On the inverse braid monoid

Vladimir Vershinin

ON THE INVERSE BRAID MONOID

V. V. VERSHININ

Abstract. Inverse braid monoid describes a structure on braids where the number of strings is not fixed. So, some strings of initial n may be deleted. In the paper we show that many properties and objects based on braid groups may be extended to the inverse braid monoids. Namely we prove an inclusion into a monoid of partial monomorphisms of a free group. This gives a solution of the word problem. Another solution is obtained by an approach similar to that of Garside. We give also the analogues of Artin presentation with two generators and Sergiescu graph-presentations.

Contents

1. Introduction

The notion of *inverse semigroup* was introduced by V. V. Wagner in 1952 [40]. By definition it means that for any element a of a semigroup (monoid) M there exists a unique element b (which is called *inverse*) such that

$$a = aba \tag{1.1}$$

and

$$b = bab. \tag{1.2}$$

The roots of this notion can be seen in the von Neumann regular rings [29] where only one condition (1.1) holds for non necessary unique b, or in the Moore-Penrose pseudoinverse for matrices [28], [30] where both conditions (1.1) and (1.2) hold (and certain supplementary conditions also).

The typical example of an inverse monoid is a monoid of partial (defined on a subset) injections of a set. For a finite set this gives us the notion of a *symmetric inverse monoid* I_n which generalizes and includes the classical symmetric group Σ_n. A presentation of symmetric inverse monoid was obtained by L. M. Popova [32], see also formulas (1.4), (1.7-1.8) below. Recently the *inverse braid monoid* IB_n was constructed by D. Easdown and T. G. Lavers [12]. It arises from a very natural operation on braids: deleting one or several strings. By the application of this procedure to braids in Br_n we get *partial* braids [12]. The multiplication of partial braids is shown at the Figure 1.1

2000 *Mathematics Subject Classification.* Primary 20F36; Secondary 20F38, 57M.
Key words and phrases. Braid, inverse braid monoid, presentation, singular braid monoid, word problem.

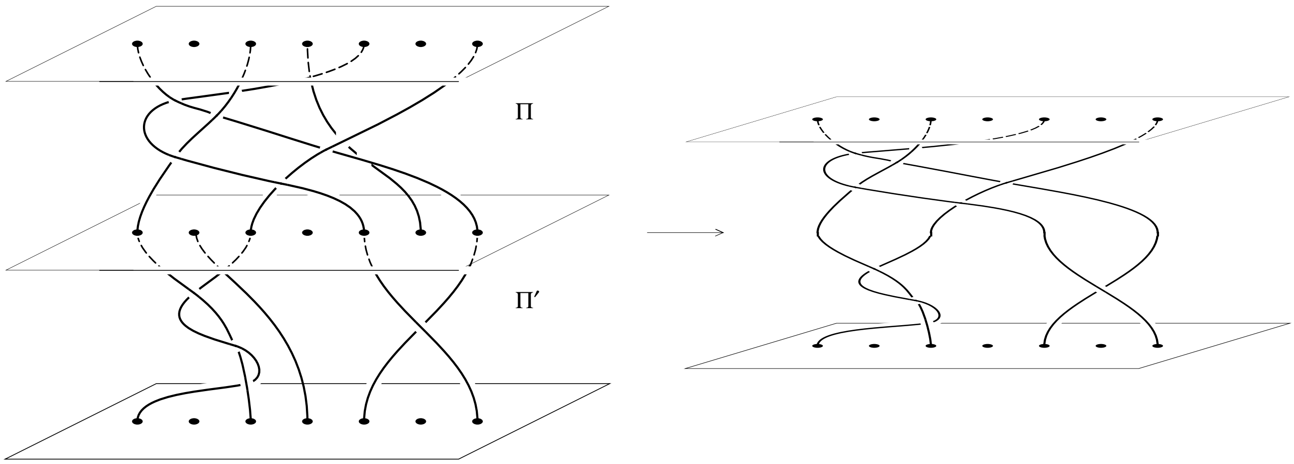

FIGURE 1.1

At the last stage it is necessary to remove any arc that does not join the upper or lower planes.

The set of all partial braids with this operation forms an inverse braid monoid IB_n.

One of the motivations to study IB_n is that it is a natural setting for the *Makanin braids*, which were also called by *smooth* braids by G. S. Makanin who first mentioned them in [24], (page 78, question 6.23), and D. L. Johnson [21], and by *Brunian* braids in the work of J. A. Berrick, F. R. Cohen, Y. L. Wong and J. Wu [6]). By the usual definition a braid is Makanin if it becomes trivial after deleting any string, see formulas (2.13 - 2.17). According to the works of Fred Cohen, Jon Berrick, Wu Jie and others Makanin braids have connections with homotopy groups of spheres. Namely the exists an exact sequence

$$1 \to Mak_{n+1}(S^2) \to Mak_n(D^2) \to Mak_n(S^2) \to \pi_{n-1}(S^2) \to 1 \tag{1.3}$$

for $n \geq 5$, where $Mak_n(D^2)$ is the group Makanin braids and $Mak_n(S^2)$ is the group of Makanin braids of the sphere S^2, see Section 3.

The purpose of this paper is to demonstrate that canonical properties of braid groups and notions based on braids often have there smooth continuation for the inverse braid monoid IB_n.

Usually the braid group Br_n is given by the following Artin presentation [3]. It has the generators σ_i, $i = 1, ..., n-1$ and two types of relations:

$$\begin{cases} \sigma_i \sigma_j &= \sigma_j \, \sigma_i, \ \text{ if } \ |i-j| > 1, \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} \, . \end{cases} \tag{1.4}$$

Classical braid groups Br_n can be defined also as the mapping class group of a disc D^2 with n points deleted (or fixed) and with its boundary fixed, or as the subgroup of the automorphism group of a free group $\operatorname{Aut} F_n$, generated by the following automorphisms:

$$\begin{cases} x_i & \mapsto x_{i+1}, \\ x_{i+1} & \mapsto x_{i+1}^{-1} x_i x_{i+1}, \\ x_j & \mapsto x_j, j \neq i, i+1. \end{cases} \tag{1.5}$$

Geometrically this action is depicted in the Figure 1.2, where x_i correspond to the canonical loops on D^2 which form the generators of the fundamental group the punctured disc.

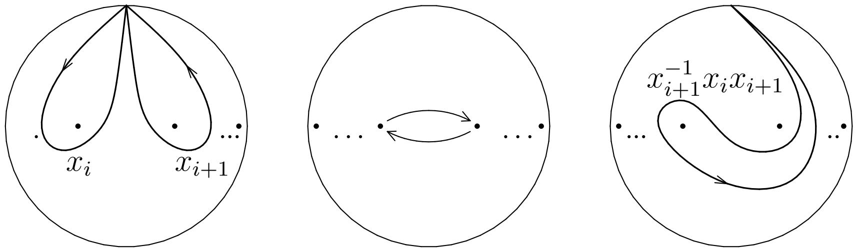

FIGURE 1.2

There exist other presentations of the braid group. Let

$$\sigma = \sigma_1\sigma_2 \dots \sigma_{n-1},$$

then the group Br_n is generated by σ_1 and σ because

$$\sigma_{i+1} = \sigma^i\sigma_1\sigma^{-i}, \quad i = 1, \dots n-2.$$

The relations for the generators σ_1 and σ are the following

$$(1.6) \qquad \begin{cases} \sigma_1\sigma^i\sigma_1\sigma^{-i} & = \sigma^i\sigma_1\sigma^{-i}\sigma_1 \ \text{ for } \ 2 \le i \le n/2, \\ \sigma^n & = (\sigma\sigma_1)^{n-1}. \end{cases}$$

The presentation (1.6) was given by Artin in the initial paper [3]. This presentation was also mentioned in the books by F. Klein [23] and by H. S. M. Coxeter and W. O. J. Moser [10].

