

$$\{(h,x), h \in (0,h_M), x \in F_h\}, \{(h,x) \in (0,h_M), x \in S_h\}, \{(h,x), h \in [0,h_M), x \in F_h \setminus \Omega_{h,\delta}\}.$$

Furthermore, there holds:

$$\varphi_h \cdot n = 0 \quad \text{on } \partial\Omega \tag{4.61}$$

and

$$\varphi_h|_{F_h} = e_3 \quad \text{on } \partial S_h. \tag{4.62}$$

Lemma 4.2. *There exist constants $0 < c < C$ so that, when $h < h_M$:*

$$\|\varphi_h\|_{L^2(F_h)} \leq C, \tag{4.63}$$

$$\|\nabla\varphi_h\|^2_{L^2(F_h)} \leq \frac{C}{h}, \tag{4.64}$$

$$\|D(\varphi_h)\|^2_{L^2(F_h)} \geq \frac{c}{h}, \tag{4.65}$$

Furthermore, for $i = 1,2,3$, denoting by exponent i cartesian coordinates, there holds:

$$\|\int_0^{h+\gamma_s(r)} \partial_h\varphi_h^i dz\|_{L^2(\partial\Omega\cap\partial\Omega_{h,\delta})} \leq C|\ln h|, \tag{4.66}$$

$$\|\int_z^{h+\gamma_s(r)} \partial_h\varphi_h^i ds\|_{L^2(\Omega_{h,\delta})} \leq C, \tag{4.67}$$

$$\|(h+\gamma_s(r))\partial_i \int_z^{h+\gamma_s(r)} \partial_h\varphi_F^i(x_1,x_2,s)ds\|_{L^2(\Omega_{h,\delta})} \leq C, \tag{4.68}$$

and

- *for $i \in \{1,2,3\}^2$, and $x \in \Omega_{h,\delta}$:*

$$\left|\partial_i\varphi_h^3 + \partial_3\varphi_h^i\right| \leq \frac{Cr}{(h+\gamma_S(r))^2}, \tag{4.69}$$

- *for* $(i,j) \in \{1,2\}^2$, *and* $x \in \Omega_{h,\delta}$*:*

$$\left| \int_z^{h+\gamma_S(r)} (\partial_i \varphi_h^j(x_1,x_2,s) + \partial_j \varphi_h^i(x_1,x_2,s)) ds \right| \leq C, \tag{4.70}$$

$$\left| (h+\gamma_S(r))\partial_i \int_z^{h+\gamma_S(r)} (\partial_i \varphi_h^j(x_1,x_2,s) + \partial_j \varphi_h^i(x_1,x_2,s)) ds \right| \leq C. \tag{4.71}$$

Lemma 4.3. *There exists a map* $(h,x) \mapsto q_h$ *which is smooth on the domain* $\{(h,x), h \in (0,h_M), x \in F_h\}$ *and which satisfies :*

- *there exists a constant positive constant* C *such that :*

$$\left| \int_{\partial\Omega} (2D(\varphi_h) - q_h I) n \cdot \varphi_h d\sigma \right| \leq C|\ln h|;$$

- *there exists a positive constant* C *such that, for any* $h \in (0,h_M)$ *and* $v \in H^1(\Omega)$ *satisfying* $v_{|S_h} \in \mathcal{R}$ *and* $v \cdot n = 0$ *on* $\partial\Omega$*, we have:*

$$\left| \int_{F_h} (\Delta\varphi_h - \nabla q_h) v \right| \leq C \left(\|D(v)\|_{L^2(F_h)} + \|v\|_{L^2(\Omega)} \right). \tag{4.72}$$

More details concerning these lemmas are provided in Appendix B.

Conclusion. Introducing the results of Lemma 4.2 and Lemma 4.3, we compute again the right-hand side of (4.55) with the help of energy estimates satisfied by u. We only point out two differences. First, there exists a universal constant C so that

$$\|u_F\|_{L^2(\partial S(t))} = \|u_S\|_{L^2(\partial S(t))} \leq C\|u_S\|_{L^2(S(t))} \leq C\|u\|_{L^2(\Omega)},$$

by taking into account that u_S is a rigid vector field. Second, one term requires a new treatment (we drop summation over index $i \in \{1,2,3\}$ for simplicity):

$$\begin{aligned}
& \int_{F(t)} u_F^i u_F^3 (\partial_i \varphi_F^3 + \partial_3 \varphi_F^i) \\
= & \int_{F(t)\cap\Omega_{h(t),2\delta}} \chi \frac{u_F^3}{\gamma_s(r)+h} u_F^i (\gamma_s(r)+h)(\partial_i \varphi_F^3 + \partial_3 \varphi_F^i) + O(\|u_F\|_{L^2}^2) \\
\leq & \|u_F^i(\gamma_s(r)+h)(\partial_i \varphi_F^3 + \partial_3 \varphi_F^i)\|_{L^2(F(t)\cap\Omega_{h(t),2\delta})} \|\frac{u_F^3}{\gamma_s(r)+h}\|_{L^2(F(t)\cap\Omega_{h(t),2\delta})} + O(\|u_F\|_{L^2}^2) \\
\leq & C\|D(u_F)\|_{L^2(F(t))} \|\chi u_F^i(\gamma_s(r)+h)(\partial_i \varphi_F^3 + \partial_3 \varphi_F^i)\|_{L^2(F(t)\cap\Omega_{h(t),2\delta})} + O(\|u_F\|_{L^2(F(t))}^2).
\end{aligned}$$

