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► **To cite this version:**

Vladimir Vershinin. On the inverse braid monoid. *Topology and its Applications*, 2009, 156 (6), pp.1153-1166. 10.1016/j.topol.2008.10.007 . hal-00795122

**HAL Id: hal-00795122**

**<https://hal.science/hal-00795122>**

Submitted on 22 Mar 2013

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# ON THE INVERSE BRAID MONOID

V. V. VERSHININ

ABSTRACT. Inverse braid monoid describes a structure on braids where the number of strings is not fixed. So, some strings of initial  $n$  may be deleted. In the paper we show that many properties and objects based on braid groups may be extended to the inverse braid monoids. Namely we prove an inclusion into a monoid of partial monomorphisms of a free group. This gives a solution of the word problem. Another solution is obtained by an approach similar to that of Garside. We give also the analogues of Artin presentation with two generators and Sergiescu graph-presentations.

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## 1. INTRODUCTION

The notion of *inverse semigroup* was introduced by V. V. Wagner in 1952 [40]. By definition it means that for any element  $a$  of a semigroup (monoid)  $M$  there exists a unique element  $b$  (which is called *inverse*) such that

$$(1.1) \quad a = aba$$

and

$$(1.2) \quad b = bab.$$

The roots of this notion can be seen in the von Neumann regular rings [29] where only one condition (1.1) holds for non necessary unique  $b$ , or in the Moore-Penrose pseudoinverse for matrices [28], [30] where both conditions (1.1) and (1.2) hold (and certain supplementary conditions also).

The typical example of an inverse monoid is a monoid of partial (defined on a subset) injections of a set. For a finite set this gives us the notion of a *symmetric inverse monoid*  $I_n$  which generalizes and includes the classical symmetric group  $\Sigma_n$ . A presentation of symmetric inverse monoid was obtained by L. M. Popova [32], see also formulas (1.4), (1.7-1.8) below. Recently the *inverse braid monoid*  $IB_n$  was constructed by D. Easdown and T. G. Lavers [12]. It arises from a very natural operation on braids: deleting one or several strings. By the application of this procedure to braids in  $Br_n$  we get *partial braids* [12]. The multiplication of partial braids is shown at the Figure 1.1

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2000 *Mathematics Subject Classification*. Primary 20F36; Secondary 20F38, 57M.

*Key words and phrases*. Braid, inverse braid monoid, presentation, singular braid monoid, word problem.

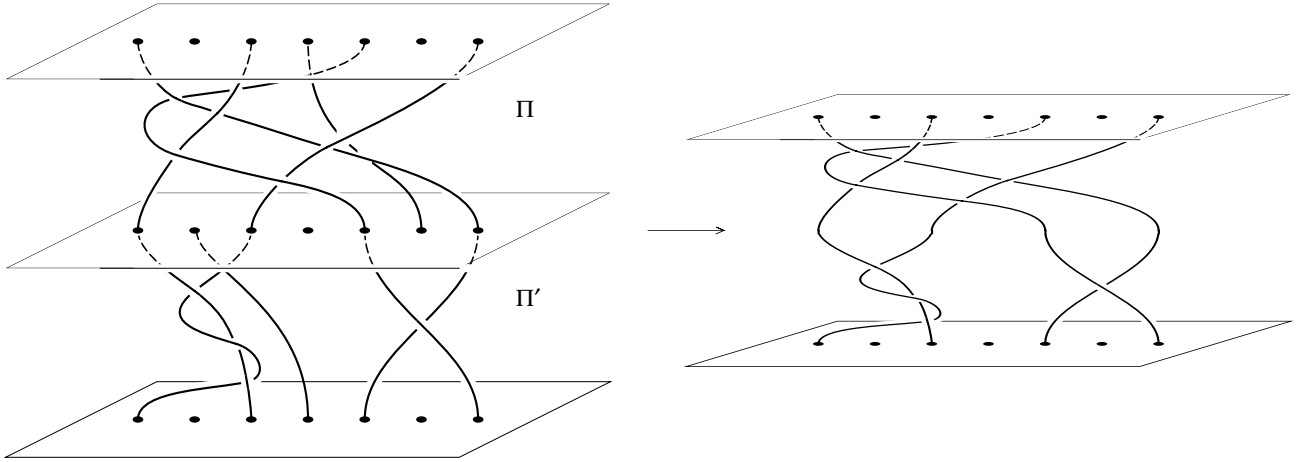


FIGURE 1.1

At the last stage it is necessary to remove any arc that does not join the upper or lower planes.

The set of all partial braids with this operation forms an inverse braid monoid  $IB_n$ .

One of the motivations to study  $IB_n$  is that it is a natural setting for the *Makanin braids*, which were also called by *smooth braids* by G. S. Makanin who first mentioned them in [24], (page 78, question 6.23), and D. L. Johnson [21], and by *Brunian braids* in the work of J. A. Berrick, F. R. Cohen, Y. L. Wong and J. Wu [6]). By the usual definition a braid is Makanin if it becomes trivial after deleting any string, see formulas (2.13 - 2.17). According to the works of Fred Cohen, Jon Berrick, Wu Jie and others Makanin braids have connections with homotopy groups of spheres. Namely the exists an exact sequence

$$(1.3) \quad 1 \rightarrow Mak_{n+1}(S^2) \rightarrow Mak_n(D^2) \rightarrow Mak_n(S^2) \rightarrow \pi_{n-1}(S^2) \rightarrow 1$$

for  $n \geq 5$ , where  $Mak_n(D^2)$  is the group Makanin braids and  $Mak_n(S^2)$  is the group of Makanin braids of the sphere  $S^2$ , see Section 3.

The purpose of this paper is to demonstrate that canonical properties of braid groups and notions based on braids often have there smooth continuation for the inverse braid monoid  $IB_n$ .

Usually the braid group  $Br_n$  is given by the following Artin presentation [3]. It has the generators  $\sigma_i$ ,  $i = 1, \dots, n - 1$  and two types of relations:

$$(1.4) \quad \begin{cases} \sigma_i \sigma_j & = \sigma_j \sigma_i, & \text{if } |i - j| > 1, \\ \sigma_i \sigma_{i+1} \sigma_i & = \sigma_{i+1} \sigma_i \sigma_{i+1}. \end{cases}$$

Classical braid groups  $Br_n$  can be defined also as the mapping class group of a disc  $D^2$  with  $n$  points deleted (or fixed) and with its boundary fixed, or as the subgroup of the automorphism group of a free group  $\text{Aut } F_n$ , generated by the following automorphisms:

$$(1.5) \quad \begin{cases} x_i & \mapsto x_{i+1}, \\ x_{i+1} & \mapsto x_{i+1}^{-1} x_i x_{i+1}, \\ x_j & \mapsto x_j, j \neq i, i + 1. \end{cases}$$

Geometrically this action is depicted in the Figure 1.2, where  $x_i$  correspond to the canonical loops on  $D^2$  which form the generators of the fundamental group the punctured disc.

