Stokes phenomenon for single-level linear differential systems: a perturbative approach

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Abstract

Given a meromorphic linear differential system with an arbitrary single level $r \geq 1$, we build a regular holomorphic perturbation which preserves the single level and we show that the Stokes-Ramis matrices of the initial system are limits of convenient products of the perturbed ones. As an application, we provide an alternative method for the effective calculation of the Stokes multipliers of the initial system illustrated on two examples. No assumption of genericity is made on the initial system.

Keywords. Linear differential system, regular perturbation, holomorphic perturbation, Stokes phenomenon, summability, Stokes multipliers

AMS subject classification. 34M03, 34M30, 34M40

Introduction

Throughout the paper, we are given a positive integer $r \geq 1$ and we consider a linear differential system (in short, a differential system or a system) of dimension $n \geq 2$ with meromorphic coefficients of order $r+1$ at the origin $0 \in \mathbb{C}$ of the form

$$(A) \qquad x^{r+1}\frac{dY}{dx} = A(x)Y \quad , \ A(x) \in M_n(\mathbb{C}\{x\}), \ A(0) \neq 0$$

together with a formal fundamental solution at 0

$$\widetilde{Y}(x) = \widetilde{F}(x)x^L e^{Q(1/x)}$$

where

- $\widetilde{F}(x) \in GL_n(\mathbb{C}[[x]][x^{-1}])$ is an invertible matrix with formal meromorphic entries in x,
- $L = \bigoplus_{j=1}^{J}(\lambda_j I_{n_j} + J_{n_j})$ where J is an integer ≥ 2, I_{n_j} denotes the identity matrix of size n_j and where

$$J_{n_j} = \begin{cases} 0 & \text{if } n_j = 1 \\ \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{bmatrix} & \text{if } n_j \geq 2 \end{cases}$$

  is an irreductible Jordan block of size n_j,
- $Q\left(\frac{1}{x}\right) = \bigoplus_{j=1}^{J} q_j\left(\frac{1}{x}\right) I_{n_j}$ where the $q_j(1/x)$'s are polynomials of maximal degree equal to r with respect to $1/x$.

In a very general system (A), the $q_j(1/x)$'s may be polynomials in a fractional power in $1/x$. However, they can always be changed into polynomials in the variable $1/x$ itself by means of an adequate finite algebraic extension $x \longmapsto x^{\nu}$, $\nu \in \mathbb{N}^*$, of the variable x. The properties in view in this paper being preserved under such algebraic extensions, we may assume, without any loss of generality, that the $q_j(1/x)$'s read as

$$q_j\left(\frac{1}{x}\right) = -\frac{a_{j,r}}{x^r} - \frac{a_{j,r-1}}{x^{r-1}} - ... - \frac{a_{j,1}}{x} \in x^{-1}\mathbb{C}\left[x^{-1}\right].$$

In addition, we suppose

(0.1) $\widetilde{F}(x) \in M_n(\mathbb{C}[[x]])$ is a formal power series in x satisfying

$$\widetilde{F}(x) = I_n + O(x^r),$$

(0.2) the eigenvalues λ_j satisfy $0 \leq \mathrm{Re}(\lambda_j) < 1$ for all $j = 1, ..., J$,

(0.3) $\lambda_1 = 0$ and $q_1 \equiv 0$.

Such conditions are not restrictive since they can always be fulfilled by means of a meromorphic gauge transformation $Y \longmapsto T(x)x^{-\lambda_1}e^{-q_1(1/x)}Y$ where $T(x)$ has explicit computable polynomial entries in x and $1/x$ (*cf.* [1]). Recall that conditions $\widetilde{F}(0) = I_n$ and $0 \leq \mathrm{Re}(\lambda_j) < 1$ guarantee the unicity of $\widetilde{F}(x)$ as formal series solution of the homological system associated with system (A) (*cf.* [1]). Conditions $\lambda_1 = 0$ and $q_1 \equiv 0$ are for notational convenience.

The assumption "*system (A) has the unique level r*" is equivalent to the conditions

(0.4)
$$\boxed{\begin{array}{l} 1.\ q_j - q_\ell \equiv 0 \text{ or with degree } r \text{ for all } j, \ell \\ \\ 2.\ \text{there exists } j \text{ such that } a_{j,r} \neq 0 \end{array}}.$$

Observe that, all over the article, no restrictive assumption is made except the assumption that the given system (A) has a unique level. In particular, we never assume that the formal monodromy L is diagonal or the Stokes values $a_{j,r}$ are distinct.

In this paper, we are interested in *regular* perturbations of system (A) of the form

$$(A^{\boldsymbol{\varepsilon}}) \qquad x^{r+1}\frac{dY}{dx} = A^{\boldsymbol{\varepsilon}}(x)Y \qquad \text{with } A^{\mathbf{1}}(x) = A(x),$$

where $\boldsymbol{\varepsilon}$ is a holomorphic multi-parameter lying in a polydisc centered at the unit $\mathbf{1} := (1, ..., 1)$ of the $\mathbb{C}$-vector space $\mathbb{C}^{p+1}$ for a convenient $p \geq 1$. Besides, we suppose that, for any value of $\boldsymbol{\varepsilon}$, system $(A^{\boldsymbol{\varepsilon}})$ has, like initial system (A), the unique level r too.

The main goal of this article is to prove that the Stokes-Ramis matrices[1] of initial system (A) are limits of convenient products of the Stokes-Ramis matrices of perturbed systems $(A^{\boldsymbol{\varepsilon}})$.

In a second time, we show how this result allows to build a method for the effective calculation of the Stokes multipliers of initial system (A) and we illustrate it on some examples.

[1] In the whole paper, we call *Stokes matrices* all the matrices providing the transition between any two asymptotic solutions whose domains of definition overlap. The name "*Stokes-Ramis matrix*" used here is reserved, according to the custom initiated by J.-P. Ramis ([5]) in the spirit of Stokes' work, to the matrices providing the transition between the sums on each side of a same anti-Stokes direction. Thereby, a Stokes-Ramis matrix is a Stokes matrix, but the converse is false in general.

The organization of the paper is as follows:

In section 1, we recall some basic definitions about the notions of the theory of summation, such as anti-Stokes directions, Stokes-Ramis matrices, etc..., which are needed.

In section 2, based on the geometry of the anti-Stokes directions of perturbed system $(A^{\boldsymbol{\varepsilon}})$, we select some Stokes matrices −defined as finite product of Stokes-Ramis matrices− which are proved to depend holomorphically on the parameter $\boldsymbol{\varepsilon}$ (theorem 2.14) and to converge to the Stokes-Ramis matrices of initial system (A) when $\boldsymbol{\varepsilon}$ goes to $\mathbf{1}$ (corollary 2.15). Let us point out that such results were already obtained by the author in the case $r = 1$ with a more specific perturbation (*cf.* [6]).

The central point of the proof of theorem 2.14 is proved in section 3. This one is based, after rank reduction, on an adequate variant of the proof of summable-resurgence theorem for single-level systems following classical Écalle's method by regular perturbation and majorant series which was given by the author in [7].

In section 4, we combine the general results obtained in section 2 with the results of [4, 7] to build an alternative method for the effective calculation of the Stokes multipliers of $\widetilde{F}(x)$. As an illustration, we develop two examples.

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1 Some definitions and notations

For the convenience of the reader, we recall here below some definitions about the notions of summation theory which are needed in this paper.

- *Anti-Stokes directions*

The *anti-Stokes directions* (*i.e.*, the singular directions) *of system* (A) (or of the full matrix $\widetilde{F}(x)$) are the directions of maximal decay of exponentials $e^{(q_j-q_\ell)(1/x)}$ with $q_j-q_\ell \not\equiv 0$. More precisely, these directions are the directions determined from 0 by the r^{th} roots of the nonzero elements of

$$\boldsymbol{\Omega} := \{a_{j,r} - a_{\ell,r} \ ; \ 1 \leq j, \ell \leq J\}.$$

Indeed, according to our hypothesis (0.4) of "*single level equal to* r", any polynomial $q_j - q_\ell \not\equiv 0$ is of degree r and reads

$$(q_j - q_\ell)\left(\frac{1}{x}\right) = -\frac{a_{j,r} - a_{\ell,r}}{x^r} + o\left(\frac{1}{x^r}\right) \quad \text{with } a_{j,r} - a_{\ell,r} \neq 0.$$

Recall that the elements $a_{j,r} - a_{\ell,r}$ of $\mathbf{\Omega}$ are called *Stokes values of system* (A). Notice that condition $a_{1,r} = 0$ implies $a_{j,r} \in \mathbf{\Omega}$ for all $j = 1, ..., J$.

Throughout the article, we refer as *a collection of anti-Stokes directions of system* (A) any set $(\theta_k)_{k=0,...,r-1} \in (\mathbb{R}/2\pi\mathbb{Z})^r$ formed by the r directions generated by a nonzero Stokes values of $\mathbf{\Omega}$ (*i.e.*, determined by its r^{th} roots).

• *Summation*

Given a non anti-Stokes direction $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ of system (A) and a choice of an argument of θ, say its principal determination $\theta^\star \in]-2\pi, 0]$ [2], we consider the sum of $\widetilde{Y}$ in the direction θ given by

$$Y_\theta(x) = s_{r;\theta}(\widetilde{F})(x) Y_{0,\theta^\star}(x)$$

where $s_{r;\theta}(\widetilde{F})(x)$ denotes the uniquely determined r-sum of $\widetilde{F}$ at θ and where $Y_{0,\theta^\star}(x)$ is the actual analytic function $Y_{0,\theta^\star}(x) := x^L e^{Q(1/x)}$ defined by the choice $\arg(x)$ close to $\theta^\star$ (denoted below $\arg(x) \simeq \theta^\star$).

Recall that $s_{r;\theta}(\widetilde{F})$ is an analytic function defined and $1/r$-Gevrey asymptotic to $\widetilde{F}$ on a germ of sector bisected by θ and opening larger than π/r.

For both practical and theoretical reasons, it is worth noting that it is often useful to rewrite $s_{r;\theta}(\widetilde{F})$ in terms of 1-sums (or Borel-Laplace sums): let us denote by $\widetilde{\boldsymbol{F}}^{[u]}(t) \in M_n(\mathbb{C}[[t]])$ with $u = 0, ..., r-1$ the *r-reduced series of* $\widetilde{F}(x)$, *i.e.*, the formal series which are uniquely determined by the relation

$$\widetilde{F}(x) = \widetilde{\boldsymbol{F}}^{[0]}(x^r) + x\widetilde{\boldsymbol{F}}^{[1]}(x^r) + ... + x^{r-1}\widetilde{\boldsymbol{F}}^{[r-1]}(x^r).$$

Then, all the $\widetilde{\boldsymbol{F}}^{[u]}$'s are 1-summable in the direction $\boldsymbol{\theta} := r\theta$ and the r-sum $s_{r;\theta}(\widetilde{F})$ is related to the 1-sums $s_{1;\boldsymbol{\theta}}(\widetilde{\boldsymbol{F}}^{[u]})$ by the relation

$$s_{r;\theta}(\widetilde{F})(x) = \sum_{u=0}^{r-1} x^u s_{1;\boldsymbol{\theta}}(\widetilde{\boldsymbol{F}}^{[u]})(x^r).$$

Recall that the 1-sum $s_{1;\boldsymbol{\theta}}(\widetilde{\boldsymbol{F}}^{[u]})(t)$ is given by the Borel-Laplace integral

$$\int_0^{\infty e^{i\boldsymbol{\theta}}} \widehat{\boldsymbol{F}}^{[u]}(\tau) e^{-\tau/t} d\tau$$

where $\widehat{\boldsymbol{F}}^{[u]}(\tau)$ denotes the Borel transform of $\widetilde{\boldsymbol{F}}^{[u]}(t)$.

[2] Any choice is convenient. However, to be compatible, on the Riemann sphere, with the usual choice $0 \leq \arg(z = 1/x) < 2\pi$ of the principal determination at infinity, we suggest to choose $-2\pi < \arg(x) \leq 0$ as principal determination about 0.

• *Stokes phenomenon and Stokes-Ramis matrices*

When $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ is an anti-Stokes direction of system (A), we consider the two lateral sums $s_{r;\theta^-}(\widetilde{F})$ and $s_{r;\theta^+}(\widetilde{F})$ of $\widetilde{F}$ at θ respectively obtained as analytic continuations of $s_{r;\theta-\eta}(\widetilde{F})$ and $s_{r;\theta+\eta}(\widetilde{F})$ to a sector with vertex 0, bisected by θ and opening π/r. Notice that such analytic continuations exist without ambiguity when $\eta > 0$ is small enough.

The *Stokes phenomenon* of system (A) stems from the fact that the sums $s_{r;\theta^-}(\widetilde{F})$ and $s_{r;\theta^+}(\widetilde{F})$ of $\widetilde{F}$ are not analytic continuations from each other in general. This defect of analyticity is quantified by the collection of *Stokes-Ramis automorphisms*

$$St_{\theta^\star} : Y_{\theta^+} \longmapsto Y_{\theta^-}$$

for all the anti-Stokes directions $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ of system (A), where Y_{θ^+} and Y_{θ^-} respectively denote the sums of $\widetilde{Y}$ defined for $\arg(x) \simeq \theta^\star$ by

$$Y_{\theta^+}(x) := s_{r;\theta^+}(\widetilde{F})(x)Y_{0,\theta^\star}(x) \quad \text{and} \quad Y_{\theta^-}(x) := s_{r;\theta^-}(\widetilde{F})(x)Y_{0,\theta^\star}(x).$$

The *Stokes-Ramis matrices* are defined as matrix representations of the $St_{\theta^\star}$'s in $GL_n(\mathbb{C})$.

Definition 1.1 (Stokes-Ramis matrices)
One calls Stokes-Ramis matrix associated with $\widetilde{Y}$ in the direction θ *the matrix of* $St_{\theta^\star}$ *in the basis* Y_{θ^+}. *We still denote it* $St_{\theta^\star}$.

Notice that the matrix $St_{\theta^\star}$ is uniquely determined by the relation

$$\boxed{Y_{\theta^-}(x) = Y_{\theta^+}(x)St_{\theta^\star} \qquad \text{for } \arg(x) \simeq \theta^\star}.$$

2 A holomorphic perturbation

In this section, we build a regular holomorphic perturbation of system (A) which preserves the single level $r \geq 1$; then, based on the geometry of the anti-Stokes directions of the perturbed system, we select some Stokes matrices −defined as convenient finite products of Stokes-Ramis matrices− and we show, on one hand, that they depend holomorphically on the parameter and, on the other hand, that they converge to the Stokes-Ramis matrices of initial system (A).

According to the normalization $\widetilde{F}(x) = I_n + O(x^r)$, the matrix $A(x)$ of system (A) reads

$$A(x) = \bigoplus_{j=1}^{J} \left[\left(ra_{j,r} + \sum_{k=1}^{r-1} ka_{j,k}x^{r-k} \right) I_{n_j} + x^r L_j \right] + B(x)$$

where $L_j := \lambda_j I_{n_j} + J_{n_j}$ denotes the j^{th} Jordan block of the matrix L of exponents of formal monodromy and where $B(x)$ is analytic at the origin $0 \in \mathbb{C}$. More precisely, splitting $B(x) = [B^{j;\ell}(x)]$ into blocks fitting the Jordan structure of L, one has

$$B^{j;\ell}(x) = \begin{cases} O(x^r) & \text{if } a_{j,r} \neq a_{\ell,r} \\ O(x^{2r}) & \text{if } a_{j,r} = a_{\ell,r} \end{cases} . \tag{2.1}$$

The holomorphic perturbation of system (A) considered below acts both on the Stokes values $a_{j,r} \neq 0$ (hence, *a fortiori*, on the set $\mathbf{\Omega}$ of all the Stokes values of system (A) and on the anti-Stokes directions of system (A) too) and on the analytic part $B(x)$.

Recall that $a_{1,r} = 0$ and the nonzero $a_{j,r}$'s are not supposed distinct. Henceforth, we denote below by $\omega_1, ..., \omega_p$ with $p \geq 1$ the distinct values of the $a_{j,r} \neq 0$ and we rewrite $\mathbf{\Omega}$ as

$$\mathbf{\Omega} = \{\omega_0 := 0\} \cup \{\omega_k - \omega_\ell \ ; \ k, \ell = 0, ..., p \text{ and } k \neq \ell\}.$$

Notice that $\omega_k - \omega_\ell \neq 0$ for all $k \neq \ell$; hence, their r^{th} roots determine all the anti-Stokes directions of system (A).

Throughout section 2, we shall use the following notations:

Notation 2.1 For any $\alpha \in \mathbb{C}$, $\rho > 0$, $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ and $\eta > 0$, we denote below by

- $D(\alpha, \rho) := \{x \in \mathbb{C} \ ; \ |x - \alpha| < \rho\}$ the open disc in $\mathbb{C}$ with midpoint α and radius ρ,
- $\overline{D}(\alpha, \rho) := \{x \in \mathbb{C} \ ; \ |x - \alpha| \leq \rho\}$ the closed disc in $\mathbb{C}$ with midpoint α and radius ρ (= the closure in $\mathbb{C}$ of $D(\alpha, \rho)$),
- $\Sigma_{\theta,\eta} := \{x \in \mathbb{C}\backslash\{0\} \ ; \ |\arg(x) - \theta| < \eta/2\}$ the open sector with vertex 0, bisected by direction θ and opening η,
- $D\Sigma_{\theta,\eta}$ the set of directions determined from 0 by all the points of $\Sigma_{\theta,\eta}$,
- $\overline{\Sigma}_{\theta,\eta} = \{x \in \mathbb{C}\backslash\{0\} \ ; \ |\arg(x) - \theta| \leq \eta/2\}$ the closure of $\Sigma_{\theta,\eta}$ in $\mathbb{C}\backslash\{0\}$,
- $D\overline{\Sigma}_{\theta,\eta}$ the set of directions determined from 0 by all the points of $\overline{\Sigma}_{\theta,\eta}$.

2.1 A perturbed system

We consider below a perturbation of system (A) of the form

$$(A^{\boldsymbol{\varepsilon}}) \qquad x^{r+1}\frac{dY}{dx} = A^{\boldsymbol{\varepsilon}}(x)Y$$

where

(1) the parameter $\boldsymbol{\varepsilon} := (\varepsilon_1, ..., \varepsilon_p, \varepsilon_{p+1})$ lies in a polydisc $\mathcal{D}_p := \prod_{k=1}^{p+1} D(1, \rho_k)$ of $\mathbb{C}^{p+1}$; precise conditions on the ρ_k's are given below,

(2) for all $\boldsymbol{\varepsilon} \in \mathcal{D}_p$, the matrix $A^{\boldsymbol{\varepsilon}}(x)$ reads

$$A^{\boldsymbol{\varepsilon}}(x) = \bigoplus_{j=1}^{J} \left[\left(r a_{j,r}^{\boldsymbol{\varepsilon}} + \sum_{k=1}^{r-1} k a_{j,k} x^{r-k} \right) I_{n_j} + x^r L_j \right] + \varepsilon_{p+1} B(x)$$

with

$$a_{j,r}^{\boldsymbol{\varepsilon}} := \begin{cases} \omega_0 = 0 & \text{if } a_{j,r} = \omega_0 \\ \omega_k \varepsilon_k & \text{if } a_{j,r} = \omega_k \text{ and } k \in \{1, ..., p\} \end{cases} .$$

Notice that systems $(A^{\boldsymbol{\varepsilon}})$ depend holomorphically on the parameter $\boldsymbol{\varepsilon}$ and coincide with system (A) for $\boldsymbol{\varepsilon} = \mathbf{1} := (1, ..., 1)$ the unit of $\mathbb{C}^{p+1}$.