where χ is given by (3.32). Because of (4.69), there holds

$$\begin{aligned}
& \|\chi u_F^i(\gamma_s(r)+h)(\partial_i\varphi_F^3+\partial_3\varphi_F^i)\|^2_{L^2(F(t)\cap\Omega_{h,2\delta})} \\
\leq\ & \int_{F(t)} |u_F^i|^2\cdot\frac{\chi}{\gamma_s(r)+h} \\
=\ & \int_{F(t)} (\partial_z u_F^i+\partial_i u_F^3)\chi u_F^i\frac{z-\gamma_s(r)-h}{\gamma_s(r)+h} - \int_{F(t)} \chi u_F^i\partial_i u_F^3\frac{z-\gamma_s(r)-h}{\gamma_s(r)+h} \\
& +O(\|u_F\|^2_{L^2(F(t))}+\|u_F\|^2_{L^2(\partial\Omega)}) \\
=\ & \int_{F(t)} \frac{u_F^3}{\gamma_s(r)+h}u_F^i\chi(\gamma_s(r)+h)\partial_i\left(\frac{z-\gamma_s(r)-h}{\gamma_s(r)+h}\right) \\
& +O\Big(\|u_F\|^2_{L^2(F(t))}+\|u_F\|^2_{L^2(\partial\Omega)}+\|D(u_F)\|_{L^2(F(t))}\|u_F\|_{L^2(F(t))}\Big) \\
=\ & O\Big(\|u_F\|^2_{L^2(F(t))}+\|u_F\|^2_{L^2(\partial\Omega)}+\|D(u_F)\|_{L^2(F(t))}\|u_F\|_{L^2(F(t))}\Big).
\end{aligned}$$

which implies that

$$\left|\int_{F(t)} u_F^i u_F^3(\partial_i\varphi_F^3+\partial_3\varphi_F^i)\right| = O\left(\|D(u_F)\|_{L^2(F(t))}(\|D(u_F)\|+\|u_F\|_{L^2(\partial\Omega)})+\|u_F\|^2_{L^2(F(t))}\right)$$

Combining with estimates similar to the previous section for the other terms, we arrive at:

$$\left|\mu_F\int_{h(0)}^{h(T)} n(s)\mathrm{d}s\right| \leq C_0(1+T). \tag{4.73}$$

Introducing the bounds of Lemma 4.2 and Lemma 4.3 into (3.20), we show there exists a constant C such that:

$$n(h)\geq\frac{C}{h}\,,\quad \forall\, h\in(0,h_M),$$

so that (4.73) yields finally :

$$|\ln(h(T))|\leq C_0(1+T)\,,\quad \forall\, T>0.$$

This ends the proof.

Acknowledgements. The last author is supported by the Project Instabilités hydrodynamiques funded partially by Mairie de Paris (program Emergence(s)) and Fondation Sciences Mathématiques de Paris.

References

[1] G. Barnocky and R. H. Davis. The influence of pressure-dependent density and viscosity on the elastohydrodynamic collision and rebound of two spheres. *J. Fluid Mech.*, 209:501–519, 1989.

[2] L. Bocquet and J. Barrat. Flow boundary conditions from nano-to micro-scales. *Soft Matter*, 3:985–693, 2007.

[3] D. Bresch, B. Desjardins, and D. Gerard-Varet. On compressible Navier-Stokes equations with density dependent viscosities in bounded domains, *J. Math. Pures Appl.(9)*, 87(2) : 227–235. 2007.

[4] C. Conca, J. A. San Martín, and M. Tucsnak. Existence of solutions for the equations modelling the motion of a rigid body in a viscous fluid. *Comm. Partial Differential Equations*, 25(5-6):1019–1042, 2000.

[5] M. Cooley and M. O'Neill. On the slow motion generated in a viscous fluid by the approach of a sphere to a plane wall or stationary sphere. *Mathematika*, 16:37–49, 1969.

[6] R.G. Cox. The motion of suspended particles almost in contact. *Int. J. Multiphase Flow*, 1:343–371,1974.

[7] P. Cumsille and T. Takahashi. Wellposedness for the system modelling the motion of a rigid body of arbitrary form in an incompressible viscous fluid. *Czechoslovak Math. J.*, 58(133)(4):961–992, 2008.

[8] R. H. Davis, J.-M. Serayssol, and E. Hinch. The elastohydrodynamic collision of two spheres. *J. Fluid Mech.*, 163:479–497, 1986.

[9] B. Desjardins and M. J. Esteban. Existence of weak solutions for the motion of rigid bodies in a viscous fluid. *Arch. Ration. Mech. Anal.*, 146(1):59–71, 1999.

[10] B. Desjardins and M. J. Esteban. On weak solutions for fluid-rigid structure interaction: compressible and incompressible models. *Comm. Partial Differential Equations*, 25(7-8):1399–1413, 2000.

[11] E. Feireisl. On the motion of rigid bodies in a viscous incompressible fluid. *J. Evol. Equ.*, 3(3):419–441, 2003. Dedicated to Philippe Bénilan.

[12] G. P. Galdi and A. L. Silvestre. Strong solutions to the Navier-Stokes equations around a rotating obstacle. *Arch. Ration. Mech. Anal.*, 176(3):331–350, 2005.

[13] D. Gérard-Varet and M. Hillairet. Regularity issues in the problem of fluid structure interaction. *Arch. Ration. Mech. Anal.*, 195(2):375–407, 2010.

[14] D. Gérard-Varet and M. Hillairet. Computation of the drag force on a sphere close to a wall: the roughness issue. *ESAIM Math. Model. Numer. Anal.*, 46(5):1201–1224, 2012.

[15] D. Gérard-Varet and M. Hillairet. Existence of weak solutions up to collision for viscous fluid-solid systems with slip. To appear in *Comm. Pure Appl. Math.*, available at `http://hal.archives-ouvertes.fr/hal-00713331`, 2012.

[16] O. Glass and F. Sueur. Uniqueness results for weak solutions of two-dimensional fluid-solid systems. arXiv:1203.2894v1, March 2012.

[17] C. Grandmont and Y. Maday. Existence de solutions d'un problème de couplage fluide-structure bidimensionnel instationnarie. *C. R. Acad. Sci. Paris Sér. I Math.*, 326(4):525–530, 1998.

[18] M. D. Gunzburger, H.-C. Lee, and G. A. Seregin. Global existence of weak solutions for viscous incompressible flows around a moving rigid body in three dimensions. *J. Math. Fluid Mech.*, 2(3):219–266, 2000.

[19] M. Hillairet. Lack of collision between solid bodies in a 2D incompressible viscous flow. *Comm. Partial Differential Equations*, 32(7-9):1345–1371, 2007.

[20] M. Hillairet, A. Lozinski, and M. Szopos. Simulation of particulate flow governed by lubrication forces and far-field hydrodynamic interactions,. *Discrete and Continuous Dynamical Systems, Series B*, 11:935 – 956, 2011.

[21] M. Hillairet and T. Takahashi. Collisions in three-dimensional fluid structure interaction problems. *SIAM J. Math. Anal.*, 40(6):2451–2477, 2009.