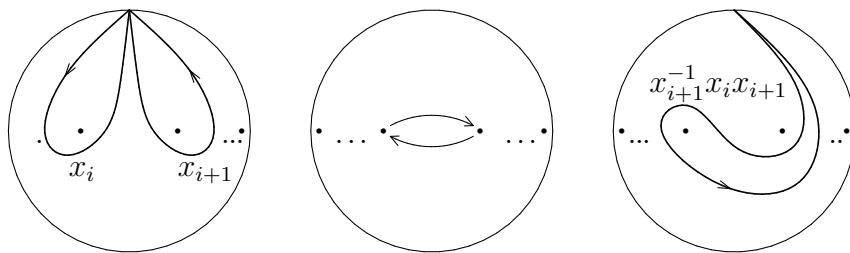


FIGURE 1.2

There exist other presentations of the braid group. Let

$$\sigma = \sigma_1 \sigma_2 \dots \sigma_{n-1},$$

then the group  $Br_n$  is generated by  $\sigma_1$  and  $\sigma$  because

$$\sigma_{i+1} = \sigma^i \sigma_1 \sigma^{-i}, \quad i = 1, \dots, n-2.$$

The relations for the generators  $\sigma_1$  and  $\sigma$  are the following

$$(1.6) \quad \begin{cases} \sigma_1 \sigma^i \sigma_1 \sigma^{-i} &= \sigma^i \sigma_1 \sigma^{-i} \sigma_1 \text{ for } 2 \leq i \leq n/2, \\ \sigma^n &= (\sigma \sigma_1)^{n-1}. \end{cases}$$

The presentation (1.6) was given by Artin in the initial paper [3]. This presentation was also mentioned in the books by F. Klein [23] and by H. S. M. Coxeter and W. O. J. Moser [10].

An interesting series of presentations was given by V. Sergiescu [34]. For every planar graph he constructed a presentation of the group  $Br_n$ , where  $n$  is the number of vertices of the graph, with generators corresponding to edges and relations reflecting the geometry of the graph. To each edge  $e$  of the graph he associates the braid  $\sigma_e$  which is a clockwise half-twist along  $e$  (see Figure 1.3). Artin's classical presentation (1.4) in this context corresponds to the graph consisting of the interval from 1 to  $n$  with the natural numbers (from 1 to  $n$ ) as vertices and with segments between them as edges.

Let  $|| : \Sigma_n \rightarrow \mathbb{Z}$  be the length function on the symmetric group with respect to the standard generators  $s_i$ : for  $x \in \Sigma_n$ ,  $|x|$  is the smallest natural number  $k$  such that  $x$  is a product of  $k$  elements of the set  $\{s_1, \dots, s_{n-1}\}$ . It is known ([8], Sect. 1, Ex. 13(b)) that two minimal expressions for an element of  $\Sigma_n$  are equivalent by using only the relations (1.4). This implies that the canonical projection  $\tau_n : Br_n \rightarrow \Sigma_n$  has a unique set-theoretic section  $r : \Sigma_n \rightarrow Br_n$  such that  $r(s_i) = \sigma_i$  for  $i = 1, \dots, n-1$  and  $r(xy) = r(x)r(y)$  whenever  $|xy| = |x| + |y|$ . The image  $r(\Sigma_n)$  under the name of *positive permutation braids* was studied by E. El-Rifai and H. R. Morton [13].

The following presentation for the inverse braid monoid was obtained in [12]. It has the generators  $\sigma_i, \sigma_i^{-1}$ ,  $i = 1, \dots, n-1$ ,  $\epsilon$ , and relations

$$(1.7) \quad \begin{cases} \sigma_i \sigma_i^{-1} = \sigma_i^{-1} \sigma_i = 1, \text{ for all } i, \\ \epsilon \sigma_i = \sigma_i \epsilon \text{ for } i \geq 2, \\ \epsilon \sigma_1 \epsilon = \sigma_1 \epsilon \sigma_1 \epsilon = \epsilon \sigma_1 \epsilon \sigma_1, \\ \epsilon = \epsilon^2 = \epsilon \sigma_1^2 = \sigma_1^2 \epsilon \end{cases}$$

and the braid relations (1.4).

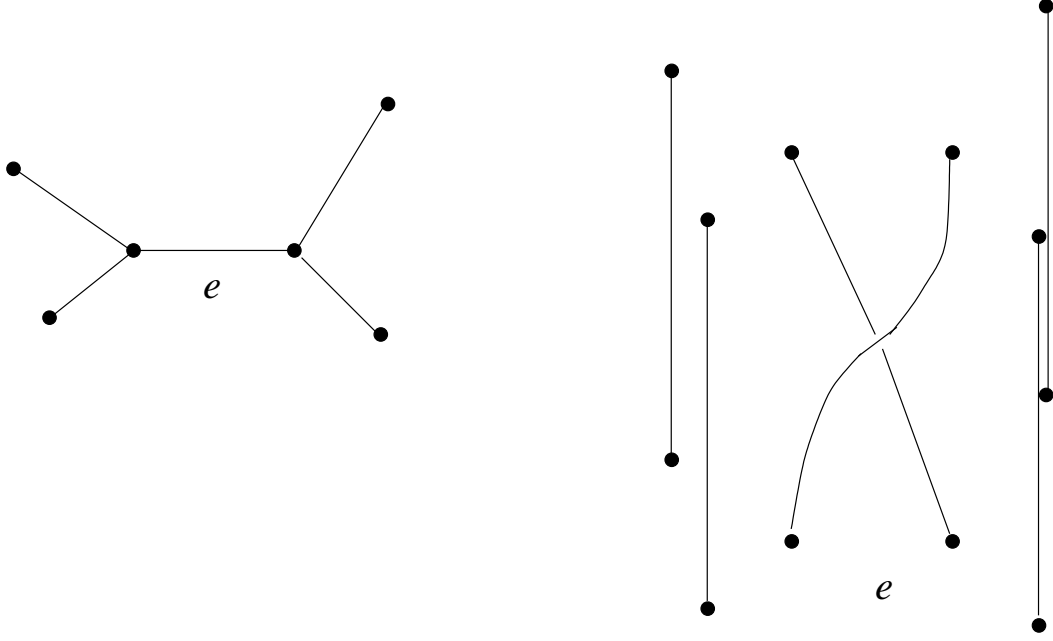


FIGURE 1.3

Geometrically the generator  $\epsilon$  means that the first string in the trivial braid is absent. If we replace the first relation in (1.7) by the following set of relations

$$(1.8) \quad \sigma_i^2 = 1, \text{ for all } i,$$

and delete the superfluous relations

$$\epsilon = \epsilon\sigma_1^2 = \sigma_1^2\epsilon,$$

we get a presentation of the symmetric inverse monoid  $I_n$  [32]. We also can simply add the relations (1.8) if we don't worry about redundant relations. We get a canonical map [12]

$$(1.9) \quad \tau_n : IB_n \rightarrow I_n$$

which is a natural extension of the corresponding map for the braid and symmetric groups.

More balanced relations for the inverse braid monoid were obtained in [19]. Let  $\epsilon_i$  denote the trivial braid with  $i$ -th string deleted, formally:

$$\begin{cases} \epsilon_1 &= \epsilon, \\ \epsilon_{i+1} &= \sigma_i^{\pm 1} \epsilon_i \sigma_i^{\pm 1}. \end{cases}$$

So, the generators are:  $\sigma_i, \sigma_i^{-1}, i = 1, \dots, n-1, \epsilon_i, i = 1, \dots, n$ , and relations are the following:

$$(1.10) \quad \left\{ \begin{array}{l} \sigma_i \sigma_i^{-1} = \sigma_i^{-1} \sigma_i = 1, \text{ for all } i, \\ \epsilon_j \sigma_i = \sigma_i \epsilon_j \text{ for } |j - i| > 1, \\ \epsilon_i \sigma_i = \sigma_i \epsilon_{i+1}, \\ \epsilon_{i+1} \sigma_i = \sigma_i \epsilon_i, \\ \epsilon_i = \epsilon_i^2, \\ \epsilon_{i+1} \sigma_i^2 = \sigma_i^2 \epsilon_{i+1} = \epsilon_{i+1}, \\ \epsilon_i \epsilon_{i+1} \sigma_i = \sigma_i \epsilon_i \epsilon_{i+1} = \epsilon_i \epsilon_{i+1}, \end{array} \right.$$

plus the braid relations (1.4).