Notice also that

$$\omega_k \varepsilon_k \in D(\omega_k, |\omega_k|\, \rho_k) \quad \text{for all } k = 1, ..., p.$$

Consequently, the radius ρ_k's, $k = 1, ..., p$, being chosen so that conditions

$(C1)$ $\overline{D}(\omega_k, |\omega_k|\, \rho_k) \cap \overline{D}(\omega_\ell, |\omega_\ell|\, \rho_\ell) = \emptyset$ for all $k, \ell = 1, ..., p$ and $k \neq \ell$,

$(C2)$ $0 \notin \overline{D}(\omega_k, |\omega_k|\, \rho_k)$ for all $k = 1, ..., p$,

be verified (such choices exist since the ω_k's are distinct in $\mathbb{C}\backslash\{0\}$ for all k), system $(A^{\boldsymbol{\varepsilon}})$ has, for all $\boldsymbol{\varepsilon} \in \mathcal{D}_p$, the unique level r and has for formal fundamental solution the matrix $\widetilde{Y}^{\boldsymbol{\varepsilon}}(x) = \widetilde{F}^{\boldsymbol{\varepsilon}}(x) x^L e^{Q^{\boldsymbol{\varepsilon}}(1/x)}$ where

- $\widetilde{F}^{\boldsymbol{\varepsilon}}(x) \in M_n(\mathbb{C}[[x]])$ is a formal power series in x satisfying $\widetilde{F}^{\boldsymbol{\varepsilon}}(0) = I_n$,
- L is the matrix of exponents of formal monodromy of initial system (A),

- $Q^{\boldsymbol{\varepsilon}}\left(\frac{1}{x}\right) = \bigoplus_{j=1}^{J} q_j^{\boldsymbol{\varepsilon}}\left(\frac{1}{x}\right) I_{n_j}$ with

$$q_j^{\boldsymbol{\varepsilon}}\left(\frac{1}{x}\right) = -\frac{a_{j,r}^{\boldsymbol{\varepsilon}}}{x^r} - \frac{a_{j,r-1}}{x^{r-1}} - ... - \frac{a_{j,1}}{x} \in x^{-1}\mathbb{C}[x^{-1}].$$

In other words, $q_j^{\boldsymbol{\varepsilon}}(1/x)$ is equal to

$$\begin{cases} 0 & \text{if } a_{j,r} = \omega_0 \\ -\frac{\omega_k \varepsilon_k}{x^r} - \frac{a_{j,r-1}}{x^{r-1}} - ... - \frac{a_{j,1}}{x} & \text{if } a_{j,r} = \omega_k \text{ and } k \in \{1, ..., p\} \end{cases}.$$

Observe that, like systems $(A^{\boldsymbol{\varepsilon}})$ and (A), the two formal fundamental solutions $\widetilde{Y}^{\boldsymbol{\varepsilon}}(x)$ and $\widetilde{Y}(x)$ coincide for $\boldsymbol{\varepsilon} = \mathbf{1}$. Observe also that, for any $\boldsymbol{\varepsilon} \in \mathcal{D}_p$, $\widetilde{Y}^{\boldsymbol{\varepsilon}}(x)$ has the same normalizations as $\widetilde{Y}(x)$. In particular, its formal series factor $\widetilde{F}^{\boldsymbol{\varepsilon}}(x)$ is uniquely determined, for all $\boldsymbol{\varepsilon} \in \mathcal{D}_p$, by the homological system associated with system $(A^{\boldsymbol{\varepsilon}})$ jointly with the initial condition $\widetilde{F}^{\boldsymbol{\varepsilon}}(0) = I_n$. Furthermore, the following condition holds for all $\boldsymbol{\varepsilon} \in \mathcal{D}_p$:

$$\begin{cases} a_{j,r} = 0 \Leftrightarrow a_{j,r}^{\boldsymbol{\varepsilon}} = 0 \Leftrightarrow q_j^{\boldsymbol{\varepsilon}} \equiv 0 \\ a_{j,r} = a_{\ell,r} \Leftrightarrow a_{j,r}^{\boldsymbol{\varepsilon}} = a_{\ell,r}^{\boldsymbol{\varepsilon}} \Leftrightarrow q_j^{\boldsymbol{\varepsilon}} \equiv q_\ell^{\boldsymbol{\varepsilon}} \end{cases}. \tag{2.2}$$

Remark 2.2 Unlike the radius ρ_k, $k = 1, ..., p$, which must be chosen so that conditions $(C1)$ and $(C2)$ hold, no condition on the radius ρ_{p+1} is imposed. In particular, we can choose it as we want.

Remark 2.3 Conclusions above on systems $(A^{\boldsymbol{\varepsilon}})$ are preserved when we replace in conditions $(C1)$ and $(C2)$ the closed discs $\overline{D}(\omega_k, |\omega_k|\,\rho_k)$ by the open discs $D(\omega_k, |\omega_k|\rho_k)$. Actually, the choice of the closed discs is to guarantee here that 0 is not an accumulation point for the set of nonzero Stokes values of systems $(A^{\boldsymbol{\varepsilon}})$ when $\boldsymbol{\varepsilon}$ runs in $\mathcal{D}_p$. As we shall see below, this point will play an essential role.

Let us now denote by $\boldsymbol{\Omega}^{\boldsymbol{\varepsilon}}$ the set of Stokes values of system $(A^{\boldsymbol{\varepsilon}})$. By construction, the set $\boldsymbol{\Omega}^{\boldsymbol{\varepsilon}}$ is deduced from the set $\boldsymbol{\Omega}$ of Stokes values of initial system (A) by replacing each nonzero Stokes value $\omega_k - \omega_\ell$ with the nonzero element $\omega_k\varepsilon_k - \omega_\ell\varepsilon_\ell$ (we set $\varepsilon_0 := 1$). Hence, for all $\boldsymbol{\varepsilon} \in \mathcal{D}_p$,

$$\boldsymbol{\Omega}^{\boldsymbol{\varepsilon}} = \{\omega_0 = 0\} \cup \{\omega_k\varepsilon_k - \omega_\ell\varepsilon_\ell \ ; \ k, \ell = 0, ..., p \text{ and } k \neq \ell\}.$$

This relation between *initial Stokes values* and *perturbed Stokes values* has a translation in terms of anti-Stokes directions.

Lemma 2.4 *Let $(\theta_k)_{k=0,\dots,r-1} \in (\mathbb{R}/2\pi\mathbb{Z})^r$ be a collection of anti-Stokes directions of initial system (A).*
Let $\mathcal{G}((\theta_k))$ be the set of Stokes values of $\mathbf{\Omega}$ generating the collection (θ_k).
Then, the image of (θ_k) by the perturbation is the set of all the anti-Stokes directions of systems $(A^{\boldsymbol{\varepsilon}})$, $\boldsymbol{\varepsilon}$ running in $\mathcal{D}_p$, generated by all the Stokes values $\omega_k\varepsilon_k - \omega_\ell\varepsilon_\ell \in \mathbf{\Omega}^{\boldsymbol{\varepsilon}}$ while $\omega_k - \omega_\ell \in \mathcal{G}((\theta_k))$.

A more precise version of lemma 2.4 is given in section 2.3, proposition 2.9. Before, we need some geometric features of the set of perturbed Stokes values.

2.2 Singular discs and singular sectors

Let us denote by

$$\mathbf{\Omega}(\mathcal{D}_p) := \bigcup_{\boldsymbol{\varepsilon}\in\mathcal{D}_p} \mathbf{\Omega}^{\boldsymbol{\varepsilon}}$$

the set of all the Stokes values of all systems $(A^{\boldsymbol{\varepsilon}})$ when $\boldsymbol{\varepsilon}$ runs in $\mathcal{D}_p$. The goal of this section is to describe some of its geometric features.

1. *Singular discs of $\mathbf{\Omega}(\mathcal{D}_p)$*

As seen before, the perturbation changes, for all $\boldsymbol{\varepsilon} \in \mathcal{D}_p$, the nonzero Stokes value $\omega_k - \omega_\ell \in \mathbf{\Omega}$ of initial system (A) into the nonzero Stokes value $\omega_k\varepsilon_k - \omega_\ell\varepsilon_\ell \in \mathbf{\Omega}^{\boldsymbol{\varepsilon}}$ of system $(A^{\boldsymbol{\varepsilon}})$. This brings us to the following definition.

Definition 2.5 (Singular disc of $\mathbf{\Omega}(\mathcal{D}_p)$)
Given a nonzero Stokes value $\omega_k - \omega_\ell \in \mathbf{\Omega}$ of initial system (A) (hence, $k \neq \ell$), we call singular disc of $\mathbf{\Omega}(\mathcal{D}_p)$ associated with $\omega_k - \omega_\ell$ *the subset $D_{\omega_k-\omega_\ell} \subset \mathbf{\Omega}(\mathcal{D}_p)$ of all the Stokes values $\omega_k\varepsilon_k - \omega_\ell\varepsilon_\ell \in \mathbf{\Omega}^{\boldsymbol{\varepsilon}}$ of all systems $(A^{\boldsymbol{\varepsilon}})$ when $\boldsymbol{\varepsilon}$ runs in $\mathcal{D}_p$.*

Notice that the set $\mathbf{\Omega}(\mathcal{D}_p)$ can be rewritten as

$$\mathbf{\Omega}(\mathcal{D}_p) = \{0\} \cup \left(\bigcup_{\omega\in\mathbf{\Omega}\backslash\{0\}} D_\omega \right)$$

Notice also that the choice of closed discs in conditions $(C1)$ and $(C2)$ (*cf.* remark 2.3) implies $0 \notin \overline{D}_\omega$ (= the closure of D_ω in $\mathbb{C}$) for all $\omega \in \mathbf{\Omega}\backslash\{0\}$.

Proposition 2.6 below gives us an explicit form of the singular discs D_ω.

Proposition 2.6 (Description of singular discs of $\mathbf{\Omega}(\mathcal{D}_p)$)
Let $\omega_k - \omega_\ell \in \mathbf{\Omega}$ be a nonzero Stokes value of initial system (A). Let $D_{\omega_k-\omega_\ell}$ be the singular disc of $\mathbf{\Omega}(\mathcal{D}_p)$ associated with $\omega_k - \omega_\ell$.
Then,

$$D_{\omega_k-\omega_\ell} = D(\omega_k - \omega_\ell, |\omega_k|\,\rho_k + |\omega_\ell|\,\rho_\ell)$$

(we set $\rho_0 := 0$).

Observe that, contrary to the discs $D(\omega_k, |\omega_k|\,\rho_k)$ (*cf.* condition $(C1)$), some of singular discs may overlap.

2. *Singular sectors of $\mathbf{\Omega}(\mathcal{D}_p)$*

We denote below by

- $\boldsymbol{\Theta}$ the set of all the directions determined from 0 by all the nonzero Stokes values of $\mathbf{\Omega}$,

and, for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$,

- $\mathbf{\Omega}_{\boldsymbol{\theta}}$ the set of all the nonzero Stokes values of $\mathbf{\Omega}$ with argument $\boldsymbol{\theta}$,
- $\mathbf{\Omega}_{\boldsymbol{\theta}}(\mathcal{D}_p) := \bigcup_{\omega\in\mathbf{\Omega}_{\boldsymbol{\theta}}} D_\omega$ the set of all the singular discs of $\mathbf{\Omega}(\mathcal{D}_p)$ associated with $\omega \in \mathbf{\Omega}_{\boldsymbol{\theta}}$. In other words, $\mathbf{\Omega}_{\boldsymbol{\theta}}(\mathcal{D}_p)$ collects all the perturbed Stokes values of $\mathbf{\Omega}(\mathcal{D}_p)$ associated with all the initial Stokes values of $\mathbf{\Omega}$ determining the given direction $\boldsymbol{\theta} \in \boldsymbol{\Theta}$.

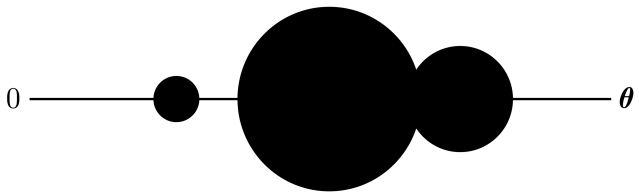

Figure 2.1 A set $\mathbf{\Omega}_{\boldsymbol{\theta}}(\mathcal{D}_p)$

According to proposition 2.6, all the singular discs D_ω with $\omega \in \mathbf{\Omega}_{\boldsymbol{\theta}}$ are centered on $\boldsymbol{\theta}$. Then, the set $\mathbf{\Omega}_{\boldsymbol{\theta}}(\mathcal{D}_p)$ is symmetrical about $\boldsymbol{\theta}$. This motivates the following definition.

Definition 2.7 (Singular sectors of $\mathbf{\Omega}(\mathcal{D}_p)$)
Given a direction $\boldsymbol{\theta} \in \mathbf{\Theta}$, we call singular sector of $\mathbf{\Omega}(\mathcal{D}_p)$ associated with $\boldsymbol{\theta}$ *the sector with minimal opening among all the sectors $\Sigma_{\boldsymbol{\theta},\eta}$ containing $\mathbf{\Omega}_{\boldsymbol{\theta}}(\mathcal{D}_p)$.* We denote it $\Sigma_{\boldsymbol{\theta},\eta(\boldsymbol{\theta})}$.

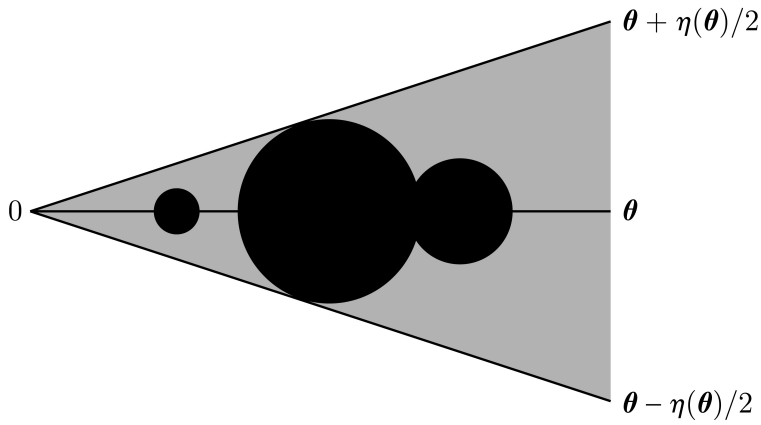

Figure 2.2 A sector $\Sigma_{\boldsymbol{\theta},\eta(\boldsymbol{\theta})}$

Proposition 2.8 below, which states some features of $\Sigma_{\boldsymbol{\theta},\eta(\boldsymbol{\theta})}$, stems from property "$0 \notin \overline{D}_\omega$ for all $\omega \in \mathbf{\Omega}\backslash\{0\}$".

Proposition 2.8 *Given a direction $\boldsymbol{\theta} \in \mathbf{\Theta}$, the following properties hold:*

(a) $\eta(\boldsymbol{\theta}) < \pi$, *i.e.,* $\Sigma_{\boldsymbol{\theta},\eta(\boldsymbol{\theta})}$ *is smaller than a half-plane.*

(b) $\eta(\boldsymbol{\theta})$ *only depends on the radius $\rho_1, ..., \rho_p$ associated with the initial Stokes values $\omega_1, ..., \omega_p$. In particular, $\eta(\boldsymbol{\theta})$ tends to 0 when the ρ_k's go to 0.*

(c) The set $D\Sigma_{\boldsymbol{\theta},\eta(\boldsymbol{\theta})}$ is the set of directions determined from 0 by all the points of $\mathbf{\Omega}_{\boldsymbol{\theta}}(\mathcal{D}_p)$.

According to proposition 2.8 (b) and calculations below, we suppose, from now on, that the radius ρ_k, $k = 1, ..., p$, are chosen so that the following conditions be verified:

$(C3)$ for all $\boldsymbol{\theta} \in \mathbf{\Theta}$, $\eta(\boldsymbol{\theta}) < \dfrac{\pi}{2}$,

$(C4)$ for all $\boldsymbol{\theta} \in \mathbf{\Theta}$, the principal determination $\boldsymbol{\theta}^\star$ of $\boldsymbol{\theta}$ and the principal determination $(\boldsymbol{\theta} - \eta(\boldsymbol{\theta})/2)^\star$ of $\boldsymbol{\theta} - \eta(\boldsymbol{\theta})/2$ satisfy

$$0 \geq \boldsymbol{\theta}^\star > (\boldsymbol{\theta} - \eta(\boldsymbol{\theta})/2)^\star > -2\pi,$$

$(C5)$ $\overline{\Sigma}_{\boldsymbol{\theta},\eta(\boldsymbol{\theta})} \cap \overline{\Sigma}_{\boldsymbol{\theta}',\eta(\boldsymbol{\theta}')} = \emptyset$ for all $\boldsymbol{\theta}, \boldsymbol{\theta}' \in \boldsymbol{\Theta}$, $\boldsymbol{\theta} \neq \boldsymbol{\theta}'$.

Notice that, once again (*cf.* remark 2.2), no condition is imposed on the last radius ρ_{p+1}.

We are now able to describe the action of the perturbation on the anti-Stokes directions of initial system (A).

2.3 Perturbation and anti-Stokes directions

The goal of this section is to give a precise description of the image of any collection $(\theta_k)_{k=0,\dots,r-1} \in (\mathbb{R}/2\pi\mathbb{Z})^r$ of anti-Stokes directions of initial system (A) by the perturbation. To this end, we base on lemma 2.4 and on the geometric features of the set $\boldsymbol{\Omega}(\mathcal{D}_p)$ stated in the previous section.

The main result of this section is the following proposition.

Proposition 2.9 *Let $(\theta_k)_{k=0,\dots,r-1} \in (\mathbb{R}/2\pi\mathbb{Z})^r$ be a collection of anti-Stokes directions of initial system (A).*
Let $\boldsymbol{\theta} := r\theta_0$ (hence, $\boldsymbol{\theta} = r\theta_k$ for all k). Then,

1. *$\boldsymbol{\theta} \in \boldsymbol{\Theta}$,*
2. *the image of the collection $(\theta_k)_{k=0,\dots,r-1}$ by the perturbation is the collection $(D\Sigma_{\theta_k,\eta(\boldsymbol{\theta})/r})_{k=0,\dots,r-1}$.*

Recall (*cf.* definition 2.7) that $\eta(\boldsymbol{\theta})$ denotes the opening of the singular sector of $\boldsymbol{\Omega}(\mathcal{D}_p)$ associated with $\boldsymbol{\theta}$.
Recall also (*cf.* lemma 2.4) that, for all $k = 0, \dots, r-1$, the directions of the set $D\Sigma_{\theta_k,\eta(\boldsymbol{\theta})/r}$ are anti-Stokes directions of systems $(A^{\boldsymbol{\varepsilon}})$, $\boldsymbol{\varepsilon}$ running in $\mathcal{D}_p$.

Proof. Obviously, $\boldsymbol{\theta}$ is the direction determined by the Stokes values of $\boldsymbol{\Omega}$ generating the collection (θ_k); hence, $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ and the set $\mathcal{G}((\theta_k))$ of lemma 2.4 coincides with the set $\boldsymbol{\Omega}_{\boldsymbol{\theta}}$ of section 2.2. Thereby, the image of (θ_k) is equal to the set of directions determined by the r^{th} roots of the elements of $\boldsymbol{\Omega}_{\boldsymbol{\theta}}(\mathcal{D}_p)$ (*cf.* lemma 2.4). Proposition 2.8 (c) ends the proof. ■

Observe that, like directions θ_k's, the sets $D\Sigma_{\theta_k,\eta(\boldsymbol{\theta})/r}$'s are regularly distribued around the origin $0 \in \mathbb{C}$.

Conditions $(C3)-(C5)$ imply some obvious properties on sectors $\Sigma_{\theta_k,\eta(\boldsymbol{\theta})/r}$ which will be useful in the following calculations.

Proposition 2.10 *With notations as above, the following properties hold:*

(a) For any collection (θ_k) *of anti-Stokes directions of initial system* (A)*,*

$$\overline{\Sigma}_{\theta_k,\eta(\boldsymbol{\theta})/r} \cap \overline{\Sigma}_{\theta_\ell,\eta(\boldsymbol{\theta})/r} = \emptyset \quad \textit{for all } k \neq \ell.$$

(b) For any collection (θ_k) *of anti-Stokes directions of initial system* (A)*, the principal determination* $\theta_k^\star$ *of* θ_k *and the principal determination* $(\theta_k - \eta(\boldsymbol{\theta})/(2r))^\star$ *of* $\theta_k - \eta(\boldsymbol{\theta})/(2r)$ *satisfy*

$$0 \geq \theta_k^\star > (\theta_k - \eta(\boldsymbol{\theta})/(2r))^\star > -2\pi \quad \textit{for all } k.$$

(c) For any two distinct collections (θ_k) *and* (θ'_ℓ) *of anti-Stokes directions of initial system* (A)*,*

$$\overline{\Sigma}_{\theta_k,\eta(\boldsymbol{\theta})/r} \cap \overline{\Sigma}_{\theta'_\ell,\eta(\boldsymbol{\theta}')/r} = \emptyset \quad \textit{for all } k \textit{ and } \ell.$$

Figure 3 below illustrates proposition 2.10 for two collections (θ_k) and (θ'_ℓ) in the case $r = 3$:

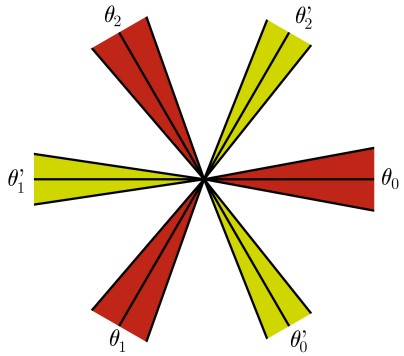

Figure 2.3

Remark 2.11 Proposition 2.10, (c) shows that the set $(D\overline{\Sigma}_{\theta_k,\eta(\boldsymbol{\theta})/r})_{k=0,\ldots,r-1}$ contains no other anti-Stokes directions of systems $(A^{\boldsymbol{\varepsilon}})$, $\boldsymbol{\varepsilon}$ running in $\mathcal{D}_p$, except those issuing from collection (θ_k) under the action of the perturbation. In particular, since systems (A) and $(A^{\boldsymbol{\varepsilon}})$ coincide for $\boldsymbol{\varepsilon} = \mathbf{1}$, the set $D\overline{\Sigma}_{\theta_k,\eta(\boldsymbol{\theta})/r}$ just contains, for all $k = 0, ..., r-1$, the direction θ_k as anti-Stokes directions of initial system (A).