[22] M. Hillairet and T. Takahashi. Blow up and grazing collision in viscous fluid solid interaction systems. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 27(1):291–313, 2010.

[23] L. Hocking. The effect of slip on the motion of a sphere close to a wall and of two adjacent sheres. *J. Eng. Mech.*, 7:207–221, 1973.

[24] K.-H. Hoffmann and V. Starovoitov. On a motion of a solid body in a viscous fluid. Two-dimensional case. *Adv. Math. Sci. Appl.*, 9(2):633–648, 1999.

[25] K.-H. Hoffmann and V. Starovoitov. Zur Bewegung einer Kugel in einer zähen Flüssigkeit. *Doc. Math.*, 5:15–21 (electronic), 2000.

[26] G. Joseph. *Collisional dynamics of macroscopic particles in a viscous fluid*. PhD thesis, California Institute of Technology, Pasadena, California, May 2003.

[27] H. Kytömaa and P. Schmid. On the collision of rigid spheres in a weakly compressible fluid. *Phys. Fluids A*, 4:2683–2689, 1992.

[28] E. Lauga, M. Brenner, and H. Stone. Microfluidics: The no-slip boundary condition. 2007.

[29] J. San Martín, V. Starovoitov, and M. Tucsnak. Global weak solutions for the two-dimensional motion of several rigid bodies in an incompressible viscous fluid. *Arch. Ration. Mech. Anal.*, 161(2):113–147, 2002.

[30] D. Serre. Chute libre d'un solide dans un fluide visqueux incompressible. Existence. *Japan J. Appl. Math.*, 4(1):99–110, 1987.

[31] F. Sueur and G. Planas. On the inviscid limit of the system "viscous incompressible fluid + rigid body" with the Navier conditions. arXiv:1206.0029, May 2012.

[32] T. Takahashi. Analysis of strong solutions for the equations modeling the motion of a rigid-fluid system in a bounded domain. *Adv. Differential Equations*, 8(12):1499–1532, 2003.

[33] T. Takahashi. Existence of strong solutions for the problem of a rigid-fluid system. *C. R. Math. Acad. Sci. Paris*, 336(5):453–458, 2003.

[34] T. Takahashi and M. Tucsnak. Global strong solutions for the two-dimensional motion of an infinite cylinder in a viscous fluid. *J. Math. Fluid Mech.*, 6(1):53–77, 2004.

A Estimates on the test-function in the slip case

In this section, we collect estimates on the test-function of Section 3 as defined in (3.33). For further computations, we define,

$$\Psi(r,z) := \Phi\left(r, \frac{z}{h+\gamma_S(r)}\right), \quad \forall\, h \in (0,h_M), \quad \forall\,(r,z) \in \Omega_{h,1}\,.$$

In particular, we note that $\Psi(r, h+\gamma_S(r)) = 1$ so that:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{r^2}{2}\Psi(r,h+\gamma_S(r))\right] &= r\\ &= \partial_r\left(\frac{r^2}{2}\Psi\right)(r,h+\gamma_S(r)) + \frac{r}{\sqrt{1-r^2}}\frac{r^2}{2}\partial_z\Psi(r,h+\gamma_S(r)).\end{aligned}$$

Thus, on $\partial S_h \cap \partial\Omega_{h,1}$, there holds :

$$\begin{aligned}n\cdot\tilde{\varphi}_h &= \frac{r^2}{2}\partial_z\Psi(r,h+\gamma_S(r)) + \frac{\sqrt{1-r^2}}{r}\partial_r\left(\frac{r^2}{2}\Psi\right)(r,h+\gamma_S(r)),\\ &= \sqrt{1-r^2} = e_3\cdot n\,.\end{aligned}$$

Before going to estimates on φ_h, we first obtain a precise decription of the size of Ψ and its derivatives. Here and in what follows we use $a \lesssim b$ to denote that $a \le Cb$ for a constant C independant of relevant parameters.

Proposition A.1. *The following inequalities hold true, for all values of $h \in (0,h_M)$ and $(r,z) \in \Omega_{h,\delta}$:*

$$|\Psi(r,z)| \lesssim 1\,, \tag{A.74}$$

$$|\partial_z\Psi(r,z)| \lesssim \frac{1}{h+\gamma_S(r)}\,, \qquad |\partial_r\Psi(r,z)| \lesssim \frac{r}{h+\gamma_S(r)}\,, \tag{A.75}$$

$$|\partial_{rr}\Psi(r,z)| \lesssim \frac{1}{h+\gamma_S(r)}\,, \qquad |\partial_{zz}\Psi(r,z)| \lesssim \frac{1}{h+\gamma_S(r)}\,, \tag{A.76}$$

$$|\partial_{rz}\Psi(r,z) + \frac{r}{h+\gamma_S(r)}\partial_z\Psi(r,z)| \lesssim \frac{r}{h+\gamma_S(r)}\,, \tag{A.77}$$

$$|\partial_{zzz}\Psi(r,z)| + |\partial_{rrz}\Psi(r,z)| \lesssim \frac{1}{(h+\gamma_S(r))^2}\,, \tag{A.78}$$

$$|\partial_{rzz}\Psi(r,z)| \lesssim \frac{r}{(h+\gamma_S(r))^2}, \qquad |\partial_{rrr}\Psi(r,z)| \lesssim \frac{r}{(h+\gamma_S(r))^2} + \frac{1}{h+\gamma_s(r)} \tag{A.79}$$

and also

$$
\begin{aligned}
|\partial_h \Psi(r,z)| &\lesssim \frac{1}{h+\gamma_S(r)}\,, && \text{(A.80)}\\
|\partial_{rh} \Psi(r,z)| &\lesssim \frac{1}{(h+\gamma_S(r))} + \frac{r}{(h+\gamma_S(r))^2}\,, && \text{(A.81)}\\
|\partial_{zzh} \Psi(r,z)| + |\partial_{rrh} \Psi(r,z)| + |\partial_{hz} \Psi(r,z)| &\lesssim \frac{1}{(h+\gamma_S(r))^2}\,, && \text{(A.82)}\\
|\partial_{rzh} \Psi(r,z)| &\lesssim \frac{1}{(h+\gamma_S(r))^2} + \frac{r}{(h+\gamma_S(r))^3}\,. && \text{(A.83)}
\end{aligned}
$$

