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Computational problems in multiple scale analysis

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Abstract

This paper discusses the computational problems that arise in the application of higher-order multiple scale methods to general multiresonant multiparameter systems. Complex amplitude equations are first analytically derived and their structure analyzed solely in the light of existing resonance conditions. A qualitative study of the complex amplitude equations is performed, using both an analytical and a geometrical approach able to furnish a complete description of the classes of motion admitted by the system and of the structure of the variational matrix governing their stability. The problem of representing the amplitude equations using real quantities is then addressed. It is shown that the use of a mixed polar-Cartesian form leads to a standard form of equation if suitable hypotheses are satisfied. This form makes it possible to overcome

problems arising in the stability analysis of (incomplete) classes of motion when the most popular polar form is used. The final section discusses the reconstitution method, consisting in alternative techniques that can be used to combine the amplitude equations on different scales into a unique equation on a single scale. Four classes of method are considered, based on the consistency or inconsistency of the approach and on the completeness or incompleteness of the terms retained in the analysis. The four methods are critically compared and general conclusions drawn. Examples are given to clarify all the procedures discussed.

1. Introduction

The multiple scale method [1] is a powerful tool for dealing with nonlinear dynamic problems. It has been widely used in recent decades to study free and forced oscillatory phenomena [2] and latterly to describe nonlinear normal modes [3]–[6] and postcritical behavior in bifurcation problems [7]–[11]. It has also been applied to discrete-time dynamical systems [12].

As is well known, the method consists in solving a chain of ordinary differential perturbative equations in which the state-variables are assumed to depend on different independent time-scales. The solvability conditions enforced at each step furnish a set of equations governing the evolution of the unknown complex amplitudes on each time-scale. Finally, the solvability equations are combined in a single equation (the so-called *amplitude equation*), according to the *reconstitution method* [13]. The multiple scale method thus transforms the original multidimensional dynamic system into a smaller equivalent problem.

In spite of the wide use of the method, some computational aspects are still unresolved, and have been addressed in the literature over the last few years. The first question that arises concerns the technique of combining the various solvability conditions. It has been proposed [14]–[21] that some terms can be omitted and that the perturbation parameter can be reabsorbed in the amplitude equation in order to obtain a simpler equation. The consequence of these choices are analyzed in detail and critically discussed in [22].

A second computational problem regards the transformation of the complex amplitudes into real quantities. The most meaningful transformation uses of the polar amplitude-phase representation. However, this leads to real equations that *are not in standard form*, giving rise to problems in analyzing the stability of the fixed points at which some real amplitudes vanish (incomplete motions). Alternative representations to overcome the problem have been proposed in literature, in particular a purely Cartesian or mixed polar-Cartesian representation, mostly with reference to specific problems [18], [23]–[27]. It is shown in [28], where general systems are instead considered, that the mixed representation leads to standard form equations if certain hypotheses are satisfied, thus avoiding the problem previously illustrated.

As a further algorithmic problem, one may wish to know in advance (i.e. *before* transforming the complex amplitude equations) whether incomplete motions are admitted by the system and whether or not a standard form of the equations exists. The problem is addressed in [29], where tools for performing qualitative analysis are implemented. Two different analytical or geometrical approaches have been developed, able to furnish a complete description of all the existing classes of motion (i.e. of the

invariant subspaces of the amplitude space). The method does not require the elaboration of the perturbation equations, but only consideration of the resonance conditions; it also furnishes information on the structure of the Jacobian matrix governing the stability of each class of motion. It therefore has an intrinsic value beyond merely checking the existence of a standard form.

The above problems do not of course exhaust the scenario of computational problems linked to the use of the multiple scale method (or, perhaps, of other asymptotic methods). For instance, in [30]–[31], the derivation of amplitude equations preserving the original conservative nature of the system is discussed, in connection with the use of first- (state-variable) or second-order (mechanical) forms of the differential equations of motion. They will not, however, be discussed here.

This paper addresses the problems illustrated above and reviews the existing literature. The paper is organized as follows. In Sect. 2 the fundamental steps of the multiple scale method are summarized and the amplitude equations derived, according to both the analytical procedure and a purely qualitative investigation. In Sect. 3 a qualitative analysis of the classes of motion is first performed, since it does not require introducing real amplitudes or discussing reconstitution procedures. These aspects are instead analyzed in Sect. 4 and 5, respectively. Illustrative examples are given in each section. Finally some conclusions are drawn in Sect. 6.

2. Amplitude equations

2.1 Analytical derivation

Let us consider a discrete dynamical system, depending on a set of control parameters $\boldsymbol{\mu}$ and harmonically driven. The equations of motion, expanded around the equilibrium position $(\mathbf{q}, \boldsymbol{\mu}) = (\mathbf{0}, \mathbf{0})$, read:

$$\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{F}(\mathbf{q}, \boldsymbol{\mu}) + E_0 \mathbf{g} e^{i\omega_0 t} \quad (1)$$

In Equation (1) \mathbf{q} are Lagrangian coordinates, \mathbf{C} and \mathbf{K} the (generally non-symmetric) damping and stiffness matrices, respectively; E_0 , \mathbf{g} and ω_0 are respectively the amplitude, shape and frequency of the harmonic excitation; \mathbf{F} is the vector collecting the nonlinear forces and the increments in damping and elastic forces due to modifications of parameters $\boldsymbol{\mu}$ (i.e. $\mathbf{F}(\mathbf{0}, \mathbf{0}) = \mathbf{F}_{,\mathbf{q}}(\mathbf{0}, \mathbf{0}) = \mathbf{F}_{,\boldsymbol{\mu}}(\mathbf{0}, \mathbf{0}) = \mathbf{0}$); finally the dot denotes differentiation with respect the time t .

The eigenvalue problem associated with the linear homogeneous part of Equation (1), evaluated at $\boldsymbol{\mu} = \mathbf{0}$, reads:

$$(\lambda^2 \mathbf{E} + \lambda \mathbf{C} + \mathbf{K})\mathbf{u} = \mathbf{0} \quad (2)$$

in which \mathbf{E} is the identity matrix. The following hypotheses are assumed to hold:

- (1) Equation (2) admits N couples of purely imaginary eigenvalues $\lambda_k = i\omega_k$, associated with $2N$ linearly independent eigenvectors \mathbf{u}_k . Cases of nilpotent matrices are thus excluded; moreover, the occurrence of zero frequencies (divergence instability) is

not taken into account. The N eigenvalues of interest are referred to as *active* eigenvalues; the remaining ones, which are assumed to have no positive real part, are considered as *passive* eigenvalues.

- (2) The N active imaginary eigenvalues, together with the external frequency ω_0 , are involved in S resonance conditions, namely:

$$\sum_{n \in \mathcal{N}^+}^N k_{sn} \omega_n = \sigma_s \quad s = 1, 2, \dots, S \quad k_{sn} \in \mathbb{Z} \quad (3)$$

where $\sigma_s \ll 1$ are small detuning parameters and $\mathcal{N}^+ = \{0, 1, \dots, N\}$. Generally the S Equations (3) are not all independent, but some of them are linear combinations of $R \leq S$ independent conditions [28]–[29]. If $k_{s0} = 0$ the resonance is called internal, if $k_{s0} \neq 0$ it is called external. If $S = 1$ the resonance is said to be simple, if $S > 1$ it is said to be multiple.