## 2. PROPERTIES OF INVERSE BRAID MONOID

The relations (1.7) look asymmetric: one generator for the idempotent part and  $n - 1$  generators for the group part. If we minimize the number of generators of the group part and take the presentation (1.6) for the braid group we get a presentation of the inverse braid monoid with generators  $\sigma_1, \sigma, \epsilon$ , and relations:

$$(2.1) \quad \left\{ \begin{array}{l} \sigma_1 \sigma_1^{-1} = \sigma_1^{-1} \sigma_1 = 1, \\ \sigma \sigma^{-1} = \sigma^{-1} \sigma = 1, \\ \epsilon \sigma^i \sigma_1 \sigma^{-i} = \sigma^i \sigma_1 \sigma^{-i} \epsilon \text{ for } 1 \leq i \leq n - 2, \\ \epsilon \sigma_1 \epsilon = \sigma_1 \epsilon \sigma_1 \epsilon = \epsilon \sigma_1 \epsilon \sigma_1, \\ \epsilon = \epsilon^2 = \epsilon \sigma_1^2 = \sigma_1^2 \epsilon, \end{array} \right.$$

plus (1.6).

Let  $\Gamma$  be a planar graph of the Sergiescu graph presentation of the braid group [34], [5]. Let us add new generators  $\epsilon_v$  which correspond to each vertex of the graph  $\Gamma$ . Geometrically it means the absence in the trivial braid of one string corresponding to the vertex  $v$ . We orient the graph  $\Gamma$  arbitrarily and so we get a starting  $v_0 = v_0(e)$  and a terminal  $v_1 = v_1(e)$  vertex for each edge  $e$ . Consider the following relations

$$(2.2) \quad \left\{ \begin{array}{l} \sigma_e \sigma_e^{-1} = \sigma_e^{-1} \sigma_e = 1, \text{ for all edges of } \Gamma, \\ \epsilon_v \sigma_e = \sigma_e \epsilon_v, \text{ if the vertex } v \text{ and the edge } e \text{ do not intersect,} \\ \epsilon_{v_0} \sigma_e = \sigma_e \epsilon_{v_1}, \text{ where } v_0 = v_0(e), v_1 = v_1(e), \\ \epsilon_{v_1} \sigma_e = \sigma_e \epsilon_{v_0}, \\ \epsilon_v = \epsilon_v^2, \\ \epsilon_{v_i} \sigma_e^2 = \sigma_e^2 \epsilon_{v_i} = \epsilon_{v_i}, \quad i = 0, 1, \\ \epsilon_{v_0} \epsilon_{v_1} \sigma_e = \sigma_e \epsilon_{v_0} \epsilon_{v_1} = \epsilon_{v_0} \epsilon_{v_1}. \end{array} \right.$$

**Theorem 2.1.** *We get a Sergiescu graph presentation of the inverse braid monoid  $IB_n$  if we add to the graph presentation of the braid group  $Br_n$  the relations (2.2).*

□

A *positive partial braid* is a element of  $IB_n$  which can be written as a word with only positive entries of the generators  $\sigma_i$ ,  $i = 1, \dots, n - 1$ .

A positive partial braid is called a *positive partial permutation braid* if it can be drawn as a geometric positive partial braid in which every pair of strings crosses at most once.

Write  $IB_n^+$  for the set of positive partial permutation braids.

**Proposition 2.1.** *If the partial braids  $b_1, b_2 \in IB_n^+$  induce the same partial permutation on their strings, then  $b_1 = b_2$ . For each  $s \in I_n$  there is a partial braid  $b \in IB_n^+$ , which induces this partial permutation:  $\tau(b) = s$ .*

*Proof.* The original arguments for  $Br_n$  are geometrical and so they translate completely to the case of partial braids. □

Let  $EF_n$  be a monoid of partial isomorphisms of a free group  $F_n$  defined as follows. Let  $a$  be an element of the symmetric inverse monoid  $I_n$ ,  $a \in I_n$ ,  $J_k = \{j_1, \dots, j_k\}$  is the image of  $a$ ,

and elements  $i_1, \dots, i_k$  belong to domain of the definition of  $a$ . The monoid  $EF_n$  consists of isomorphisms

$$\langle x_{i_1}, \dots, x_{i_k} \rangle \rightarrow \langle x_{j_1}, \dots, x_{j_k} \rangle$$

expressed by

$$f_a : x_i \mapsto w_i^{-1} x_{a(i)} w_i,$$

if  $i$  is among  $i_1, \dots, i_k$  and not defined otherwise and  $w_i$  is a word on  $x_{j_1}, \dots, x_{j_k}$ . The composition of  $f_a$  and  $g_b$ ,  $a, b \in I_n$  is defined for  $x_i$  belonging to the domain of  $a \circ b$ . We put  $x_{j_m} = 1$  in a word  $w_i$  if  $x_{j_m}$  does not belong to the domain of definition of  $g$ . We define a map  $\phi_n$  from  $IB_n$  to  $EF_n$  expanding the canonical inclusion

$$Br_n \rightarrow \text{Aut } F_n$$

by the condition that  $\phi_n(\epsilon)$  as a partial isomorphism of  $F_n$  is given by the formula

$$(2.3) \quad \phi(\epsilon)(x_i) = \begin{cases} x_i & \text{if } i \geq 2, \\ \text{not defined,} & \text{if } i = 1. \end{cases}$$

Using the presentation (1.7) we see that  $\phi_n$  is correctly defined homomorphism of monoids

$$\phi_n : IB_n \rightarrow EF_n.$$

**Theorem 2.2.** *The homomorphism  $\phi_n$  is a monomorphism.*

*Proof.* Monoid  $IB_n$  as a set is a disjoint union of copies of braid groups  $Br_k$ ,  $k = 0, \dots, n$ . (See [19] for the exact formula of this splitting of  $IB_n$  as a groupoid.) Each copy of the group  $Br_k$  is identified by the numbers of inputs of strings  $i_1, \dots, i_k$  and outputs of them  $j_1, \dots, j_k$ . Let  $I_k = \{i_1, i_2, \dots, i_k\}$ ,  $i_1 < i_2 < \dots < i_k$ ,  $J_k = \{j_1, j_2, \dots, j_k\}$ ,  $j_1 < j_2 < \dots < j_k$ , and let  $Br(I_k, J_k)$  be the corresponding copy of the braid group. So

$$(2.4) \quad IB_n = \coprod_{I_k, J_k \subset \{1, \dots, n\}} Br(I_k, J_k).$$

Define a homomorphism

$$\psi(I_k, J_k) : Br_k \rightarrow EF_n.$$

Let  $\gamma(I_k)$  be the homomorphism  $F_n \rightarrow F_k$  defined by

$$(2.5) \quad \begin{cases} x_{i_l} & \mapsto x_l, \\ x_s & \mapsto e \text{ if } s \notin I_k. \end{cases}$$