Proof. At first, we note that

$$
\begin{aligned}
|\partial_r \alpha_S| + |\partial_r \alpha_P| &\lesssim r,\\
|\partial_{rrr} \alpha_S| + |\partial_{rr} \alpha_S| + |\partial_{rrr} \alpha_P| + |\partial_{rr} \alpha_P| &\lesssim 1,
\end{aligned}
$$

This yields, by (3.21)-(3.25), that we have:

$$
|\partial_r \Phi(r,t)| + |\partial_{rtt} \Phi(t,r)| + |\partial_{tr} \Phi(t,r)| \lesssim r\,, \quad \forall\,(r,t) \in (0,\delta) \times (0,1)\,.
$$

and

$$
|\partial_t \Phi(r,t)| + |\partial_{rrr} \Phi(t,r)| + |\partial_{rrt} \Phi(t,r)| + |\partial_{rr} \Phi(t,r)| \lesssim 1\,.
$$

We remark also that

$$
|\partial_{ttt} \Phi(t,r)| + |\partial_{tt} \Phi(t,r)| \lesssim (\alpha_P + \alpha_S) \lesssim (h + \gamma_S(r)).
$$

We introduce these bounds implicitly in the following computations.

For instance, we have:

$$
\begin{aligned}
\partial_r \Psi(r,z) &= \partial_r \Phi\left(r, \frac{z}{h+\gamma_S(r)}\right) - \frac{z\gamma_S'(r)}{(h+\gamma_S(r))^2} \partial_t \Phi\left(r, \frac{z}{(h+\gamma_S(r)}\right)\\
&\lesssim \frac{r}{(h+\gamma_S(r))}\,.
\end{aligned}
$$

We obtain similarly (A.75)–(A.79) after tedious calculations. The only computation we make precise is the one concerning $\partial_{rz}\Psi$. Differentiating the last equality w.r.t. z, we get :

$$
\partial_{rz} \Psi(r,z) = \frac{1}{h+\gamma_S(r)} \partial_{rt} \Phi - \frac{\gamma_S'(r)}{(h+\gamma_S(r))^2} \partial_t \Phi - \frac{z\gamma_S'(r)}{(h+\gamma_S(r))^3} \partial_{tt} \Phi,
$$

where :

$$
\partial_z \Psi(r,z) = \frac{1}{(h+\gamma_S(r))} \partial_t \Phi\,.
$$

Thus,

$$
\begin{aligned}
\partial_{rz} \Psi(r,z) + \frac{r}{h+\gamma_S(r)} \partial_z \Psi(r,z) &= \frac{1}{(h+\gamma_S(r))} \partial_{rt} \Phi + \frac{r-\gamma_S'(r)}{(h+\gamma_S(r))^2} \partial_t \Phi\\
&- \frac{z\gamma_S'(r)}{(h+\gamma_S(r))^3} \partial_{tt} \Phi\,.
\end{aligned}
$$

where we dominate :

$$\left|r-\gamma_S'(r)\right| \lesssim r^3\,.$$

Combining with the previous estimates on the derivative of Φ, we obtain the expected result.

Concerning derivatives w.r.t. h the computations are similar. First, we note that :

$$|\partial_r^k\partial_h\alpha_S(r)|+|\partial_r^k\partial_h\alpha_P(r)|\lesssim 1\,,\quad \forall\, k\in\mathbb{N}\,.$$

Correspondingly, we obtain that

$$|\partial_r^k\partial_t^l\partial_h\Phi|\lesssim 1\,,\quad \forall (k,l)\in\mathbb{N}^2\,.$$

Then, we obtain, for instance :

$$\begin{aligned}\partial_{rh}\Psi(r,z) &= \partial_{rh}\Phi-\frac{z}{(h+\gamma_S(r))^2}\partial_{rt}\Phi+2\frac{z\gamma_S'(r)}{(h+\gamma_S(r))^3}\partial_t\Phi\\ &- \frac{z\gamma_S'(r)}{(h+\gamma_S(r))^2}\partial_{th}\Phi+\frac{z^2\gamma_S'(r)}{(h+\gamma_S(r))^4}\partial_{tt}\Phi\end{aligned}$$

in which, we plug previous estimates on Φ and its derivative. This yields :

$$|\partial_{rh}\Psi(r,z)|\lesssim\frac{1}{(h+\gamma_S(r))}+\frac{r}{(h+\gamma_S(r))^2}\,.$$

Other estimates are obtained similarly. □

The rest of this appendix is devoted to the proofs of lemmas 3.2 and 3.3. They rely on the following lemma whose proof can be found in [13]:

Lemma A.2. *Given $(p,q)\in(0,\infty)^2$, when h goes to 0, the quantity*

$$\int_0^\delta\frac{|x|^p dx}{(h+|x|^2)^q}$$

behaves like
(1) $h^{\frac{p+1}{2}-q}$, if $p+1<2q$,
(2) $\ln h$, if $p+1=2q$,
(3) C, if $p+1>2q$,
with C a constant depending on (p,q).

A.1 Proof of Lemma 3.2

First, we recall that, by construction φ_h is smooth outside $\Omega_{h,\delta}$. Thus we have get that

$$\|\nabla^k\varphi_h\|_{L^\infty(F_h\setminus\Omega_{h,\delta})}\lesssim 1\,,\quad\forall\, k\in\mathbb{N}\,.$$

Hence, we focus on $\Omega_{h,\delta}$ and replace φ_h with $\widetilde{\varphi}_h$ in what remains. We keep notations of the previous lemma. In particular, we have by definition :

$$\widetilde{\varphi}_h=\frac{1}{2}\left(-\partial_z\Psi(r,z)re_r+\frac{1}{r}\partial_r\left[r^2\Psi(r,z)\right]e_z\right),\qquad\text{in }\Omega_{h,\delta}.$$

Sobolev bounds. First, applying the bounds of the previous lemma, we get:

$$\begin{aligned}\|\widetilde{\varphi}_h\|_{L^2(\Omega_{h,\delta})} &\lesssim \|r\partial_z\Psi\|_{L^2(\Omega_{h,\delta})} + \|\frac{1}{r}\partial_r\left[r^2\Psi(r,z)\right]\|_{L^2(\Omega_{h,\delta})} \\ &\lesssim \left[1+\int_0^\delta \frac{r^3\mathrm{d}r}{(r^2+h)}\right]^{\frac{1}{2}} \lesssim 1.\end{aligned}$$