- (3) The amplitude E_0 of the external excitation is assumed to be of order 1 if $\omega_0 \neq \omega_k$ ($k = 1, 2, \dots, N$) (*hard resonant excitation*); otherwise it is assumed to be small, of the order of a perturbation parameter ε , if it is equal to one of the active frequencies (*soft resonant excitation*).

The integer $M := N + R$, equal to the number of critical eigenvalues, counted in pairs, plus the number of independent resonance conditions, is referred to as the *linear codimension* of the problem, as is usual in bifurcation problems [32]. It is equal to the number of degeneracy conditions of the linear operator of Equations (1) (see e.g. [22]); therefore, in the control parameter space, M is the codimension of the manifold on which the assumed spectral properties (1) and (2) are satisfied.

A solution to Equation (1) is sought for $(\mathbf{q}, \boldsymbol{\mu}) \rightarrow (\mathbf{0}, \mathbf{0})$. To this end, an ordering of the (small) parameters $\boldsymbol{\mu}$ is made:

$$\boldsymbol{\mu} = \varepsilon \hat{\boldsymbol{\mu}}, \quad \hat{\boldsymbol{\mu}} = O(1), \quad (4)$$

and the Lagrangian \mathbf{b} coordinates are expanded in series of ε around $\varepsilon = 0$:

$$\mathbf{q} = \varepsilon \mathbf{q}_1 + \varepsilon^2 \mathbf{q}_2 + \dots \quad (5)$$

Several temporal scales $t_k = \varepsilon^k t$ ($k = 0, 1, \dots$) are introduced, so that $d/dt = d_0 + \varepsilon d_1 + \varepsilon^2 d_2 + \dots$, with $d_k := \partial/\partial t_k$. By substituting Equations (4, 5) in Equation (1), expanding it and separately vanishing terms with the same powers of ε , the perturbative equations are obtained. In the hard resonant case they read:

$$\begin{aligned} \mathbb{L} \mathbf{q}_1 &= E_0 \mathbf{g} e^{i\omega_0 t} \\ \mathbb{L} \mathbf{q}_2 &= \mathcal{F}_2(\mathbf{q}_1; \hat{\boldsymbol{\mu}}) - 2d_0 d_1 \mathbf{q}_1 \\ \mathbb{L} \mathbf{q}_3 &= \mathcal{F}_3(\mathbf{q}_1, \mathbf{q}_2; \hat{\boldsymbol{\mu}}) - [2d_0 d_1 \mathbf{q}_2 + (d_1^2 + 2d_0 d_2) \mathbf{q}_1] \\ &\dots \end{aligned} \quad (6)$$

where $\mathbb{L} := \mathbf{E}d_0^2 + \mathbf{C}d_0 + \mathbf{K}$ and \mathcal{F}_j denotes the j th-order terms in the Mac Laurin series expansion of \mathbf{F} in terms of \mathbf{q} and μ . Equations (6₁) admit the (so-called generating) solution

$$\mathbf{q}_1 = \sum_{n \in \mathcal{N}^+} A_n(t_1, t_2, \dots) \mathbf{u}_n e^{i\omega_n t_0} + c.c. \quad (7)$$

where A_0 is the (constant) amplitude of the forced response, \mathbf{u}_0 the shape, and A_n ($n = 1, 2, \dots, N$) are complex functions of the slow times; moreover, *c.c.* stands for complex conjugate and i is the imaginary unit. In the soft resonance case the external excitation must be shifted to a higher perturbative equation in order to prevent \mathbf{q}_1 growing to infinity.

To solve higher-order perturbation equations, solvability conditions must be imposed at each step, requiring the resonant terms on the right side (i.e. terms that, according to Equations (3) are of frequencies ω_k) to be orthogonal to the N left eigenvectors \mathbf{v}_k dual of \mathbf{u}_k . The solvability conditions lead to sets of N non-autonomous first-order differential equations on scales t_1, t_2, \dots in N unknowns $\mathbf{A} := \{A_1, A_2, \dots, A_n\}^T$. They assume the following form:

$$\begin{aligned} d_1 \mathbf{A} &= \mathbf{F}_1(\mathbf{A}, A_0; \hat{\boldsymbol{\mu}}) \\ d_2 \mathbf{A} &= \mathbf{F}_2(\mathbf{A}, A_0; \hat{\boldsymbol{\mu}}) + \beta_1(\mathbf{A}, A_0; \hat{\boldsymbol{\mu}}) d_1 \mathbf{A} + \beta_2 d_1^2 \mathbf{A} + \beta_3 (d_1 \mathbf{A})^2 \\ &\dots \end{aligned} \quad (8)$$

where β_2, β_3 are matrices and \mathbf{F}_i ($i = 1, 2$) vectors.

The solvability Equations (8) govern the evolution of amplitudes \mathbf{A} on different slow time-scales. However, they can be combined in a single equation by returning to the true time t , according to the *reconstitution method* [13]. By accounting for $d\mathbf{A}/dt = +\varepsilon d_1 \mathbf{A} + \varepsilon^2 d_2 \mathbf{A} + \dots$, and using Equation (8₁) in Equation (8₂), it follows that

$$\begin{aligned} \dot{\mathbf{A}} &= \varepsilon \mathbf{F}_1(\mathbf{A}, A_0; \hat{\boldsymbol{\mu}}) + \varepsilon^2 [\mathbf{F}_2(\mathbf{A}, A_0; \hat{\boldsymbol{\mu}}) + \beta_1(\mathbf{A}, A_0; \hat{\boldsymbol{\mu}}) \mathbf{F}_1(\mathbf{A}, A_0; \hat{\boldsymbol{\mu}}) \\ &\quad + \beta_2 \mathbf{F}_{1,\mathbf{A}}(\mathbf{A}, A_0; \hat{\boldsymbol{\mu}}) \mathbf{F}_1(\mathbf{A}, A_0; \hat{\boldsymbol{\mu}}) + \beta_3 \mathbf{F}_1^2(\mathbf{A}, A_0; \hat{\boldsymbol{\mu}})] + O(\varepsilon^3) \end{aligned} \quad (9)$$

where an index after a comma denotes differentiation. Equations (9) are known as *reconstituted* amplitude equations; they constitute a (generally) small set of differential equations able to capture the asymptotic dynamics of the original larger system. In this respect the MSM works as a *reduction method*, similarly, for instance, to the Center Manifold Method. Equations (9), however, still contain the perturbation parameter ε , so that they are an *asymptotic representation* of a reduced dynamical system:

$$\dot{\mathbf{A}} = \mathbf{G}(\mathbf{A}, A_0; \boldsymbol{\mu}) \quad (10)$$

which is not known in closed form. It is customary to eliminate the parameter ε by reabsorbing it into the amplitudes and control parameters (i.e., by formally putting $\varepsilon = 1$). The consequence of this choice will be discussed in Sect. 5.

2.2 Qualitative derivation

Since the amplitude equations are obtained by zeroing resonant terms at each step, they depend only on the resonance conditions in Equations (3). Therefore, the form of the amplitude equations (i.e. to within the coefficients) can be predicted *before expanding and solving* the perturbation equations. The procedure is useful both to check the analytical results, and in view of a subsequent qualitative analysis.