2.4 Initial vs perturbed Stokes-Ramis matrices

In this section, we consider a collection $(\theta_k)_{k=0,\dots,r-1}$ of anti-Stokes directions of initial system (A). Let $(D\Sigma_{\theta_k,\eta(\boldsymbol{\theta})/r})_{k=0,\dots,r-1}$ be its image by the perturbation. Recall that $\boldsymbol{\theta} = r\theta_k$ for any k (*cf.* proposition 2.9).

According to condition $(C3)$ and proposition 2.10 above, there exists $\eta \in]\eta(\boldsymbol{\theta}), \pi/2[$ such that, for all $k = 0, ..., r-1$,

1. $\Sigma_{\theta_k,\eta(\boldsymbol{\theta})/r} \subsetneqq \Sigma_{\theta_k,\eta/r} \subsetneqq \Sigma_{\theta_k,(\pi-\eta)/r} \subsetneqq \Sigma_{\theta_k,\pi/r}$,
2. the principal determination $(\theta_k - \eta/(2r))^\star$ of $\theta_k - \eta/(2r)$ satisfies
$$0 \geq \theta_k^\star > (\theta_k - \eta(\boldsymbol{\theta})/(2r))^\star > (\theta_k - \eta/(2r))^\star > -2\pi,$$
3. $\overline{\Sigma}_{\theta_k,\eta/r} \cap \overline{\Sigma}_{\theta_\ell,\eta/r} = \emptyset$ for all $\ell \neq k$,
4. for any collection (θ'_ℓ) of anti-Stokes directions of initial system (A) distinct of (θ_k),
$$\overline{\Sigma}_{\theta_k,\eta/r} \cap \overline{\Sigma}_{\theta'_\ell,\eta(\boldsymbol{\theta}')/r} = \emptyset \quad \text{for all } \ell = 0, ..., r-1.$$

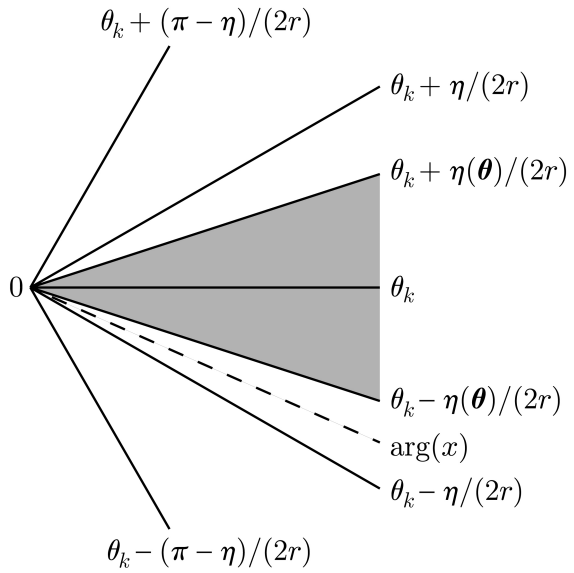

Figure 2.4 Sector $\Sigma_{\theta_k,\eta(\boldsymbol{\theta})/r}$ and associated directions

Notice that point 1. results from the choice η in $]\eta(\boldsymbol{\theta}), \pi/2[$ and that points 2.–4. hold as soon as η is close enough to $\eta(\boldsymbol{\theta})$. Notice also that

points 3.–4. guarantee that the set $D\overline{\Sigma}_{\theta_k,\eta/r}$ contains no other anti-Stokes directions of systems $(A^{\boldsymbol{\varepsilon}})$, $\boldsymbol{\varepsilon}$ running in $\mathcal{D}_p$, except those of $D\Sigma_{\theta_k,\eta(\boldsymbol{\theta})/r}$.

Let $k \in \{0, ..., r-1\}$ and fix, for now, $\boldsymbol{\varepsilon} \in \mathcal{D}_p$. According to points 1.–4. above, the directions $\theta_k \pm \eta/(2r)$ are not anti-Stokes directions of system $(A^{\boldsymbol{\varepsilon}})$. Thereby, (*cf.* section 1, page 5), the r-sums $s_{r;\theta_k\pm\eta/(2r)}(\widetilde{F}^{\boldsymbol{\varepsilon}})$ are defined and analytic on a same germ of sector $\Sigma_{\theta_k,(\pi-\eta)/r}$. Consequently, the sums

$$Y^{\boldsymbol{\varepsilon}}_{\theta_k\pm\eta/(2r)}(x) := s_{r;\theta_k\pm\eta/(2r)}(\widetilde{F}^{\boldsymbol{\varepsilon}})(x)x^L e^{Q^{\boldsymbol{\varepsilon}}(1/x)}$$

are related, for $\arg(x) \in \left]\left(\theta_k - \frac{\eta}{2r}\right)^\star, \left(\theta_k - \frac{\eta(\boldsymbol{\theta})}{2r}\right)^\star\right[$ and x close enough to $0 \in \mathbb{C}$, by the relation

$$\boxed{Y^{\boldsymbol{\varepsilon}}_{\theta_k-\eta/(2r)}(x) = Y^{\boldsymbol{\varepsilon}}_{\theta_k+\eta/(2r)}(x)\mathfrak{S}^{\boldsymbol{\varepsilon}}_{\theta_k^\star}} \quad . \tag{2.3}$$

The matrix $\mathfrak{S}^{\boldsymbol{\varepsilon}}_{\theta_k^\star} \in GL_n(\mathbb{C})$ denotes the *(perturbed) connection matrix between* $Y^{\boldsymbol{\varepsilon}}_{\theta_k+\eta/(2r)}$ *and* $Y^{\boldsymbol{\varepsilon}}_{\theta_k-\eta/(2r)}$; it is uniquely determined by identity (2.3). Furthermore, remark 2.11 and points 1. and 3.–4. above imply that the Stokes matrix[3] $\mathfrak{S}^{\boldsymbol{\varepsilon}}_{\theta_k^\star}$ is defined as a (finite) product of Stokes-Ramis matrices associated with $\widetilde{Y}^{\boldsymbol{\varepsilon}}$ in the anti-Stokes directions of system $(A^{\boldsymbol{\varepsilon}})$ contained in $D\Sigma_{\theta_k,\eta(\boldsymbol{\theta})/r}$. In particular, for $\boldsymbol{\varepsilon} = \mathbf{1}$, we have $\mathfrak{S}^{\mathbf{1}}_{\theta_k^\star} \equiv St_{\theta_k^\star}$ the Stokes-Ramis matrix of initial system (A) associated with $\widetilde{Y}$ in the direction θ_k. Indeed, $Y^{\mathbf{1}}_{\theta_k\pm\eta/(2r)}(x) = Y_{\theta_k\pm\eta/(2r)}(x) = Y_{\theta_k^\pm}(x)$.

The aim of this section (and of this article) is to study the holomorphic dependence in $\boldsymbol{\varepsilon}$ of the Stokes matrices $\mathfrak{S}^{\boldsymbol{\varepsilon}}_{\theta_k^\star}$ (see theorem 2.14 below). To this end, we must, first of all, answer the following questions:

(a) Is there a germ Σ_k of sector

$$\left\{x \in \mathbb{C}^* \text{ such that } \left(\theta_k - \frac{\eta}{2r}\right)^\star < \arg(x) < \left(\theta_k - \frac{\eta(\boldsymbol{\theta})}{2r}\right)^\star\right\}$$

on which the r-sums $s_{r;\theta_k\pm\eta/(2r)}(\widetilde{F}^{\boldsymbol{\varepsilon}})$ are defined for all $\boldsymbol{\varepsilon} \in \mathcal{D}_p$?

(b) If such a Σ_k exists, what can be said about the holomorphy of functions $\boldsymbol{\varepsilon} \longmapsto s_{r;\theta_k\pm\eta/(2r)}(\widetilde{F}^{\boldsymbol{\varepsilon}})(x)$, x fixed in Σ_k? More precisely, are those functions holomorphic on all $\mathcal{D}_p$?

[3] *cf.* note 1.

As seen in section 1, page 5, we write the r-sums $s_{r;\theta_k\pm\eta/(2r)}(\widetilde{F}^{\boldsymbol{\varepsilon}})(x)$ as

$$s_{r;\theta_k\pm\eta/(2r)}(\widetilde{F}^{\boldsymbol{\varepsilon}})(x) = \sum_{u=0}^{r-1} x^u s_{1;\boldsymbol{\theta}\pm\eta/2}(\widetilde{\boldsymbol{F}}^{\boldsymbol{\varepsilon}[u]})(x^r) \tag{2.4}$$

where the $\widetilde{\boldsymbol{F}}^{\boldsymbol{\varepsilon}[u]}$'s denote the r-reduced series of $\widetilde{F}^{\boldsymbol{\varepsilon}}$. Let us admit for the moment the following lemma which yields some properties on Borel transforms of the $\widetilde{\boldsymbol{F}}^{\boldsymbol{\varepsilon}[u]}$'s.

Lemma 2.12 *Let $\widehat{\boldsymbol{F}}^{\boldsymbol{\varepsilon}[u]}(\tau)$ denote the Borel transform of $\widetilde{\boldsymbol{F}}^{\boldsymbol{\varepsilon}[u]}(t)$ with respect to t.*
Let V^+ (resp. V^-) be a domain in $\mathbb{C}$ defined by the data of an open disc centered at $0 \in \mathbb{C}$ and an open sector in $\mathbb{C}$ with vertex 0 and bisected by direction $\boldsymbol{\theta} + \eta/2$ (resp. $\boldsymbol{\theta} - \eta/2$).
Suppose that the closures $\overline{V}^+$ of V^+ and $\overline{V}^-$ of V^- in $\mathbb{C}$ satisfy

$$\overline{V}^+ \cap \overline{D}_\omega = \emptyset \quad \text{and} \quad \overline{V}^- \cap \overline{D}_\omega = \emptyset$$

for all the nonzero Stokes values $\omega \in \boldsymbol{\Omega}$ (recall that $\overline{D}_\omega$ denotes the closure in $\mathbb{C}$ of the singular disc D_ω).
Then,

1. *Domain V^+*
   (a) *For all $u = 0, ..., r-1$, the function $(\tau, \boldsymbol{\varepsilon}) \longmapsto \widehat{\boldsymbol{F}}^{\boldsymbol{\varepsilon}[u]}(\tau)$ is holomorphic on $V^+ \times \mathcal{D}_p$.*
   (b) *There exist $C^+, K^+ > 0$ such that inequality*

$$\left|\widehat{\boldsymbol{F}}^{\boldsymbol{\varepsilon}[u]}(\tau)\right| \leq C^+ e^{K^+|\tau|}$$

   holds for all $u = 0, ..., r-1$, all $\tau \in V^+$ and all $\boldsymbol{\varepsilon} \in \mathcal{D}_p$.

2. *Domain V^-*
   (a) *For all $u = 0, ..., r-1$, the function $(\tau, \boldsymbol{\varepsilon}) \longmapsto \widehat{\boldsymbol{F}}^{\boldsymbol{\varepsilon}[u]}(\tau)$ is holomorphic on $V^- \times \mathcal{D}_p$.*
   (b) *There exist $C^-, K^- > 0$ such that inequality*

$$\left|\widehat{\boldsymbol{F}}^{\boldsymbol{\varepsilon}[u]}(\tau)\right| \leq C^- e^{K^-|\tau|}$$

   holds for all $u = 0, ..., r-1$, all $\tau \in V^-$ and all $\boldsymbol{\varepsilon} \in \mathcal{D}_p$.

Observe that the existence of domains V^+ and V^- is guaranteed, on one hand, by the fact that $0 \notin \overline{D}_\omega$ for all $\omega \in \mathfrak{Q}\backslash\{0\}$ (*cf.* page 10) and, on the other hand, by the fact that the choice of η implies $\boldsymbol{\theta} \pm \eta/2 \notin D\overline{\Sigma}_{\boldsymbol{\theta}',\eta(\boldsymbol{\theta}')}$ for all $\boldsymbol{\theta}' \in \boldsymbol{\Theta}$ (*cf.* points 1.–4. above).

We will prove lemma 2.12 (in fact, a stronger statement) in section 3.

The following proposition gives a positive answer to questions (a) and (b) previously given.

Proposition 2.13 *Let $k \in \{0, ..., r-1\}$.*

1. *For all $\boldsymbol{\varepsilon} \in \mathcal{D}_p$, the functions $x \longmapsto s_{r;\theta_k \pm \eta/(2r)}(\widetilde{F}^{\,\boldsymbol{\varepsilon}})(x)$ are all defined and holomorphic on the sector*

$$\Sigma_k := \left\{ x \in \mathbb{C}^*; |x| < K_r \text{ and } \left(\theta_k - \frac{\eta}{2r}\right)^\star < \arg(x) < \left(\theta_k - \frac{\eta(\boldsymbol{\theta})}{2r}\right)^\star \right\}$$

where $K_r := \min\left(\sqrt[r]{\frac{1}{K^-}}, \sqrt[r]{\frac{1}{K^+}}\right)$.

2. *For all $x \in \Sigma_k$, the functions $\boldsymbol{\varepsilon} \longmapsto s_{r;\theta_k \pm \eta/(2r)}(\widetilde{F}^{\,\boldsymbol{\varepsilon}})(x)$ are holomorphic on $\mathcal{D}_p$.*

Proof. 1. Let $\boldsymbol{\varepsilon} \in \mathcal{D}_p$. According to lemma 2.12, the 1-sum $s_{1;\boldsymbol{\theta}+\eta/2}(\widetilde{\boldsymbol{F}}^{\boldsymbol{\varepsilon}[u]})(t)$ (resp. $s_{1;\boldsymbol{\theta}-\eta/2}(\widetilde{\boldsymbol{F}}^{\boldsymbol{\varepsilon}[u]})(t)$) is defined and holomorphic, for all $u = 0, ..., r-1$, on the sector

$$\Sigma_{\boldsymbol{\theta}+\eta/2}\left(\frac{1}{K^+}\right) := \left\{ t \in \mathbb{C}^*; |t| < \frac{1}{K^+} \text{ and } \left|\arg(t) - \boldsymbol{\theta} - \frac{\eta}{2}\right| < \frac{\pi}{2} \right\}$$

$$\left(\text{resp. } \Sigma_{\boldsymbol{\theta}-\eta/2}\left(\frac{1}{K^-}\right) := \left\{ t \in \mathbb{C}^*; |t| < \frac{1}{K^-} \text{ and } \left|\arg(t) - \boldsymbol{\theta} + \frac{\eta}{2}\right| < \frac{\pi}{2} \right\}\right).$$

Thereby, the choice of η (*cf.* points 1.–4. page 15) implies that the 1-sums $s_{1;\boldsymbol{\theta}\pm\eta/2}(\widetilde{\boldsymbol{F}}^{\boldsymbol{\varepsilon}[u]})(t)$ are defined and holomorphic, for all $u = 0, ..., r-1$, on the same sector

$$\Sigma := \left\{ t \in \mathbb{C}^*; |t| < K \text{ and } \left(\boldsymbol{\theta} - \frac{\eta}{2}\right)^\star < \arg(t) < \left(\boldsymbol{\theta} - \frac{\eta(\boldsymbol{\theta})}{2}\right)^\star \right\}$$

where

$$K = \min\left(\frac{1}{K^-}, \frac{1}{K^+}\right).$$

Observe that, since constants K^+ and K^- are independent of $\boldsymbol{\varepsilon}$, sector Σ is independent of $\boldsymbol{\varepsilon}$ too. Point 1. follows from identity (2.4).

2. Fix now $x \in \Sigma_k$. According to identity (2.4), it is sufficient to show that, for any $u = 0, ..., r-1$, the functions $\boldsymbol{\varepsilon} \longmapsto s_{1;\boldsymbol{\theta}\pm\eta/2}(\widetilde{\boldsymbol{F}}^{\,\boldsymbol{\varepsilon}[u]})(x^r)$ are holomorphic on $\mathcal{D}_p$.
For all $\boldsymbol{\varepsilon} \in \mathcal{D}_p$, the 1-sums $s_{1;\boldsymbol{\theta}\pm\eta/2}(\widetilde{\boldsymbol{F}}^{\,\boldsymbol{\varepsilon}[u]})(x^r)$ are given by the Borel-Laplace integrals

$$s_{1;\boldsymbol{\theta}\pm\eta/2}(\widetilde{\boldsymbol{F}}^{\,\boldsymbol{\varepsilon}[u]})(x^r) = \int_0^{\infty e^{i(\boldsymbol{\theta}\pm\eta/2)}} \widehat{\boldsymbol{F}}^{\,\boldsymbol{\varepsilon}[u]}(\tau) e^{-\tau/x^r} d\tau = \int_0^{+\infty} \widehat{\boldsymbol{G}}_{\pm}^{\boldsymbol{\varepsilon}[u]}(\tau) d\tau$$

where

$$\widehat{\boldsymbol{G}}_{\pm}^{\boldsymbol{\varepsilon}[u]}(\tau) = \widehat{\boldsymbol{F}}^{\,\boldsymbol{\varepsilon}[u]}(\tau e^{i(\boldsymbol{\theta}\pm\eta/2)}) e^{-\tau \exp(i(\boldsymbol{\theta}\pm\eta/2))/x^r}.$$

Since $\tau e^{i(\boldsymbol{\theta}\pm\eta/2)} \in V^{\pm}$ for all $\tau \geq 0$, lemma 2.12 applies:

- for all $\tau \geq 0$, the functions $\boldsymbol{\varepsilon} \longmapsto \widehat{\boldsymbol{G}}_{\pm}^{\boldsymbol{\varepsilon}[u]}(\tau)$ are holomorphic on $\mathcal{D}_p$,
- for all $\tau \geq 0$ and all $\boldsymbol{\varepsilon} \in \mathcal{D}_p$,

$$\begin{aligned} \left| \widehat{\boldsymbol{G}}_{\pm}^{\boldsymbol{\varepsilon}[u]}(\tau) \right| &\leq \left| \widehat{\boldsymbol{F}}^{\,\boldsymbol{\varepsilon}[u]}(\tau e^{i(\boldsymbol{\theta}\pm\eta/2)}) \right| e^{-\tau \operatorname{Re}(\exp(i(\boldsymbol{\theta}\pm\eta/2))/x^r)} \\ &\leq C^{\pm} e^{-\tau(\operatorname{Re}(\exp(i(\boldsymbol{\theta}\pm\eta/2))/x^r) - K^{\pm})} := M_{\pm}(\tau). \end{aligned}$$

Obviously, $M_{\pm}$ does not depend on $\boldsymbol{\varepsilon}$. Furthermore, the choice "$x \in \Sigma_k$" implying $x^r \in \Sigma$, the functions $\tau \longmapsto M_{\pm}(\tau)$ are integrable on $[0; +\infty[$. Then, from Lebesgues dominated convergence theorem, the functions $\boldsymbol{\varepsilon} \longmapsto s_{1;\boldsymbol{\theta}\pm\eta/2}(\widetilde{\boldsymbol{F}}^{\,\boldsymbol{\varepsilon}[u]})(x^r)$ are holomorphic on $\mathcal{D}_p$. ■

We are now able to state the two main theoretical results of this paper.

Theorem 2.14 *Let $k \in \{0, ..., r-1\}$.*
Then, the function $\boldsymbol{\varepsilon} \longmapsto \mathfrak{S}_{\theta_k^\star}^{\boldsymbol{\varepsilon}}$ is holomorphic on $\mathcal{D}_p$.