An interesting series of presentations was given by V. Sergiescu [34]. For every planar graph he constructed a presentation of the group Br_n, where n is the number of vertices of the graph, with generators corresponding to edges and relations reflecting the geometry of the graph. To each edge e of the graph he associates the braid σ_e which is a clockwise half-twist along e (see Figure 1.3). Artin's classical presentation (1.4) in this context corresponds to the graph consisting of the interval from 1 to n with the natural numbers (from 1 to n) as vertices and with segments between them as edges.

Let $|\ | : \Sigma_n \to \mathbb{Z}$ be the length function on the symmetric group with respect to the standard generators s_i: for $x \in \Sigma_n$, $|x|$ is the smallest natural number k such that x is a product of k elements of the set $\{s_1, ..., s_{n-1}\}$. It is known ([8], Sect. 1, Ex. 13(b)) that two minimal expressions for an element of Σ_n are equivalent by using only the relations (1.4). This implies that the canonical projection $\tau_n : Br_n \to \Sigma_n$ has a unique set-theoretic section $r : \Sigma_n \to Br_n$ such that $r(s_i) = \sigma_i$ for $i = 1, ..., n-1$ and $r(xy) = r(x)\,r(y)$ whenever $|xy| = |x| + |y|$. The image $r(\Sigma_n)$ under the name of *positive permutation braids* was studied by E. El-Rifai and H. R. Morton [13].

The following presentation for the inverse braid monoid was obtained in [12]. It has the generators σ_i, σ_i^{-1}, $i = 1, \dots, n-1$, ϵ, and relations

$$(1.7) \qquad \begin{cases} \sigma_i\sigma_i^{-1} = \sigma_i^{-1}\sigma_i = 1, \ \text{for} \ \text{ all } i, \\ \epsilon\sigma_i = \sigma_i\epsilon \ \text{ for } i \ge 2, \\ \epsilon\sigma_1\epsilon = \sigma_1\epsilon\sigma_1\epsilon = \epsilon\sigma_1\epsilon\sigma_1, \\ \epsilon = \epsilon^2 = \epsilon\sigma_1^2 = \sigma_1^2\epsilon \end{cases}$$

and the braid relations (1.4).

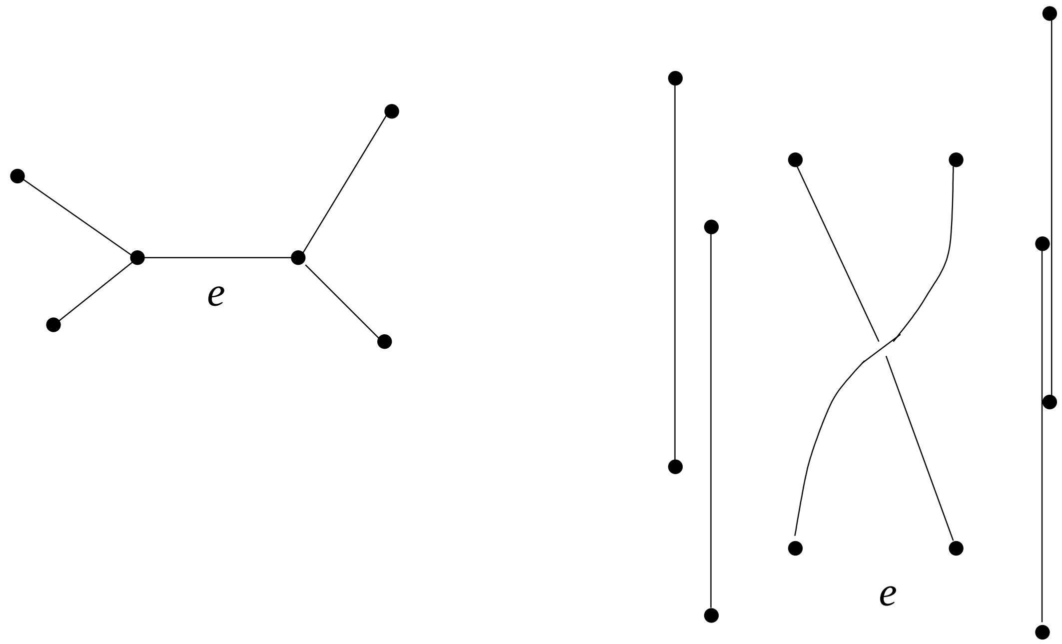

FIGURE 1.3

Geometrically the generator ϵ means that the first string in the trivial braid is absent.

If we replace the first relation in (1.7) by the following set of relations

$$\sigma_i^2 = 1, \text{ for all } i, \tag{1.8}$$

and delete the superfluous relations

$$\epsilon = \epsilon\sigma_1^2 = \sigma_1^2\epsilon,$$

we get a presentation of the symmetric inverse monoid I_n [32] . We also can simply add the relations (1.8) if we don't worry about redundant relations. We get a canonical map [12]

$$\tau_n : IB_n \to I_n \tag{1.9}$$

which is a natural extension of the corresponding map for the braid and symmetric groups.

More balanced relations for the inverse braid monoid were obtained in [19]. Let ϵ_i denote the trivial braid with i-th string deleted, formally:

$$\begin{cases} \epsilon_1 & = \epsilon, \\ \epsilon_{i+1} & = \sigma_i^{\pm 1}\epsilon_i\sigma_i^{\pm 1}. \end{cases}$$

So, the generators are: σ_i, σ_i^{-1}, $i = 1, \dots, n-1$, ϵ_i, $i = 1, \dots, n$, and relations are the following:

$$\begin{cases} \sigma_i\sigma_i^{-1} = \sigma_i^{-1}\sigma_i = 1, \text{ for all } i, \\ \epsilon_j\sigma_i = \sigma_i\epsilon_j \text{ for } |j-i| > 1, \\ \epsilon_i\sigma_i = \sigma_i\epsilon_{i+1}, \\ \epsilon_{i+1}\sigma_i = \sigma_i\epsilon_i, \\ \epsilon_i = \epsilon_i^2, \\ \epsilon_{i+1}\sigma_i^2 = \sigma_i^2\epsilon_{i+1} = \epsilon_{i+1}, \\ \epsilon_i\epsilon_{i+1}\sigma_i = \sigma_i\epsilon_i\epsilon_{i+1} = \epsilon_i\epsilon_{i+1}, \end{cases} \tag{1.10}$$

plus the braid relations (1.4).