The m -th amplitude equation obeys the following simple rule: it contains all the products (up to the higher nonlinearity accounted for in the analysis) of amplitudes A_n and their complex conjugate \bar{A}_n so that the sum of the associated frequencies $\pm\omega_n$ is equal to ω_m , according to all the Equations (3). It therefore has the following form [29]:

$$\dot{A}_m = \mathcal{L}_m \left(A_m, A_m (A_n \bar{A}_n), \dots, \prod_{n \in \mathcal{N}^\pm} A_n^{l_{smn}} e^{\mp i \sigma_n t}, \dots \right) \quad m = 1, 2, \dots, N \quad (11)$$

in which the parameter ε has been reabsorbed, \mathcal{L}_m is a complex linear operator with constant coefficients (depending on $\hat{\mu}$), $A_{-n} := \bar{A}_n$ ($n = 0, 1, \dots, N$) and $\mathcal{N}^\pm = \{\pm 0, \pm 1, \dots, \pm N\}$, by formally distinguishing ± 0 . The exponents $l_{smn} \in \mathbb{N}$ are obtained by solving simple algebraic problems [28] – [29].

As an example, if the resonance condition $\omega_i = 2\omega_j$ is considered, the A_i -equation contains terms such as A_j^2 , $A_j^2 (A_i \bar{A}_i)$, $A_j^2 (A_j \bar{A}_j)$ and so on. Since $\omega_j = \omega_i - \omega_j$, the A_j -equation contains terms as $A_i \bar{A}_j$, $A_i \bar{A}_j (A_i \bar{A}_i)$, $A_i \bar{A}_j (A_j \bar{A}_j)$, and so on. All these terms will be referred to as *proper resonant terms*, since they are associated with the resonance conditions (3); terms independent of Equations (3), such as A_m , $A_m (A_n \bar{A}_n), \dots$, in Equation (11) will instead be referred to as *improper resonant terms*. The proper terms of lower order will be called *primary resonant terms* (e.g. A_j^2 and $A_i \bar{A}_j$ in the example), while terms of higher orders will be named *secondary resonant terms* (which are drawn by the former by simple rules, as the example has shown).

In Table 1 the explicit form (truncated at the ε^3 -order terms) of Equation (11) is given for a number of simple ($S = 1$) resonances, with the primary resonant terms typed in bold. The rules for assembling such equations for multiple ($S > 1$) resonances are given in [28]–[29].

3. Qualitative analysis of classes of motion

The form (11) of the amplitude equations is sufficient to perform the following qualitative analysis.

When a system oscillates at a steady or unsteady state, all the resonant modes generally contribute to the motion, i.e. all the amplitudes A_n are different from zero. However, depending on the type of resonance (and on the initial conditions) some amplitudes are allowed to vanish, so that the system oscillates in a smaller number of

Table 1. Amplitude equations for simple resonances

ORDER-2 RESONANCES	
$\omega_i = 2\omega_j$	$\dot{A}_i = \mathcal{L}_i(A_i; \mathbf{A}_j^2)$ $\dot{A}_j = \mathcal{L}_j(A_j; \mathbf{A}_i \bar{\mathbf{A}}_j)$
$\omega_i = \omega_j + \omega_k$	$\dot{A}_i = \mathcal{L}_i(A_i; \mathbf{A}_j \mathbf{A}_k)$ $\dot{A}_j = \mathcal{L}_j(A_j; \mathbf{A}_i \bar{\mathbf{A}}_k)$ $A_k \bar{\mathcal{L}}_k(A_k; \mathbf{A}_i \bar{\mathbf{A}}_j)$
ORDER-1 RESONANCE	
$\omega_i = \omega_j$	$\dot{A}_i = \mathcal{L}_i(A_i, A_i^2 \bar{A}_i, A_i A_j \bar{A}_j; \mathbf{A}_i \bar{\mathbf{A}}_i \mathbf{A}_j, \mathbf{A}_j^2 \bar{\mathbf{A}}_j, \mathbf{A}_j^2 \bar{\mathbf{A}}_i)$ $\dot{A}_j = \mathcal{L}_j(A_j; A_j^2 \bar{A}_j, A_j A_i \bar{A}_i; \mathbf{A}_j \bar{\mathbf{A}}_j \mathbf{A}_i, \mathbf{A}_i^2 \bar{\mathbf{A}}_i, \mathbf{A}_i^2 \bar{\mathbf{A}}_j)$
ORDER-3 RESONANCES	
$\omega_i = 3\omega_j$	$\dot{A}_i = \mathcal{L}_i(A_i; A_i^2 \bar{A}_i, A_i A_j \bar{A}_j; \mathbf{A}_j^3)$ $\dot{A}_j = \mathcal{L}_j(A_j; A_j^2 \bar{A}_j, A_j A_i \bar{A}_i; \mathbf{A}_i \bar{\mathbf{A}}_j^2)$
$\omega_i = 2\omega_j + \omega_k$	$\dot{A}_i = \mathcal{L}_i(A_i; A_i^2 \bar{A}_i, A_i A_j \bar{A}_j, A_i A_k \bar{A}_k; \bar{\mathbf{A}}_i \mathbf{A}_j \mathbf{A}_k)$ $\dot{A}_j = \mathcal{L}_j(A_j; A_j^2 \bar{A}_j, A_j A_i \bar{A}_i, A_j A_k \bar{A}_k; \mathbf{A}_i^2 \bar{\mathbf{A}}_k)$ $\dot{A}_k = \mathcal{L}_k(A_k; A_k^2 \bar{A}_k, A_k A_i \bar{A}_i, A_k A_j \bar{A}_j; \mathbf{A}_i^2 \bar{\mathbf{A}}_j)$
$\omega_i = \omega_j + \omega_h + \omega_k$	$\dot{A}_i = \mathcal{L}_i(A_i; A_i^2 \bar{A}_i, A_i A_j \bar{A}_j, A_i A_h \bar{A}_h, A_i A_k \bar{A}_k; \mathbf{A}_j \mathbf{A}_h \bar{\mathbf{A}}_k)$ $\dot{A}_j = \mathcal{L}_j(A_j; A_j^2 \bar{A}_j, A_j A_i \bar{A}_i, A_j A_h \bar{A}_h, A_j A_k \bar{A}_k; \mathbf{A}_i \bar{\mathbf{A}}_h \bar{\mathbf{A}}_k)$ $\dot{A}_h = \mathcal{L}_h(A_h; A_h^2 \bar{A}_h, A_h A_i \bar{A}_i, A_h A_j \bar{A}_j, A_h A_k \bar{A}_k; \mathbf{A}_i \bar{\mathbf{A}}_j \bar{\mathbf{A}}_k)$ $\dot{A}_k = \mathcal{L}_k(A_k; A_k^2 \bar{A}_k, A_k A_i \bar{A}_i, A_k A_j \bar{A}_j, A_k A_h \bar{A}_h; \mathbf{A}_i \bar{\mathbf{A}}_j \bar{\mathbf{A}}_h)$

modes. Such motions are referred to as (*incomplete*) *classes of motion*. From a geometrical point of view, the trajectories of complete motions belong to the whole $2N$ -dimensional state-space $(\Re(A_n), \Im(A_n))$ whereas incomplete motions only belong to subspaces.