We define a homomorphism

$$\beta(J_k) : F_k \rightarrow F_n$$

as an inclusion

$$\beta(J_k)(x_l) = x_{j_l}, \quad l = 1, \dots, k.$$

For each automorphism  $\alpha : F_k \rightarrow F_k$ ,  $\alpha \in Br_k$ , its image  $\psi(I_k, J_k)(\alpha)$  in  $EF_n$  is defined as a composition

$$\psi(I_k, J_k)(\alpha) = \beta(J_k) \alpha \gamma(I_k),$$

we compose from right to left as for functions. Homomorphism  $\psi(I_k, J_k)$  is a monomorphism. Consider the following diagram

$$(2.6) \quad \begin{array}{ccc} Br_k & \xrightarrow{Id} & Br_k \\ \downarrow \rho & & \downarrow \psi(I_k, J_k) \\ Br(I_k, J_k) & \xrightarrow{\phi_n} & EF_n \end{array}$$

where the left hand map  $\rho$  is the bijection. Let us prove that the diagram commutes. Consider a generator of  $Br_k$ , say  $\sigma_1$ . We denote  $\rho(\sigma_1)$  by  $\sigma(i_1, i_2; j_1, j_2) \in IB_n$ . This is the positive partial braid where the string starting at  $i_1$  goes to  $j_1$  and the string starting at  $i_2$  goes to  $j_2$ . There is no strings starting before  $i_1$ , between  $i_1$  and  $i_2$ , ending before  $j_1$  and between  $j_1$  and  $j_2$ . Suppose that  $i_1 < j_2 < i_2 < j_1$ , the other cases can be considered the same way. The partial braid  $\sigma(i_1, i_2; j_1, j_2) \in IB_n$  as an element of the inverse braid monoid can be expressed as a word on generators in the following form:

$$\sigma(i_1, i_2; j_1, j_2) = \sigma_{i_1} \sigma_{i_1+1} \dots \sigma_{i_2} \dots \sigma_{j_1-1} \sigma_{i_2-2} \dots \sigma_{j_1} \epsilon_{i_1} \epsilon_{i_1+1} \dots \epsilon_{j_2-1} \epsilon_{j_2+1} \dots \epsilon_{j_1-1}.$$

Note that the expression  $\sigma_{i_2-2} \dots \sigma_{j_1}$  is present in the formula only if  $i_2 - 2 \geq j_2$ . We denote it also as consisting of the two parts:

$$\sigma(i_1, i_2; j_1, j_2) = \sigma \epsilon.$$

Let us study the action of  $\sigma(i_1, i_2; j_1, j_2)$  on the generators of the free group. We have:

$$\sigma(x_{i_1}) = x_{j_2}$$

and then apply the action of the part  $\epsilon$ :

$$\epsilon(x_{j_2}) = x_{j_2}.$$

Also we have:

$$\sigma(x_l) = x_{j_2}^{-1} x_l x_{j_2} \text{ for } i_1 < l < i_2.$$

After the application of  $\epsilon$  we obtain  $\sigma(i_1, i_2; j_1, j_2)(x_l) = e$ . We have

$$\sigma(x_{i_2}) = x_{j_2}^{-1} x_{i_2-1}^{-1} \dots x_{j_1+1}^{-1} x_{j_1} x_{j_1+1} \dots x_{i_2-1} x_{j_2}$$

and then apply the action of the part  $\epsilon$ :

$$\epsilon(x_{j_2}^{-1} x_{i_2-1}^{-1} \dots x_{j_1+1}^{-1} x_{j_1} x_{j_1+1} \dots x_{i_2-1} x_{j_2}) = x_{j_2}^{-1} x_{j_1} x_{j_2}.$$

We get exactly the action of the image of  $\sigma_1$  by the composition of the canonical inclusion and the map  $\psi(I_k, J_k)$ . The diagram (2.6) commutes. So,  $\phi_n$  is also a monomorphism. The different copies of  $Br(I_k, J_k)$  of  $IB_n$  do not intersect in  $EF_n$ . So,

$$\phi_n : IB_n \rightarrow EF_n$$

is a monomorphism. □

**Theorem 2.3.** *The monomorphism  $\phi_n$  gives a solution of the word problem for the inverse braid monoid in the presentations (1.4), (1.7), (1.10), (2.2) and (2.1).*

*Proof.* As for the braid group it follows from the fact that two words represent the same element of the monoid iff they have the same action on the finite set of generators of the free group  $F_n$ . □

Theorem 2.2 gives also a possibility to interpret the inverse braid monoid as a monoid of isotopy classes of maps. As usual consider a disc  $D^2$  with  $n$  fixed points. Denote the set of these points by  $Q_n$ . The fundamental group of  $D^2$  with these points deleted is isomorphic to  $F_n$ . Consider homeomorphisms of  $D^2$  onto a copy of the same disc with the condition that only  $k$  points of  $Q_n$ ,  $k \leq n$  (say  $i_1, \dots, i_k$ ) are mapped bijectively onto the  $k$  points (say  $j_1, \dots, j_k$ ) of the second copy of  $D^2$ . Consider the isotopy classes of such homeomorphisms and denote the set of them by  $IM_n(D^2)$ . Evidently it is a monoid.

**Theorem 2.4.** *The monoids  $IB_n$  and  $IM_n(D^2)$  are isomorphic.*



*Proof.* The same way as in the proof for the braid group using Alexander's trick we associate a partial braid to an element of  $IM_n(D^2)$  and prove that it is an isomorphism.  $\square$

These considerations can be generalized to the following definition. Consider a surface  $S_{g,b,n}$  of the genus  $g$  with  $b$  boundary components and the set  $Q_n$  of  $n$  fixed points. Let  $f$  be a homeomorphism of  $S_{g,b,n}$  which maps  $k$  points,  $k \leq n$ , from  $Q_n$ :  $\{i_1, \dots, i_k\}$  to  $k$  points  $\{j_1, \dots, j_k\}$  also from  $Q_n$ . The same way let  $h$  be a homeomorphism of  $S_{g,b,n}$  which maps  $l$  points,  $l \leq n$ , from  $Q_n$ , say  $\{s_1, \dots, s_l\}$  to  $l$  points  $\{t_1, \dots, t_l\}$  again from  $Q_n$ . Consider the intersection of the sets  $\{j_1, \dots, j_k\}$  and  $\{s_1, \dots, s_l\}$ , let it be the set of cardinality  $m$ , it may be empty. Then the composition of  $f$  and  $h$  maps  $m$  points of  $Q_n$  to  $m$  points (may be different) of  $Q_n$ . If  $m = 0$  then the composition have no relation to the set  $Q_n$ . Denote the set of isotopy classes of such maps by  $\mathcal{IM}_{g,b,n}$ . Composition defines a structure of monoid on  $\mathcal{IM}_{g,b,n}$ .

**Proposition 2.2.** *The monoid  $\mathcal{IM}_{g,b,n}$  is inverse.*

*Proof.* Each element of  $\mathcal{IM}_{g,b,n}$  is represented by a homeomorphism  $h$  of  $S_{g,b,n}$ . So, take an inverse of  $h$  and get the identities (1.1) and (1.2).  $\square$

We call the monoid  $\mathcal{IM}_{g,b,n}$  the *inverse mapping class monoid*. If  $g = 0$  and  $b = 1$  we get the inverse braid monoid. In the general case  $\mathcal{IM}_{g,b,n}$  the role of the empty braid plays the mapping class group  $\mathcal{M}_{g,b}$  (without fixed points).

We remind that a monoid  $M$  is *factorisable* if  $M = EG$  where  $E$  is a set of idempotents of  $M$  and  $G$  is a subgroup of  $M$ .

**Proposition 2.3.** *The monoid  $\mathcal{IM}_{g,b,n}$  can be written in the form*

$$\mathcal{IM}_{g,b,n} = E\mathcal{M}_{g,b,n},$$

where  $E$  is a set of idempotents of  $\mathcal{IM}_{g,b,n}$  and  $\mathcal{M}_{g,b,n}$  is the corresponding mapping class group. So this monoid is factorisable.

*Proof.* An element of  $\mathcal{IM}_{g,b,n}$  is represented by a homeomorphism  $h$  of  $S_{g,b,n}$  which maps  $k$  points,  $k \leq n$ , from  $Q_n$ :  $\{i_1, \dots, i_k\}$  to  $k$  points  $\{j_1, \dots, j_k\}$  from  $Q_n$ . In the isotopy class of  $h$  we find a homeomorphism  $h_1$  which maps arbitrarily  $Q_n \setminus \{i_1, \dots, i_k\}$  to  $Q_n \setminus \{j_1, \dots, j_k\}$ . Necessary idempotent element is the isotopy class of the identity homeomorphism which fixes only the points  $\{i_1, \dots, i_k\}$ .  $\square$

Let  $\Delta$  be the Garside's fundamental word in the braid group  $Br_n$  [18]. It can be defined by the formula:

$$\Delta = \sigma_1 \dots \sigma_{n-1} \sigma_1 \dots \sigma_{n-2} \dots \sigma_1 \sigma_2 \sigma_1.$$

If we use Garside's notation  $\Pi_t \equiv \sigma_1 \dots \sigma_t$ , then  $\Delta \equiv \Pi_{n-1} \dots \Pi_1$ .