Proof. Let $k \in \{0, ..., r-1\}$ and $x \in \Sigma_k$. According to proposition 2.13, 1., the Stokes matrices $\mathfrak{S}_{\theta_k^\star}^{\boldsymbol{\varepsilon}}$ are uniquely determined, for all $\boldsymbol{\varepsilon} \in \mathcal{D}_p$, by the relation

$$Y_{\theta_k - \eta/(2r)}^{\boldsymbol{\varepsilon}}(x) = Y_{\theta_k + \eta/(2r)}^{\boldsymbol{\varepsilon}}(x) \mathfrak{S}_{\theta_k^\star}^{\boldsymbol{\varepsilon}} \tag{2.3}$$

where

$$Y_{\theta_k \pm \eta/(2r)}^{\boldsymbol{\varepsilon}}(x) = s_{r;\theta_k \pm \eta/(2r)}(\widetilde{F}^{\boldsymbol{\varepsilon}})(x) x^L e^{Q^{\boldsymbol{\varepsilon}}(1/x)}.$$

Since $\boldsymbol{\varepsilon} \longmapsto Q^{\boldsymbol{\varepsilon}}(1/x)$ is obvious holomorphic on $\mathcal{D}_p$, proposition 2.13, 2., implies that the functions $\boldsymbol{\varepsilon} \longmapsto Y^{\boldsymbol{\varepsilon}}_{\theta_k \pm \eta/(2r)}(x)$ are also holomorphic on $\mathcal{D}_p$.
On the other hand, for any $\boldsymbol{\varepsilon} \in \mathcal{D}_p$, the matrix $Y^{\boldsymbol{\varepsilon}}_{\theta_k \pm \eta/(2r)}$ is a formal fundamental solution of system $(A^{\boldsymbol{\varepsilon}})$. Thereby, $Y^{\boldsymbol{\varepsilon}}_{\theta_k \pm \eta/(2r)}(x) \neq 0$ for all $\boldsymbol{\varepsilon} \in \mathcal{D}_p$ and, consequently, the functions $\boldsymbol{\varepsilon} \longmapsto Y^{\boldsymbol{\varepsilon}}_{\theta_k \pm \eta/(2r)}(x)^{-1}$ are again holomorphic on $\mathcal{D}_p$. Identity

$$\mathfrak{S}^{\boldsymbol{\varepsilon}}_{\theta_k^\star} = Y^{\boldsymbol{\varepsilon}}_{\theta_k + \eta/(2r)}(x)^{-1} Y^{\boldsymbol{\varepsilon}}_{\theta_k - \eta/(2r)}(x)$$

ends the proof. ■

Theorem 2.14 obviously leads to the following result which tells us that the Stokes-Ramis matrices $St_{\theta_k^\star}$ of initial system (A) are limits of the Stokes matrices $\mathfrak{S}^{\boldsymbol{\varepsilon}}_{\theta_k^\star}$.

Corollary 2.15 *Let $k \in \{0, ..., r-1\}$. Then,*

$$\boxed{\lim_{\boldsymbol{\varepsilon} \to \mathbf{1}} \mathfrak{S}^{\boldsymbol{\varepsilon}}_{\theta_k^\star} = St_{\theta_k^\star}} \quad . \tag{2.5}$$

Relations (2.5) will be applied in section 4 with a more specific perturbation in order to provide a method of effective calculation of the Stokes multipliers of $\widetilde{F}(x)$. Before, let us end the proof of theorem 2.14 by proving lemma 2.12.

3 Proof of lemma 2.12

Recall that the formal Borel transformation is an isomorphism from the $\mathbb{C}$-differential algebra $\left(\mathbb{C}[[t]], +, \cdot, t^2 \frac{d}{dt}\right)$ to the $\mathbb{C}$-differential algebra $(\delta\mathbb{C} \oplus \mathbb{C}[[\tau]], +, *, \tau\cdot)$ that changes ordinary product $\cdot$ into convolution product $*$ and changes derivation $t^2 \frac{d}{dt}$ into multiplication by τ. It also changes multiplication by $\frac{1}{t}$ into derivation $\frac{d}{d\tau}$.
Recall also that the formal Borel transform $\widehat{g}(\tau)$ of an analytic function $g(t) \in \mathbb{C}\{t\}$ at 0 defines an entire function on all $\mathbb{C}$ with exponential growth at infinity.

Lemma 2.12 obviously stems from the following theorem.

Theorem 3.1 *Let $\widehat{\boldsymbol{F}}^{\,\boldsymbol{\varepsilon}[u]}(\tau)$ denote the Borel transform of $\widetilde{\boldsymbol{F}}^{\,\boldsymbol{\varepsilon}[u]}(t)$ with respect to t.*
Let V be a domain in $\mathbb{C}$ defined by the data of an open disc centered at $0 \in \mathbb{C}$ and an open sector in $\mathbb{C}$ with vertex 0.
Suppose that the closure $\overline{V}$ of V in $\mathbb{C}$ satisfies $\overline{V} \cap \overline{D}_\omega = \emptyset$ for all the nonzero

Stokes values ω of $\boldsymbol{\Omega}$ (recall that $\overline{D}_\omega$ denotes the closure in $\mathbb{C}$ of the singular disc D_ω of $\boldsymbol{\Omega}(\mathcal{D}_p)$, see section 2.2, point 1).
Then,

1. *for all $u = 0, ..., r-1$, the function $(\tau, \boldsymbol{\varepsilon}) \longmapsto \widehat{\boldsymbol{F}}^{\,\boldsymbol{\varepsilon}[u]}(\tau)$ is holomorphic on $V \times \mathcal{D}_p$,*

2. *there exist $C, K > 0$ such that inequality*

$$\left|\widehat{\boldsymbol{F}}^{\,\boldsymbol{\varepsilon}[u]}(\tau)\right| \leq Ce^{K|\tau|}$$

holds for all $u = 0, ..., r-1$, all $\tau \in V$ and all $\boldsymbol{\varepsilon} \in \mathcal{D}_p$.

Notice that the existence of domain V is guaranteed by the fact that $0 \notin \overline{D}_\omega$ for all $\omega \in \boldsymbol{\Omega}\backslash\{0\}$ (*cf.* page 10).

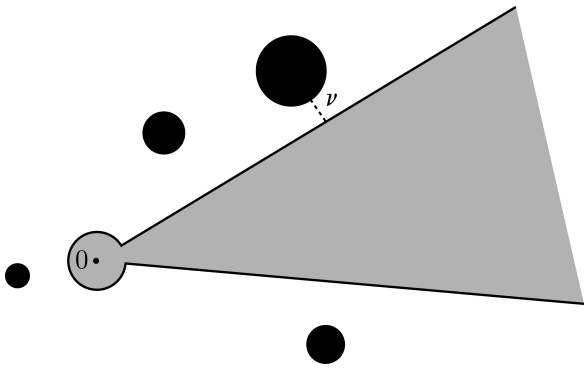

Figure 3.1 A domain V and the set of singular discs of $\boldsymbol{\Omega}(\mathcal{D}_p)$

The proof of theorem 3.1 is based, after rank reduction, on an adequate variant of the proof of summable-resurgence theorem for single-level systems following classical Écalle's method by regular perturbation and majorant series which was given by the author in [7].

Remark 3.2 Since any of the column-blocks of $\widetilde{F}^{\,\boldsymbol{\varepsilon}}(x)$ associated with the Jordan structure of L (matrix of exponents of formal monodromy of system (A) and, by construction, matrix of exponents of formal monodromy of any system $(A^{\boldsymbol{\varepsilon}})$ too) can be positionned at the first place by means of a same

permutation P (hence, independent of $\boldsymbol{\varepsilon}$) acting on the columns of $\widetilde{Y}^{\boldsymbol{\varepsilon}}(x)$ [4], relation

$$\widetilde{F}^{\boldsymbol{\varepsilon}}(x) = \widetilde{\boldsymbol{F}}^{\boldsymbol{\varepsilon}[0]}(x^r) + x\widetilde{\boldsymbol{F}}^{\boldsymbol{\varepsilon}[1]}(x^r) + ... + x^{r-1}\widetilde{\boldsymbol{F}}^{\boldsymbol{\varepsilon}[r-1]}(x^r)$$

shows that it is sufficient to prove theorem 3.1 in restriction to the column-blocks $\widetilde{\boldsymbol{f}}^{\boldsymbol{\varepsilon}[u]}$ formed by the first n_1 (= dimension of the first Jordan block of L) columns of the $\widetilde{\boldsymbol{F}}^{\boldsymbol{\varepsilon}[u]}$'s.

3.1 Rank reduction

Let $\widetilde{\boldsymbol{f}}^{\boldsymbol{\varepsilon}}(t)$ be the $rn \times n_1$-matrix of $M_{rn,n_1}(\mathbb{C}[[t]])$ defined, for any $\boldsymbol{\varepsilon} \in \mathcal{D}_p$, by

$$\widetilde{\boldsymbol{f}}^{\boldsymbol{\varepsilon}}(t) := \begin{bmatrix} \widetilde{\boldsymbol{f}}^{\boldsymbol{\varepsilon}[0]}(t) \\ \vdots \\ \widetilde{\boldsymbol{f}}^{\boldsymbol{\varepsilon}[r-1]}(t) \end{bmatrix}.$$

Observe that condition $\widetilde{F}^{\boldsymbol{\varepsilon}}(x) = I_n + O(x^r)$ implies $\widetilde{\boldsymbol{f}}^{\boldsymbol{\varepsilon}}(t) = I_{rn,n_1} + O(t)$ where I_{rn,n_1} denotes the first n_1 columns of the identity matrix of size rn.

By definition of rank reduction, the r-reduced system $(\boldsymbol{A}^{\boldsymbol{\varepsilon}})$ associated with system $(A^{\boldsymbol{\varepsilon}})$ admits, for all $\boldsymbol{\varepsilon} \in \mathcal{D}_p$, a formal fundamental solution whose the first n_1 columns of its formal series factor are equal to the n_1 columns of $\widetilde{\boldsymbol{f}}^{\boldsymbol{\varepsilon}}(t)$ (*cf.* [3]). Thereby, normalizations of $\widetilde{Y}^{\boldsymbol{\varepsilon}}(x)$ (= formal fundamental solution of $(A^{\boldsymbol{\varepsilon}})$) implies that $\widetilde{\boldsymbol{f}}^{\boldsymbol{\varepsilon}}(t)$ is uniquely determined by the first n_1 columns of the homological system associated with system $(\boldsymbol{A}^{\boldsymbol{\varepsilon}})$ jointly with the initial condition $\widetilde{\boldsymbol{f}}^{\boldsymbol{\varepsilon}}(0) = I_{rn,n_1}$. This brings us to proposition 3.3 below.

Before to state it, recall that the matrix $A^{\boldsymbol{\varepsilon}}(x)$ of system $(A^{\boldsymbol{\varepsilon}})$ reads

$$A^{\boldsymbol{\varepsilon}}(x) = \bigoplus_{j=1}^{J} \left[\left(ra^{\boldsymbol{\varepsilon}}_{j,r} + \sum_{k=1}^{r-1} k a_{j,k} x^{r-k} \right) I_{n_j} + x^r L_j \right] + \varepsilon_{p+1} B(x)$$

where $L_j = \lambda_j I_{n_j} + J_{n_j}$ denotes the j^{th} Jordan block of L and where $B(x) = [B^{j;\ell}(x)] \in M_n(\mathbb{C}\{x\})$ satisfies normalizations

$$B^{j;\ell}(x) = \begin{cases} O(x^r) & \text{if } a^{\boldsymbol{\varepsilon}}_{j,r} \neq a^{\boldsymbol{\varepsilon}}_{\ell,r} \\ O(x^{2r}) & \text{if } a^{\boldsymbol{\varepsilon}}_{j,r} = a^{\boldsymbol{\varepsilon}}_{\ell,r} \end{cases} \tag{2.1}$$

for all $j, \ell = 1, ..., J$ and all $\boldsymbol{\varepsilon} \in \mathcal{D}_p$. Recall also that $\lambda_1 = a^{\boldsymbol{\varepsilon}}_{1,r} = 0$ and $a_{1,k} = 0$ for all $k = 1, ..., r-1$.

[4] The new formal fundamental solution reads $\widetilde{Y}^{\boldsymbol{\varepsilon}}(x)P = \widetilde{F}^{\boldsymbol{\varepsilon}}(x)Px^{P^{-1}LP}e^{P^{-1}Q^{\boldsymbol{\varepsilon}}(1/x)P}$.

Proposition 3.3 *Let us denote by* $\boldsymbol{A}^{\boldsymbol{\varepsilon}[u]}(t)$ *(resp.* $\boldsymbol{B}^{[u]}(t)$*) with* $u = 0, ..., r-1$ *the* r*-reduced series of* $A^{\boldsymbol{\varepsilon}}(x)$ *(resp.* $B(x)$*).*
Then, for all $\boldsymbol{\varepsilon} \in \mathcal{D}_p$*, the formal series* $\widetilde{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}}(t) \in M_{rn,n_1}(\mathbb{C}[[t]])$ *is uniquely determined by the system*

$$\boxed{rt^2\frac{d\boldsymbol{f}}{dt} = \boldsymbol{A}^{\boldsymbol{\varepsilon}}(t)\boldsymbol{f} - t\boldsymbol{f}\,J_{n_1}} \tag{3.1}$$

jointly with the initial condition $\widetilde{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}}(0) = I_{rn,n_1}$*, where the matrix* $\boldsymbol{A}^{\boldsymbol{\varepsilon}}(t) \in M_{rn}(\mathbb{C}\{t\})$ *is defined by*

$$\boldsymbol{A}^{\boldsymbol{\varepsilon}}(t) = \begin{bmatrix} \boldsymbol{A}^{\boldsymbol{\varepsilon}[0]}(t) & t\boldsymbol{A}^{\boldsymbol{\varepsilon}[r-1]}(t) & \cdots & \cdots & t\boldsymbol{A}^{\boldsymbol{\varepsilon}[1]}(t) \\ \boldsymbol{A}^{\boldsymbol{\varepsilon}[1]}(t) & \boldsymbol{A}^{\boldsymbol{\varepsilon}[0]}(t) & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \boldsymbol{A}^{\boldsymbol{\varepsilon}[0]}(t) & t\boldsymbol{A}^{\boldsymbol{\varepsilon}[r-1]}(t) \\ \boldsymbol{A}^{\boldsymbol{\varepsilon}[r-1]}(t) & \cdots & \cdots & \boldsymbol{A}^{\boldsymbol{\varepsilon}[1]}(t) & \boldsymbol{A}^{\boldsymbol{\varepsilon}[0]}(t) \end{bmatrix} - \bigoplus_{u=0}^{r-1} utI_n$$

with

$$\boldsymbol{A}^{\boldsymbol{\varepsilon}[0]}(t) = \bigoplus_{j=1}^{J} \left(ra_{j,r}^{\boldsymbol{\varepsilon}} I_{n_j} + tL_j\right) + \varepsilon_{p+1}\boldsymbol{B}^{[0]}(t)$$

and

$$\boldsymbol{A}^{\boldsymbol{\varepsilon}[u]}(t) = \bigoplus_{j=1}^{J}(r-u)a_{j,r-u}I_{n_j} + \varepsilon_{p+1}\boldsymbol{B}^{[u]}(t) \quad \textit{for all } u = 1, ..., r-1.$$

Furthermore, splitting the matrix $\boldsymbol{B}^{[u]}(t) = [\boldsymbol{B}^{[u]j;\ell}(t)] \in M_n(\mathbb{C}\{t\})$ *into blocks fitting the Jordan structure of* L*, normalizations (2.1) imply*

$$\boldsymbol{B}^{[u]j;\ell}(t) = \begin{cases} O(t) & \text{if } a_{j,r}^{\boldsymbol{\varepsilon}} \neq a_{\ell,r}^{\boldsymbol{\varepsilon}} \\ O(t^2) & \text{if } a_{j,r}^{\boldsymbol{\varepsilon}} = a_{\ell,r}^{\boldsymbol{\varepsilon}} \end{cases} \tag{3.2}$$

for all $u = 0, ..., r-1$ *and all* $j, \ell = 1, ..., J$*.*

Let us now denote by $\widehat{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}}(\tau)$ the Borel transform of $\widetilde{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}}(t)$ with respect to t. In sections below, we shall prove, by applying Écalle's method to system (3.1), that, for any domain V as in theorem 3.1,

(a) the function $(\tau, \boldsymbol{\varepsilon}) \longmapsto \widehat{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}}(\tau)$ is well-defined and holomorphic on $V \times \mathcal{D}_p$,

(b) there exist $C, K > 0$ such that inequality $\left|\widehat{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}}(\tau)\right| \leq Ce^{K|\tau|}$ holds for all $\tau \in V$ and $\boldsymbol{\varepsilon} \in \mathcal{D}_p$.

Observe that those two points obviously lead to theorem 3.1.

Calculations below are rather similar to those detailed in [7, § 3.2] to prove the summable-resurgence theorem for single-level systems. Furthermore, they generalize calculations made in [6] in the case of perturbed level-one systems.

Throughout the rest of the paper, we use the following notation.

Notation 3.4 Given a matrix M split into blocks fitting the Jordan structure of L, we denote by $M^{j;\bullet}$ the j^{th} row-block of M. Thereby, $M^{j;\bullet}$ is a $n_j \times p$-matrix when M is a $n \times p$-matrix.

3.2 Regular perturbation

Following J. Écalle ([2]), we consider, instead of system (3.1), the regularly perturbed system

$$rt^2\frac{d\boldsymbol{f}}{dt} = \boldsymbol{A}^{\boldsymbol{\varepsilon}}(t,\alpha)\boldsymbol{f} - t\boldsymbol{f}\,J_{n_1} \tag{3.3}$$

obtained by substituting $\alpha\boldsymbol{B}^{[u]}$ for $\boldsymbol{B}^{[u]}$ for all $u = 0, ..., r-1$ in the matrix $\boldsymbol{A}^{\boldsymbol{\varepsilon}}(t)$ of system (3.1).

Like in [7], an identification of equal powers in α shows that system (3.3) admits, for all $\boldsymbol{\varepsilon} \in \mathcal{D}_p$, a unique formal solution of the form

$$\widetilde{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}}(t,\alpha) = \sum_{m\geq 0} \widetilde{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}}_{\,m}(t)\alpha^m$$

satisfying $\widetilde{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}}_{\,0}(t) = I_{rn,n_1}$ and $\widetilde{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}}_{\,m}(t) \in M_{rn,n_1}(\mathbb{C}[[t]])$ for all $m \geq 1$. The following lemma yields some precisions on the $\widetilde{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}}_{\,m}$'s.

Lemma 3.5 *Let $\boldsymbol{\varepsilon} \in \mathcal{D}_p$. Split $\widetilde{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}}_{\,m}(t) = \left[\widetilde{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}[0]}_{\,m}(t), ..., \widetilde{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}[r-1]}_{\,m}(t)\right]$ into r blocks of size $n \times n_1$ like $\widetilde{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}}(t)$ and denote by*

$$\widetilde{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}}_{\,m,j}(t) := \begin{bmatrix} \widetilde{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}[0]j;\bullet}_{\,m}(t) \\ \vdots \\ \widetilde{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}[r-1]j;\bullet}_{\,m}(t) \end{bmatrix} \quad \textit{for all } j = 1, ..., J$$

the $rn_j \times n_1$-matrix formed by all the j^{th} row-blocks of the $\widetilde{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}[u]}_{\,m}(t)$'s (see notation 3.4).

Then, the components $\widetilde{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}}_{m,j}(t) \in M_{rn_j,n_1}(\mathbb{C}[[t]])$ *are uniquely determined, for all* $m \geq 1$ *and* $j = 1, ..., J$*, as formal solutions of systems*

$$rt^2\frac{d\widetilde{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}}_{m,j}}{dt} - \boldsymbol{A}^{\boldsymbol{\varepsilon}}_j\widetilde{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}}_{m,j} - t\boldsymbol{A}_j\widetilde{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}}_{m,j} = \varepsilon_{p+1}\boldsymbol{B}_j(t)\widetilde{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}}_{m-1} - t\widetilde{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}}_{m,j}J_{n_1} \tag{3.4}$$

where

$$\boldsymbol{B}_j(t) := \begin{bmatrix} \boldsymbol{B}^{[0]j;\bullet}(t) & t\boldsymbol{B}^{[r-1]j;\bullet}(t) & \cdots & \cdots & t\boldsymbol{B}^{[1]j;\bullet}(t) \\ \boldsymbol{B}^{[1]j;\bullet}(t) & \boldsymbol{B}^{[0]j;\bullet}(t) & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \boldsymbol{B}^{[0]j;\bullet}(t) & t\boldsymbol{B}^{[r-1]j;\bullet}(t) \\ \boldsymbol{B}^{[r-1]j;\bullet}(t) & \cdots & \cdots & \boldsymbol{B}^{[1]j;\bullet}(t) & \boldsymbol{B}^{[0]j;\bullet}(t) \end{bmatrix}$$

is a $rn_j \times rn$*-matrix with analytic entries at* $0 \in \mathbb{C}$ *and where the matrices* $\boldsymbol{A}^{\boldsymbol{\varepsilon}}_j$ *and* $\boldsymbol{A}_j$ *are the* $rn_j \times rn_j$*-constant matrices defined by*

- $\boldsymbol{A}^{\boldsymbol{\varepsilon}}_j := \begin{bmatrix} ra^{\boldsymbol{\varepsilon}}_{j,r} & 0 & \cdots & 0 \\ (r-1)a_{j,r-1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ a_{j,1} & \cdots & (r-1)a_{j,r-1} & ra^{\boldsymbol{\varepsilon}}_{j,r} \end{bmatrix} \otimes I_{n_j}$ *and*

- $\boldsymbol{A}_j := \begin{bmatrix} 0 & a_{j,1} & \cdots & (r-1)a_{j,r-1} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & a_{j,1} \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \otimes I_{n_j} + \bigoplus_{u=0}^{r-1}(L_j - uI_{n_j}).$

Furthermore, according to normalizations (3.2), the following relations

$$\widetilde{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}}_{2m-1,j}(t) = O(t^m) \quad \textit{and} \quad \widetilde{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}}_{2m,j}(t) = \begin{cases} O(t^m) & \textit{if } a^{\boldsymbol{\varepsilon}}_{j,r} = 0 \\ O(t^{m+1}) & \textit{if } a^{\boldsymbol{\varepsilon}}_{j,r} \neq 0 \end{cases} \tag{3.5}$$

hold for all $m \geq 1$ *and* $j = 1, ..., J$*.*

Notice that the matrix $\boldsymbol{A}^{\boldsymbol{\varepsilon}}_j$ is invertible when $a^{\boldsymbol{\varepsilon}}_{j,r} \neq 0$. Notice also that relation (2.2) implies $\boldsymbol{A}^{\boldsymbol{\varepsilon}}_j = 0$ and $\boldsymbol{A}_j = \bigoplus_{u=0}^{r-1}\left(L_j - uI_{n_j}\right)$ when $a^{\boldsymbol{\varepsilon}}_{j,r} = 0$.