Formally, a class of motion \mathcal{C} is defined as follows:

$$\begin{aligned} \mathcal{C} &= \{A_i, i \in \mathcal{M} \mid A_j = 0 \forall j \notin \mathcal{M} \Rightarrow \mathcal{L}_j \equiv 0 \forall A_i\} \\ \mathcal{M} &= \{i_1, i_2, \dots, i_M\} \quad M \leq N \end{aligned} \quad (12)$$

In other words: class \mathcal{C} is a set of amplitudes A_i such that, by zeroing in Equation (11) all the amplitudes A_j not included in the set, the associated right-hand members \mathcal{L}_j identically vanish for all values of the amplitudes of the class. Therefore, if non-trivial initial conditions are imposed only on the amplitudes of the class, then only these amplitudes participate in the motion. In other words, the amplitudes of a class span an invariant subspace of the state-space. In the following the locution *M-class* will be used to describe a class of M amplitudes. If the system is forced, since A_0 cannot be made equal to zero by initial conditions, then A_0 necessarily belongs to all the existing classes.

Classes of (periodic) motions comprising a unique amplitude (the so-called monomodal solutions) are also known as *nonlinear normal modes* and have been extensively studied for nonresonant systems [33]. Classes comprising one or more amplitudes (multimode solutions) could be considered as generalizations for resonant systems of such nonlinear modes.

In order to analyze the behavior of a system, it is helpful to determine all the existing classes of motion. A procedure is illustrated here to address this problem qualitatively.

3.1 An analytical method

In order to simplify the search for the classes of motion of a resonant system, the role of the different terms entering Equation (11) is studied. The objective of the analysis is to ascertain the existence of *reduced equations* in which some terms are omitted, and which are still able to give all the qualitative information concerning the complete system.

First of all, the role of the improper resonant terms (i.e. terms not associated with any resonance conditions) is examined. It is interesting to note that if $S = 0$, i.e. *if the system is non-resonant, all subsets of $\{A_n\}$ are classes of motion*. This follows from the fact that the m -th Equation (11) contains in this case only the improper terms, which are all proportional to A_m ; therefore, the definition (Equation (12)) holds for any choice of \mathcal{M} . This circumstance corroborates the hypothesis that the classes of motion are intrinsically connected to the resonance conditions. However, when a resonant system is considered (i.e. $S \neq 0$), the role of the improper terms is stated by the following [29]:

Theorem 1 The improper resonant terms are unessential for evaluating classes of motion.

It should be noted that although improper terms are unessential to the existence of classes of motion, *they generally affect the motion itself*. Improper terms thus have a *quantitative* rather than a *qualitative* influence on the motion.

By focusing on proper resonant terms, the role of primary and secondary terms is now examined. The following Theorem [29], holds:

Theorem 2 The secondary resonant terms are unessential for evaluating classes of motion.

Comments similar to those for relevant improper terms hold for secondary terms; in particular, it follows from Theorem 2 that [29]:

Corollary 1 Asymptotic approximations of an order higher than the order at which all the primary terms have appeared do not modify the classes of motion of a resonant system unless new resonances emerge.

Therefore, higher-order approximations only modify the quantitative characters of motion if no new resonances are encountered. In contrast, if higher-order resonances arise in the given set of frequencies, then some of the lower-order classes could disappear. If this happens the effects of the higher-order resonance terms manifest themselves after a very long time. This means that the lower-order classes of motion remain virtually unchanged for a long time until higher-order resonance terms become meaningful and destroy them [29].

From Theorems 1 and 2 it ensues that the classes of motion of a system $\dot{A}_m = \mathcal{L}_m(\cdot)$ can be determined by reduced equations $\dot{A}_m = \mathcal{L}_m^*(\cdot)$ in which only primary resonant terms are retained (i.e. only bold terms in Table 1 should be taken into account in the examples shown there). Therefore, in the following, reference will be made to the reduced system, and the asterisk will be dropped. A practical method to evaluate all the classes of motion admitted by a multiresonant system is now illustrated. It is based on the following [29]:

Theorem 3 A set of amplitudes $\mathcal{C} = \{A_i\}$ $i \in \mathcal{M}$ (necessarily including A_0 , if it is different from zero) is a class of motion for the multiresonant system (11) if and only if

$$\sum_{j \neq \mathcal{M}} |k_{sj}| \neq 1 \quad \forall s \quad (13)$$

Theorem 3 holds for generic systems. However, when nonlinearities are lacking, some classes of motion can exist in addition to those admitted by Equation (13) even if the condition is violated.

Theorem 3 suggests the following algorithm to evaluate all the classes of motion of a (generic) multiresonant system.

Algorithm 1 (RCM, Resonant Coefficients Method): (1) The table $[k_{sn}]$ ($n = 0, 1, \dots, N$) of the resonance coefficients is built up; (2) a set of M columns (always including column k_{s0}) is canceled: if the remaining part of the table does not contain the string $(0, \dots, 0, \pm 1, 0, \dots, 0)$ on any row, then the amplitudes associated with those columns are an M -class of motion for the system; (3) all the combinations of M columns are tried and step (2) is repeated.

An Example of the RCM is given in Section 3.3.

The RCM shows that any resonance condition works as a *constraint* on classes by inhibiting some of them and allowing others. Only classes that are not constrained by any resonance condition are admitted by the multiresonant system. Therefore, *the higher the number of resonance conditions, the lower the number of classes of motion admitted*. A degree of constraint can be attributed to each resonance, equal to the number of classes it inhibits. The index allows the resonances to be ordered hierarchically: resonances with a lower degree of constraint are usually unessential to the evaluation of the classes, but general rules are not yet available [29].

3.2 A geometrical method

Here, a geometrical approach is instead followed, based on a set representation of the classes. Some definitions are preliminarily given.

The set $\mathcal{S} := \{A_n | n \in \mathcal{N}^+\}$ of all the amplitudes involved in the S resonances (3) is called the *state-space* of the dynamical system $\dot{A}_m = \mathcal{L}_m(\mathcal{S})$ (Equation (11)). The set $\mathcal{S}_s \subseteq \mathcal{S}$ of the amplitudes A_n associated with the frequencies ω_n involved in the

s -th resonance is called the *elementary s -th space*, or the *s -th resonance space*; by remembering Equations (3) it is $\mathcal{S}_s := \{A_n | k_{sn} \neq 0\}$. A set $\mathcal{C} = \{A_i, i \in \mathcal{M} := \{i_1, i_2, \dots, i_M\}\}$ is an *M -class of motion* for the system if it is an *invariant subspace* of \mathcal{S} (see Equation 12). If the forcing amplitude $A_0 \in \mathcal{S}$ then necessarily $A_0 \in \mathcal{C}$, since it cannot vanish. As particular cases, the null set $\{0\}$ and the state space \mathcal{S} are classes of motion, containing no components or all the components, respectively. The classes of motion admitted by the s -th resonance (i.e. the invariant subspaces of \mathcal{S}_s) are called the *elementary classes of motion* $\mathcal{C}_{s_1}, \mathcal{C}_{s_2}, \dots$, of the s -th resonance. The set of all the elementary classes of motion \mathcal{C}_{s_i} is called the *family of classes of motion* of the s -th resonance $\mathcal{F}_s := \{\mathcal{C}_{s_1}, \mathcal{C}_{s_2}, \dots\}$; the family contains the null-set. The classes of motion admitted by two (or more) resonances, e.g. the r -th and the s -th, are the invariant subspaces of $\mathcal{S}_r \cup \mathcal{S}_s$; they are referred to as *multiple classes of motion* $\mathcal{C}_{rs_1}, \mathcal{C}_{rs_2}, \dots$, and their totality as *the multiple family* \mathcal{F}_{rs} .