Then, in cylindrical coordinates, we obtain:

$$|\nabla\widetilde{\varphi}_h| \lesssim |\partial_z\Psi| + |r\partial_{zz}\Psi| + |\partial_r\Psi| + |r\partial_{rr}\Psi| + \frac{r^2}{h+\gamma_s(r)}, \tag{A.84}$$

where we used (A.77) to get rid of $\partial_{rz}\Psi$ terms. Consequently :

$$\|\nabla\widetilde{\varphi}_h\|^2_{L^2(\Omega_{h,\delta})} \lesssim 1 + \int_0^\delta \frac{r\mathrm{d}r}{(h+\gamma_S(r))} \lesssim |\ln(h)|\,.$$

On the other hand, we have

$$\|\partial_2\varphi_h^2\|^2_{L^2(\Omega_{h,\delta})} = \frac{1}{4}\left\|\left(1-\frac{r^2|\sin(\theta)|^2}{(h+\gamma_S(r))^2}\right)\partial_z\Psi(r,z)\right\|^2_{L^2(\Omega_{h,\delta})} + O(1)$$

Let now $\varepsilon > 0$ small enough. There holds:

$$\begin{aligned}\|\partial_2\varphi_h^2\|^2_{L^2(\Omega_{h,\delta})} &\geq \frac{1}{8}\|\partial_z\Psi(r,z)\|^2_{L^2(\Omega_{h,\delta})\cap\{|\theta|<\varepsilon\}} + O(1) \\ &\geq c\int_0^1\int_0^\delta |\partial_t\Phi(r,t)|^2\frac{r\mathrm{d}r\,\mathrm{d}t}{(h+\gamma_S(r))} + O(1).\end{aligned}$$

where $c = c_\varepsilon$ is a small positive constant. Finally, as $\partial_t\Phi$ is bounded from below, we get

$$\|\partial_2\varphi_h^2\|^2_{L^2(\Omega_{h,\delta})} \geq c'\int_0^1\int_0^\delta \frac{r\mathrm{d}r\,\mathrm{d}t}{(h+\gamma_S(r))} + O(1) \geq c|\ln h|.$$

Combining the above estimate yields :

$$c|\ln h| \leq \|D(\varphi_F)\|^2_{L^2(F(t))} \leq \|\nabla\varphi_F\|^2_{L^2(F(t))} \leq C|\ln h|.$$

This completes the proof of (3.36)-(3.38).

Boundary estimates. We proceed with estimate (3.39). By standard identities of differential geometry (see [3]), we have

$$2D(\widetilde{\varphi}_h)n\times n = \partial_n\widetilde{\varphi}_h\times n + (\widetilde{\varphi}_h - e_3)\times n.$$

Thus, we only need to prove that

$$\|\beta_S\partial_n\widetilde{\varphi}_h\times n + (\beta_S+1)\widetilde{\varphi}_h\times n\|_{L^2(\partial S_h\cap\partial\Omega_{h,2\delta})} \leq C.$$

In the aperture $\Omega_{h,2\delta}$ we recall that the normal to ∂S_h directed toward S_h satisfies $n = -re_r + ((1+h)-z)e_z$. This yields:

$$\begin{aligned} 2\widetilde{\varphi}_h \times n &= \left(-\partial_z\Psi\ re_r + \frac{1}{r}\partial_r\left[r^2\Psi\right]e_z\right) \times (-re_r + ((1+h)-z)e_z)\,, \\ &= \left(r((1+h)-z)\partial_z\Psi - \partial_r\left[r^2\Psi\right]\right)e_\theta\,. \end{aligned}$$

Note here that, on ∂S_h, we have $(1+h)-z-1 \lesssim r^2$ and that

$$\partial_r\left[r^2\Psi\right] \lesssim r = O(1)\,.$$

Here and in the end of this proof $O(1)$ means bounded in $L^2(\partial S_h \cap \partial\Omega_{h,2\delta})$. This yields:

$$\widetilde{\varphi}_h \times n = \frac{r}{2}\partial_z\Psi e_\theta + O(1)\,.$$

On the other hand, in the aperture $\Omega_{h,\delta}$ we have:

$$\begin{aligned} 2\,\partial_n\widetilde{\varphi}_h &= (r\partial_{rz}(\Psi r) - ((1+h)-z)r\partial_{zz}\Psi)\,e_r \\ &+ \left(-r\partial_r\left[\frac{1}{r}\partial_r[r^2\Psi]\right] + ((1+h)-z)\partial_z\left[\frac{1}{r}\partial_r[r^2\Psi]\right]\right)e_z. \end{aligned}$$

Introducing the estimates of Proposition A.1 we get in particular:

$$\begin{aligned} \partial_z\left[\frac{1}{r}\partial_r[r^2\Psi]\right] &= 2\partial_z\Psi + r\partial_{rz}\Psi \\ &= \left(2 - \frac{r^2}{h+\gamma_S(r)}\right)\partial_z\Psi + O(1) = \frac{2h}{h+\gamma_s(r)}\partial_z\Psi + O(1). \end{aligned}$$

Similarly, we have:

$$r\partial_{rz}(\Psi r) = r^2\partial_{rz}\Psi + r\partial_z\Psi = -r\partial_z\Psi + r\left(-\frac{r^2}{h+\gamma_s(r)} + 2\right)\partial_z\Psi + O(1) = -r\partial_z\Psi + O(1),$$

using that

$$r\left(-\frac{r^2}{h+\gamma_s(r)} + 2\right)\partial_z\Psi = O\left(\frac{rh}{(h+\gamma_s(r))^2}\right) \tag{A.85}$$

is uniformly bounded in L^2 (use case 3 of Lemma A.2 with $p=3$, $q=4$). Then,

$$2\partial_n\widetilde{\varphi}_h = -(r\partial_z\Psi + r\partial_{zz}\Psi)e_r\ +\ \frac{2h}{h+\gamma_s(r)}\partial_z\Psi e_z\ +\ O(1).$$