2. Properties of inverse braid monoid

The relations (1.7) look asymmetric: one generator for the idempotent part and $n-1$ generators for the group part. If we minimize the number of generators of the group part and take the presentation (1.6) for the braid group we get a presentation of the inverse braid monoid with generators σ_1, σ, ϵ, and relations:

$$\begin{cases} \sigma_1\sigma_1^{-1} = \sigma_1^{-1}\sigma_1 = 1, \\ \sigma\sigma^{-1} = \sigma^{-1}\sigma = 1, \\ \epsilon\sigma^i\sigma_1\sigma^{-i} = \sigma^i\sigma_1\sigma^{-i}\epsilon \ \text{ for } 1 \leq i \leq n-2, \\ \epsilon\sigma_1\epsilon = \sigma_1\epsilon\sigma_1\epsilon = \epsilon\sigma_1\epsilon\sigma_1, \\ \epsilon = \epsilon^2 = \epsilon\sigma_1^2 = \sigma_1^2\epsilon, \end{cases} \tag{2.1}$$

plus (1.6).

Let Γ be a planar graph of the Sergiescu graph presentation of the braid group [34], [5]. Let us add new generators ϵ_v which correspond to each vertex of the graph Γ. Geometrically it means the absence in the trivial braid of one string corresponding to the vertex v. We orient the graph Γ arbitrarily and so we get a starting $v_0 = v_0(e)$ and a terminal $v_1 = v_1(e)$ vertex for each edge e. Consider the following relations

$$\begin{cases} \sigma_e\sigma_e^{-1} = \sigma_e^{-1}\sigma_e = 1, \ \text{for all edges of } \Gamma, \\ \epsilon_v\sigma_e = \sigma_e\epsilon_v, \ \ \text{if the vertex } \ v \text{ and the edge } e \text{ do not intersect}, \\ \epsilon_{v_0}\sigma_e = \sigma_e\epsilon_{v_1}, \ \text{where } v_0 = v_0(e), \ v_1 = v_1(e), \\ \epsilon_{v_1}\sigma_e = \sigma_e\epsilon_{v_0}, \\ \epsilon_v = \epsilon_e^2, \\ \epsilon_{v_i}\sigma_e^2 = \sigma_e^2\epsilon_{v_i} = \epsilon_{v_i}, \quad i = 0,1, \\ \epsilon_{v_0}\epsilon_{v_1}\sigma_e = \sigma_e\epsilon_{v_0}\epsilon_{v_1} = \epsilon_{v_0}\epsilon_{v_1}. \end{cases} \tag{2.2}$$

Theorem 2.1. *We get a Sergiescu graph presentation of the inverse braid monoid IB_n if we add to the graph presentation of the braid group Br_n the relations (2.2).*

□

A *positive partial braid* is a element of IB_n which can be written as a word with only positive entries of the generators σ_i, $i = 1, \dots, n-1$.

A positive partial braid is called a *positive partial permutation braid* if it can be drawn as a geometric positive partial braid in which every pair of strings crosses at most once.

Write IB_n^+ for the set of positive partial permutation braids.

Proposition 2.1. *If the partial braids b_1, $b_2 \in IB_n^+$ induce the same partial permutation on their strings, then $b_1 = b_2$. For each $s \in I_n$ there is a partial braid $b \in IB_n^+$, which induces this partial permutation: $\tau(b) = s$.*

Proof. The original arguments for Br_n are geometrical and so they translate completely to the case of partial braids. □

Let EF_n be a monoid of partial isomorphisms of a free group F_n defined as follows. Let a be an element of the symmetric inverse monoid I_n, $a \in I_n$, $J_k = \{j_1, \dots, j_k\}$ is the image of a, and elements $i_1, \dots, i_k$ belong to domain of the definition of a. The monoid EF_n consists of isomorphisms

$$< x_{i_1}, \dots, x_{i_k} > \to < x_{j_1}, \dots, x_{j_k} >$$

expressed by

$$f_a : x_i \mapsto w_i^{-1} x_{a(i)} w_i,$$

if i is among $i_1, \dots, i_k$ and not defined otherwise and w_i is a word on $x_{j_1}, \dots, x_{j_k}$. The composition of f_a and g_b, $a, b \in I_n$ is defined for x_i belonging to the domain of $a \circ b$. We put $x_{j_m} = 1$ in a word w_i if x_{j_m} does not belong to the domain of definition of g. We define a map ϕ_n from IB_n to EF_n expanding the canonical inclusion

$$Br_n \to \operatorname{Aut} F_n$$

by the condition that $\phi_n(\epsilon)$ as a partial isomorphism of F_n is given by the formula

$$\phi(\epsilon)(x_i) = \begin{cases} x_i \text{ if } i \geq 2, \\ \text{not defined, if } i = 1. \end{cases} \tag{2.3}$$

Using the presentation (1.7) we see that ϕ_n is correctly defined homomorphism of monoids

$$\phi_n : IB_n \to EF_n.$$

Theorem 2.2. *The homomorphism ϕ_n is a monomorphism.*

Proof. Monoid IB_n as a set is a disjoint union of copies of braid groups Br_k, $k = 0, \dots n$. (See [19] for the exact formula of this splitting of IB_n as a groupoid.) Each copy of the group Br_k is identified by the numbers of inputs of strings $i_1, \dots, i_k$ and outputs of them $j_1, \dots, j_k$. Let $I_k = \{i_1, i_2, \dots, i_k\}$, $i_1 < i_2 < \dots < i_k$, $J_k = \{j_1, j_2, \dots, j_k\}$, $j_1 < j_2 < \dots < j_k$, and let $Br(I_k, J_k)$ be the corresponding copy of the braid group. So

$$IB_n = \amalg_{I_k, J_k \subset \{1, \dots n\}} Br(I_k, J_k). \tag{2.4}$$

Define a homomorphism

$$\psi(I_k, J_k) : Br_k \to EF_n.$$

Let $\gamma(I_k)$ be the homomorphism $F_n \to F_k$ defined by

$$\begin{cases} x_{i_l} & \mapsto x_l, \\ x_s & \mapsto e \ \text{ if } s \notin I_k. \end{cases} \tag{2.5}$$