The elementary classes of motion of the most common resonances are obtained for inspection of the reduced equation of motion in Table 1. They are represented in Table 2 by closed curves (sets) surrounding the amplitudes, or by lines, to make the diagram more clear. For instance, in the $\omega_i = 2\omega_j$ case only one 1-class exists, while in the $\omega_i = \omega_i \pm \omega_j$ case all the three 1-classes are admitted, but no 2-classes exist. If one of the frequencies in the Table is the external frequency (i.e. $\omega_j \equiv \omega_0, A_j \equiv A_0$), the elementary classes not containing A_j disappear.

Table 2. Elementary classes of motion

	$\omega_i = 2\omega_j$	$\omega_i = \omega_j \pm \omega_k$
<i>Quadratic resonances</i>		
<i>Cubic resonances</i>	$\omega_i = \omega_j$	$\omega_i = 3\omega_j$
	$\omega_i = 2\omega_j \pm \omega_k$	$\omega_i = \omega_j \pm \omega_h \pm \omega_k$

The general case of multiple ($S > 1$) resonance is now analyzed. Each of the S resonance conditions (3) entails the existence of some elementary classes of motion, according to the results previously discussed. The question is to evaluate how these elementary classes interact, in order to furnish new multiple classes. To solve this problem the following strategy is adopted. First, the interaction between two families of classes of motion, \mathcal{F}_1 and \mathcal{F}_2 , associated with $s = 1$ and $s = 2$ respectively, is studied and a new (multiple) family, \mathcal{F}_{12} , is determined. The latter is combined with the \mathcal{F}_3 -family associated with $s = 3$ and the multiple family is up-dated to \mathcal{F}_{123} . The process is stopped when the last family $s = S$ has been considered, namely $\mathcal{F}_{12\dots S}$. The task thus essentially consists in analyzing the simplest problem $S = 2$. Reference will initially be made to internally resonant systems.

Let us consider two families $\mathcal{F}_A = \{C_{A_1}, C_{A_2}, \dots\}$ and $\mathcal{F}_B = \{C_{B_1}, C_{B_2}, \dots\}$, each associated with an internal resonance condition. Let \mathcal{A} and \mathcal{B} be the spaces of the two resonances. Moreover, let us denote by $\mathcal{W} := \mathcal{A} \cap \mathcal{B}$, $\mathcal{U} := \mathcal{A} \setminus \mathcal{W}$ and $\mathcal{V} := \mathcal{B} \setminus \mathcal{W}$ the subsets of $\mathcal{A} \cup \mathcal{B}$ so that $\mathcal{A} = \mathcal{U} \cup \mathcal{W}$ and $\mathcal{B} = \mathcal{V} \cup \mathcal{W}$ (Fig. (1_a)). The conditions under which \mathcal{C}_{AB} (Fig. (1_b)) is a multiple class of the $(\mathcal{A}, \mathcal{B})$. resonances are stated by the following [29]:

Theorem 4 Given two families of unforced classes of motion, \mathcal{F}_A and \mathcal{F}_B , of spaces \mathcal{A} and \mathcal{B} , respectively, a subset, \mathcal{C}_{AB} of $\mathcal{A} \cup \mathcal{B}$ is a class of the multiple \mathcal{F}_{AB} family if and only if $\mathcal{C}_{AB} \cap \mathcal{A}$ is an elementary class of \mathcal{F}_A and $\mathcal{C}_{AB} \cap \mathcal{B}$ is an elementary class of \mathcal{F}_B .

When an external excitation acts on the system, Theorem 4 must be slightly modified. Since forced classes of motion necessarily contain the forced amplitude A_0 , classes satisfying the conditions of Theorem 4 but not containing A_0 are not forced multiple classes. This is stated in the following:

Corollary 2 If one or both the \mathcal{F}_A and \mathcal{F}_B families of Theorem 4 are associated with an external resonance condition, then the subset \mathcal{C}_{AB} is a forced multiple class if and only if the conditions of Theorem 4 are satisfied and the forced amplitude A_0 is contained in \mathcal{C}_{AB} .

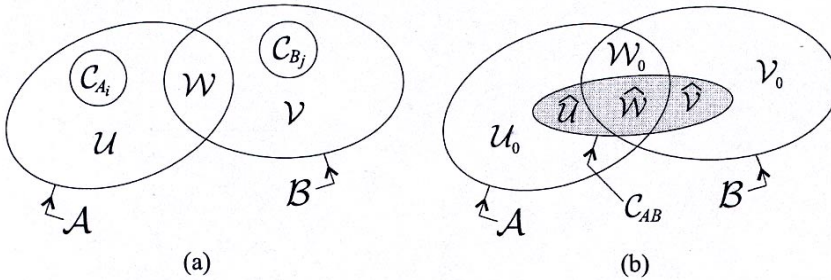


Figure 1. Interaction between two families of elementary classes of motion: (a) dominions $(\mathcal{A}, \mathcal{B})$, sub-sets $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ and elementary classes (C_{A_i}, C_{B_j}) ; (b) multiple class \mathcal{C}_{AB}

The above results lead to the following practical method to find classes of motion:

Algorithm 2 (CDM, Class Diagram Method): (1) the elementary class diagrams \mathcal{F}_i of each resonance condition are drawn from Table 2; (2) multiple class diagrams are built up by composing in sequence diagrams $\mathcal{F}_{12}, \mathcal{F}_{123}, \mathcal{F}_{12\dots}\mathcal{S}$. To compose two diagrams, \mathcal{F}_A ed \mathcal{F}_B , the following strategy is adopted: (a) the individual classes are considered and either Theorem 4 or its Corollary 2 is applied to verify whether they still survive as multiple classes; (b) the possible merging of some classes of \mathcal{F}_A and \mathcal{F}_B is checked by analyzing all the combinations among the classes.

An Example of CDM is given in Section 3.3.