**Proposition 2.4.** *The generators  $\epsilon_i$  commute with  $\Delta$  in the following way:*

$$\epsilon_i \Delta = \Delta \epsilon_{n+1-i}.$$

*Proof.* Direct calculation using the second, third and the fourth relations in (1.10).  $\square$

**Proposition 2.5.** *The center of  $IB_n$  consists of the union of the center of the braid group  $Br_n$  (generated by  $\Delta^2$ ) and the empty braid  $\emptyset = \epsilon_1 \dots \epsilon_n$ .*

*Proof.* The given element lie in the center. Suppose that there are other ones. Let  $c$  be one of them. It does not belong to  $Br_n$ , because its center is already taken into account. It is a partial braid with starting points  $I_k = \{i_1, \dots, i_k\}$  and ending points  $J_k = \{j_1, \dots, j_k\}$   $k < n$ .

Take the one-string partial braid  $x$  that starts in the complement of  $J_k$  and ends in  $I_k$ . Then  $cx$  is the empty braid, while  $xc$  is not.  $\square$

Let  $\mathcal{E}$  be the monoid generated by one idempotent generator  $\epsilon$ .

**Proposition 2.6.** *The abelianization of  $IB_n$  is isomorphic to  $\mathcal{E} \oplus \mathbb{Z}$ . The canonical map*

$$a : IB_n \rightarrow \mathcal{E} \oplus \mathbb{Z}$$

is given by the formula:

$$\begin{cases} a(\epsilon_i) = \epsilon, \\ a(\sigma_i) = 1. \end{cases}$$

$\square$

Let  $\epsilon_{k+1,n}$  denote the partial braid with the trivial first  $k$  strings and the absent rest  $n - k$  strings. It can be expressed using the generator  $\epsilon$  or the generators  $\epsilon_i$  as follows

$$(2.7) \quad \epsilon_{k+1,n} = \epsilon \sigma_{n-1} \dots \sigma_{k+1} \epsilon \sigma_{n-1} \dots \sigma_{k+2} \epsilon \dots \epsilon \sigma_{n-1} \sigma_{n-2} \epsilon \sigma_{n-1} \epsilon,$$

$$(2.8) \quad \epsilon_{k+1,n} = \epsilon_{k+1} \epsilon_{k+2} \dots \epsilon_n,$$

It was proved in [12] the every partial braid has a representative of the form

$$(2.9) \quad \sigma_{i_1} \dots \sigma_1 \dots \sigma_{i_k} \dots \sigma_k \epsilon_{k+1,n} x \epsilon_{k+1,n} \sigma_k \dots \sigma_{j_k} \dots \sigma_1 \dots \sigma_{j_1},$$

$$(2.10) \quad k \in \{0, \dots, n\}, x \in Br_k, 0 \leq i_1 < \dots < i_k \leq n - 1 \text{ and } 0 \leq j_1 < \dots < j_k \leq n - 1.$$

Note that in the formula (2.9) we can delete one of the  $\epsilon_{k+1,n}$ , but we shall use the form (2.9) because of convenience: two symbols  $\epsilon_{k+1,n}$  serve as markers to distinguish the elements of  $Br_k$ . We can put the element  $x \in Br_k$  in the Markov normal form [25] and get the corresponding *Markov normal form for the inverse braid monoid  $IB_n$* . The same way for the Garside normal form.

Let us remind the mains point of Garside's construction. Essential role in Garside work plays the monoid of *positive* braids  $Br_n^+$ , that is the monoid which has a presentation with generators  $\sigma_i$ ,  $i = 1, \dots, n$  and relations (1.4). In other words each element of this monoid can be represented as a word on the elements  $\sigma_i$ ,  $i = 1, \dots, n$  with no entrances of  $\sigma_i^{-1}$ . Two positive words  $V$  and  $W$  in the alphabet  $\{\sigma_i, (i = 1, \dots, n - 1)\}$  will be said to be *positively equal* if they are equal as elements of  $Br_n^+$ . Usually this is written as  $V \doteq W$ .

Among positive words on the alphabet  $\{\sigma_1 \dots \sigma_n\}$  let us introduce a lexicographical ordering with the condition that  $\sigma_1 < \sigma_2 < \dots < \sigma_n$ . For a positive word  $V$  the *base* of  $V$  is the smallest positive word which is positively equal to  $V$ . The base is uniquely determined. If a positive word  $V$  is prime to  $\Delta$ , then for the base of  $V$  the notation  $\bar{V}$  will be used.

**Theorem 2.5.** *Every word  $W$  in  $IBr_n$  can be uniquely written in the form*

$$(2.11) \quad \sigma_{i_1} \dots \sigma_1 \dots \sigma_{i_k} \dots \sigma_k \epsilon_{k+1,n} x \epsilon_{k+1,n} \sigma_k \dots \sigma_{j_k} \dots \sigma_1 \dots \sigma_{j_1},$$

$$(2.12) \quad k \in \{0, \dots, n\}, x \in Br_k, 0 \leq i_1 < \dots < i_k \leq n - 1 \text{ and } 0 \leq j_1 < \dots < j_k \leq n - 1.$$

where  $x$  is written in the Garside normal form for  $Br_k$

$$\Delta^m \bar{V},$$

where  $m$  is an integer.

*Proof.* Note that the elements  $\sigma_{i_1} \dots \sigma_1 \dots \sigma_{i_k} \dots \sigma_k$  and  $\sigma_k \dots \sigma_{j_k} \dots \sigma_1 \dots \sigma_{j_1}$  are uniquely determined by a given element of  $IB_n$  (written as a word  $W$  in the alphabet  $A = \{\sigma_i, \sigma_i^{-1}, i = 1, \dots, n-1, \epsilon\}$ ). Then Theorem follows from the existence of the Garside normal form for  $Br_k$ .  $\square$

Theorem 2.5 is evidently true also for the presentation with  $\epsilon_i, i = 1, \dots, n$ . In this case the elements  $\epsilon_{k+1, n}$  are expressed by (2.8).

The form of a word  $W$  established in this theorem we call the *Garside left normal form for the inverse braid monoid*  $IB_n$  and the index  $m$  we call the *power* of  $W$ . The same way the *Garside right normal form for the inverse braid monoid* is defined and the corresponding variant of Theorem 2.5 is true.

**Theorem 2.6.** *The necessary and sufficient condition that two words in  $IB_n$  are equal is that their Garside normal forms are identical. The Garside normal form gives a solution to the word problem in the braid group.*

*Proof.* As we noted in the proof of the previous Theorem the elements  $\sigma_{i_1} \dots \sigma_1 \dots \sigma_{i_k} \dots \sigma_k$  and  $\sigma_k \dots \sigma_{j_k} \dots \sigma_1 \dots \sigma_{j_1}$  are uniquely determined. Also in [12] (implicitly) there was given an algorithm how to obtain the form (2.9) for an arbitrary word  $W$  in the alphabet  $A$ . Then combining it with the Garside algorithm we get a solution of the word problem for the inverse braid monoid.  $\square$

Garside normal form for the braid groups was precized in the subsequent works of S. I. Adyan [2], W. Thurston [14], E. El-Rifai and H. R. Morton [13]. Namely, there was introduced the *left-greedy form* (in the terminology of W. Thurston [14])

$$\Delta^t A_1 \dots A_k,$$

where  $A_i$  are the successive possible longest *fragments of the word*  $\Delta$  (in the terminology of S. I. Adyan [2]) or *positive permutation braids* (in the terminology of E. El-Rifai and H. R. Morton [13]). Certainly, the same way the *right-greedy form* is defined. These greedy forms are defined for the inverse braid monoid the same way.