We define a homomorphism

$$\beta(J_k) : F_k \to F_n$$

as an inclusion

$$\beta(J_k)(x_l) = x_{j_l}, \quad l = 1, \dots, k.$$

For each automorphism $\alpha : F_k \to F_k$, $\alpha \in Br_k$, its image $\psi(I_k, J_k)(\alpha)$ in EF_n is defined as a composition

$$\psi(I_k, J_k)(\alpha) = \beta(J_k)\, \alpha\, \gamma(I_k),$$

we compose from right to left as for functions. Homomorphism $\psi(I_k, J_k)$ is a monomorphism. Consider the following diagram

$$\begin{array}{ccc} Br_k & \xrightarrow{Id} & Br_k \\ \downarrow \rho & & \downarrow \psi(I_k, J_k) \\ Br(I_k, J_k) & \xrightarrow{\phi_n} & EF_n \end{array} \tag{2.6}$$

where the left hand map ρ is the bijection. Let us prove that the diagram commutes. Consider a generator of Br_k, say σ_1. We denote $\rho(\sigma_1)$ by $\sigma(i_1, i_2; j_1, j_2) \in IB_n$. This is the positive partial braid where the string starting at i_1 goes to j_1 and the string starting at i_2 goes to j_2. There is no strings starting before i_1, between i_1 and i_2, ending before j_1 and between j_1 and j_2. Suppose that $i_1 < j_2 < i_2 < j_1$, the other cases can be considered the same way. The partial braid $\sigma(i_1, i_2; j_1, j_2) \in IB_n$ as an element of the inverse braid monoid can be expressed as a word on generators in the following form:

$$\sigma(i_1, i_2; j_1, j_2) = \sigma_{i_1}\sigma_{i_1+1}\dots\sigma_{i_2}\dots\sigma_{j_1-1}\sigma_{i_2-2}\dots\sigma_{j_1}\epsilon_{i_1}\epsilon_{i_1+1}\dots\epsilon_{j_2-1}\epsilon_{j_2+1}\dots\epsilon_{j_1-1}.$$

Note that the expression $\sigma_{i_2-2}\dots\sigma_{j_1}$ is present in the formula only if $i_2 - 2 \geq j_2$. We denote it also as consisting of the two parts:

$$\sigma(i_1, i_2; j_1, j_2) = \sigma\epsilon.$$

Let us study the action of $\sigma(i_1, i_2; j_1, j_2)$ on the generators of the free group. We have:

$$\sigma(x_{i_1}) = x_{j_2}$$

and then apply the action of the part ϵ:

$$\epsilon(x_{j_2}) = x_{j_2}.$$

Also we have:

$$\sigma(x_l) = x_{j_2}^{-1}x_l x_{j_2} \quad \text{for} \quad i_1 < l < i_2.$$

After the application of ϵ we obtain $\sigma(i_1, i_2; j_1, j_2)(x_l) = e$. We have

$$\sigma(x_{i_2}) = x_{j_2}^{-1}x_{i_2-1}^{-1}\dots x_{j_1+1}^{-1}x_{j_1}x_{j_1+1}\dots x_{i_2-1}x_{j_2}$$

and then apply the action of the part ϵ:

$$\epsilon(x_{j_2}^{-1}x_{i_2-1}^{-1}\dots x_{j_1+1}^{-1}x_{j_1}x_{j_1+1}\dots x_{i_2-1}x_{j_2}) = x_{j_2}^{-1}x_{j_1}x_{j_2}.$$

We get exactly the action of the image of σ_1 by the composition of the canonical inclusion and the map $\psi(I_k, J_k)$. The diagram (2.6) commutes. So, ϕ_n is also a monomorphism. The different copies of $Br(I_k, J_k)$ of IB_n do not intersect in EF_n. So,

$$\phi_n : IB_n \to EF_n$$

is a monomorphism. □

Theorem 2.3. *The monomorphism ϕ_n gives a solution of the word problem for the inverse braid monoid in the presentations (1.4), (1.7), (1.10), (2.2) and (2.1).*

Proof. As for the braid group if follows from the fact that two words represent the same element of the monoid iff they have the same action on the finite set of generators of the free group F_n. □

Theorem 2.2 gives also a possibility to interpret the inverse braid monoid as a monoid of isotopy classes of maps. As usual consider a disc D^2 with n fixed points. Denote the set of these points by Q_n. The fundamental group of D^2 with these points deleted is isomorphic to F_n. Consider homeomorphisms of D^2 onto a copy of the same disc with the condition that only k points of Q_n, $k \leq n$ (say $i_1, \dots, i_k$) are mapped bijectively onto the k points (say $j_1, \dots, j_k$) of the second copy of D^2. Consider the isotopy classes of such homeomorphisms and denote the set of them by $IM_n(D^2)$. Evidently it is a monoid.

Theorem 2.4. *The monoids IB_n and $IM_n(D^2)$ are isomorphic.*

Proof. The same way as in the proof for the braid group using Alexander's trick we associate a partial braid to an element of $IM_n(D^2)$ and prove that it is an isomorphism. □

These considerations can be generalized to the following definition. Consider a surface $S_{g,b,n}$ of the genus g with b boundary components and the set Q_n of n fixed points. Let f be a homeomorphism of $S_{g,b,n}$ which maps k points, $k \leq n$, from Q_n: $\{i_1, \dots, i_k\}$ to k points $\{j_1, \dots, j_k\}$ also from Q_n. The same way let h be a homeomorphism of $S_{g,b,n}$ which maps l points, $l \leq n$, from Q_n, say $\{s_1, \dots, s_l\}$ to l points $\{t_1, \dots, t_l\}$ again from Q_n. Consider the intersection of the sets $\{j_1, \dots, j_k\}$ and $\{s_1, \dots, s_l\}$, let it be the set of cardinality m, it may be empty. Then the composition of f and h maps m points of Q_n to m points (may be different) of Q_n. If $m = 0$ then the composition have no relation to the set Q_n. Denote the set of isotopy classes of such maps by $\mathcal{IM}_{g,b,n}$. Composition defines a structure of monoid on $\mathcal{IM}_{g,b,n}$.

Proposition 2.2. *The monoid $\mathcal{IM}_{g,b,n}$ is inverse.*

Proof. Each element of $\mathcal{IM}_{g,b,n}$ is represented by a homeomorphism h of $S_{g,b,n}$. So, take an inverse of h and get the identities (1.1) and (1.2). □

We call the monoid $\mathcal{IM}_{g,b,n}$ the *inverse mapping class monoid.* If $g = 0$ and $b = 1$ we get the inverse braid monoid. In the general case $\mathcal{IM}_{g,b,n}$ the role of the empty braid plays the mapping class group $\mathcal{M}_{g,b}$ (without fixed points).

We remind that a monoid M is *factorisable* if $M = EG$ where E is a set of idempotents of M and G is a subgroup of M.

Proposition 2.3. *The monoid $\mathcal{IM}_{g,b,n}$ can be written in the form*

$$\mathcal{IM}_{g,b,n} = E\mathcal{M}_{g,b,n},$$

where E is a set of idempotents of $\mathcal{IM}_{g,b,n}$ and $\mathcal{M}_{g,b,n}$ is the corresponding mapping class group. So this monoid is factorisable.

Proof. An element of $\mathcal{IM}_{g,b,n}$ is represented by a homeomorphism h of $S_{g,b,n}$ which maps k points, $k \leq n$, from Q_n: $\{i_1, \dots, i_k\}$ to k points $\{j_1, \dots, j_k\}$ from Q_n. In the isotopy class of h we find a homeomorphism h_1 which maps arbitrarily $Q_n \setminus \{i_1, \dots, i_k\}$ to $Q_n \setminus \{j_1, \dots, j_k\}$. Necessary idempotent element is the isotopy class of the identity homeomorphism which fixes only the points $\{i_1, \dots, i_k\}$. □

Let Δ be the Garside's fundamental word in the braid group Br_n [18]. It can be defined by the formula:

$$\Delta = \sigma_1 \dots \sigma_{n-1}\sigma_1 \dots \sigma_{n-2} \dots \sigma_1\sigma_2\sigma_1.$$

If we use Garside's notation $\Pi_t \equiv \sigma_1 \dots \sigma_t$, then $\Delta \equiv \Pi_{n-1} \dots \Pi_1$.