The geometrical theory previously developed also permits us to gain insight into the structure of the Jacobian matrix $\mathbf{J} = [J_{mn}]$, by simply examining the diagram of the classes of motion. To this end, let us denote by $\mathbf{X} = \{X_i\}$, $\mathbf{Y} = \{Y_j\}$ and $\mathbf{Z} = \{Z_k\}$ three subsets of the state-space $\mathcal{S} = \{A_n\} = \{\mathbf{X}, \mathbf{Y}, \mathbf{Z}\}$, having the following properties: \mathbf{X} is the set of the (active) amplitudes participating in the steady (periodic or bi-periodic) motion $\mathcal{S}^0 := \{\mathbf{X}^0, \mathbf{0}, \mathbf{0}\}$ whose stability is under analysis; \mathbf{Y} is the set of (passive resonant) amplitudes present in all the *elementary* spaces \mathcal{S}_s which contain at least one amplitude X_i ; \mathbf{Z} is the set of all the (passive non-resonant) remaining amplitudes. The following Theorem holds [29]:

Theorem 5 The variational equation based on the steady solution $\mathcal{S}^0 = \{\mathbf{X}^0, \mathbf{0}, \mathbf{0}\}$ is uncoupled in the variables $\delta\mathbf{X}$, $\delta\mathbf{Y}$ and $\delta\mathbf{Z}$, namely:

$$\delta\dot{\mathbf{X}} = \mathbf{J}_{\mathbf{X}}\delta\mathbf{X} \quad \delta\dot{\mathbf{Y}} = \mathbf{J}_{\mathbf{Y}}\delta\mathbf{Y} \quad \delta\dot{\mathbf{Z}} = \mathbf{J}_{\mathbf{Z}}\delta\mathbf{Z} \quad (14)$$

with $\mathbf{J}_{\mathbf{Z}}$ diagonal.

According to Theorem 5, there exist three different classes of perturbation, each of which is independent and which possibly lead \mathcal{S}^0 to become unstable, namely: \mathbf{X} -perturbations, or *in-class perturbations*; \mathbf{Y} -perturbations, or *out-of-class resonant perturbations*; \mathbf{Z} -perturbations, or *out-of-class non-resonant perturbations*. The three classes of perturbation describe different forms of instability. If $\mathbf{J}_{\mathbf{X}}$ has at least one unstable eigenvalue (or Floquet multiplier), the perturbed motion evolves (at least initially) in the same class $\{\mathbf{X} \neq \mathbf{0}, \mathbf{Y} = \mathbf{Z} = \mathbf{0}\}$. If $\mathbf{J}_{\mathbf{Y}}$ is instead unstable, the perturbed motion leaves the class, since the amplitudes \mathbf{Y} , directly involved in the resonances with \mathbf{X} , are triggered. Finally, if the eigenvalues of $\mathbf{J}_{\mathbf{Z}}$ are unstable, the non-resonant perturbations \mathbf{Z} grow in time.

A deeper analysis of the structure of the variational equations reveals some additional interesting aspects. It has been observed in the literature [26], that the Jacobian matrix $\mathbf{J}_{\mathbf{Y}}$ is often composed of diagonal blocks, i.e. there exists some uncoupling among the out-of-class resonant perturbations $\delta\mathbf{Y}$. General rules about such uncoupling are drawn directly by the geometrical method illustrated, according to the following Theorem [29]. The Theorem refers to the generic case in which all terms in the

equations are assumed not to be zero. If some terms vanish, further uncouplings can be present.

Theorem 6 Given two subsets \mathbf{Y}_1 and \mathbf{Y}_2 of \mathbf{Y} , such that $\mathbf{Y} = \{\mathbf{Y}_1, \mathbf{Y}_2\}$, the perturbations $\delta\mathbf{Y}_1$ and $\delta\mathbf{Y}_2$ are uncoupled in the variational equation (14₂) if and only if there exists a multiple class \mathcal{C}^* containing all the amplitudes \mathbf{X} and the subset \mathbf{Y}_1 .

Theorem 6 has the following consequence: if the smallest class \mathcal{C}^* that contains \mathbf{X} and *at least* one amplitude Y_j also contains *all* the amplitudes \mathbf{Y} , then the Jacobian matrix \mathbf{J}_Y is full. In contrast, if such a class contains only *some* of the \mathbf{Y} , then the matrix \mathbf{J}_Y is block-diagonal. An example of the application of Theorem 6 is given in Section 3.3.

3.3 Examples

As an example, a system is considered in which $N = 5$ frequencies are involved in three independent resonance conditions:

$$\omega_1 = 2\omega_2, \quad \omega_2 = \omega_3 + 2\omega_5, \quad \omega_3 = \omega_4 \quad (15)$$

From Equation (15) a further (dependent) condition is derived:

$$\omega_2 = \omega_4 + 2\omega_5 \quad (16)$$

so that four resonance conditions must be accounted for. Applying the RCM, the table of resonance coefficients reads:

ω_1	ω_2	ω_3	ω_4	ω_5	
1	-2	0	0	0	
0	1	-1	0	-2	
0	0	1	-1	0	
0	1	0	-1	-2	(17)

By canceling the first column no strings are found containing a unique digit ± 1 and all zeros, and therefore $\{A_1\}$ is a 1-class. In contrast, if the second column is canceled, such a string is found on the first row, and $\{A_2\}$ is therefore *not* a 1-class. Similarly, $\{A_1, A_2\}$ is a 2-class, but $\{A_1, A_3\}$ is not since the third resonance condition inhibits it. By repeatedly applying the test, the following classes of motion are found: $\{A_1\}$, $\{A_5\}$, $\{A_1, A_2\}$, $\{A_1, A_5\}$, $\{A_3, A_4\}$, $\{A_1, A_3, A_4\}$, $\{A_1, A_2, A_3, A_4\}$, in addition, of course, to the complete class.

Applying the CDM, the spaces of the resonances (15) are $\mathcal{S}_1 = \{A_1, A_2\}$, $\mathcal{S}_2 = \{A_2, A_3, A_5\}$, and $\mathcal{S}_4 = \{A_3, A_4\}$, while for the resonance (16) the space is $\mathcal{S}_3 = \{A_2, A_4, A_5\}$. The \mathcal{F}_r ($r = 1, 2, 3, 4$) families of elementary classes are obtained from Table 2 and are drawn in Fig. 2_a in thin lines. First, the interaction between the \mathcal{F}_1 and \mathcal{F}_2 families is studied and the multiple \mathcal{F}_{12} (Fig. 2_b) family is built up as follows. The elementary monomodal solutions $\{A_1\}$, $\{A_3\}$ and $\{A_5\}$ all survive the interaction since they are external to $\mathcal{S}_1 \cap \mathcal{S}_2 = \{A_2\}$, in contrast, the monomodal solution $\{A_2\}$ disappears since it