Recalling that $n = -re_r + ((1+h)-z)e_z$, and still applying (A.85), we obtain:

$$\partial_n\widetilde{\varphi}_h \times n = \frac{r}{2}\left(\partial_{zz}\Psi + \partial_z\Psi\right)e_\theta + O(1).$$

Finally, we obtain:

$$\beta_S\partial_n\widetilde{\varphi}_h \times n + (\beta_S+1)\tilde{\varphi}_h \times n = \frac{r\beta_S}{2}\left(\partial_{zz}\Psi + \left(\frac{1}{\beta_S}+2\right)\partial_z\Psi\right) + O(1)\,.$$

By our choice of α_S, it follows that (see (3.29))

$$\partial_{zz}\Psi(r, h+\gamma_S(r)) + \left(\frac{1}{\beta_S}+2\right)\partial_z\Psi(r, h+\gamma_S(r)) = O(1).$$

This yields (3.39).

h-derivatives. For $\partial_h \widetilde{\varphi}_h$, we note that, for $i = 1, 2, 3$ we have:

$$|\partial_h \widetilde{\varphi}_h^i| \lesssim |r\partial_{zh}\Psi| + |\partial_h \Psi| + |r\partial_{rh}\Psi| \lesssim \frac{r}{(h+\gamma_S(r))^2} + \frac{1}{(h+\gamma_S(r))}\,.$$

Consequently, as integrating with respect to the z-variable reduces to remove one power of $(h+\gamma_S(r))$ in the denominator, we get

$$\|\int_z^{h+\gamma_s(r)} \partial_h \widetilde{\varphi}_F^i ds\|^2_{L^2(F_h\cap\Omega_{h,\delta})} \lesssim 1 + \int_0^\delta \frac{r^3 \mathrm{d}r}{(h+\gamma_S(r))} \lesssim 1\,.$$

and

$$\|\int_0^{h+\gamma_s(r)} \partial_h \widetilde{\varphi}_F^i dz\|^2_{L^2(\partial\Omega\cap\partial\Omega_{h,\delta})} \lesssim 1 + \int_0^\delta \frac{r^3 \mathrm{d}r}{(h+\gamma_S(r))^2} \lesssim C|\ln h|\,.$$

Finally, we also compute: for $i = 1, 2$:

$$\begin{aligned} |\partial_i \partial_h \widetilde{\varphi}_h| &\lesssim |\partial_{zh}\Psi| + |r\partial_{rzh}\Psi| + |\partial_{rh}\Psi| + |r\partial_{rrh}\Psi| \\ &\lesssim \frac{1}{(h+\gamma_S(r))^2} \end{aligned}$$

so that for the same reason as above, we have:

$$|(h+\gamma_s(r))\partial_i \int_z^{h+\gamma_s(r)} \partial_h \widetilde{\varphi}_h^i(x_1, x_2, s) ds| \lesssim 1\,, \qquad \text{for } i = 1, 2$$

and

$$\|(h+\gamma_s(r))\partial_i \int_z^{h+\gamma_s(r)} \partial_h \widetilde{\varphi}_F^i(x_1, x_2, s) ds\|_{L^2(F_h\cap\Omega_{h,\delta})} \lesssim 1, \qquad \text{for } i = 1, 2, 3\,.$$

This ends the proof of (4.66)–(4.68).

Pointwise bounds. We easily get that on $\Omega_{h,\delta}$

$$|\nabla\varphi_h| \lesssim |r\partial_{rz}\Psi| + |\partial_z\Psi| + |r\partial_{rr}\Psi| + |\partial_r\Psi| + |r\partial_{zz}\Psi| \lesssim \frac{1}{h+\gamma_S(r)},$$

from which we conclude:

$$\left|\int_z^{h+\gamma_S(r)} (\partial_i \varphi_h^j(x_1, x_2, s) + \partial_j \varphi_h^i(x_1, x_2, s)) ds\right| \lesssim 1, \text{ for } (i, j) \in \{1, 2, 3\}^2.$$

Similarly, we get:

$$|\partial_1\nabla\varphi_h| + |\partial_2\nabla\varphi_h| \lesssim |r\partial_{rrr}\Psi| + |\partial_{rr}\Psi| + |r\partial_{rrz}\Psi| + |\partial_{rz}\Psi| + |\partial_{zz}\Psi| \lesssim \frac{r}{(h+\gamma_S(r))^2} + \frac{1}{h+\gamma_s}.$$

and we conclude:

$$\left|(h+\gamma_S(r))\partial_i \int_z^{h+\gamma_S(r)} (\partial_i \varphi_F^j(x_1, x_2, s) + \partial_j \varphi_F^i(x_1, x_2, s)) ds\right| \lesssim 1.$$

This ends the proof of our lemma.

A.2 Proof of Lemma 3.3

Boundary integrals We focus on the domain $\Omega_{h,\delta}$ and we replace φ_h with $\widetilde{\varphi}_h$ in the proof because everything is bounded elsewhere. First, it turns out that the first identity is independant of q_h. Indeed, applying boundary conditions (3.31) we obtain :

$$\begin{aligned}\int_{\partial\Omega_{h,\delta}\cap\partial\Omega}(2D(\widetilde{\varphi}_h)-q_h)n\cdot\widetilde{\varphi}_h\mathrm{d}\sigma &= \int_{\partial\Omega_{h,\delta}\cap\partial\Omega}[(2D(\widetilde{\varphi}_h)n)\times n]\cdot(\widetilde{\varphi}_h\times n)\mathrm{d}\sigma\\ &= -\frac{1}{\beta_\Omega}\int_{\partial\Omega_{h,\delta}\cap\partial\Omega}|\widetilde{\varphi}_h\times n|^2\mathrm{d}\sigma\,.\end{aligned}$$

Similarly, introducing boundary conditions (3.31) and estimate (3.39), we compute:

$$\begin{aligned}\int_{\partial\Omega_{h,\delta}\cap\partial S_h}(2D(\widetilde{\varphi}_h)-q_h)n\cdot(e_3-\widetilde{\varphi}_h)\mathrm{d}\sigma &= \int_{\partial\Omega_{h,\delta}\cap\partial S_h}[(2D(\widetilde{\varphi}_h)n)\times n]\cdot((e_3-\widetilde{\varphi}_h)\times n)\mathrm{d}\sigma\\ &= \frac{1}{\beta_S}\int_{\partial\Omega_{h,\delta}\cap\partial S_h}|(e_3-\widetilde{\varphi}_h)\times n|^2\mathrm{d}\sigma+O(\|(e_3-\widetilde{\varphi}_h)\times n\|_{L^2(\partial S_h\cap\Omega_{h,\delta})})\,.\end{aligned}$$