Proposition 2.4. *The generators ϵ_i commute with Δ in the following way:*

$$\epsilon_i\Delta = \Delta\epsilon_{n+1-i}.$$

Proof. Direct calculation using the second, third and the forth relations in (1.10). □

Proposition 2.5. *The center of IB_n consists of the union of the center of the braid group Br_n (generated by Δ^2) and the empty braid $\varnothing = \epsilon_1 \dots \epsilon_n$.*

Proof. The given element lie in the center. Suppose that there are other ones. Let c be one of them. It does not belong to Br_n, because its center is already taken into account. It is a partial braid with starting points $I_k = \{i_1, \dots, i_k\}$ and ending points $J_k = \{j_1, \dots, j_k\}$ $k < n$.

Take the one-string partial braid x that starts in the complement of J_k and ends in I_k. Then cx is the empty braid, while xc is not. □

Let $\mathcal{E}$ be the monoid generated by one idempotent generator ϵ .

Proposition 2.6. *The abelianization of IB_n is isomorphic to $\mathcal{E} \oplus \mathbb{Z}$. The canonical map*

$$a : IB_n \to \mathcal{E} \oplus \mathbb{Z}$$

is given by the formula:

$$\begin{cases} a(\epsilon_i) = \epsilon, \\ a(\sigma_i) = 1. \end{cases}$$

□

Let $\epsilon_{k+1,n}$ denote the partial braid with the trivial first k strings and the absent rest $n-k$ strings. It can be expressed using the generator ϵ or the generators ϵ_i as follows

$$\epsilon_{k+1,n} = \epsilon\sigma_{n-1}\dots\sigma_{k+1}\epsilon\sigma_{n-1}\dots\sigma_{k+2}\epsilon\dots\epsilon\sigma_{n-1}\sigma_{n-2}\epsilon\sigma_{n-1}\epsilon, \tag{2.7}$$

$$\epsilon_{k+1,n} = \epsilon_{k+1}\epsilon_{k+2}\dots\epsilon_n, \tag{2.8}$$

It was proved in [12] the every partial braid has a representative of the form

$$\sigma_{i_1}\dots\sigma_1\dots\sigma_{i_k}\dots\sigma_k\epsilon_{k+1,n}x\epsilon_{k+1,n}\sigma_k\dots\sigma_{j_k}\dots\sigma_1\dots\sigma_{j_1}, \tag{2.9}$$

$$k \in \{0,\dots,n\}, x \in Br_k, 0 \le i_1 < \dots < i_k \le n-1 \text{ and } 0 \le j_1 < \dots < j_k \le n-1. \tag{2.10}$$

Note that in the formula (2.9) we can delete one of the $\epsilon_{k+1,n}$, but we shall use the form (2.9) because of convenience: two symbols $\epsilon_{k+1,n}$ serve as markers to distinguish the elements of Br_k. We can put the element $x \in Br_k$ in the Markov normal form [25] and get the corresponding *Markov normal form for the inverse braid monoid IB_n*. The same way for the Garside normal form.

Let us remind the mains point of Garside's construction. Essential role in Garside work plays the monoid of *positive* braids Br_n^+, that is the monoid which has a presentation with generators σ_i, $i = 1, ..., n$ and relations (1.4). In other words each element of this monoid can be represented as a word on the elements σ_i, $i = 1, ..., n$ with no entrances of σ_i^{-1}. Two positive words V and W in the alphabet $\{\sigma_i, (i = 1, \dots, n-1)\}$ will be said to be *positively equal* if they are equal as elements of Br_n^+. Usually this is written as $V \doteq W$.

Among positive words on the alphabet $\{\sigma_1 \dots \sigma_n\}$ let us introduce a lexicographical ordering with the condition that $\sigma_1 < \sigma_2 < \dots < \sigma_n$. For a positive word V the *base* of V is the smallest positive word which is positively equal to V. The base is uniquely determined. If a positive word V is prime to Δ, then for the base of V the notation $\overline{V}$ will be used.

Theorem 2.5. *Every word W in IBr_n can be uniquely written in the form*

$$\sigma_{i_1}\dots\sigma_1\dots\sigma_{i_k}\dots\sigma_k\epsilon_{k+1,n}x\epsilon_{k+1,n}\sigma_k\dots\sigma_{j_k}\dots\sigma_1\dots\sigma_{j_1}, \tag{2.11}$$

$$k \in \{0,\dots,n\}, x \in Br_k, 0 \le i_1 < \dots < i_k \le n-1 \textit{ and } 0 \le j_1 < \dots < j_k \le n-1. \tag{2.12}$$

where x is written in the Garside normal form for Br_k

$$\Delta^m\overline{V},$$

where m is an integer.

Proof. Note that the elements $\sigma_{i_1}\dots\sigma_1\dots\sigma_{i_k}\dots\sigma_k$ and $\sigma_k\dots\sigma_{j_k}\dots\sigma_1\dots\sigma_{j_1}$ are uniquely determined by a given element of IB_n (written as a word W in the alphabet $A = \{\sigma_i, \sigma_i^{-1}, i = 1,\dots,n-1, \epsilon\}$). Then Theorem follows from the existence of the Garside normal form for Br_k. ☐

Theorem 2.5 is evidently true also for the presentation with ϵ_i, $i = 1,\dots n$. In this case the elements $\epsilon_{k+1,n}$ are expressed by (2.8).

The form of a word W established in this theorem we call the *Garside left normal form for the inverse braid monoid* IB_n and the index m we call the *power* of W. The same way the *Garside right normal form for the inverse braid monoid* is defined and the corresponding variant of Theorem 2.5 is true.

Theorem 2.6. *The necessary and sufficient condition that two words in IB_n are equal is that their Garside normal forms are identical. The Garside normal form gives a solution to the word problem in the braid group.*

Proof. As we noted in the proof of the previous Theorem the elements $\sigma_{i_1}\dots\sigma_1\dots\sigma_{i_k}\dots\sigma_k$ and $\sigma_k\dots\sigma_{j_k}\dots\sigma_1\dots\sigma_{j_1}$ are uniquely determined. Also in [12] (implicitly) there was given an algorithm how to obtain the form (2.9) for an arbitrary word W in the alphabet A. Then combining it with the Garside algorithm we get a solution of the word problem for the inverse braid monoid. ☐

Garside normal form for the braid groups was precized in the subsequent works of S. I. Adyan [2], W. Thurston [14], E. El-Rifai and H. R. Morton [13]. Namely, there was introduced the *left-greedy form* (in the terminology of W. Thurston [14])

$$\Delta^t A_1 \dots A_k,$$

where A_i are the successive possible longest *fragments of the word* Δ (in the terminology of S. I. Adyan [2]) or *positive permutation braids* (in the terminology of E. El-Rifai and H. R. Morton [13]). Certainly, the same way the *right-greedy form* is defined. These greedy forms are defined for the inverse braid monoid the same way.