is not a class of \mathcal{F}_1 . Similarly, the set $\{A_1, A_2\}$ is a multiple class of \mathcal{F}_{12} while the sets $\{A_2, A_3\}$ and $\{A_2, A_3, A_5\}$, are not. To check if other classes emerge from the interaction, combinations between the elementary classes must be considered. Thus, two new bimodal solutions $\{A_1, A_3\}$ and $\{A_1, A_5\}$, a new trimodal solution $\{A_1, A_2, A_3\}$ and, finally, a new four-modal solution $\{A_1, A_2, A_3, A_5\}$ are found to be multiple classes. They are drawn in Fig. 2_b as thicker curves. As a second step, the interaction between the \mathcal{F}_4 - and \mathcal{F}_{12} -families is studied and the new \mathcal{F}_{124} -family (Fig. 2_c) is built up as follows. Among the monomodal solutions, $\{A_1\}$ and $\{A_5\}$ survive, while $\{A_3\}$ disappears. Among the bimodal solutions, $\{A_1, A_2\}$, $\{A_1, A_5\}$ and $\{A_3, A_4\}$ still exist, while $\{A_1, A_3\}$ is destroyed. Similarly, $\{A_1, A_2, A_3\}$ and $\{A_1, A_2, A_3, A_5\}$ disappear. New classes, however, are formed by merging some elementary classes of \mathcal{F}_{12} and \mathcal{F}_4 namely: $\{A_1, A_3, A_4\}$, $\{A_1, A_2, A_3, A_4\}$ and $\{A_1, A_2, A_3, A_4, A_5\}$, drawn in heavy type in Fig. 2_c, all of which satisfy Theorem 4. In contrast, $\{A_3, A_4, A_5\}$ and $\{A_1, A_3, A_4, A_5\}$ are not multiple classes, since their intersection with $\mathcal{S}_{12} = \mathcal{S}_1 \cup \mathcal{S}_2$, i.e. $\{A_3, A_5\}$ and $\{A_1, A_3, A_5\}$, are not elementary classes of \mathcal{F}_{12} . The last graph thus obtained (Fig. 2_c) represents all the classes of motion existing under the resonance conditions (15). It is easy to check that the dependent condition (16) does not alter the scenario of Fig. 2_c. Classes of motion found using the CDM coincide with those obtained using the RCM.

As a second (and third) example the system considered above is taken, and an external resonance is added, namely:

$$\omega_0 = \omega_1 \quad \text{or} \quad \omega_0 = 2\omega_1 + \omega_5 \quad (18)$$

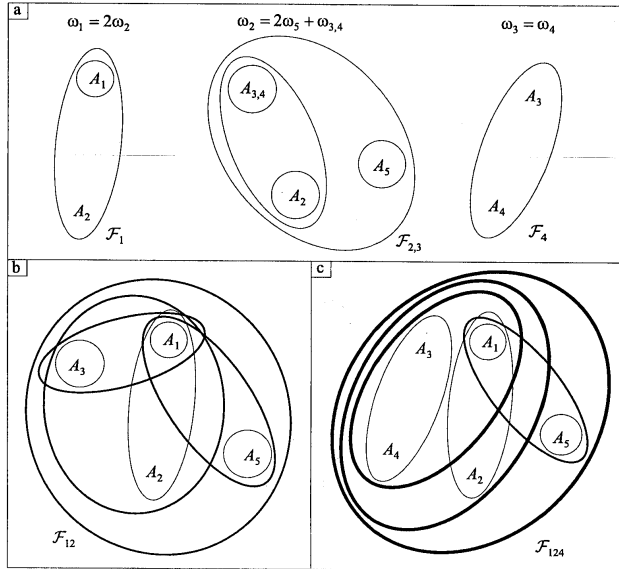


Figure 2. Class of motion diagram for a system undergoing $S = 4$ internal resonance conditions: (a) elementary families \mathcal{F}_r ($r = 1, 2, 3, 4$) of dominions \mathcal{D}_r ; (b) \mathcal{F}_{12} multiple family, resulting from the interaction between \mathcal{F}_1 and \mathcal{F}_2 ; (c) overall multiple family \mathcal{F}_{124} .

These entail the dependent conditions $\omega_0 = 2\omega_2$ and $\omega_2 = \omega_4 + 2\omega_5$, $\omega_2 = \omega_1 \pm 2\omega_3 \mp \omega_4$, respectively; however, as in the previous example, it can be seen that they do not affect the classes of motion. A new space $\mathcal{S}_0 = \{A_0, A_1\}$ or $\mathcal{S}_0 = \{A_0, A_1, A_5\}$ must be added to those in Fig. 2_a. The relevant \mathcal{F}_0 elementary families are plotted in Figs. 3_{a1} and 3_{a2}. To build up the multiple classes of motion, \mathcal{F}_{0124} , the interaction between the \mathcal{F}_{124} (drawn in Fig. 2_c) and \mathcal{F}_0 families must be analyzed. Resonance (18₁) is first studied. Since the forced amplitude A_0 cannot vanish, it must be contained in every multiple class; the same property therefore holds for A_1 , which is connected to A_0 in \mathcal{F}_0 . Consequently, all the classes of \mathcal{F}_{124} not containing A_1 , are destroyed by the interaction with \mathcal{F}_0 . The scenario in Fig. 2_c is thus simplified in Fig. 3_{b1}. It should be noted that although the classes $\{A_3, A_4\}$ and $\{A_5\}$ would satisfy the conditions of Theorem 4, they are not multiple classes because of Corollary 2. To sum up, in the $\omega_0 = \omega_1$ case, six classes still exist, since the excitation only inhibits the bi-modal $\{A_3, A_4\}$ and monomodal $\{A_5\}$ solutions of \mathcal{F}_{124} .

Resonance (18₂) is then analyzed. The \mathcal{F}_0 -family associated with Equation (18₂) is drawn in Fig. 3_{a2}. Unlike the example in Fig. 3_{a1} it admits the incomplete $\{A_0, A_5\}$ class of motion. From Corollary 2 all the classes of \mathcal{F}_{124} not containing A_5 are disrupted, so that only $\{A_0, A_5\}$ and $\{A_0, A_1, A_5\}$ (drawn in thin lines in Fig. 2_{b2}) survive. The possible merging among classes is then analyzed. Class $\{A_1, A_2\}$ cannot merge with classes of \mathcal{F}_0 , since $\{A_1, A_2, A_5\}$ is not a class of \mathcal{F}_{124} . For similar reasons $\{A_1, A_3, A_4\}$ cannot merge. However, $\{A_3, A_4\}$ merges with both $\{A_0, A_5\}$ and $\{A_0, A_1, A_5\}$. The following

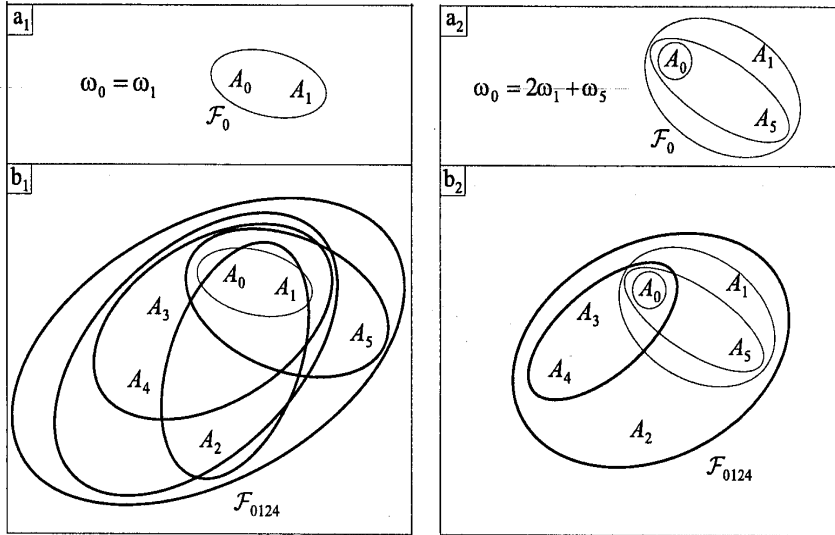


Figure 3. Class of motion diagram for a system undergoing three independent internal resonances as in Fig. 2 plus one external resonance as in Fig. a₁ or Fig. a₂: (a₁, a₂): elementary family \mathcal{F}_0 of dominion \mathcal{D}_0 ; (b₁, b₂): overall multiple family \mathcal{F}_{0124} , resulting from the interaction between the \mathcal{F}_{124} -family of Fig. 2_c and the \mathcal{F}_0 -family of Fig. a₁ or Fig. a₂, respectively.

classes therefore exist in addition to the complete classes: $\{A_0, A_5\}$, $\{A_0, A_1, A_5\}$, $\{A_0, A_1, A_3, A_4, A_5\}$ (drawn in thick lines).