On the boundaries, there holds

$$\widetilde{\varphi}_h=-\frac{P_1(r)}{(h+\gamma_S(r))}re_r\quad\text{in}\quad\partial\Omega_{h,\delta}\cap\partial\Omega,\quad|\widetilde{\varphi}_h|\lesssim\frac{r}{(h+\gamma_S(r))}\quad\text{in}\quad\partial\Omega_{h,\delta}\cap\partial S_h,$$

which implies that

$$|\ln(h)|\lesssim\int_0^\delta\frac{r^3dr}{(h+\gamma_S(r))^2}\lesssim\|\widetilde{\varphi}_h\times n\|^2_{L^2(\partial\Omega_{h,\delta}\cap\partial\Omega)}=\|\widetilde{\varphi}_h\|^2_{L^2(\partial\Omega_{h,\delta}\cap\partial\Omega)}$$

and which implies that

$$|\ln h|\lesssim\int_{\partial\Omega_{h,\delta}\cap\partial S_h}(2D(\widetilde{\varphi}_h)-q_h)n\cdot(e_3-\widetilde{\varphi}_h)\mathrm{d}\sigma-\int_{\partial\Omega\cap\partial\Omega_{h,\delta}}(D(\widetilde{\varphi}_h)-q_hI)n\cdot\varphi_hd\sigma\lesssim|\ln(h)|.$$

Construction of q_h***.*** Again, we focus on the domain $\Omega_{h,\delta}$ where $\varphi_h=\widetilde{\varphi}_h$ to compute q_h. The construction is extended to the whole F_h by a standard truncation argument.

Given $h>0$, let define

$$2q_h\triangleq-\left(r\partial_{rz}\Psi+2\partial_z\Psi+\int_0^r\partial_{zzz}\Psi(s,z)sds\right).$$

Obviously, the map $(h,x)\mapsto q_h(x)$ is C^1 on $\{h\in(0,h_M),\ x\in F_h\}$. and we have $\Delta\widetilde{\varphi}_h-\nabla q_h=f_he_3$ where

$$|f_h|\lesssim|r\partial_{rrr}\Psi|+|\partial_{rr}\Psi|+\left|\frac{\partial_r\Psi}{r}\right|+|r\partial_{rzz}\Psi|+|\partial_{zz}\Psi|\lesssim\frac{1}{(h+\gamma_s(r))}$$

Then, for arbitrary $h\in(0,h_M)$ and $v\in H^1(F_h)$ we introduce χ the truncation function as defined in (3.32) and we split :

$$\int_{F_h}(\Delta\varphi_h-\nabla q_h)v=\int_{F_h}\chi(\Delta\varphi_h-\nabla q_h)v+\int_{F_h}(1-\chi)(\Delta\varphi_h-\nabla q_h)v=I+II.$$

Then, we have:

$$\begin{aligned} |I| &= \int_{F_h\cap\Omega_{h,2\delta}} |\chi f_h||v_3| \\ &\leq \|(h+\gamma_S)f_h\|_{L^2(F_h\cap\Omega_{h,2\delta})}\|\frac{v_3}{\gamma_s(r)+h}\|_{L^2(F_h\cap\Omega_{h,2\delta})} \end{aligned}$$

where a standard Poincaré inequality yields:

$$\|\frac{v_3}{\gamma_s(r)+h}\|_{L^2(F_h\cap\Omega_{h,2\delta})} \lesssim \|\partial_3 v_3\|_{L^2(F_h\cap\Omega_{h,\delta})} \lesssim \|D(v)\|_{L^2(F_h\cap\Omega_{h,\delta})}.$$

As for the second integral, we introduce P_zR^i for $i=1,2,3$, which satisfy :

$$P_zR^i = \int_z^{h+\gamma_s(r)} (1-\chi)(\Delta\varphi_h^i - \partial_i q_h) \quad \text{for } r<1,\ z<r+\gamma_s(r) \tag{A.86}$$

$$P_zR^i = \int_z^{h+1} (1-\chi)(\Delta\varphi_h^i - \partial_i q_h) \quad \text{otherwise.} \tag{A.87}$$

Repeating the computations of $I_1(t)$ and $I_2(t)$ in Section 3.3, we obtain (the summation over index i is implicit) :

$$|II| = \int_{F_h} (\partial_3 v^i + \partial_i v^3)P_zR^i + \int_{F_h} v^3\partial_i P_zR^i - \int_{\partial\Omega} (v^3n^i + v^in^3)P_zR^i \mathrm{d}\sigma\,.$$

As $(\Delta\varphi_h - \nabla q_h)$ is uniformly bounded outside $\Omega_{h,\delta}$, applying a Cauchy Schwarz inequality to dominate K_2 and combining with K_1 concludes the proof :

$$\left|\int_{F_h} (\Delta\varphi_h - \nabla q_h)\cdot v\right| \lesssim \|D(v)\|_{L^2(F_h)} + \|v\|_{L^2(\partial\Omega)}.$$

B Estimates on the test function in the mixed case

In this appendix, we collect estimates on the test function of Section 4. As previously, we define,

$$\Psi(r,z) := \Phi\left(r, \frac{z}{h+\gamma_S(r)}\right), \quad \forall\, h\in(0,h_M), \quad \forall\,(r,z)\in\Omega_{h,1}\,.$$

The only point which differs from the test-function in the slip case is the decription of this Ψ. Combining these estimates in the same manner as in the previous appendix, we obtain Lemma 4.2. As the computations are completely similar, all the proofs are left to the reader.