As a result of relations (3.5), the series $\widetilde{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}}(t,\alpha)$ can be rewritten as a series in t with polynomial coefficients in α. Consequently, $\widetilde{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}}(t) = \widetilde{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}}(t,1)$

(by unicity of $\widetilde{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}}(t)$ and $\widetilde{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}}(t,1)$) and, for all α, the series $\widetilde{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}}(t,\alpha)$ admits a formal Borel transform $\boldsymbol{\varphi}^{\boldsymbol{\varepsilon}}(\tau,\alpha)$ with respect to t of the form

$$\boldsymbol{\varphi}^{\boldsymbol{\varepsilon}}(\tau,\alpha) = \delta I_{rn,n_1} + \sum_{m\geq 1} \boldsymbol{\varphi}^{\boldsymbol{\varepsilon}}_m(\tau)\alpha^m$$

where $\boldsymbol{\varphi}^{\boldsymbol{\varepsilon}}_m(\tau) \in M_{rn,n_1}(\mathbb{C}[[\tau]])$ denotes, for all $m \geq 1$, the Borel transform of $\widetilde{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}}_m(t)$. In particular, the Borel transform $\widehat{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}}(\tau)$ reads formally as

$$\widehat{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}}(\tau) = \boldsymbol{\varphi}^{\boldsymbol{\varepsilon}}(\tau,1) = \sum_{m\geq 1} \boldsymbol{\varphi}^{\boldsymbol{\varepsilon}}_m(\tau) \quad \text{for all } \boldsymbol{\varepsilon} \in \mathcal{D}_p.$$

The two following results give us some properties of the $\boldsymbol{\varphi}^{\boldsymbol{\varepsilon}}_m$'s. The first one obviously stems from lemma 3.5.

Lemma 3.6 *Let $\boldsymbol{\varepsilon} \in \mathcal{D}_p$. Split as before $\boldsymbol{\varphi}^{\boldsymbol{\varepsilon}}_m(\tau) = \left[\boldsymbol{\varphi}^{\boldsymbol{\varepsilon}[0]}_m(\tau), ..., \boldsymbol{\varphi}^{\boldsymbol{\varepsilon}[r-1]}_m(\tau)\right]$ into r blocks of size $n \times n_1$ and denote by*

$$\boldsymbol{\varphi}^{\boldsymbol{\varepsilon}}_{m,j}(\tau) := \begin{bmatrix} \boldsymbol{\varphi}^{\boldsymbol{\varepsilon}[0]j;\bullet}_m(\tau) \\ \vdots \\ \boldsymbol{\varphi}^{\boldsymbol{\varepsilon}[r-1]j;\bullet}_m(\tau) \end{bmatrix} \quad \textit{for all } j = 1, ..., J$$

the $rn_j \times n_1$-matrix formed by all the j^{th} row-blocks of the $\boldsymbol{\varphi}^{\boldsymbol{\varepsilon}[u]}_m(\tau)$'s.
Then, for all $m \geq 1$, the components $\boldsymbol{\varphi}^{\boldsymbol{\varepsilon}}_{m,j}(\tau) \in M_{rn_j,n_1}(\mathbb{C}[[\tau]])$ are iteratively determined, for all $j = 1, ..., J$, as solutions of systems

$$\boxed{R^{\boldsymbol{\varepsilon}}_j \frac{d\boldsymbol{\varphi}^{\boldsymbol{\varepsilon}}_{m,j}}{d\tau} = S_j \boldsymbol{\varphi}^{\boldsymbol{\varepsilon}}_{m,j} + \frac{d}{d\tau}\left(\widehat{\boldsymbol{B}_j} * \boldsymbol{\varphi}^{\boldsymbol{\varepsilon}}_{m-1}\right) - \boldsymbol{\varphi}^{\boldsymbol{\varepsilon}}_{m,j} J_{n_1}} \tag{3.6}$$

where $\boldsymbol{\varphi}^{\boldsymbol{\varepsilon}}_0 := \delta I_{rn,n_1}$, $\widehat{\boldsymbol{B}_j}$ denotes the Borel transform of $\boldsymbol{B}_j$ and where the $rn_j \times rn_j$-matrices $R^{\boldsymbol{\varepsilon}}_j$ and S_j are defined by

- $R^{\boldsymbol{\varepsilon}}_j = \begin{bmatrix} r(\tau - a^{\boldsymbol{\varepsilon}}_{j,r}) & 0 & \cdots & 0 \\ (r-1)a_{j,r-1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ a_{j,1} & \cdots & (r-1)a_{j,r-1} & r(\tau - a^{\boldsymbol{\varepsilon}}_{j,r}) \end{bmatrix} \otimes I_{n_j}$ *and*

- $S_j = \begin{bmatrix} 0 & a_{j,1} & \cdots & (r-1)a_{j,r-1} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & a_{j,1} \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \otimes I_{n_j} + \bigoplus_{u=0}^{r-1} (L_j - (u+r)I_{n_j}).$

Lemma 3.6 implies the following proposition.

Proposition 3.7 *Let V a domain as in theorem 3.1.*
Then, the function $(\tau, \boldsymbol{\varepsilon}) \longmapsto \boldsymbol{\varphi}^{\boldsymbol{\varepsilon}}_m(\tau)$ is holomorphic on $V \times \mathcal{D}_p$ for all $m \geq 1$.

Proof. Since the $\boldsymbol{B}_j$'s are analytic at $0 \in \mathbb{C}$, their Borel transforms $\widehat{\boldsymbol{B}_j}$ are entire functions on all $\mathbb{C}$. Consequently, normalizations (3.2) imply that the only singularities in $\mathbb{C}$ of systems (3.6) are the Stokes values $a^{\boldsymbol{\varepsilon}}_{j,r} \neq 0$ of $\boldsymbol{\Omega}(\mathcal{D}_p)$. Proposition 3.7 follows from the fact that domain V never meets $\boldsymbol{\Omega}(\mathcal{D}_p)\backslash\{0\}$. ■

To prove theorem 3.1, we are left to show that

(a) the function $(\tau, \boldsymbol{\varepsilon}) \longmapsto \widehat{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}}(\tau) = \boldsymbol{\varphi}^{\boldsymbol{\varepsilon}}(\tau, 1) = \sum_{m \geq 1} \boldsymbol{\varphi}^{\boldsymbol{\varepsilon}}_m(\tau)$ is well-defined and holomorphic on $V \times \mathcal{D}_p$,

(b) there exist $C, K > 0$ such that inequality $\left|\widehat{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}}(\tau)\right| \leq Ce^{K|\tau|}$ holds for all $\tau \in V$ and $\boldsymbol{\varepsilon} \in \mathcal{D}_p$.

These two points are proved below by using a technique of majorant series satisfying a convenient system. Of course, there exist many possible majorant systems. Here, we make explicit a possible one.

3.3 Majorant series

Let ν denote the minimal distance between the elements of V and the elements of $\boldsymbol{\Omega}(\mathcal{D}_p)\backslash\{0\}$ (*cf.* figure 3.1). According to condition "$\overline{V} \cap \overline{D}_\omega = \emptyset$ for all $\omega \in \boldsymbol{\Omega}\backslash\{0\}$" (*cf.* theorem 3.1), we have $\nu > 0$.

Let $\boldsymbol{g} = \left[\boldsymbol{g}^{[0]}, ..., \boldsymbol{g}^{[r-1]}\right]$ be a $rn \times n_1$-matrix split as previously into r blocks of size $n \times n_1$. Let $\boldsymbol{g}_j$ denote the $rn_j \times n_1$-matrix formed by all the j^{th} row-blocks of the $\boldsymbol{g}^{[u]}$'s:

$$\boldsymbol{g}_j := \begin{bmatrix} \boldsymbol{g}^{[0]j;\bullet} \\ \vdots \\ \boldsymbol{g}^{[r-1]j;\bullet} \end{bmatrix} \quad \text{for all } j = 1, ..., J.$$

In the case where $\boldsymbol{g} = I_{rn,n_1}$, we simply denote by I^j_{rn,n_1} in place of $\boldsymbol{g}_j$.

Let us now consider, for $j = 1, ..., J$, the regularly perturbed linear system

$$(3.7)\quad \begin{cases} C_j(\boldsymbol{g}_j - I^j_{rn,n_1}) = (I_r \otimes J_{n_j})\boldsymbol{g}_j + \boldsymbol{g}_j J_{n_1} - 2I^j_{rn,n_1} J_{n_1} \\ \qquad\qquad\qquad\qquad +\alpha(\rho_{p+1}+1)\dfrac{|\boldsymbol{B}_j|(t)}{t}\boldsymbol{g} \quad \text{if } a_{j,r} = 0 \\ (\mathcal{R}_j - t\mathcal{S}_j)\boldsymbol{g}_j = t\boldsymbol{g}_j J_{n_1} + \alpha(\rho_{p+1}+1)\,|\boldsymbol{B}_j|(t)\boldsymbol{g} \qquad \text{if } a_{j,r} \neq 0 \end{cases}$$

where

- $|\boldsymbol{B}_j|(t)$ denotes the series $\boldsymbol{B}_j(t)$ in which the coefficients of the powers of t are replaced by their module,
- C_j is an invertible constant $rn_j \times rn_j$-diagonal matrix with positive entries which will be adequatly chosen below (see proposition 3.9),
- $\mathcal{R}_j$ and $\mathcal{S}_j$ are the $rn_j \times rn_j$-constant matrices defined by

$$\mathcal{R}_j := \begin{bmatrix} \nu & 0 & \cdots & 0 \\ -|a_{j,r-1}| & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ -|a_{j,1}| & \cdots & -|a_{j,r-1}| & \nu \end{bmatrix} \otimes I_{n_j}$$

and

$$\mathcal{S}_j := \begin{bmatrix} 0 & |a_{j,1}| & \cdots & |a_{j,r-1}| \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & |a_{j,1}| \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \otimes I_{n_j} + \bigoplus_{u=0}^{r-1} \left(\left| \frac{\lambda_j}{r} - \frac{u}{r} - 1 \right| I_{n_j} + J_{n_j} \right).$$

Recall that λ_j denotes the eigenvalue of the j^{th} Jordan block L_j of L.

Notice that the constant $\rho_{p+1}+1$ satisfies $|\varepsilon_{p+1}| \leq \rho_{p+1}+1$ for all $\boldsymbol{\varepsilon} \in \mathcal{D}_p$. Notice also that system (3.7) depends on the domain V but not on the parameter $\boldsymbol{\varepsilon}$.

Up to the constant $\rho_{p+1}+1$, system (3.7) is the majorant system used in [7] to prove summable-resurgence theorem for single-level systems. Hence, by adapting calculations made in [7, § 3.2.2], we can prove the following lemma.

Lemma 3.8 *The Borel transformed system of system (3.7) admits, for $\alpha = 1$, a unique solution of the form*

$$\widehat{\boldsymbol{g}}(\tau) = \delta I_{rn,n_1} + \sum_{m \geq 1} \boldsymbol{\Phi}_m(\tau) \in M_{rn,n_1}(\mathbb{C}[[\tau]])$$

which is entire on all $\mathbb{C}$ *with exponential growth at infinity. Furthermore, using notations as above, the components* $\mathbf{\Phi}_{m,j}(\tau) \in M_{rn_j,n_1}(\mathbb{C}[[\tau]])$ *of* $\mathbf{\Phi}_m(\tau)$ *are iteratively determined, for all* $m \geq 1$ *and* $j = 1, ..., J$*, as solutions of systems:*

- *Case* $a_{j,r} = 0$*:*

$$C_j \mathbf{\Phi}_{m,j} = (I_r \otimes J_{n_j})\mathbf{\Phi}_{m,j} + \mathbf{\Phi}_{m,j} J_{n_1} + (\rho_{p+1} + 1)\frac{d}{d\tau}\left(\widehat{|\boldsymbol{B}_j|} * \mathbf{\Phi}_{m-1}\right).$$

- *Case* $a_{j,r} \neq 0$*:*

$$\mathcal{R}_j \frac{d\mathbf{\Phi}_{m,j}}{d\tau} = \mathcal{S}_j \mathbf{\Phi}_{m,j} + \mathbf{\Phi}_{m,j} J_{n_1} + (\rho_{p+1} + 1)\frac{d}{d\tau}\left(\widehat{|\boldsymbol{B}_j|} * \mathbf{\Phi}_{m-1}\right).$$

We set $\mathbf{\Phi}_0 := \delta I_{rn,n_1}$*.*
In particular, $\mathbf{\Phi}_m(\tau)$ *is an entire function on all* $\mathbb{C}$ *and lies in* $M_{rn,n_1}(\mathbb{R}^+\{\tau\})$ *for all* $m \geq 1$*.*

The following proposition shows that $\widehat{\boldsymbol{g}}$ defines a convenient majorant series of the $\widehat{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}}$'s.

Proposition 3.9 *Let* a *be a constant such that* $|\arg(\tau)| \leq a$ *for all* $\tau \in V$*. Let*

$$C_j = \frac{1}{\max\limits_{1 \leq j \leq J} \exp(2a\,|\mathrm{Im}\,\lambda_j|)} \bigoplus_{u=0}^{r-1} \left(1 - \mathrm{Re}\left(\frac{\lambda_j}{r} - \frac{u}{r}\right)\right) I_{n_j}.$$

Then, for all $m \geq 1$*,* $\tau \in V$*,* $\boldsymbol{\varepsilon} \in \mathcal{D}_p$ *and* $j = 1, ..., J$*, the following inequalities hold:*

$$\left|\boldsymbol{\varphi}^{\boldsymbol{\varepsilon}}_{m,j}(\tau)\right| \leq \mathbf{\Phi}_{m,j}(|\tau|) \tag{3.8}$$

In particular, the series

$$\widehat{\boldsymbol{g}}(|\tau|) = \sum_{m \geq 1} \mathbf{\Phi}_m(|\tau|)$$

is a majorant series of $\widehat{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}}(\tau)$ *for any* $\boldsymbol{\varepsilon} \in \mathcal{D}_p$*.*

Proposition 3.9 is proved by applying Grönwall lemma to systems defining the $\boldsymbol{\varphi}^{\boldsymbol{\varepsilon}}_{m,j}$'s and the $\mathbf{\Phi}_{m,j}$'s. Calculations are similar to those detailed in [7, § 3.2.2] and are left to the reader. However, note that the constant K which appears in [7] is equal to 1 in our case. Indeed, according to the definition of domain V, the "optimal" path γ_τ from 0 to any $\tau \in V$ is here the straigth line $[0, \tau]$.

Remark 3.10 Like system (3.7), the majorant series $\widehat{\boldsymbol{g}}(|\tau|)$ depends on domain V but not on the parameter $\boldsymbol{\varepsilon}$. This is the key point of the proof of theorem 3.1 as we shall see in the next section 3.4.

We are now able to prove theorem 3.1.

3.4 Proof of theorem 3.1

Recall that we must prove the two following points:

(a) the function $(\tau, \boldsymbol{\varepsilon}) \longmapsto \widehat{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}}(\tau) = \sum_{m\geq 1} \boldsymbol{\varphi}_m^{\boldsymbol{\varepsilon}}(\tau)$ is well-defined and holomorphic on $V \times \mathcal{D}_p$,

(b) there exist $C, K > 0$ such that inequality $\left|\widehat{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}}(\tau)\right| \leq Ce^{K|\tau|}$ holds for all $\tau \in V$ and $\boldsymbol{\varepsilon} \in \mathcal{D}_p$.

According to propositions 3.7 and 3.9 and remark 3.10, the series

$$(\tau, \boldsymbol{\varepsilon}) \longmapsto \widehat{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}}(\tau) = \sum_{m\geq 1} \boldsymbol{\varphi}_m^{\boldsymbol{\varepsilon}}(\tau)$$

is a series of holomorphic functions on $V \times \mathcal{D}_p$ which normally converges on all the compact sets of $V \times \mathcal{D}_p$. Hence, point (a).

As for point (b), it stems from inequality $\left|\widehat{\boldsymbol{f}}^{\,\boldsymbol{\varepsilon}}(\tau)\right| \leq \widehat{\boldsymbol{g}}(|\tau|)$ (proposition 3.9) and from the fact that $\widehat{\boldsymbol{g}}$ has exponential growth at infinity (lemma 3.8).

This ends the proof of theorem 3.1.

4 Effective calculation of Stokes multipliers

In this section, we are given a collection $(\theta_k)_{k=0,\ldots,r-1} \in (\mathbb{R}/2\pi\mathbb{Z})^r$ of anti-Stokes directions of system (A) and we consider, for all k, the Stokes-Ramis matrix $St_{\theta_k^\star}$ associated with $\widetilde{Y}(x)$ in the direction θ_k (*cf.* definition 1.1). Split $St_{\theta_k^\star} = [St_{\theta_k^\star}^{j;\ell}]$ into blocks fitting the Jordan structure of the matrix L of exponents of formal monodromy (hence, $St_{\theta_k^\star}^{j;\ell}$ is a $n_j \times n_\ell$-matrix). Split $\widetilde{F}(x)$ in the same way and denote by $\widetilde{F}^{\bullet;\ell}(x)$ its ℓ^{th} column-block (recall that $\widetilde{F}^{\bullet;1}(x) = \widetilde{f}(x)$).

The matrix $St^{j;j}_{\theta^\star_k}$ is the identity matrix I_{n_j} of size n_j and, for $j \neq \ell$, the matrix $St^{j;\ell}_{\theta^\star_k}$ is zero as soon as θ_k is not a direction of maximal decay of exponential $e^{(q_j - q_\ell)(1/x)}$. When θ_k is a direction of maximal decay of exponential $e^{(q_j - q_\ell)(1/x)}$ (hence, $j \neq \ell$ and the Stokes value $a_{j,r} - a_{\ell,r}$ generates the collection (θ_k)), the entries of $St^{j;\ell}_{\theta^\star_k}$ are called *Stokes multipliers of* $\widetilde{F}^{\bullet;\ell}(x)$ *in the direction* θ_k.

The goal of this section is to build a method for the effective calculation of the Stokes multipliers of $\widetilde{F}(x)$ based on the results of the holomorphic perturbation of system (A) stated in section 2.

As in section 3 (*cf.* remark 3.2), we restrict our study to the calculation of the Stokes multipliers of the first column-block $\widetilde{f}(x)$ of $\widetilde{F}(x)$. Henceforth, we denote by $st^{j;\bullet}_{\theta^\star_k}$ in place of $St^{j;1}_{\theta^\star_k}$ and we suppose that (θ_k) is a collection of anti-Stokes directions of system (A) associated with $\widetilde{f}(x)$ (otherwise, $st^{j;\bullet}_{\theta^\star_k} = 0$ for all k and j). Recall that such a collection (θ_k) is generated by (at least) one of the Stokes values $\omega_1, ..., \omega_p$ (= the distinct values of the $a_{j,r} \neq 0$, *cf.* the beginning of section 2).

4.1 Stokes multipliers and connection constants

Let Ω denote the set of Stokes values $\omega_1, ..., \omega_p$. For any $\omega \in \Omega$, we call *front of* ω the set of polynomials $q_j(1/x)$ with leading term $-\omega/x^r$. According to the hypothesis (0.4) of single level equal to r, the front of ω is a singleton

$$\left\{ -\frac{\omega}{x^r} + \dot{q}_\omega\left(\frac{1}{x}\right) \right\}$$

where $\dot{q}_\omega \equiv 0$ or $\dot{q}_\omega(1/x)$ is a polynomial in $1/x$ of degree $\leq r-1$ and with no constant term. When $\dot{q}_\omega \equiv 0$, the Stokes value $\omega \in \Omega$ is said to be *with monomial front*. Notice that, in the case $r = 1$, *all* the Stokes values of Ω are with monomial front.