Let us consider the elements  $m \in IB_n$  satisfying the equation:

$$(2.13) \quad \epsilon_i m = \epsilon_i.$$

Geometrically this means that removing the string (if it exists) that starts at the point with the number  $i$  we get a trivial braid on the rest  $n-1$  strings. It is equivalent to the condition

$$(2.14) \quad m \epsilon_{\tau(m)(i)} = \epsilon_{\tau(m)(i)},$$

where  $\tau$  is the canonical map to the symmetric monoid (1.9). With the exception of  $\epsilon_i$  itself all such elements belong to  $Br_n$ . We call such braids as  *$i$ -Makanin* and denote the subgroup of  *$i$ -Makanin braids* by  $A_i$ . The subgroups  $A_i, i = 1, \dots, n$ , are conjugate

$$(2.15) \quad A_i = \sigma_{i-1}^{-1} \dots \sigma_1^{-1} A_1 \sigma_1 \dots \sigma_{i-1}$$

free subgroups. The group  $A_1$  is freely generated by the set  $\{x_1, \dots, x_{n-1}\}$  [21], where

$$(2.16) \quad x_i = \sigma_{i-1}^{-1} \dots \sigma_1^{-1} \sigma_1^2 \sigma_1 \dots \sigma_{i-1}.$$

The intersection of all subgroups of  *$i$ -Makanin braids* is the group of *Makanin braids*

$$(2.17) \quad Mak_n = \bigcap_{i=1}^n A_i.$$

That is the same as  $m \in Mak_n$  if and only if the equation (2.13) holds for all  $i$ .

## 3. MONOIDS OF PARTIAL GENERALIZED BRAIDS

Construction of partial braids can be applied to various generalizations of braids, namely to those where geometric or diagrammatic construction of braids takes place. Let  $S_g$  be a surface of genus  $g$  probably with boundary components and punctures. We consider partial braids lying in a layer between two such surfaces:  $S_g \times I$  and take a set of isotopy classes of such braids. We get a monoid of partial braid of a surface  $S_g$ , denote it by  $IB_n(S_g)$ . An interesting case is when the surface is a sphere  $S^2$ . So our partial braids are lying in a layer between two concentric spheres. It was proved by O. Zariski [41] and then rediscovered by E. Fadell and J. Van Buskirk [15] that the braid group of a sphere has a presentation with generators  $\sigma_i$ ,  $i = 1, \dots, n-1$ , the same as for the classical braid group satisfying the braid relations (1.4) and the following sphere relation:

$$(3.1) \quad \sigma_1 \sigma_2 \dots \sigma_{n-2} \sigma_{n-1}^2 \sigma_{n-2} \dots \sigma_2 \sigma_1 = 1.$$

**Theorem 3.1.** *We get a presentation of the monoid  $IB_n(S^2)$  if we add to the presentation (1.7) or the presentation (1.10) of  $IB_n$  the sphere relation (3.1). It is a factorisable inverse monoid.*

*Proof.* Essentially it is the same as for  $IB_n$ . Denote temporarily by  $M_n$  the monoid defined by the presentation and  $IB_n(S^2)$  denotes the monoid of homotopy classes. We already used that every word in the alphabet  $A$  is congruent (using the relations (1.7) to a word of the form (2.9). Now note that for the sphere inverse braid monoid the alphabet is the same and relations for  $IB_n$  are included into the set of relations for  $IB_n(S^2)$ . As in [12] the evident map

$$\Psi : M_n \rightarrow IB_n(S^2)$$

is defined and proved that it is onto. Let us prove that  $\Psi$  is a monomorphism. Suppose that for two words  $W_1, W_2 \in M_n$  we have

$$\Psi(W_1) = \Psi(W_2).$$

That means that the corresponding braids are isotopic. Using relations (1.7) transform the words  $W_1, W_2$  into the form (2.9)

$$\sigma(i_1, \dots, i_k; k) \epsilon_{k+1, n} x \epsilon_{k+1, n} \sigma(k; j_1, \dots, j_k).$$

Then the corresponding fragments  $\sigma(i_1, \dots, i_k; k)$  and  $\sigma(k; j_1, \dots, j_k; k)$  for  $W_1$  and  $W_2$  coincide. The elements  $x_1$  of  $W_1$  and  $x_2$  of  $W_2$ , which are the words on  $\sigma_1, \dots, \sigma_k$ , correspond after  $\Psi$  to homotopic braids on  $k$  strings on the sphere  $S^2$ . So  $x_1$  can be transformed into  $x_2$  using relations for the braid groups  $Br_k(S^2)$ . The words  $W_1$  and  $W_2$  represent the same element in  $M_n$ .  $\square$

Another example here is the braid group of a punctured disc which is isomorphic to the Artin-Brieskorn braid group of the type  $B$  [9], [37]. With respect to the classical braid group it has an extra generator  $\tau$  and the relations of type  $B$ :

$$(3.2) \quad \begin{cases} \tau \sigma_1 \tau \sigma_1 & = \sigma_1 \tau \sigma_1 \tau, \\ \sigma_i \sigma_j & = \sigma_j \sigma_i, \text{ if } |i - j| > 1, \end{cases}$$

Denote by  $IBB_n$  the monoid of partial braids of the type  $B$ .

**Theorem 3.2.** *We get a presentation of the monoid  $IBB_n$  if we add to the presentation (1.7) or the presentation (1.10) of  $IB_n$  one generator  $\tau$ , the type B relation (3.2) and the following relations*

$$(3.3) \quad \begin{cases} \tau\tau^{-1} = \tau^{-1}\tau = 1, \\ \epsilon_1\tau = \tau\epsilon_1 = \epsilon_1. \end{cases}$$

*It is a factorisable inverse monoid.*

*Proof.* The same as for  $IB_n$ . □

**Remark 3.1.** *Theorem 3.3 can be easily generalized for partial braids in handlebodies [35].*

The same way as for  $IB_n$  the notion of Makanin braids can be defined for any surface and we get  $Mak_n(S_g) \subset IB_n(S_g)$ . The group of Makanin braids for the sphere was used in the exact sequence (1.3).

Let  $BP_n$  be the braid-permutation group of R. Fenn, R. Rimányi and C. Rourke [17]. It is defined as a subgroup of  $\text{Aut } F_n$ , generated by both sets of the automorphisms  $\sigma_i$  of (1.5) and  $\xi_i$  of the following form:

$$(3.4) \quad \begin{cases} x_i & \mapsto x_{i+1}, \\ x_{i+1} & \mapsto x_i, \\ x_j & \mapsto x_j, j \neq i, i+1, \end{cases}$$

R. Fenn, R. Rimányi and C. Rourke proved that this group is given by the set of generators:  $\{\xi_i, \sigma_i, i = 1, 2, \dots, n-1\}$  and relations:

$$\begin{cases} \xi_i^2 & = 1, \\ \xi_i\xi_j & = \xi_j\xi_i, \text{ if } |i-j| > 1, \\ \xi_i\xi_{i+1}\xi_i & = \xi_{i+1}\xi_i\xi_{i+1}. \end{cases}$$

The symmetric group relations

$$\begin{cases} \sigma_i\sigma_j & = \sigma_j\sigma_i, \text{ if } |i-j| > 1, \\ \sigma_i\sigma_{i+1}\sigma_i & = \sigma_{i+1}\sigma_i\sigma_{i+1}. \end{cases}$$

The braid group relations

$$(3.5) \quad \begin{cases} \sigma_i\xi_j & = \xi_j\sigma_i, \text{ if } |i-j| > 1, \\ \xi_i\xi_{i+1}\sigma_i & = \sigma_{i+1}\xi_i\xi_{i+1}, \\ \sigma_i\sigma_{i+1}\xi_i & = \xi_{i+1}\sigma_i\sigma_{i+1}. \end{cases}$$

The mixed relations for the braid-permutation group

R. Fenn, R. Rimányi and C. Rourke also gave a geometric interpretation of  $BP_n$  as a group of *welded braids*.

We consider the image of monoid  $I_n$  in  $\text{End } F_n$  by the map defined by the formulas (3.4), (2.3). We take also the monoid  $IB_n$  lying in  $\text{End } F_n$  under the map  $\phi_n$  of Theorem (2.2). We define the *braid-permutation* monoid as a submonoid of  $\text{End } F_n$  generated by both images of  $IB_n$  and  $I_n$  and denote it by  $IBP_n$ . It can be also defined by the diagrams of partial welded braids.