Let us consider the elements $m \in IB_n$ satisfying the equation:

$$\epsilon_i m = \epsilon_i. \tag{2.13}$$

Geometrically this means that removing the string (if it exists) that starts at the point with the number i we get a trivial braid on the rest $n-1$ strings. It is equivalent to the condition

$$m\epsilon_{\tau(m)(i)} = \epsilon_{\tau(m)(i)}, \tag{2.14}$$

where τ is the canonical map to the symmetric monoid (1.9). With the exception of ϵ_i itself all such elements belong to Br_n. We call such braids as i-*Makanin* and denote the subgroup of i-Makanin braids by A_i. The subgroups A_i, $i = 1,\dots,n$, are conjugate

$$A_i = \sigma_{i-1}^{-1}\dots\sigma_1^{-1}A_1\sigma_1\dots\sigma_{i-1} \tag{2.15}$$

free subgroups. The group A_1 is freely generated by the set $\{x_1,\dots,x_{n-1}\}$ [21], where

$$x_i = \sigma_{i-1}^{-1}\dots\sigma_1^{-1}\sigma_1^2\sigma_1\dots\sigma_{i-1}. \tag{2.16}$$

The intersection of all subgroups of i-Makanin braids is the group of Makanin braids

$$Mak_n = \cap_{i=1}^n A_i. \tag{2.17}$$

That is the same as $m \in Mak_n$ if and only if the equation (2.13) holds for all i.

3. Monoids of partial generalized braids

Construction of partial braids can be applied to various generalizations of braids, namely to those where geometric or diagrammatic construction of braids takes place. Let S_g be a surface of genus g probably with boundary components and punctures. We consider partial braids lying in a layer between two such surfaces: $S_g \times I$ and take a set of isotopy classes of such braids. We get a monoid of partial braid of a surface S_g, denote it by $IB_n(S_g)$. An interesting case is when the surface is a sphere S^2. So our partial braids are lying in a layer between two concentric spheres. It was proved by O. Zariski [41] and then rediscovered by E. Fadell and J. Van Buskirk [15] that the braid group of a sphere has a presentation with generators σ_i, $i = 1, ..., n-1$, the same as for the classical braid group satisfying the braid relations (1.4) and the following sphere relation:

$$\sigma_1\sigma_2 \dots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2} \dots \sigma_2\sigma_1 = 1. \tag{3.1}$$

Theorem 3.1. *We get a presentation of the monoid $IB_n(S^2)$ if we add to the presentation (1.7) or the presentation (1.10) of IB_n the sphere relation (3.1). It is a factorisable inverse monoid.*

Proof. Essentially it is the same as for IB_n. Denote temporarily by M_n the monoid defined by the presentation and $IB_n(S^2)$ denotes the monoid of homotopy classes. We already used that every word in the alphabet A is congruent (using the relations (1.7) to a word of the form (2.9). Now note that for the sphere inverse braid monoid the alphabet is the same and relations for IB_n are included into the set of relations for $IB_n(S^2)$. As in [12] the evident map

$$\Psi : M_n \to IB_n(S^2)$$

is defined and proved that it is onto. Let us prove that Ψ is a monomorphism. Suppose that for two words $W_1, W_2 \in M_n$ we have

$$\Psi(W_1 = \Psi(W_2).$$

That means that the corresponding braids are isotopic. Using relations (1.7) transform the words W_1, W_2 into the form (2.9)

$$\sigma(i_1, \dots i_k; k)\epsilon_{k+1,n}x\epsilon_{k+1,n}\sigma(k; j_1, \dots j_k).$$

Then the corresponding fragments $\sigma(i_1, \dots i_k; k)$ and $\sigma(k, j_1, \dots j_k; k)$ for W_1 and W_2 coincide. The elements x_1 of W_1 and x_2 of W_2, which are the words on $\sigma_1, \dots, \sigma_k$, correspond after Ψ to homotopic braids on k strings on the sphere S^2. So x_1 can be transformed into x_2 using relations for the braid groups $Br_k(S^2)$. The words W_1 and W_2 represent the same element in M_n. □

Another example here is the braid group of a punctured disc which is isomorphic to the Artin-Brieskorn braid group of the type B [9], [37]. With respect to the classical braid group it has an extra generator τ and the relations of type B:

$$\begin{cases} \tau\sigma_1\tau\sigma_1 &= \sigma_1\tau\sigma_1\tau, \\ \sigma_i\sigma_j &= \sigma_j\,\sigma_i, \quad \text{if} \ \ |i-j| > 1, \end{cases} \tag{3.2}$$

Denote by IBB_n the monoid of partial braids of the type B.

Theorem 3.2. *We get a presentation of the monoid IBB_n if we add to the presentation (1.7) or the presentation (1.10) of IB_n one generator τ, the type B relation (3.2) and the following relations*

$$\begin{cases} \tau\tau^{-1} = \tau^{-1}\tau = 1, \\ \epsilon_1\tau = \tau\epsilon_1 = \epsilon_1. \end{cases} \tag{3.3}$$

It is a factorisable inverse monoid.

Proof. The same as for IB_n. □

Remark 3.1. *Theorem 3.3 can be easily generalized for partial braids in handlebodies* [35].

The same way as for IB_n the notion of Makanin braids can be defined for any surface and we get $Mak_n(S_g) \subset IB_n(S_g)$. The group of Makanin braids for the sphere was used in the exact sequence (1.3).

Let BP_n be the braid-permutation group of R. Fenn, R. Rimányi and C. Rourke [17]. It is defined as a subgroup of $\operatorname{Aut} F_n$, generated by both sets of the automorphisms σ_i of (1.5) and ξ_i of the following form:

$$\begin{cases} x_i & \mapsto x_{i+1}, \\ x_{i+1} & \mapsto x_i, \\ x_j & \mapsto x_j, j \neq i, i+1, \end{cases} \tag{3.4}$$

R. Fenn, R. Rimányi and C. Rourke proved that this group is given by the set of generators: $\{\xi_i, \sigma_i, \quad i = 1, 2, ..., n-1\}$ and relations:

$$\begin{cases} \xi_i^2 & = 1, \\ \xi_i\xi_j & = \xi_j\xi_i, \quad \text{if} \quad |i-j| > 1, \\ \xi_i\xi_{i+1}\xi_i & = \xi_{i+1}\xi_i\xi_{i+1}. \end{cases}$$

The symmetric group relations

$$\begin{cases} \sigma_i\sigma_j & = \sigma_j\sigma_i, \text{ if } |i-j| > 1, \\ \sigma_i\sigma_{i+1}\sigma_i & = \sigma_{i+1}\sigma_i\sigma_{i+1}. \end{cases}$$

The braid group relations

$$\begin{cases} \sigma_i\xi_j & = \xi_j\sigma_i, \text{ if } |i-j| > 1, \\ \xi_i\xi_{i+1}\sigma_i & = \sigma_{i+1}\xi_i\xi_{i+1}, \\ \sigma_i\sigma_{i+1}\xi_i & = \xi_{i+1}\sigma_i\sigma_{i+1}. \end{cases} \tag{3.5}$$

The mixed relations for the braid-permutation group

R. Fenn, R. Rimányi and C. Rourke also gave a geometric interpretation of BP_n as a group of *welded braids.*

We consider the image of monoid I_n in $\operatorname{End} F_n$ by the map defined by the formulas (3.4), (2.3). We take also the monoid IB_n lying in $\operatorname{End} F_n$ under the map ϕ_n of Theorem (2.2). We define the *braid-permutation* monoid as a submonoid of $\operatorname{End} F_n$ generated by both images of IB_n and I_n and denote it by IBP_n. It can be also defined by the diagrams of partial welded braids.