In the two previous papers [4] (case $r = 1$) and [7] (case $r \geq 2$), M. Loday-Richaud and the author displayed explicit formulæ between the Stokes multipliers of $\widetilde{f}(x)$ associated with the Stokes values $\omega \in \Omega$ [5] with *monomial front* (hence, *all* the Stokes multipliers of $\widetilde{f}(x)$ when $r = 1$) and the connection constants given, in the Borel plane, by the *right analytic continuation* (see [4, § 3.4] for a precise definition) of the Borel transforms $\widehat{\boldsymbol{f}}^{[u]}(\tau)$ at $\tau = \omega$. Recall that such formulæ exist too when ω has a non-monomial front, but

[5] *i.e.*, in the directions generated by the Stokes values $\omega \in \Omega$.

require to first reduce ω into a Stokes value with monomial front by means of a convenient change of the variable x in initial system (A) (*cf.* [7, § 4.3.2]).

Thereby, the effective calculation of the Stokes multipliers of $\widetilde{f}(x)$ can be reduced to the effective calculation of the connection constants of the $\widehat{\boldsymbol{f}}^{[u]}(\tau)$. As an illustration, we develop below three typical examples.

Example 4.1 Let us consider the system

$$x^2\frac{dY}{dx} = \begin{bmatrix} 0 & 0 & 0 \\ x^2 & 1+\dfrac{x}{4} & 0 \\ -2x^3 & x & 3 \end{bmatrix} Y \tag{4.1}$$

together with the formal fundamental solution $\widetilde{Y}(x) = \widetilde{F}(x)x^L e^{Q(1/x)}$ where

- $Q\left(\dfrac{1}{x}\right) = \operatorname{diag}\left(0, -\dfrac{1}{x}, -\dfrac{3}{x}\right)$,
- $L = \operatorname{diag}\left(0, \dfrac{1}{4}, 0\right)$,
- $\widetilde{F}(x) = \begin{bmatrix} 1 & 0 & 0 \\ \widetilde{f}_2(x) & 1 & 0 \\ \widetilde{f}_3(x) & * & 1 \end{bmatrix} \in M_3(\mathbb{C}[[x]])$ satisfies $\widetilde{F}(0) = I_3$,

$$\widetilde{f}_2(x) = -x^2 - \frac{7}{4}x^3 + O(x^3) \quad \text{and} \quad \widetilde{f}_3(x) = x^3 + O(x^3).$$

System (4.1) has the unique level 1 and $\Omega = \{1, 3\}$. Then, the direction $\theta = 0$ is the unique anti-Stokes direction of system (4.1) associated with the first column $\widetilde{f}(x)$ of $\widetilde{F}(x)$. The Stokes-Ramis matrix St_0 in this direction reads

$$St_0 = \begin{bmatrix} 1 & 0 & 0 \\ st_0^2 & 1 & 0 \\ st_0^3 & * & 1 \end{bmatrix}.$$

Furthermore, according to [4, thm. 4.3], the Stokes multiplier st_0^2 (resp. st_0^3) is related to the connection constant $k_{1,+}^2$ (resp. $k_{3,+}^3$) of $\widehat{f}(\xi)$ at the point $\xi = 1$ (resp. $\xi = 3$) by the relation

$$st_0^2 = \frac{(1+i)\pi\sqrt{2}}{\Gamma\left(\dfrac{3}{4}\right)} k_{1,+}^2 \qquad (\text{resp. } st_0^3 = 2i\pi k_{3,+}^3). \tag{4.2}$$

Since the formal series $\widetilde{f_2}(x)$ and $\widetilde{f_3}(x)$ satisfy the equations

$$\begin{cases} x^2\dfrac{d\widetilde{f_2}}{dx} - \left(1+\dfrac{x}{4}\right)\widetilde{f_2} = x^2 \\ \\ x^2\dfrac{d\widetilde{f_3}}{dx} - 3\widetilde{f_3} = -2x^3 + x\widetilde{f_2} \end{cases},$$

their Borel transforms $\widehat{f_2}(\xi)$ and $\widehat{f_3}(\xi)$ are the unique solutions of the system

$$\begin{cases} (\xi-1)\dfrac{d\widehat{f_2}}{dx} + \dfrac{3}{4}\widehat{f_2} = 1 \qquad , \ \widehat{f_2}(0) = 0 \\ \\ (\xi-3)\widehat{f_3} = -\xi^2 + 1 * \widehat{f_2} \end{cases}.$$

Hence, for all $|\xi| < 1$,

$$\begin{cases} \widehat{f_2}(\xi) = \dfrac{4}{3} - \dfrac{4}{3}(1-\xi)^{-3/4} \\ \\ \widehat{f_3}(\xi) = \dfrac{-3\xi^2 + 4\xi - 12 + 12(1-\xi)^{1/4}}{3(\xi-3)} \end{cases}$$

(we chose a determination of the logarithm such that $\ln(\xi) \in \mathbb{R}$ for $\xi > 0$). Thereby, the connection matrices $K_{1,+}$ and $K_{3,+}$ of $\widehat{f}(\xi)$ at the points $\xi = 1$ and $\xi = 3$ are given by

$$K_{1,+} = \begin{bmatrix} 0 \\ k^2_{1,+} = \dfrac{2\sqrt{2}}{3}(1+i) \\ 0 \end{bmatrix} \quad \text{and} \quad K_{3,+} = \begin{bmatrix} 0 \\ 0 \\ k^3_{3,+} = -9 + 2^{7/4}(1+i) \end{bmatrix}.$$

Then, identities (4.2) imply

$$\boxed{st^2_0 = \frac{8i\pi}{3\Gamma\left(\dfrac{3}{4}\right)} \qquad st^3_0 = 2i\pi(2^{7/4} - 9 + 2^{7/4}i)} \tag{4.3}$$

Observe that, in this example, the choice of a triangular matrix for system (4.1) allows us to explicitly write the Borel transform $\widehat{f}(\xi)$ and, consequently, to calculate the exact values of the Stokes multipliers st^2_0 and st^3_0. Of course, such a case is anecdotal and, in a more general situation, *i.e.*, for systems for which the matrices are not triangular, such exact calculations are not possible anymore. Nevertheless, we can *always* determine an *approximation* of the connection constants −hence, of the Stokes multipliers− by using a technique of successive analytic continuations like shown below.

Example 4.2 Let us now consider the system

$$x^2\frac{dY}{dx} = \begin{bmatrix} 0 & 0 & x \\ x & 1+\frac{x}{2} & 0 \\ 0 & -x & 2+\frac{x}{3} \end{bmatrix} Y \tag{4.4}$$

together with the formal fundamental solution $\widetilde{Y}(x) = \widetilde{F}(x)x^L e^{Q(1/x)}$ where

- $Q\left(\frac{1}{x}\right) = \text{diag}\left(0, -\frac{1}{x}, -\frac{2}{x}\right)$,
- $L = \text{diag}\left(0, \frac{1}{2}, \frac{1}{3}\right)$,
- $\widetilde{F}(x) \in M_3(\mathbb{C}[[x]])$ satisfies $\widetilde{F}(x) = I_3 + O(x)$.

As in example 4.1, system (4.4) is a level-one system and $\theta = 0$ is its unique anti-Stokes direction associated with the first column $\widetilde{f}(x)$ of $\widetilde{F}(x)$ (we have $\Omega = \{1, 2\}$). The Stokes-Ramis matrix St_0 reads

$$St_0 = \begin{bmatrix} 1 & 0 & 0 \\ st_0^2 & 1 & 0 \\ st_0^3 & * & 1 \end{bmatrix}$$

and, according to [4, thm. 4.3], the Stokes multiplier st_0^2 (resp. st_0^3) is related to the connection constant $k_{1,+}^2$ (resp. $k_{2,+}^3$) of $\widehat{f}(\xi)$ at the point $\xi = 1$ (resp. $\xi = 2$) by the relation

$$st_0^2 = 2\sqrt{\pi}k_{1,+}^2 \qquad (\text{resp. } st_0^3 = \frac{\pi(\sqrt{3}+i)}{\Gamma\left(\frac{2}{3}\right)}k_{2,+}^3). \tag{4.5}$$

Let $\widetilde{f}(x) = \begin{bmatrix} \widetilde{f_1}(x) \\ \widetilde{f_2}(x) \\ \widetilde{f_3}(x) \end{bmatrix}$. Since the $\widetilde{f_j}$'s are formal series solutions of the system

$$\begin{cases} x^2\dfrac{d\widetilde{f_1}}{dx} = x\widetilde{f_3} \\ x^2\dfrac{d\widetilde{f_2}}{dx} - \left(1+\dfrac{x}{2}\right)\widetilde{f_2} = x\widetilde{f_1} \\ x^2\dfrac{d\widetilde{f_3}}{dx} - \left(2+\dfrac{x}{3}\right)\widetilde{f_3} = -x\widetilde{f_2} \end{cases},$$

their Borel transforms $\widehat{f_j}$'s satisfy the differential equations

$$\begin{cases} \xi\dfrac{d\widehat{f_1}}{d\xi} = -\widehat{f_1} + \widehat{f_3} \\ (\xi-1)\dfrac{d\widehat{f_2}}{d\xi} = \widehat{f_1} - \dfrac{1}{2}\widehat{f_2} \\ (\xi-2)\dfrac{d\widehat{f_3}}{d\xi} = -\widehat{f_2} - \dfrac{2}{3}\widehat{f_3} \end{cases} .$$

Consequently, the Borel transform $\widehat{f}(\xi) = \begin{bmatrix} \widehat{f_1}(\xi) \\ \widehat{f_2}(\xi) \\ \widehat{f_3}(\xi) \end{bmatrix}$ of $\widetilde{f}(x)$ is an analytic solution on the open disc $D(0,1)$ of the system

$$\frac{dZ}{d\xi} = \begin{bmatrix} -\frac{1}{\xi} & 0 & \frac{1}{\xi} \\ \frac{1}{\xi-1} & -\frac{1}{2(\xi-1)} & 0 \\ 0 & -\frac{1}{\xi-2} & -\frac{2}{3(\xi-2)} \end{bmatrix} Z \tag{4.6}$$

which has two regular singular points at $\xi = 1$ and $\xi = 2$. More precisely, system (4.6) reads near $\xi = 1$ as

$$(\xi-1)\frac{dZ}{d\xi} = C_1(\xi-1)Z \quad \text{with } C_1(\xi) := \begin{bmatrix} -\frac{\xi}{\xi+1} & 0 & \frac{\xi}{\xi+1} \\ 1 & -\frac{1}{2} & 0 \\ 0 & -\frac{\xi}{\xi-1} & -\frac{2\xi}{3(\xi-1)} \end{bmatrix} \tag{4.7}$$

and near $\xi = 2$ as

$$(\xi-2)\frac{dZ}{d\xi} = C_2(\xi-2)Z \quad \text{with } C_2(\xi) := \begin{bmatrix} -\frac{\xi}{\xi+2} & 0 & \frac{\xi}{\xi+2} \\ \frac{\xi}{\xi+1} & -\frac{\xi}{2(\xi+1)} & 0 \\ 0 & -1 & -\frac{2}{3} \end{bmatrix} . \tag{4.8}$$

Notice that $C_1(\xi-1)$ is analytic on the open disc $D(1,1)$ and $C_2(\xi-2)$ is analytic on the open disc $D(2,1)$. Following Wasow ([8]), we consider the two matrices

$$D_1 := \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad D_2 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{3}{2} & 1 \end{bmatrix}$$

so that

$$M_1 := D_1^{-1}C_1(0)D_1 = \mathrm{diag}\left(0, -\frac{1}{2}, 0\right) \quad \text{and}$$
$$M_2 := D_2^{-1}C_2(0)D_2 = \mathrm{diag}\left(0, 0, -\frac{2}{3}\right).$$

Hence, choosing as before a determination of the logarithm such that $\ln(\xi) \in \mathbb{R}$ for $\xi > 0$, system (4.7) (resp. system (4.8)) has for fundamental solution at $\xi = 1$ (resp. $\xi = 2$) a matrix of the form

$$Z_1(\xi - 1) = D_1 G_1(\xi - 1)(\xi - 1)^{M_1}$$
$$(\text{resp. } Z_2(\xi - 2) = D_2 G_2(\xi - 2)(\xi - 2)^{M_2})$$

where $G_1(\xi - 1) \in M_3(\mathbb{C}\{\xi - 1\})$ (resp. $G_2(\xi - 2) \in M_3(\mathbb{C}\{\xi - 2\})$) is analytic on the open disc $D(1, 1)$ (resp. $D(2, 1)$) and satisfies $G_1(0) = I_3$ (resp. $G_2(0) = I_3$). More precisely,

- the first and the third columns of $Z_1(\xi - 1)$ are analytic on $D(1, 1)$; the second column of $Z_1(\xi - 1)$ reads as

$$\begin{bmatrix} 0 \\ (\xi - 1)^{-1/2} \\ 0 \end{bmatrix} + (\xi - 1)^{1/2} g_1(\xi - 1)$$

  with $g_1(\xi - 1)$ analytic on $D(1, 1)$,

- the two first columns of $Z_2(\xi - 2)$ are analytic on $D(2, 1)$; the third column of $Z_2(\xi - 2)$ reads as

$$\begin{bmatrix} 0 \\ 0 \\ (\xi - 2)^{-2/3} \end{bmatrix} + (\xi - 2)^{1/3} g_2(\xi - 2)$$

  with $g_2(\xi - 2)$ analytic on $D(2, 1)$.

Following Cauchy's theorem, the right analytic continuation of $\widehat{f}$ (still denoted $\widehat{f}$) at the point $\xi = 1$ (resp. $\xi = 2$) is a solution of system (4.7) (resp. system (4.8)). Thereby, there exists a unique matrix

$$S_1 := \begin{bmatrix} \sigma_1^1 \\ \sigma_1^2 \\ \sigma_1^3 \end{bmatrix} \in M_{3,1}(\mathbb{C}) \qquad (\text{resp. } S_2 := \begin{bmatrix} \sigma_2^1 \\ \sigma_2^2 \\ \sigma_2^3 \end{bmatrix} \in M_{3,1}(\mathbb{C}))$$

such that $\widehat{f}(\xi) = Z_1(\xi - 1)S_1$ for all $\xi \in D(1,1)\backslash\{1\}$ (resp. $\widehat{f}(\xi) = Z_2(\xi - 2)S_2$ for all $\xi \in D(2,1)\backslash\{2\}$). In particular, calculations above shows that the connection constant $k^2_{1,+}$ (resp. $k^3_{2,+}$) is equal to σ_1^2 (resp. σ_2^3), and, consequently, identities (4.5) imply

$$st_0^2 = 2\sqrt{\pi}\sigma_1^2 \quad \text{and} \quad st_0^3 = \frac{\pi(\sqrt{3}+i)}{\Gamma\left(\frac{2}{3}\right)}\sigma_2^3.$$

We are left to evaluate σ_1^2 and σ_2^3. According to the geometry of the "convergence discs" $D(0,1)$, $D(1,1)$ and $D(2,1)$ (see figure 4.1 below), we evaluate, on one hand, $\widehat{f}(\xi)$ and $Z_1(\xi-1)$ at the point $\xi = 1/2$ and, on the other hand, $Z_1(\xi - 1)$ and $Z_2(\xi - 2)$ at the point $\xi = 3/2$.

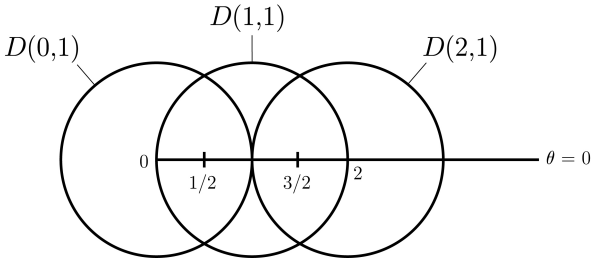

Figure 4.1

Then,

$$S_1 = Z_1\left(-\frac{1}{2}\right)^{-1}\widehat{f}\left(\frac{1}{2}\right) \quad \text{and} \quad S_2 = Z_2\left(-\frac{1}{2}\right)^{-1} Z_1\left(\frac{1}{2}\right) S_1.$$

Observe that, by definition of the right analytic continuation, $Z_1(-1/2)$ and $Z_2(-1/2)$ are evaluate at the point $-1/2$ such that $\arg(-1/2) = -\pi$. Hence, one can check that

$$\begin{aligned} \sigma_1^2 &\approx 0.46823766i \\ \sigma_2^3 &\approx 3.05123307 + 2.39083857i \end{aligned}$$

and, consequently,

$$\boxed{\begin{aligned} st_0^2 &\approx 1.6598593i \\ st_0^3 &\approx 6.714284368 + 16.68631306i \end{aligned}}.$$

This method by successive analytic continuations still holds for systems with a single arbitrary level $r \geq 2$. However, the calculations may be much more difficult when one of the singular points of Ω generating the collection (θ_k) is with non-monomial front.

Example 4.3 Here below, we consider the system

$$x^3 \frac{dY}{dx} = \begin{bmatrix} 0 & 0 & x^2 \\ x^2 & 1+x & 0 \\ 0 & x^2 & 2+\dfrac{x^2}{2} \end{bmatrix} Y \tag{4.9}$$

together with the formal fundamental solution $\widetilde{Y}(x) = \widetilde{F}(x) x^L e^{Q(1/x)}$ where

- $Q\left(\dfrac{1}{x}\right) = \text{diag}\left(0, -\dfrac{1}{2x^2} - \dfrac{1}{x}, -\dfrac{1}{x^2}\right)$ (hence, the system has the unique level 2, $\Omega = \{1/2, 1\}$ and the front of $1/2$ (resp. 1) is non-monomial (resp. monomial)),
- $L = \text{diag}\left(0, 0, \dfrac{1}{2}\right)$,
- $\widetilde{F}(x) \in M_3(\mathbb{C}[[x]])$ satisfies $\widetilde{F}(x) = I_3 + O(x^2)$.

As before, we denote by $\widetilde{f}(x)$ the first column of $\widetilde{F}(x)$. We also denote by $\widetilde{\boldsymbol{f}}^{\,[u]}(t)$ with $u = 0, 1$ the 2-reduced series of $\widetilde{f}(x)$.
Let $(\theta_0 = 0, \theta_1 = -\pi)$ be the unique collection of anti-Stokes directions of system (4.9) associated with $\widetilde{f}(x)$. For all $k \in \{0, 1\}$, the corresponding Stokes-Ramis matrix St_{θ_k} reads

$$St_{\theta_k} = \begin{bmatrix} 1 & 0 & 0 \\ st^2_{\theta_k} & 1 & 0 \\ st^3_{\theta_k} & * & 1 \end{bmatrix}.$$

We are just interested below in the calculation of the Stokes multipliers $st^3_{\theta_k}$'s associated with the Stokes value with monomial front 1. According to [7, cor. 4.5], the $st^3_{\theta_k}$'s are related to the connection constants $k^{[u]}_{1,+}$ of $\widehat{\boldsymbol{f}}^{\,[u]}(\tau)$ at the point $\tau = 1$ by the relations

$$\begin{cases} st^3_0 = \dfrac{(1+i)\pi\sqrt{2}}{\Gamma\left(\dfrac{3}{4}\right)} k^{[0]}_{1,+} - (4-4i)\Gamma\left(\dfrac{3}{4}\right) k^{[1]}_{1,+} \\ \\ st^3_{-\pi} = \dfrac{(-1+i)\pi\sqrt{2}}{\Gamma\left(\dfrac{3}{4}\right)} k^{[0]}_{1,+} + (4+4i)\Gamma\left(\dfrac{3}{4}\right) k^{[1]}_{1,+} \end{cases}.$$

To determine an approximation of the $k_{1,+}^{[u]}$'s, we can proceed, like in example 4.2, by a method of successive analytic continuations. According to proposition 3.3,

$$\widetilde{\boldsymbol{f}}(t) := \begin{bmatrix} \widetilde{\boldsymbol{f}}^{[0]}(t) \\ \widetilde{\boldsymbol{f}}^{[1]}(t) \end{bmatrix} \in M_{6,1}(\mathbb{C}[[t]])$$

is a formal series solution of the system

$$2t^2 \frac{d\mathbf{Y}}{dt} = \begin{bmatrix} 0 & 0 & t & 0 & 0 & 0 \\ t & 1 & 0 & 0 & t & 0 \\ 0 & t & 2+\frac{t}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -t & 0 & t \\ 0 & 1 & 0 & t & 1-t & 0 \\ 0 & 0 & 0 & 0 & t & 2-\frac{t}{2} \end{bmatrix} \mathbf{Y}.$$

By adapting calculations of previous example 4.2, one can check that the Borel transform $\widehat{\boldsymbol{f}}(\tau)$ of $\widetilde{\boldsymbol{f}}(t)$ is an analytic solution on the open disc $D(0,1/2)$ of the system

$$\frac{dZ}{d\tau} = \begin{bmatrix} -\frac{1}{\tau} & 0 & \frac{1}{2\tau} & 0 & 0 & 0 \\ \frac{1}{2\tau-1} & -\frac{2}{2\tau-1} & 0 & 0 & \frac{1}{2\tau-1} & 0 \\ 0 & \frac{1}{2(\tau-1)} & -\frac{3}{4(\tau-1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{3}{2\tau} & 0 & \frac{1}{2\tau} \\ \frac{1}{(2\tau-1)^2} & -\frac{2}{(2\tau-1)^2} & 0 & \frac{1}{2\tau-1} & -\frac{2(3\tau-2)}{(2\tau-1)^2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2(\tau-1)} & -\frac{5}{4(\tau-1)} \end{bmatrix} Z \tag{4.10}$$

which has an *irregular* singular point at the Stokes value with *non-monomial front* $\tau = 1/2$ and a *regular* singular point at the Stokes value with *monomial front* $\tau = 1$. More precisely, system (4.10) reads near $\tau = 1$ as

$$(\tau - 1)\frac{dZ}{d\tau} = C_1(\tau - 1)Z \tag{4.11}$$

where $C_1(\tau - 1)$ is the analytic matrix on the open disc $D(1,1/2)$ defined by

$$C_1(\tau) := \begin{bmatrix} -\frac{\tau}{\tau+1} & 0 & \frac{\tau}{2(\tau+1)} & 0 & 0 & 0 \\ \frac{\tau}{2\tau+1} & -\frac{2\tau}{2\tau+1} & 0 & 0 & \frac{\tau}{2\tau+1} & 0 \\ 0 & \frac{1}{2} & -\frac{3}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{3\tau}{2(\tau+1)} & 0 & \frac{\tau}{2(\tau+1)} \\ \frac{\tau}{(2\tau+1)^2} & -\frac{2\tau}{(2\tau+1)^2} & 0 & \frac{\tau}{2\tau+1} & -\frac{2\tau(3\tau+1)}{(2\tau+1)^2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{5}{4} \end{bmatrix}.$$

As in example 4.2, we consider the matrix

$$D_1 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{5} & 1 \end{bmatrix}$$

so that

$$M_1 := D_1^{-1} C_1(0) D_1 = \operatorname{diag}\left(0, 0, -\frac{3}{4}, 0, 0, -\frac{5}{4}\right).$$

Then, system (4.11) has a fundamental solution of the form $Z_1(\tau - 1) = D_1 G_1(\tau - 1)(\tau - 1)^{M_1}$ where $G_1(\tau - 1) \in M_6(\mathbb{C}\{\tau - 1\})$ is analytic on the open disc $D(1, 1/2)$ and satisfies $G_1(0) = I_6$ (*cf.* [8]). More precisely,

- the third column of $Z_1(\tau - 1)$ reads

$$\begin{bmatrix} 0 \\ 0 \\ (\tau - 1)^{-3/4} \\ 0 \\ 0 \\ 0 \end{bmatrix} + (\tau - 1)^{1/4} g_{1,3}(\tau - 1)$$

  where $g_{1,3}(\tau - 1)$ is analytic on $D(1, 1/2)$,

- the sixth column of $Z_1(\tau - 1)$ reads

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ (\tau - 1)^{-5/4} \end{bmatrix} + (\tau - 1)^{-1/4} g_{1,6}(\tau - 1)$$

  where $g_{1,6}(\tau - 1)$ is analytic on $D(1, 1/2)$,

- the four other columns of $Z_1(\tau - 1)$ are analytic on $D(1, 1/2)$.