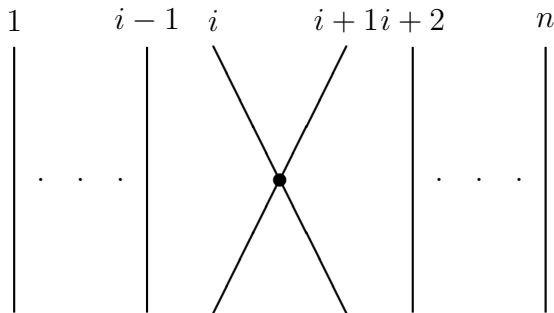


FIGURE 3.1

**Theorem 3.3.** *We get a presentation of the monoid  $IBP_n$  if we add to the presentation of  $BP_n$  the generator  $\epsilon$ , relations (1.7) and the analogous relations between  $\xi_i$  and  $\epsilon$ , or generators  $\epsilon_i$ ,  $1 \leq i \leq n$  relations (1.10) and the analogous relations between  $\xi_i$  and  $\epsilon_i$ . It is a factorisable inverse monoid.*

*Proof.* The same as for  $BP_n$ . □

The virtual braids [38] can be defined by the plane diagrams with real and virtual crossings. The corresponding Reidemeister moves are the same as for the welded braids of the braid-permutation group with one exception. The forbidden move corresponds to the last mixed relation for the braid-permutation group. This allows to define the partial virtual braids and the corresponding monoid  $IVB_n$ . So the mixed relation for  $IVB_n$  have the form:

$$(3.6) \quad \begin{cases} \sigma_i \xi_j &= \xi_j \sigma_i, \text{ if } |i - j| > 1, \\ \xi_i \xi_{i+1} \sigma_i &= \sigma_{i+1} \xi_i \xi_{i+1}. \end{cases}$$

The mixed relations for virtual braids

**Theorem 3.4.** *We get a presentation of the monoid  $IVB_n$  if we delete the last mixed relation in the presentation of  $IBP_n$ , that is replace the relations (3.5) by (3.6) It is a factorisable inverse monoid. The canonical epimorphism*

$$IVB_n \rightarrow IBP_n$$

*is evidently defined.*

The *singular braid monoid*  $SB_n$  or *Baez–Birman monoid* [4], [7] is defined as a monoid with generators  $\sigma_i, \sigma_i^{-1}, x_i$ ,  $i = 1, \dots, n - 1$ , and relations

$$(3.7) \quad \begin{cases} \sigma_i \sigma_j = \sigma_j \sigma_i, \text{ if } |i - j| > 1, \\ x_i x_j = x_j x_i, \text{ if } |i - j| > 1, \\ x_i \sigma_j = \sigma_j x_i, \text{ if } |i - j| \neq 1, \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \\ \sigma_i \sigma_{i+1} x_i = x_{i+1} \sigma_i \sigma_{i+1}, \\ \sigma_{i+1} \sigma_i x_{i+1} = x_i \sigma_{i+1} \sigma_i, \\ \sigma_i \sigma_i^{-1} = \sigma_i^{-1} \sigma_i = 1. \end{cases}$$

In pictures  $\sigma_i$  corresponds to the canonical generator of the braid group and  $x_i$  represents an intersection of the  $i$ th and  $(i + 1)$ st strand as in Figure 3.1. The singular braid monoid on two

strings is isomorphic to  $\mathbb{Z} \oplus \mathbb{Z}^+$ . The construction of  $SB_n$  is geometric, so we can easily get the analogous monoid of partial singular braids  $PSB_n$ .

**Theorem 3.5.** *We get a presentation of the monoid  $PSB_n$  if we add to the presentation of  $SB_n$  the generators  $\epsilon_i$ ,  $1 \leq i \leq n$ , relations (1.10) and the analogous relations between  $x_i$  and  $\epsilon_i$ .*

*Proof.* The same as for  $BP_n$ . □

**Remark 3.2.** *The monoid  $PSB_n$  is not neither factorisable nor inverse.*

The construction of braid groups on graphs [20], [16] is geometrical so, the same way as for the classical braid groups we can define *partial braids on a graph  $\Gamma$*  and the *monoid of partial braids on a graph  $\Gamma$*  which will be evidently inverse, so we call it as *inverse braid monoid on the graph  $\Gamma$*  and we denote it as  $IB_n\Gamma$ .

#### 4. PARTIAL BRAIDS AND BRAIDED MONOIDAL CATEGORIES

The system of braid groups  $Br_n$  is equipped with the standard pairings

$$\mu : Br_k \times Br_l \rightarrow Br_{k+l}.$$

It may be constructed by means of adding  $l$  extra strings to the initial  $k$ . If  $\sigma'_i$  are the generators of  $Br_k$ ,  $\sigma''_j$  are the generators of  $Br_l$  and  $\sigma_r$  are the generators of  $Br_{k+l}$ , then the map  $\mu$  can be expressed in the form

$$\begin{aligned} \mu(\sigma'_i, e) &= \sigma_i, \quad 1 \leq i \leq k-1, \\ \mu(e, \sigma''_j) &= \sigma_{j+k}, \quad 1 \leq j \leq l-1. \end{aligned}$$

The same geometric construction allows to extend this pairing to a pairing for the inverse braid monoids.

$$\mu : IB_k \times IB_l \rightarrow IB_{k+l}$$

such that the following diagram commutes

$$(4.1) \quad \begin{array}{ccc} Br_k \times Br_l & \xrightarrow{\mu} & Br_{k+l} \\ \downarrow \kappa & & \downarrow \kappa \\ IB_k \times IB_l & \xrightarrow{\mu} & IB_{k+l}. \end{array}$$

The vertical lines denote here the canonical inclusions. For the generators  $\epsilon_i$  we have:

$$\begin{aligned} \mu(\epsilon'_i, e) &= \epsilon_i, \quad 1 \leq i \leq k, \\ \mu(e, \epsilon''_j) &= \epsilon_{j+k}, \quad 1 \leq j \leq l. \end{aligned}$$

A strict monoidal (tensor) category  $\mathcal{B}$  is defined in a standard way. Its objects  $\{\bar{0}, \bar{1}, \dots\}$  correspond to integers from 0 to infinity and morphisms are defined by the formula:

$$(4.2) \quad \text{hom}(\bar{k}, \bar{l}) = \begin{cases} Br_k, & \text{if } k = l, \\ \emptyset, & \text{if } k \neq l. \end{cases}$$

The product in  $\mathcal{B}$  is defined on objects by the sum of numbers and on morphisms, by the pairing  $\mu$ . The category  $\mathcal{B}$ , generated by the braid groups, is a *braided monoidal category* as defined by A. Joyal and R. Street [22].