Proposition B.1. *The following inequalities hold true, for all $h \in (0, h_M)$ and $(r,z) \in \Omega_{h,\delta}$:*

$$|\Psi(r,z)| \lesssim 1\,, \tag{B.88}$$

$$|\partial_z \Psi(r,z)| \lesssim \frac{1}{h+\gamma_S(r)}\,, \qquad |\partial_r \Psi(r,z)| \lesssim \frac{r}{h+\gamma_S(r)}\,, \tag{B.89}$$

$$|\partial_{rr} \Psi(r,z)| \lesssim \frac{1}{h+\gamma_S(r)}\,, \qquad |\partial_{zz} \Psi(r,z)| \lesssim \frac{1}{(h+\gamma_S(r)^2}\,, \tag{B.90}$$

$$|\partial_{rz} \Psi(r,z)| \lesssim \frac{r}{(h+\gamma_S(r))^2}\,, \qquad |\partial_{zzz} \Psi(r,z)| \lesssim \frac{1}{(h+\gamma_S(r))^3} \tag{B.91}$$

$$|\partial_{rrz} \Psi(r,z)| \lesssim \frac{1}{(h+\gamma_S(r))^2}\,, \qquad |\partial_{rzz} \Psi(r,z)| \lesssim \frac{r}{(h+\gamma_S(r))^3}\,, \tag{B.92}$$

$$|\partial_{rrr} \Psi(r,z)| \lesssim \frac{r}{(h+\gamma_S(r))^2}\,, \qquad |\partial_{rrrz} \Psi(r,z)| \lesssim \frac{r}{(h+\gamma_S(r))^3}\,, \tag{B.93}$$

$$\tag{B.94}$$

and also

$$|\partial_h \Psi(r,z)| \lesssim \frac{1}{h+\gamma_S(r)}\,, \qquad |\partial_{rh} \Psi(r,z)| \lesssim \frac{r}{(h+\gamma_S(r))^2}\,, \tag{B.95}$$

$$|\partial_{rrh} \Psi(r,z)| \lesssim \frac{1}{(h+\gamma_S(r))^2}\,, \qquad |\partial_{hz} \Psi(r,z)| \lesssim \frac{1}{(h+\gamma_S(r))^2} \tag{B.96}$$

$$|\partial_{rzh} \Psi(r,z)| \lesssim \frac{r}{(h+\gamma_S(r))^3} + \frac{1}{(h+\gamma_S(r))^2} \tag{B.97}$$

$$|\partial_{zzh} \Psi(r,z)| \leq \frac{1}{(h+\gamma_S(r))^3}\,. \tag{B.98}$$

We also need a different way to construct the pressure q_h in order to prove Lemma 4.3. Again, we construct q_h only in the aperture domain $\Omega_{h,\delta}$, the extension to F_h being done by a standard truncation. So, for $h > 0$ we set :

$$2q_h \triangleq r\partial_{rz}\Psi + 2\partial_z \Psi - \int_0^r \partial_{zzz}\Psi s \mathrm{d}s\,.$$

In this way, we obtain $\Delta\varphi_h - \nabla q_h = f^r e_r + f^z e_z$ where the f_i's are bounded outside $\Omega_{h,\delta}$ and, in $\Omega_{h,\delta}$, we have:

$$|f^z| \lesssim |r\partial_{rrr}\Psi| + |\partial_{rr}\Psi| + \left|\frac{\partial_r \Psi}{r}\right| \lesssim \frac{1}{(h+\gamma_S(r))} \tag{B.99}$$

$$|f^r| \lesssim |r\partial_{rrz}\Psi| + |\partial_{rz}\Psi| \lesssim \frac{r}{(h+\gamma_S(r))^2} \tag{B.100}$$

$$|\partial_r f^r| + \left|\frac{f^r}{r}\right| \lesssim |r\partial_{rrrz}\Psi| + |\partial_{rrz}\Psi| + \left|\frac{\partial_{rz}\Psi}{r}\right| \lesssim \frac{1}{(h+\gamma_S(r))^2}. \tag{B.101}$$

Then, for any $v \in H^1(\Omega)$ such that $v \cdot n = 0$ on $\partial\Omega$ and $v \in \mathcal{R}$ on S_h we have :

$$\int_{F_h} (\Delta\varphi_h - \nabla q_h) \cdot v = \int_{F_h} f^z v^3 + \int_{F_h} f^r e_r \cdot v.$$

First, we apply Poincaré inequality, as $v^3 = 0$ on $\partial\Omega \cap \partial\Omega_{h,\delta}$, together with (B.99). This yields:

$$\begin{aligned}\left|\int_{F_h} f^z v_3\right| &\lesssim \left|\int_{F_h\cap\Omega_{h,\delta}} f^z(h+\gamma_S(r))\frac{v_3}{(h+\gamma_S(r))}\right| + \|v\|_{L^2(F_h)} \\ &\lesssim \|D(v)\|_{L^2(F_h)} + \|v\|_{L^2(F_h)}\end{aligned}$$

For the second integral: we write $f^r e_r = f^1 e_1 + f^2 e_2$, and split the integral, with χ as defined in (3.32) :

$$\int_{F_h} f^i v_i = \int_{F_h} \chi f^i v^i + O(\|v\|_{L^2(F_h)}),$$

as f^r is uniformly bounded outside $\Omega_{h,\delta}$. Then, we introduce $P_z f^i$, for $i = 1, 2$, which satisfies:

$$P_z f^i = \int_0^z \chi f^i$$

Through integration by parts, we get (with the Einstein convention of repeated indices):

$$\begin{aligned}\int_{F_h} \chi f^i v^i &= \int_{F_h\cap\Omega_{h,2\delta}} P_z f^i(\partial_3 v^i + \partial_i v^3) + \int_{F_h\cap\Omega_{h,2\delta}} \frac{v_3}{h+\gamma_S(r)}(h+\gamma_S(r))\left(\partial_i P_z f^i\right) \\ &\quad - \int_{\partial S_h\cap\Omega_{h,2\delta}} P_z f^i(v^i n^3 + v^3 n^i) \triangleq I + II + III\end{aligned}$$

Thanks to the bounds (B.100)-(B.101), we get:

$$|I| + |II| \lesssim \|D(v)\|_{L^2(F_h)}.$$

As regards the last integral, we simply write, because $v_{|S_h} \in \mathcal{R}$,

$$|III| \lesssim \|P_z f^i\|_{L^1(\partial S_h)} \|v\|_{L^\infty(\partial S_h)} \lesssim \|v\|_{L^2(S_h)} \lesssim \|v\|_{L^2(\Omega)}$$

This ends the proof.