Since the right analytic continuation of $\widehat{\boldsymbol{f}}(\tau)$ (still denoted $\widehat{\boldsymbol{f}}(\tau)$) at the point

$\tau = 1$ is a solution of system (4.11), there exists a unique matrix

$$S_1 = \begin{bmatrix} \sigma_1^1 \\ \sigma_1^2 \\ \sigma_1^3 \\ \sigma_1^4 \\ \sigma_1^5 \\ \sigma_1^6 \end{bmatrix} \in M_{6,1}(\mathbb{C})$$

such that $\widehat{\boldsymbol{f}}(\tau) = Z_1(\tau - 1)S_1$ for all $\tau \in D(1, 1/2)\backslash\{1\}$. Thereby, the connection constants $k_{1,+}^{[0]3}$ and $k_{1,+}^{[1]3}$ are given by

$$k_{1,+}^{[0]3} = \sigma_1^3 \quad \text{and} \quad k_{1,+}^{[1]3} = \sigma_1^6.$$

The two constants σ_1^3 and σ_1^6 can be numerically evaluate in a similar way as example 4.2 by considering the analytic continuation of $\widehat{\boldsymbol{f}}$ from the disc $D(0, 1/2)$ (= the disc of convergence of $\widehat{\boldsymbol{f}}(\tau)$) to the disc $D(1, 1/2)$ (= the disc of "convergence"of $Z_1(\tau-1)$) through any disc of the form $D(1/2-ia, a)$ with $a > 0$.

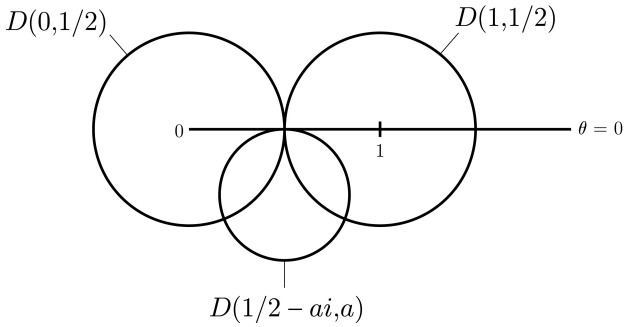

Figure 4.2

Notice that, for any $a > 0$, the point $\alpha = 1/2 - ia$ is an ordinary point of system (4.10); hence, any of its fundamental solution is analytic on the disc $D(\alpha, a)$. Notice also that the choice of such a disc is due to the fact that we must bypass the irregular singular point $\tau = 1/2$ of system (4.10) to the right to connect $D(0, 1/2)$ and $D(1, 1/2)$.

The two previous examples 4.2 and 4.3 bring us to the following general remark.

Remark 4.4 Let (θ_k) be a collection of anti-Stokes directions of system (A) associated with the first column-block $\widetilde{f}(x)$ of $\widetilde{F}(x)$. Let us assume that this collection is generated by $s \geq 2$ Stokes values of Ω, say $\omega_1, \omega_2, ..., \omega_s$ with $|\omega_1| < |\omega_2| < ... < |\omega_s|$.
Fix $\ell \in \{2, ..., s\}$ and suppose that ω_ℓ has a *monomial* front (recall that such a condition can always be fulfilled by means of a convenient change of the variable x in system (A)). Then, as shown in examples 4.2 and 4.3 above, the connection constants of the $\widehat{\boldsymbol{f}}^{\,[u]}(\tau)$'s at ω_ℓ can be evaluate as follows:

(a) evaluate all or part of the connection constants at the intermediate Stokes values $\omega_1, ..., \omega_{\ell-1}$ who have a monomial front,

(b) bypass all or part of the intermediate Stokes values $\omega_1, ..., \omega_{\ell-1}$ to the right (always those with a non-monomial front and possibly the others).

With a numerical point of view, these two methods pose some problems. Indeed, point (a) requires to handle fundamental solutions at regular singular points (see example 4.2) and their numerical evaluations are much more difficult than those of fundamental solutions at ordinary points. As for point (b), if it allows to avoid handling fundamental solutions at regular singular points as point (a) by focusing on fundamental solutions at ordinary points, it significantly increases the number of intermediate numerical evaluations which can degrade the precision of the results obtained (see example 4.3).

In section 4.2 below, we build an alternative method for the effective calculation of Stokes multipliers in order to get around all these difficults. This method is based on a perturbation of system (A) in which each perturbed Stokes value generates its *own* collection of anti-Stokes directions.

4.2 Effective calculation and perturbation

We consider here below a collection (θ_k) of anti-Stokes directions of system (A) associated with the first column-block $\widetilde{f}(x)$ of $\widetilde{F}(x)$. As previously, we denote by Ω the set of nonzero Stokes values $\omega_1, ..., \omega_p$ of system (A) associated with $\widetilde{f}(x)$. We also denote by $\Omega_{(\theta_k)}$ the set of Stokes values of Ω generating the collection (θ_k).

The goal of this section is to build a method for effective calculation of the Stokes multipliers of $\widetilde{f}(x)$ in the directions θ_k, $k = 0, ..., r-1$, when the cardinal $\sharp\Omega_{(\theta_k)}$ of $\Omega_{(\theta_k)}$ is ≥ 2 (hence, $p \geq 2$ too).

4.2.1 Case of two Stokes values

1. Setting the problem

In this section, we suppose $\sharp\Omega_{(\theta_k)} = 2$, *i.e.*, just two Stokes values of Ω, say ω_1 and ω_2, generate the collection (θ_k). We also suppose, without loss of generality, that $|\omega_1| < |\omega_2|$.

Then, as collection of anti-Stokes directions of the full matrix $\widetilde{F}(x)$, the collection (θ_k) is generated by the three Stokes values ω_1, ω_2 and $\omega_2 - \omega_1$ and possibly by the Stokes values of the form

- $\omega_j - \omega_k$ with $\omega_j \in \Omega_{(\theta_k)}$, $\omega_k \notin \Omega_{(\theta_k)}$ and $\arg(\omega_k) = r\theta_0 - \pi$

or of the form

- $\omega_j - \omega_k$ with $\omega_j, \omega_k \notin \Omega_{(\theta_k)}$, *i.e.*, distinct of ω_1 and ω_2

if they exist.

2. A perturbed system

Let us now fix $\delta > 0$ and let us consider, for all $\varepsilon \in [0, \delta]$, the system (A^ε) in which the initial Stokes value ω_2 of system (A) is replaced by $\omega_2 e^{-ir\varepsilon}$. Let Ω^ε denote the set deduced from Ω by replacing ω_2 by $\omega_2 e^{-ir\varepsilon}$ too. Then, for all $\varepsilon \in [0, \delta]$, Ω^ε is the set of nonzero Stokes values of system (A^ε) associated with the first column-block $\widetilde{f}^\varepsilon(x)$ of $\widetilde{F}^\varepsilon(x)$ (we resume the perturbed notations as section 2).

Observe that, for δ small enough, the set of systems $(A^\varepsilon)_{\varepsilon\in[0,\delta]}$ defines a sub-perturbation $\mathcal{P}_\delta(A)$ of the holomorphic perturbation of system (A) studied in section 2. In particular, the image of (θ_k) by $\mathcal{P}_\delta(A)$ is a subset of $(D\Sigma_{\theta_k,\eta(\boldsymbol{\theta})/r})_{k=0,\dots,r-1}$ (*cf.* proposition 2.9) and corollay 2.15 tells us that the corresponding Stokes matrices $\mathfrak{S}^\varepsilon_{\theta_k^\star}$ (see page 16) tend, for all $k = 0, ..., r-1$, to the initial Stokes-Ramis matrices $St_{\theta_k^\star}$ when ε goes to 0.

Lemmas 4.5 and 4.6 below allow us to precise this last result by making explicit the image of (θ_k) by $\mathcal{P}_\delta(A)$ as well as the form of the matrices $\mathfrak{S}^\varepsilon_{\theta_k^\star}$.

Lemma 4.5 (Action of $\mathcal{P}_\delta(A)$ on the collection (θ_k))
Given $\varepsilon \in [0, \delta]$, the collection (θ_k) of initial system (A) splits into the following collections of anti-Stokes directions of system (A^ε):

1. *the collection (θ_k) which is generated by the Stokes value ω_1 and possibly by all the Stokes values of the form $\omega_1 - \omega_k$ with $\arg(\omega_k) = r\theta_0 - \pi$ or of the form $\omega_j - \omega_k$ with $\omega_j, \omega_k \notin \Omega_{(\theta_k)}$ if they exist,*

2. *the collection* $(\theta_k - \varepsilon)$ *which is generated by the Stokes value* $\omega_2 e^{-ir\varepsilon}$,

3. *the collection* $(\theta_{k,\varepsilon})$ *which is generated by the Stokes value* $\omega_2 e^{-ir\varepsilon} - \omega_1$,

4. *the possible* $\ell \geq 1$ *collections* $(\theta_{k,1,\varepsilon})$, ..., $(\theta_{k,\ell,\varepsilon})$ *which are generated by all the Stokes values of the form* $\omega_2 e^{-ir\varepsilon} - \omega_k$ *with* $\arg(\omega_k) = r\theta_0 - \pi$ *if they exist.*

Furthermore, for all $\varepsilon \neq 0$, *the principal determinations* $\theta^\star \in]-2\pi, 0]$ *of all these directions* θ *satisfy*

$$-\frac{2k\pi}{r} \geq \theta_k^\star > \theta_{k,1,\varepsilon}^\star > ... > \theta_{k,\ell,\varepsilon}^\star > (\theta_k - \varepsilon)^\star > \theta_{k,\varepsilon}^\star > -\frac{2(k+1)\pi}{r} \tag{4.12}$$

for all $k = 0, ..., r-1$ *(the chosen order on the* $\theta_{k,s,\varepsilon}^\star$ *is to fix ideas).*

Observe that, among all the collections above, collections (θ_k) and $(\theta_k - \varepsilon)$ are, for all $\varepsilon \in [0, \delta]$, the unique collections of anti-Stokes directions of system (A^ε) associated with $\widetilde{f}^\varepsilon(x)$. Moreover, for $\varepsilon \neq 0$, they are both generated by *just one* Stokes value of Ω^ε.

For any direction $*$ of lemma 4.5, we denote by St_{**}^ε the corresponding Stokes-Ramis matrix. Then, according to inequalities (4.12), the following lemma holds.

Lemma 4.6 (Description of the Stokes matrices $\mathfrak{S}_{\theta_k^\star}^\varepsilon$)
Let $k \in \{0, ..., r-1\}$ *and* $\varepsilon \in [0, \delta]$, $\varepsilon \neq 0$.
Let M_k^ε *be the matrix defined by* $M_k^\varepsilon := St_{\theta_{k,1,\varepsilon}^\star}^\varepsilon St_{\theta_{k,2,\varepsilon}^\star}^\varepsilon ... St_{\theta_{k,\ell,\varepsilon}^\star}^\varepsilon$ *when the collections* $(\theta_{k,s,\varepsilon})$*'s exist and by* $M_k^\varepsilon := I_n$ *otherwise.*
Then,

$$\boxed{\mathfrak{S}_{\theta_k^\star}^\varepsilon = St_{\theta_k^\star}^\varepsilon M_k^\varepsilon St_{(\theta_k - \varepsilon)^\star}^\varepsilon St_{\theta_{k,\varepsilon}^\star}^\varepsilon}.$$

We shall now precise the structure of the Stokes-Ramis matrices St_{**}^ε of lemma 4.6 above. As before, we split all these matrices into blocks $St_{**}^{\varepsilon j;\ell}$ of size $n_j \times n_\ell$ (recall that n_j denotes the size of the j^{th} Jordan block of the matrix L of exponents of formal monodromy of initial system (A)). Then, lemma 4.5 implies:

Lemma 4.7 (Structure of the Stokes-Ramis matrices St_{**}^ε)
Let $k \in \{0, ..., r-1\}$ *and* $\varepsilon \in [0, \delta]$, $\varepsilon \neq 0$.
Recall that the $a_{j,r}$*'s denote, for all* $j = 1, ..., J$, *the Stokes values of initial system* (A) *associated with* $\widetilde{f}^\varepsilon(x)$.
Then,

1. *Stokes-Ramis matrix* $St^{\varepsilon}_{\theta^\star_k}$*:*

   - $St^{\varepsilon j;j}_{\theta^\star_k} = I_{n_j}$ *for all* j,
   - $St^{\varepsilon j;1}_{\theta^\star_k} = 0$ *as soon as* $j \neq 1$ *and* $a_{j,r} \neq \omega_1$,
   - $St^{\varepsilon j;\ell}_{\theta^\star_k} = 0$ *as soon as* $j \neq \ell$ *and* $a_{\ell,r} = \omega_2$.

2. *Stokes-Ramis matrix* $St^{\varepsilon}_{(\theta_k-\varepsilon)^\star}$*:*

   - $St^{\varepsilon j;j}_{(\theta_k-\varepsilon)^\star} = I_{n_j}$ *for all* j,
   - *for* $j \neq \ell$, $St^{\varepsilon j;\ell}_{(\theta_k-\varepsilon)^\star} = 0$ *as soon as* $a_{j,r} \neq \omega_2$ *or* $\ell \neq 1$.

3. *Stokes-Ramis matrix* $St^{\varepsilon}_{\theta^\star_{k,\varepsilon}}$ *:*

   - $St^{\varepsilon j;j}_{\theta^\star_{k,\varepsilon}} = I_{n_j}$ *for all* j,
   - *for* $j \neq \ell$, $St^{\varepsilon j;\ell}_{\theta^\star_{k,\varepsilon}} = 0$ *as soon as* $a_{j,r} \neq \omega_2$ *or* $a_{\ell,r} \neq \omega_1$.

4. *Stokes-Ramis matrices* $St^{\varepsilon}_{\theta^\star_{k,s,\varepsilon}}$, $s = 1, .., \ell$*:*

   - $St^{\varepsilon j;j}_{\theta^\star_{k,s,\varepsilon}} = I_{n_j}$ *for all* j,
   - *for* $j \neq \ell$, $St^{\varepsilon j;\ell}_{\theta^\star_{k,s,\varepsilon}} = 0$ *as soon as* $\ell = 1$ *or* $a_{\ell,r} = \omega_2$.

Let us now denote by $st^{\varepsilon j;\bullet}_{\theta^\star_k}$ (resp. $st^{\varepsilon j;\bullet}_{(\theta_k-\varepsilon)^\star}$) in place of $St^{\varepsilon j;1}_{\theta^\star_k}$ (resp. $St^{\varepsilon j;1}_{(\theta_k-\varepsilon)^\star}$). The entries of $st^{\varepsilon j;\bullet}_{\theta^\star_k}$ (resp. $st^{\varepsilon j;\bullet}_{(\theta_k-\varepsilon)^\star}$) for j such that $a_{j,r} = \omega_1$ (resp. $a_{j,r} = \omega_2$) are the perturbed Stokes multipliers of $\widetilde{f}^{\,\varepsilon}(x)$ in the direction θ_k (resp. $\theta_k - \varepsilon$).

As a result of the various structures of the Stokes-Ramis matrices $St^{\varepsilon}_{*^\star}$ given in lemma 4.7 above, lemma 4.6 and corollary 2.15 imply the following proposition.

Proposition 4.8 (Initial vs perturbed Stokes multipliers)
For all $j \in \{1, ..., J\}$ *such that* $a_{j,r} \in \Omega_{(\theta_k)} = \{\omega_1, \omega_2\}$*, the initial Stokes multipliers* $st^{j;\bullet}_{\theta^\star_k}$ *of* $\widetilde{f}(x)$ *are related, for all* $k \in \{0, ..., r-1\}$*, to the perturbed Stokes multipliers* $st^{\varepsilon j;\bullet}_{\theta^\star_k}$ *and* $st^{\varepsilon j;\bullet}_{(\theta_k-\varepsilon)^\star}$ *of* $\widetilde{f}^{\,\varepsilon}(x)$ *by the relations*

$$\boxed{\begin{array}{ll} st^{j;\bullet}_{\theta^\star_k} = \lim_{\varepsilon\to 0} st^{\varepsilon j;\bullet}_{\theta^\star_k} & \text{if } a_{j,r} = \omega_1 \\ st^{j;\bullet}_{\theta^\star_k} = \lim_{\varepsilon\to 0} st^{\varepsilon j;\bullet}_{(\theta_k-\varepsilon)^\star} & \text{if } a_{j,r} = \omega_2 \end{array}} \quad . \tag{4.13}$$

Remark 4.9 In practice, relations (4.13) are rather difficult to apply since the perturbed Stokes multipliers, like the initial Stokes multipliers, can not be displayed in general. Nevertheless, proposition 4.8 tells us that, for ε small enough, the perturbed Stokes multipliers provide a "good"approximation of the initial Stokes multipliers.

As an illustration of proposition 4.8, we shall develop below two typical examples.

3. Examples

Here below, we revisit the two previous examples 4.1 and 4.2 with the point of view of the perturbative method. More precisely, we perturb each of systems (4.1) and (4.4) as above; then we "evaluate" the perturbed Stokes multipliers and we compare the values of initial Stokes multipliers obtained by proposition 4.8 with those previously obtained in examples 4.1 and 4.2.

Those two examples illustrate the two situations that may occur with our perturbative method (see remark 4.9). In the first one, we are able to calculate the exact values of the perturbed Stokes multipliers for any value of ε; hence, we can apply relations (4.13) as they are. As before, this case is, of course, anecdotal but it is worth to be treated. In the second one, such exact calculations are not possible anymore. In that case, we have to calculate an approximate value of the perturbed Stokes multipliers for some small values of ε, say of the form $\varepsilon = 10^{-m}$ with $m \geq 1$.