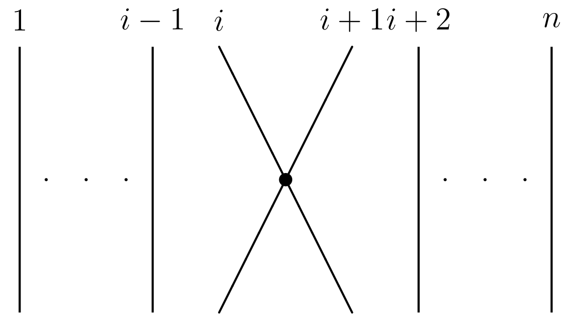

FIGURE 3.1

Theorem 3.3. *We get a presentation of the monoid IBP_n if we add to the presentation of BP_n the generator ϵ, relations (1.7) and the analogous relations between ξ_i and ϵ, or generators ϵ_i, $1 \le i \le n$ relations (1.10) and the analogous relations between ξ_i and ϵ_i. It is a factorisable inverse monoid.*

Proof. The same as for BP_n. □

The virtual braids [38] can be defined by the plane diagrams with real and virtual crossings. The corresponding Reidemeister moves are the same as for the welded braids of the braid-permutation group with one exception. The forbidden move corresponds to the last mixed relation for the braid-permutation group. This allows to define the partial virtual braids and the corresponding monoid IVB_n. So the mixed relation for IVB_n have the form:

$$\begin{cases} \sigma_i\xi_j & = \xi_j\sigma_i, \text{ if } |i-j| > 1, \\ \xi_i\xi_{i+1}\sigma_i & = \sigma_{i+1}\xi_i\xi_{i+1}. \end{cases} \tag{3.6}$$

The mixed relations for virtual braids

Theorem 3.4. *We get a presentation of the monoid IVB_n if we delete the last mixed relation in the presentation of IBP_n, that is replace the relations (3.5) by (3.6) It is a factorisable inverse monoid. The canonical epimorphism*

$$IVB_n \to IBP_n$$

is evidently defined.

The *singular braid monoid SB_n* or *Baez–Birman monoid* [4], [7] is defined as a monoid with generators $\sigma_i, \sigma_i^{-1}, x_i$, $i = 1, \dots, n-1$, and relations

$$\begin{cases} \sigma_i\sigma_j = \sigma_j\sigma_i, \text{ if } |i-j| > 1, \\ x_ix_j = x_jx_i, \text{ if } |i-j| > 1, \\ x_i\sigma_j = \sigma_jx_i, \text{ if } |i-j| \neq 1, \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \\ \sigma_i\sigma_{i+1}x_i = x_{i+1}\sigma_i\sigma_{i+1}, \\ \sigma_{i+1}\sigma_ix_{i+1} = x_i\sigma_{i+1}\sigma_i, \\ \sigma_i\sigma_i^{-1} = \sigma_i^{-1}\sigma_i = 1. \end{cases} \tag{3.7}$$

In pictures σ_i corresponds to the canonical generator of the braid group and x_i represents an intersection of the ith and $(i+1)$st strand as in Figure 3.1. The singular braid monoid on two strings is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}^+$. The constructions of SB_n is geometric, so we can easily get the analogous monoid of partial singular braids PSB_n.

Theorem 3.5. *We get a presentation of the monoid PSB_n if we add to the presentation of SB_n the generators ϵ_i, $1 \leq i \leq n$, relations (1.10) and the analogous relations between x_i and ϵ_i.*

Proof. The same as for BP_n. □

Remark 3.2. *The monoid PSB_n is not neither factorisable nor inverse.*

The construction of braid groups on graphs [20], [16] is geometrical so, the same way as for the classical braid groups we can define *partial braids on a graph* Γ and the *monoid of partial braids on a graph* Γ which will be evidently inverse, so we call it as *inverse braid monoid on the graph* Γ and we denote it as $IB_n\Gamma$.

4. Partial braids and braided monoidal categories

The system of braid groups Br_n is equipped with the standard pairings

$$\mu : Br_k \times Br_l \to Br_{k+l}.$$

It may be constructed by means of adding l extra strings to the initial k. If σ_i' are the generators of Br_k, σ_j'' are the generators of Br_l and σ_r are the generators of Br_{k+l}, then the map μ can be expressed in the form

$$\mu(\sigma_i', e) = \sigma_i, \ 1 \leq i \leq k-1,$$
$$\mu(e, \sigma_j'') = \sigma_{j+k}, \ 1 \leq j \leq l-1.$$

The same geometric construction allows to extend this pairing to a pairing for the inverse braid monoids.

$$\mu : IB_k \times IB_l \to IB_{k+l}$$

such that the following diagram commutes

$$\begin{array}{ccc} Br_k \times Br_l & \xrightarrow{\mu} & Br_{k+l} \\ \downarrow \kappa & & \downarrow \kappa \\ IB_k \times IB_l & \xrightarrow{\mu} & IB_{k+l}. \end{array} \tag{4.1}$$

The vertical lines denote here the canonical inclusions. For the generators ϵ_i we have:

$$\mu(\epsilon_i', e) = \epsilon_i, \ 1 \leq i \leq k,$$

$$\mu(e, \epsilon_j'') = \epsilon_{j+k}, \ 1 \leq j \leq l.$$

A strict monoidal (tensor) category $\mathcal{B}$ is defined in a standard way. Its objects $\{\overline{0}, \overline{1}, ...\}$ correspond to integers from 0 to infinity and morphisms are defined by the formula:

$$\hom(\overline{k}, \overline{l}) = \begin{cases} Br_k, & \text{if } \ k = l, \\ \varnothing, & \text{if } \ k \neq l. \end{cases} \tag{4.2}$$

The product in $\mathcal{B}$ is defined on objects by the sum of numbers and on morphisms, by the pairing μ. The category $\mathcal{B}$, generated by the braid groups, is a *braided monoidal category* as defined by A. Joyal and R. Street [22].

The following system of elements

$$\sigma_m \dots \sigma_1 \sigma_{m+1} \dots \sigma_2 \dots \sigma_{n+m-1} \dots \sigma_n \in Br_{m+n}$$

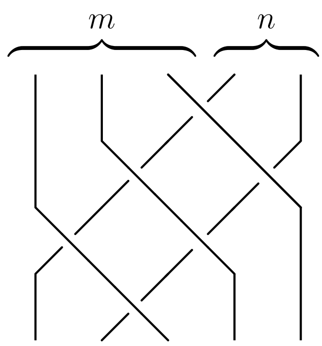

FIGURE 4.1

defines a braiding c in $\mathcal{B}$. Graphically it is depicted in Figure 4.1.