The following system of elements

$$\sigma_m \dots \sigma_1 \sigma_{m+1} \dots \sigma_2 \dots \sigma_{n+m-1} \dots \sigma_n \in Br_{m+n}$$

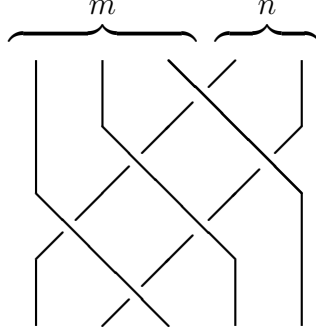


FIGURE 4.1

defines a braiding  $c$  in  $\mathcal{B}$ . Graphically it is depicted in Figure 4.1.

The same way we define a strict monoidal category  $\mathcal{IB}$  with the same objects as for  $\mathcal{B}$  and morphisms

$$(4.3) \quad \text{hom}(\bar{k}, \bar{l}) = \begin{cases} IB_k, & \text{if } k = l, \\ \emptyset, & \text{if } k \neq l. \end{cases}$$

The canonical inclusions

$$(4.4) \quad \kappa_n : Br_n \rightarrow IB_n$$

define a functor

$$\mathcal{K} : \mathcal{B} \rightarrow \mathcal{IB}.$$

The image of the braiding  $c$  by the functor  $\mathcal{K} : \mathcal{B} \rightarrow \mathcal{IB}$  is a braiding in the category  $\mathcal{IB}$ .

Geometrically the fact that the braiding for  $\mathcal{B}$  defines also a braiding for partial braids is easily seen from Figure 4.1.

**Proposition 4.1.** *The image of the braiding  $c$  in the category  $\mathcal{B}$  by the functor  $\mathcal{K}$  is a braiding in the category  $\mathcal{IB}$ , so it becomes a braided monoidal category and the functor  $\mathcal{K}$  becomes a morphism between the braided monoidal categories.*

*Proof.* By definition, the naturality of the braiding  $\mathcal{K}(c)$  (which we denote by the same symbol  $c$ ) means that the following equality

$$c_{\bar{m}, \bar{n}} \cdot \mu(b'_m, b''_n) = \mu(b''_n, b'_m) \cdot c_{\bar{m}, \bar{n}}, \quad b'_m \in Br_m, \quad b''_n \in Br_n,$$

is fulfilled. This amounts to the expression

$$c_{\bar{m}, \bar{n}} \cdot \mu(b'_m, b''_n) \cdot c_{\bar{m}, \bar{n}}^{-1} = \mu(b''_n, b'_m),$$

which means that the conjugation by the element  $c_{\bar{m}, \bar{n}}$  transforms the elements of  $IB_m \times IB_n$ , lying canonically in  $IB_{m+n}$ , into the corresponding elements of  $IB_n \times IB_m$ . The elements  $c_{\bar{m}, \bar{n}}$  define a braiding for the category  $\mathcal{B}$ , so, for checking the naturality of  $c$  in  $\mathcal{IB}$  it remains to verify the naturality for the generators  $\epsilon_i$ ,  $1 \leq i \leq m$ ,  $m+1 \leq i \leq m+n$ . Let us consider the corresponding conjugation:

$$\sigma_m \dots \sigma_1 \sigma_{m+1} \dots \sigma_2 \dots \sigma_{n+m-1} \dots \sigma_n \epsilon_i \sigma_n^{-1} \dots \sigma_{n+m-1}^{-1} \dots \sigma_2^{-1} \sigma_{m+1}^{-1} \sigma_1^{-1} \dots \sigma_m^{-1}.$$

When  $i > n$ , we move  $\epsilon_i$  back, using the relation

$$\epsilon_i \sigma_i = \sigma_i \epsilon_{i+1}.$$



We have:

$$\begin{aligned} & \sigma_m \dots \sigma_1 \sigma_{m+1} \dots \sigma_2 \dots \sigma_{n+m-1} \dots \sigma_n \epsilon_i \sigma_n^{-1} \dots \sigma_{n+m-1}^{-1} \dots \sigma_2^{-1} \sigma_{m+1}^{-1} \sigma_1^{-1} \dots \sigma_m^{-1} = \\ & \sigma_m \dots \sigma_1 \sigma_{m+1} \dots \sigma_2 \dots \sigma_{n+m-1} \dots \sigma_{i+1} \sigma_i \epsilon_{i-1} \sigma_{i-1} \sigma_{i-1}^{-1} \dots \sigma_{n+m-1}^{-1} \dots \sigma_2^{-1} \sigma_{m+1}^{-1} \sigma_1^{-1} \dots \sigma_m^{-1} = \\ & \dots = \epsilon_{i-n}. \end{aligned}$$

When  $i < n$ , we move  $\epsilon_i$  back using the relation

$$\epsilon_{i+1} \sigma_i = \sigma_i \epsilon_i.$$

We have:

$$\begin{aligned} & \sigma_m \dots \sigma_1 \sigma_{m+1} \dots \sigma_2 \dots \sigma_{n+m-1} \dots \sigma_n \epsilon_i \sigma_n^{-1} \dots \sigma_{n+m-1}^{-1} \dots \sigma_2^{-1} \sigma_{m+1}^{-1} \sigma_1^{-1} \dots \sigma_m^{-1} = \\ & \sigma_m \dots \sigma_1 \sigma_{m+1} \dots \sigma_2 \dots \sigma_i \sigma_{i+m-1} \dots \sigma_{i+2} \sigma_{i+1} \sigma_i \epsilon_i \sigma_i^{-1} \sigma_{i+1}^{-1} \dots \sigma_{i+m-1}^{-1} \dots \sigma_2^{-1} \sigma_{m+1}^{-1} \sigma_1^{-1} \dots \sigma_m^{-1} = \\ & \sigma_m \dots \sigma_1 \sigma_{m+1} \dots \sigma_2 \dots \sigma_i \sigma_{i+m-1} \dots \sigma_{i+2} \sigma_{i+1} \epsilon_{i+1} \sigma_{i+1}^{-1} \dots \sigma_{i+m-1}^{-1} \sigma_i^{-1} \dots \sigma_2^{-1} \sigma_{m+1}^{-1} \sigma_1^{-1} \dots \sigma_m^{-1} = \\ & = \epsilon_{i+m}. \end{aligned}$$

The conditions of coherence are fulfilled because they are true for  $\mathcal{B}$ . □

Let  $BIB$  denote the classifying spaces of the limit inverse braid monoid. As usual, the pairings  $\mu_{m,n}$  define a monoid structure on the disjoint sum of the classifying spaces of  $IB_n$ :

$$\coprod_{n \geq 0} BIB_n.$$

**Proposition 4.2.** *The canonical maps*

$$BIB \rightarrow \Omega B(\coprod_{n \geq 0} BIB_n)$$

*induce isomorphisms in homology*

$$H_*(BIB; A) \rightarrow H_*((\Omega B(\coprod_{n \geq 0} BIB_n))_0; A),$$

*with any (constant) coefficients. So,*

$$BIB^+ \cong (\Omega B(\coprod_{n \geq 0} BIB_n))_0.$$

The proof is the same as that of Theorem 3.2.1 and Corollary 3.2.2 in [1] or (which is essentially the same) based directly on [26]. The braiding  $c$  gives the necessary homotopy commutativity for the  $H$ -spaces  $\coprod_{n \geq 0} BIB_n$ .

**Theorem 4.1.** *The homomorphisms  $\kappa_n$  induce morphisms of braided monoidal categories*

$$\mathcal{B} \xrightarrow{\kappa} \mathcal{IB}$$

*and the corresponding double loop maps*

$$\Omega^2 S^2 \longrightarrow \Omega B(\coprod_{n \geq 0} BIB_n).$$

The proof follows from the fact that the classifying space of a braided monoidal category is a double loop space after group completion.

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