Example 4.10 We consider, for $\varepsilon > 0$ small enough, the perturbed system

$$x^2 \frac{dY}{dx} = \begin{bmatrix} 0 & 0 & 0 \\ x^2 & 1 + \dfrac{x}{4} & 0 \\ -2x^3 & x & 3e^{-i\varepsilon} \end{bmatrix} Y \tag{4.14}$$

of system (4.1) (*cf.* example 4.1) together with the formal fundamental solution $\widetilde{Y}^{\,\varepsilon}(x) = \widetilde{F}^{\,\varepsilon}(x) x^L e^{Q^\varepsilon(1/x)}$ where

- $Q^\varepsilon\left(\dfrac{1}{x}\right) = \operatorname{diag}\left(0, -\dfrac{1}{x}, -\dfrac{3e^{-i\varepsilon}}{x}\right)$,
- $L = \operatorname{diag}\left(0, \dfrac{1}{4}, 0\right)$,
- $\widetilde{F}^{\,\varepsilon}(x) = \begin{bmatrix} 1 & 0 & 0 \\ \widetilde{f}_2^{\,\varepsilon}(x) & 1 & 0 \\ \widetilde{f}_3^{\,\varepsilon}(x) & * & 1 \end{bmatrix} \in M_3(\mathbb{C}[[x]])$ satisfies $\widetilde{F}^{\,\varepsilon}(x) = I_3 + O(x^2)$.

System (4.14) has the unique level 1 and its anti-Stokes directions associated with the first column $\widetilde{f}^{\,\varepsilon}(x)$ of $\widetilde{F}^{\,\varepsilon}(x)$ are the direction $\theta = 0$ generated by the Stokes value 1 and the direction $\theta = -\varepsilon$ generated by the Stokes value $3e^{-i\varepsilon}$. The corresponding Stokes-Ramis matrices St_0^ε and $St_{-\varepsilon}^\varepsilon$ read

$$St_0^\varepsilon = \begin{bmatrix} 1 & 0 & 0 \\ st_0^{\varepsilon 2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad St_{-\varepsilon}^\varepsilon = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ st_{-\varepsilon}^{\varepsilon 3} & 0 & 1 \end{bmatrix}$$

where, according to [4, thm. 4.3], the Stokes multiplier $st_0^{\varepsilon 2}$ (resp. $st_{-\varepsilon}^{\varepsilon 3}$) is related to the connection constant $k_{1,+}^{\varepsilon 2}$ (resp. $k_{3e^{-i\varepsilon},+}^{\varepsilon 3}$) of $\widehat{f}^{\,\varepsilon}(\xi)$ at the point $\xi = 1$ (resp. $\xi = 3e^{-i\varepsilon}$) by the relation

$$st_0^{\varepsilon 2} = \frac{(1+i)\pi\sqrt{2}}{\Gamma\left(\frac{3}{4}\right)} k_{1,+}^{\varepsilon 2} \qquad (\text{resp. } st_{-\varepsilon}^{\varepsilon 3} = 2i\pi k_{3e^{-i\varepsilon},+}^{\varepsilon 3}).$$

As in example 4.1, the connection constants $k_{1,+}^{\varepsilon 2}$ and $k_{3e^{-i\varepsilon},+}^{\varepsilon 3}$ and, consequently, the Stokes multipliers $st_0^{\varepsilon 2}$ and $st_{-\varepsilon}^{\varepsilon 3}$, can be explicitly calculate. More precisely, by adapting the calculations made in example 4.1, one can check that the Borel transforms $\widehat{f}_2^{\,\varepsilon}(\xi)$ of $\widetilde{f}_2^{\,\varepsilon}(x)$ and $\widehat{f}_3^{\,\varepsilon}(\xi)$ of $\widetilde{f}_3^{\,\varepsilon}(x)$ read

$$\begin{cases} \widehat{f}_2^{\,\varepsilon}(\xi) = \dfrac{4}{3} - \dfrac{4}{3}(1-\xi)^{-3/4} \\ \\ \widehat{f}_3^{\,\varepsilon}(\xi) = \dfrac{-3\xi^2 + 4\xi - 12 + 12(1-\xi)^{1/4}}{3(\xi - 3e^{-i\varepsilon})} \end{cases}$$

for all $|\xi| < 1$. Hence,

$$\begin{cases} k_{1,+}^{\varepsilon 2} = \dfrac{2\sqrt{2}}{3}(1+i) \\ \\ k_{3e^{-i\varepsilon},+}^{\varepsilon 3} = -9e^{-2i\varepsilon} + 4e^{-i\varepsilon} - 4 + 4(1-3e^{-i\varepsilon})^{1/4} \end{cases}$$

(recall that we chose a determination of the logarithm such that $\ln(\xi) \in \mathbb{R}$ for $\xi > 0$) and, consequently,

$$\boxed{\begin{cases} st_0^{\varepsilon 2} = \dfrac{8i\pi}{3\Gamma\left(\frac{3}{4}\right)} \\ \\ st_{-\varepsilon}^{\varepsilon 3} = 2i\pi(-9e^{-2i\varepsilon} + 4e^{-i\varepsilon} - 4 + 4(1-3e^{-i\varepsilon})^{1/4}) \end{cases}}.$$

Now, we apply proposition 4.8: the Stokes multipliers st_0^2 and st_0^3 of initial system (4.1) are given by

$$st_0^2 = \lim_{\varepsilon\to 0} st_0^{\varepsilon 2} \quad \text{and} \quad st_0^3 = \lim_{\varepsilon\to 0} st_{-\varepsilon}^{\varepsilon 3}.$$

Then, we get

$$st_0^2 = \frac{8i\pi}{3\Gamma\left(\frac{3}{4}\right)} \quad \text{and} \quad st_0^3 = 2i\pi(2^{7/4} - 9 + 2^{7/4}i)$$

which are the same values as those calculated in example 4.1.

Example 4.11 Let us now consider, for $\varepsilon > 0$ small enough, the perturbed system

$$x^2\frac{dY}{dx} = \begin{bmatrix} 0 & 0 & x \\ x & 1+\frac{x}{2} & 0 \\ 0 & -x & 2e^{-i\varepsilon}+\frac{x}{3} \end{bmatrix} Y \tag{4.15}$$

of system (4.4) (*cf.* example 4.2) together with the formal fundamental solution $\widetilde{Y}^{\,\varepsilon}(x) = \widetilde{F}^{\,\varepsilon}(x)x^L e^{Q^\varepsilon(1/x)}$ where

- $Q^\varepsilon\left(\frac{1}{x}\right) = \operatorname{diag}\left(0, -\frac{1}{x}, -\frac{2e^{-i\varepsilon}}{x}\right)$,
- $L = \operatorname{diag}\left(0, \frac{1}{2}, \frac{1}{3}\right)$,
- $\widetilde{F}^{\,\varepsilon}(x) \in M_3(\mathbb{C}[[x]])$ satisfies $\widetilde{F}^{\,\varepsilon}(x) = I_3 + O(x)$.

System (4.15) is again a level-one system; its anti-Stokes directions associated with the first column $\widetilde{f}^{\,\varepsilon}(x)$ of $\widetilde{F}^{\,\varepsilon}(x)$ are the directions $\theta = 0$ and $\theta = -\varepsilon$ (we have $\Omega^\varepsilon = \{1, 2e^{-i\varepsilon}\}$) and the corresponding Stokes-Ramis matrices St_0^ε and $St_{-\varepsilon}^\varepsilon$ read

$$St_0^\varepsilon = \begin{bmatrix} 1 & 0 & 0 \\ st_0^{\varepsilon 2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad St_{-\varepsilon}^\varepsilon = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ st_{-\varepsilon}^{\varepsilon 3} & 0 & 1 \end{bmatrix}.$$

Furthermore, according to [4, thm. 4.3], the Stokes multiplier $st_0^{\varepsilon 2}$ (resp. $st_{-\varepsilon}^{\varepsilon 3}$) is related to the connection constant $k_{1,+}^{\varepsilon 2}$ (resp. $k_{2e^{-i\varepsilon},+}^{\varepsilon 3}$) of $\widehat{f}^{\,\varepsilon}(\xi)$ at

the point $\xi = 1$ (resp. $\xi = 2e^{-i\varepsilon}$) by the relation

$$st_0^{\varepsilon 2} = 2\sqrt{\pi} k_{1,+}^{\varepsilon 2} \qquad (\text{resp. } st_{-\varepsilon}^{\varepsilon 3} = \frac{\pi(\sqrt{3}+i)}{\Gamma\left(\frac{2}{3}\right)} k_{2e^{-i\varepsilon},+}^{\varepsilon 3}).$$

To evaluate the connection constants $k_{1,+}^{\varepsilon 2}$ and $k_{2e^{-i\varepsilon},+}^{\varepsilon 3}$, we procceed like in example 4.2. First, we check that, for all ε, the Borel transform $\widehat{f}^{\,\varepsilon}(\xi)$ of $\widetilde{f}^{\,\varepsilon}(x)$ is an analytic solution on the open disc $D(0,1)$ of the system

$$\frac{dZ}{d\xi} = \begin{bmatrix} -\frac{1}{\xi} & 0 & \frac{1}{\xi} \\ \frac{1}{\xi-1} & -\frac{1}{2(\xi-1)} & 0 \\ 0 & -\frac{1}{\xi-2} & -\frac{2}{3(\xi-2e^{-i\varepsilon})} \end{bmatrix} Z \tag{4.16}$$

which has two regular singular points at $\xi = 1$ and $\xi = 2e^{-i\varepsilon}$. More precisely, system (4.16) reads near $\xi = 1$ as

$$(\xi-1)\frac{dZ}{d\xi} = C_1(\xi-1)Z \tag{4.17}$$

$$\text{with } C_1(\xi) := \begin{bmatrix} -\frac{\xi}{\xi+1} & 0 & \frac{\xi}{\xi+1} \\ 1 & -\frac{1}{2} & 0 \\ 0 & -\frac{\xi}{\xi-2e^{-i\varepsilon}+1} & -\frac{2\xi}{3(\xi-2e^{-i\varepsilon}+1)} \end{bmatrix}$$

and near $\xi = 2e^{-i\varepsilon}$ as

$$(\xi-2e^{-i\varepsilon})\frac{dZ}{d\xi} = C_2(\xi-2e^{-i\varepsilon})Z \tag{4.8}$$

$$\text{with } C_2(\xi) := \begin{bmatrix} -\frac{\xi}{\xi+2e^{-i\varepsilon}} & 0 & \frac{\xi}{\xi+2e^{-i\varepsilon}} \\ \frac{\xi}{\xi-1+2e^{-i\varepsilon}} & -\frac{\xi}{2(\xi-1+2e^{-i\varepsilon})} & 0 \\ 0 & -1 & -\frac{2}{3} \end{bmatrix}.$$

Notice that $C_1(\xi-1)$ is analytic on the open disc $D(1,1)$ and $C_2(\xi-2e^{-i\varepsilon})$ is analytic on the open disc $D(2e^{-i\varepsilon}, r_\varepsilon)$ with $r_\varepsilon := |2e^{-i\varepsilon}-1| > 1$ for all $\varepsilon > 0$. Next, we define a fundamental solution $Z_1^\varepsilon(\xi-1)$ (resp. $Z_2^\varepsilon(\xi-2e^{-i\varepsilon})$) of system (4.16) (resp. system (4.17)) in the same way as in example 4.2 (see page 35) and we consider the unique matrix S_1^ε (resp. S_2^ε) of $M_{3,1}(\mathbb{C})$ such that the right analytic continuation of $\widehat{f}^{\,\varepsilon}(\xi)$ (still denoted $\widehat{f}^{\,\varepsilon}(\xi)$) at the point $\xi = 1$ (resp. $\xi = 2e^{-i\varepsilon}$) reads $\widehat{f}^{\,\varepsilon}(\xi) = Z_1^\varepsilon(\xi-1)S_1^\varepsilon$ for all $\xi \in D(1,1)\backslash\{1\}$ (resp. $\widehat{f}^{\,\varepsilon}(\xi) = Z_2^\varepsilon(\xi-2e^{-i\varepsilon})S_2^\varepsilon$ for all $\xi \in D(2e^{-i\varepsilon}, r_\varepsilon)\backslash\{2e^{-i\varepsilon}\}$). Then, we

can check that $k^{\varepsilon 2}_{1,+} = \sigma_1^{\varepsilon 2}$ the second entry of S_1^ε and $k^{\varepsilon 3}_{2e^{-i\varepsilon},+} = \sigma_2^{\varepsilon 3}$ the third entry of S_2^ε. Hence,

$$st_0^{\varepsilon 2} = 2\sqrt{\pi}\sigma_1^{\varepsilon 2} \quad \text{and} \quad st_{-\varepsilon}^{\varepsilon 3} = \frac{\pi(\sqrt{3}+i)}{\Gamma\left(\frac{2}{3}\right)}\sigma_2^{\varepsilon 3}.$$

Following table 4.1 gives us some approximations of $\sigma_0^{\varepsilon 2}$ and $st_0^{\varepsilon 2}$ for different values of $\varepsilon = 10^{-m}$. All the approximations of $\sigma_0^{\varepsilon 2}$ are calculated, like in example 4.2, from the relation

$$S_1^\varepsilon = Z_1^\varepsilon\left(-\frac{1}{2}\right)^{-1}\widehat{f}^{\,\varepsilon}\left(\frac{1}{2}\right)$$

where $\arg(-1/2) = -\pi$ (we connect the discs of "convergence" $D(0,1)$ of $\widehat{f}^{\,\varepsilon}(\xi)$ and $D(1,1)$ of $Z_1^\varepsilon(\xi-1)$ to the right). Notice that the number of intermediate calculations needed for the determination of the connection matrix S_1^ε is identical to the one of example 4.2 for the determination of the connection matrix S_1. Notice also, by comparing the values of the $st_0^{\varepsilon 2}$'s with the value of st_0^2 obtained in example 4.2, that the perturbed Stokes multiplier $st_0^{\varepsilon 2}$ provides a "good" approximation of the initial Stokes multiplier st_0^2 as soon as $\varepsilon \leq 10^{-6}$.

ε	$\sigma_0^{\varepsilon 2}$	$st_0^{\varepsilon 2}$
10^{-1}	$0.0704 + 0.47437249i$	$0.2496 + 1.6816067i$
10^{-2}	$0.00709 + 0.46829947i$	$0.0251 + 1.6600784i$
10^{-3}	$7.09 \times 10^{-4} + 0.46823828i$	$0.00251 + 1.6598615i$
10^{-4}	$7.09 \times 10^{-5} + 0.46823767i$	$2.51 \times 10^{-4} + 1.6598593i$
10^{-5}	$7.09 \times 10^{-6} + 0.46823766i$	$2.51 \times 10^{-5} + 1.6598593i$
10^{-6}	$7.09 \times 10^{-7} + 0.46823766i$	$2.51 \times 10^{-6} + 1.6598593i$
10^{-7}	$7.09 \times 10^{-8} + 0.46823766i$	$2.51 \times 10^{-7} + 1.6598593i$
10^{-8}	$7.09 \times 10^{-9} + 0.46823766i$	$2.51 \times 10^{-8} + 1.6598593i$
10^{-9}	$7.09 \times 10^{-10} + 0.46823766i$	$2.51 \times 10^{-9} + 1.6598593i$
10^{-10}	$7.09 \times 10^{-11} + 0.46823766i$	$2.51 \times 10^{-10} + 1.6598593i$

Table 4.1

Let us now evaluate the constants $\sigma_2^{\varepsilon 3}$. For any $\varepsilon > 0$, the radius $r_\varepsilon = |2e^{-i\varepsilon} - 1|$ of the disc of "convergence" $D(2e^{-i\varepsilon}, r_\varepsilon)$ of $Z_2^\varepsilon(\xi - 2e^{-i\varepsilon})$ is > 1. Thereby, $D(0,1) \cap D(2e^{-i\varepsilon}, r_\varepsilon) \neq \emptyset$ and any value a_ε of $]2 - r_\varepsilon, 1[$ satisfies $a_\varepsilon e^{-i\varepsilon} \in D(0,1) \cap D(2e^{-i\varepsilon}, r_\varepsilon)$.

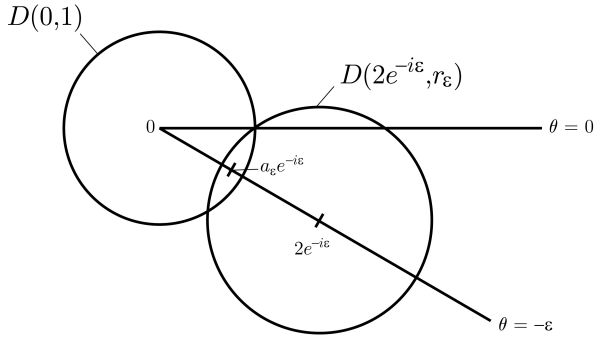

Figure 4.3

Hence, the matrix S_2^ε is uniquely determined by the relation

$$S_2^\varepsilon = Z_2^\varepsilon((a_\varepsilon - 2)e^{-i\varepsilon})^{-1}\,\widehat{f}^{\,\varepsilon}\left(a_\varepsilon e^{-i\varepsilon}\right)$$

where $\arg((a_\varepsilon - 2)e^{-i\varepsilon}) = -\varepsilon - \pi$. Let us choose for example a_ε as the midpoint of $]2-r_\varepsilon, 1[$; then one can check that the perturbed Stokes multiplier $st_{-\varepsilon}^{\varepsilon 3}$ provides a "good" approximation of the initial Stokes multiplier st_0^3 as soon as $\varepsilon \leq 10^{-6}$. Observe here that, contrary to the calculation of the connection matrix S_1^ε made above, the number of intermediate calculations needed for the determination of S_2^ε is much lower than the one of example 4.2 for the determination of the connection matrix S_2.

By adapting the calculations made above to system (4.9) (*cf.* example 4.3), we can evidently get an approximate value of the Stokes multipliers st_0^3 and $st_{-\pi}^3$ of example 4.3. In particular, note that this method allows to replace the path of right analytic continuation given in figure 4.2 by a "simpler" path similar to the one of figure 4.3 and, therefore, to significantly reduce the number of intermediate calculations needed to determine the adequate connection matrix.

4.2.2 General case

Let us now suppose that $\sharp\Omega_{(\theta_k)} \geq 2$, *i.e.*, there exist $s \in \{2, ..., p\}$ Stokes values of Ω, say $\omega_1, \omega_2, ..., \omega_s$, generating the collection (θ_k). Without loss of generality, we also suppose $|\omega_1| < |\omega_2| < ... < |\omega_s|$.

The method previously detailed in the case $\sharp\Omega_{(\theta_k)} = 2$ can be extended to our present case by considering, for $\varepsilon > 0$ small enough, the system (A^ε) in which the initial Stokes value ω_ℓ of system (A) is replaced, for all $\ell = 1, ..., s$, by the perturbed Stokes value

$$\omega_\ell^\varepsilon := \omega_\ell \exp\left(-ir\frac{\ell-1}{s-1}\varepsilon\right).$$

Notice that, for all $\ell = 1, ..., s$, the Stokes value ω_ℓ^ε generates its own collection $\left(\theta_k - \dfrac{\ell-1}{s-1}\varepsilon\right)$ of anti-Stokes directions of system (A^ε).

Then, one can prove the following proposition which generalizes proposition 4.8.

Proposition 4.12 (Initial vs perturbed Stokes multipliers)
For all $j \in \{1, ..., J\}$ such that $a_{j,r} \in \Omega_{(\theta_k)} = \{\omega_1, \omega_2, ..., \omega_s\}$, the initial Stokes multipliers $st^{j;\bullet}_{\theta_k^\star}$ of $\widetilde{f}(x)$ are related, for all $k \in \{0, ..., r-1\}$, to the perturbed Stokes multipliers $st^{\varepsilon j;\bullet}_{(\theta_k - \frac{\ell-1}{s-1}\varepsilon)^\star}$, $\ell = 1, ..., s$, of $\widetilde{f}^\varepsilon(x)$ by the relations

$$\boxed{st^{j;\bullet}_{\theta_k^\star} = \lim_{\varepsilon\to 0} st^{\varepsilon j;\bullet}_{(\theta_k - \frac{\ell-1}{s-1}\varepsilon)^\star} \qquad \text{if } a_{j,r} = \omega_\ell}.$$

4.2.3 Conclusion and directions for further research

In the two previous sections 4.2.1 and 4.2.2, we presented and illustrated an alternative method for the effective calculation of the Stokes multipliers of $\widetilde{f}(x)$ (hence, of the full matrix $\widetilde{F}(x)$). This method, based on a perturbation of system (A) in which each nonzero Stokes value of Ω generates its *own* collection of anti-Stokes directions, has the two following main interests.

1. It allows to avoid all the difficults stated in remark 4.4 which can occur with a "direct" method, *i.e.*, without perturbation.

2. It shows that it suffices to build and to develop algorithms to evaluate, in a given anti-Stokes direction associated with $\widetilde{f}(x)$, the Stokes multipliers associated with the nearest (to the origin $0 \in \mathbb{C}$) Stokes values of Ω.

The construction of such algorithms is a direction of our further researchs.

Another direction of research is related to the perturbative method presented in this paper: how choose ε to guarantee that the perturbed Stokes multipliers would be approximate values of the initial Stokes multipliers with a precision set in advance?

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