The same way we define a strict monoidal category $\mathcal{IB}$ with the same objects as for $\mathcal{B}$ and morphisms

$$\hom(\overline{k},\overline{l}) = \begin{cases} IB_k, & \text{if} \quad k = l, \\ \varnothing, & \text{if} \quad k \neq l. \end{cases} \tag{4.3}$$

The canonical inclusions

$$\kappa_n : Br_n \to IB_n \tag{4.4}$$

define a functor

$$\mathcal{K} : \mathcal{B} \to \mathcal{IB}.$$

The image of the braiding c by the functor $\mathcal{K} : \mathcal{B} \to \mathcal{IB}$ is a braiding in the category $\mathcal{IB}$.

Geometrically the fact that the braiding for $\mathcal{B}$ defines also a braiding for partial braids is easily seen from Figure 4.1.

Proposition 4.1. *The image of the braiding c in the category $\mathcal{B}$ by the functor $\mathcal{K}$ is a braiding in the category $\mathcal{IB}$, so it becomes a braided monoidal category and the functor $\mathcal{K}$ becomes a morphism between the braided monoidal categories.*

Proof. By definition, the naturality of the braiding $\mathcal{K}(c)$ (which we denote by the same symbol c) means that the following equality

$$c_{\overline{m},\overline{n}} \cdot \mu(b'_m, b''_n) = \mu(b''_n, b'_m) \cdot c_{\overline{m},\overline{n}}, \quad b'_m \in Br_m, \quad b''_n \in Br_n,$$

is fulfilled. This amounts to the expression

$$c_{\overline{m},\overline{n}} \cdot \mu(b'_m, b''_n) \cdot c^{-1}_{\overline{m},\overline{n}} = \mu(b''_n, b'_m),$$

which means that the conjugation by the element $c_{\overline{m},\overline{n}}$ transforms the elements of $IB_m \times IB_n$, lying canonically in IB_{m+n}, into the corresponding elements of $IB_n \times IB_m$. The elements $c_{\overline{m},\overline{n}}$ define a braiding for the category $\mathcal{B}$, so, for checking the naturality of c in $\mathcal{IB}$ it remains to verify the naturality for the generators ϵ_i, $1 \leq i \leq m$, $m+1 \leq i \leq m+n$. Let us consider the corresponding conjugation:

$$\sigma_m \dots \sigma_1 \sigma_{m+1} \dots \sigma_2 \dots \sigma_{n+m-1} \dots \sigma_n \epsilon_i \sigma_n^{-1} \dots \sigma_{n+m-1}^{-1} \dots \sigma_2^{-1} \sigma_{m+1}^{-1} \sigma_1^{-1} \dots \sigma_m^{-1}.$$

When $i > n$, we move ϵ_i back, using the relation

$$\epsilon_i \sigma_i = \sigma_i \epsilon_{i+1}.$$

We have:

$$\sigma_m ... \sigma_1 \sigma_{m+1} ... \sigma_2 ... \sigma_{n+m-1} ... \sigma_n \epsilon_i \sigma_n^{-1} ... \sigma_{n+m-1}^{-1} ... \sigma_2^{-1} \sigma_{m+1}^{-1} \sigma_1^{-1} ... \sigma_m^{-1} =$$

$$\sigma_m ... \sigma_1 \sigma_{m+1} ... \sigma_2 ... \sigma_{n+m-1} ... \sigma_{i+1} \sigma_i \epsilon_{i-1} \sigma_{i-1} \sigma_{i-1}^{-1} ... \sigma_{n+m-1}^{-1} ... \sigma_2^{-1} \sigma_{m+1}^{-1} \sigma_1^{-1} ... \sigma_m^{-1} =$$

$$... = \epsilon_{i-n}.$$

When $i < n$, we move ϵ_i back using the relation

$$\epsilon_{i+1} \sigma_i = \sigma_i \epsilon_i.$$

We have:

$$\sigma_m ... \sigma_1 \sigma_{m+1} ... \sigma_2 ... \sigma_{n+m-1} ... \sigma_n \epsilon_i \sigma_n^{-1} ... \sigma_{n+m-1}^{-1} ... \sigma_2^{-1} \sigma_{m+1}^{-1} \sigma_1^{-1} ... \sigma_m^{-1} =$$

$$\sigma_m ... \sigma_1 \sigma_{m+1} ... \sigma_2 ... \sigma_i \sigma_{i+m-1} ... \sigma_{i+2} \sigma_{i+1} \sigma_i \epsilon_i \sigma_i^{-1} \sigma_{i+1}^{-1} ... \sigma_{i+m-1}^{-1} ... \sigma_2^{-1} \sigma_{m+1}^{-1} \sigma_1^{-1} ... \sigma_m^{-1} =$$

$$\sigma_m ... \sigma_1 \sigma_{m+1} ... \sigma_2 ... \sigma_i \sigma_{i+m-1} ... \sigma_{i+2} \sigma_{i+1} \epsilon_{i+1} \sigma_{i+1}^{-1} ... \sigma_{i+m-1}^{-1} \sigma_i^{-1} ... \sigma_2^{-1} \sigma_{m+1}^{-1} \sigma_1^{-1} ... \sigma_m^{-1} =$$

$$= \epsilon_{i+m}.$$

The conditions of coherence are fulfilled because they are true for $\mathcal{B}$. □

Let BIB denote the classifying spaces of the limit inverse braid monoid. As usual, the pairings $\mu_{m,n}$ define a monoid structure on the disjoint sum of the classifying spaces of IB_n:

$$\amalg_{n \geq 0} BIB_n.$$

Proposition 4.2. *The canonical maps*

$$BIB \to \Omega B(\amalg_{n \geq 0} BIB_n)$$

induce isomorphisms in homology

$$H_*(BIB; A) \to H_*((\Omega B(\amalg_{n \geq 0} BIB_n))_0; A),$$

with any (constant) coefficients. So,

$$BIB^+ \cong (\Omega B(\amalg_{n \geq 0} BIB_n))_0.$$

The proof is the same as that of Theorem 3.2.1 and Corollary 3.2.2 in [1] or (which is essentially the same) based directly on [26]. The braiding c gives the necessary homotopy commutativity for the H-spaces $\amalg_{n \geq 0} BIB_n$.

Theorem 4.1. *The homomorphisms κ_n induce morphisms of braided monoidal categories*

$$\mathcal{B} \xrightarrow{\mathcal{K}} \mathcal{IB}$$

and the corresponding double loop maps

$$\Omega^2 S^2 \longrightarrow \Omega B(\amalg_{n \geq 0} BIB_n).$$

The proof follows from the fact that the classifying space of a braided monoidal category is a double loop space after group completion.

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Département des Sciences Mathématiques, Université Montpellier II, Place Eugéne Bataillon, 34095 Montpellier cedex 5, France

E-mail address: vershini@math.univ-montp2.fr

Sobolev Institute of Mathematics, Novosibirsk, 630090, Russia

E-mail address: versh@math.nsc.ru