To illustrate some applications of Theorems 5 and 6 let us consider the diagram in Fig. 2_c. First, let us analyze the stability of a steady motion $\mathbf{X} = \{A_1, A_2\}$. From the elementary resonance spaces in Fig. 2_a, it follows that $\mathbf{Y} = \{A_3, A_4, A_5\}$ (since A_3 and A_5 are in resonance with A_2 in Eq. (15₂), A_4 in Eq. (16)) and $\mathbf{Z} = \{0\}$. Theorem 5 states that the variational equations uncouple in the perturbations $\{\delta A_1, \delta A_2\}$ and $\{\delta A_3, \delta A_4, \delta A_5\}$. Moreover, Theorem 6 states that, since a class \mathcal{C}_1^* exists (see Fig. 2_c), containing $\{A_1, A_2\}$ and $\mathbf{Y}_1 = \{A_3, A_4\}$ the variational equations in $\{\delta A_3, \delta A_4, \delta A_5\}$ are uncoupled in two blocks $\{\delta A_3, \delta A_4\}$ and $\{\delta A_5\}$.

As a second example, the stability of a steady solution $\mathbf{X} = \{A_1, A_5\}$ is analyzed. From the elementary classes of resonance (15) and (16) it follows that $\mathbf{Y} = \{A_2, A_3, A_4\}$, $\mathbf{Z} = \{0\}$. Therefore, according to Theorem 5, $\{\delta A_1, \delta A_5\}$, $\{\delta A_2, \delta A_3, \delta A_4\}$ are uncoupled in the variational equation. Moreover, since the only class that contains $\{A_1, A_5\}$ also contains $\{A_2, A_3, A_4\}$, the perturbations δA_2 , δA_3 and δA_4 are coupled in the variational equation in accordance with Theorem 6.

Finally, as a third example of Fig. 2_c, if $\mathbf{X} = \{A_3, A_4\}$, then $\mathbf{Y} = \{A_2, A_5\}$, $\mathbf{Z} = \{A_1\}$; since there exists a class \mathcal{C}_1^* containing $\{A_3, A_4\}$ and $\mathbf{Y}_1 = \{A_2\}$ it follows that \mathbf{J}_Y uncouples in two blocks (Theorem 6).

4. The standard form of amplitude equations

4.1 The Polar form

The complex amplitude Equations (11) govern the evolution of a $2N$ -dimensional dynamical system. They are usually stated in real form by adopting the polar representation $A_n := \frac{1}{2}a_n e^{i\theta_n}$ ($n \in \mathcal{N}^+$) where a_n are real amplitudes and θ_n are phases. However, since complex amplitudes are involved in the equations, periodic motions *are not* their fixed points. In fact, while the real amplitudes a_n remain constant during a periodic motion, the phases θ_n vary linearly in time, as a result of the tuning of the nearly-resonant linear frequencies. To make the mathematical problem easier, the $2N$ amplitude-phase equations are combined and a (generally) reduced system is obtained. This step requires the introduction of S phase-differences γ_s , which remain constant during a periodic motion. Therefore, the fixed points of the (a_n, γ_s) -Reduced Amplitude Equations actually *are* the periodic motions of the original system. The phases γ_s naturally appear in the procedure, as a consequence of the form (11) of the amplitude equations discussed in Sect. 2.2.

They turn out to be equal to:

$$\gamma_s := \sum_{n \in \mathcal{N}^+} k_{sn} \theta_n + \sigma_s t \quad s = 1, 2, \dots, S \quad (19)$$

where k_{sn} are the resonance coefficients appearing in Equation (3). However, due to the linear dependence of the resonance conditions (Equation (3)), only R of them are independent of the remaining ones; e.g. $\{\gamma_s\} = \{\gamma_r, \gamma_q\}$ with $r = 1, 2, \dots, R$ and $q = R + 1, R + 2, \dots, S$.

The reduced amplitude equations assume the following form [28]:

$$\begin{aligned} \dot{a}_m &= \Re_m \left(a_m a_n^2, \dots, \prod_{n \in \mathcal{N}^\pm} a_n^{l_{smn}} \exp(\mp i \gamma_s), \dots \right), \quad m = 1, 2, \dots, N \\ \prod_{j \in J_r} a_j \dot{\gamma}_r &= \sum_{n \in \mathcal{N}^+} k_{rm} \left(\prod_{\substack{j \in J_r \\ j \neq m}} a_j \right) \Im_m \left(a_m a_n^2, \dots, \prod_{n \in \mathcal{N}^\pm} a_n^{l_{smn}} \exp(\mp i \gamma_s), \dots \right) + \sigma_r \\ J_r &= \{j | k_{rj} \neq 0\}, \quad r = 1, 2, \dots, R \end{aligned} \quad (20)$$

in which $\Re_m := \text{Re}[\mathcal{L}_m]$. Equations (20) are a set of $M := N + R$ equations in the (a_m, γ_r) unknown; after solving them, the modulation of the phases θ_n is evaluated by means of an additional set of N equations, uncoupled from Equations (20).

In conclusion, although the asymptotic motion develops in a $2N$ -dimensional space, its *essential* aspects (e.g., the existence of periodic motions and their stability) are described in a smaller M -dimensional space, equal to the *codimension of the problem*; the remaining $2N - M$ dimensions govern *complementary* aspects of the motion (e.g., the frequency corrections), not affecting the qualitative character of the solution.

Nevertheless, the $a\gamma$ -form (Equations (20)) entails some computational difficulty. The equations appear in a *non-standard* form, since the γ -equations admit some amplitudes a_h as factors which cannot be eliminated if a_h identically vanish in some classes of motion. In such cases some γ -equations are identically satisfied and the relevant γ 's remain undetermined. From a geometrical point of view such fixed points are *not-isolated points*, lying on a manifold in the (a_n, γ_r) space. When the stability of these points is analyzed, the *standard method fails*, since some coefficients of the time-derivative $\delta \dot{\gamma}_r$ vanish in the variational equation, so that more complex *ad hoc* methods must be employed. To overcome this drawback, it is customary in the literature to return to the complex amplitude equations and: (a) to express the amplitudes A_n in Cartesian form, (b) to multiply the equations by *suitable time-exponential factors* (so as to render the equations autonomous) and finally (c) to build up the $2N \times 2N$ variational equation. In these new variables the non-isolated fixed points of the $a\gamma$ -form equations reduce to isolated points, so that the standard variational equation procedure works well. The method therefore calls for a change of variables and for an enlargement of the dimensions of the problem. An alternative procedure that allows the dimension M to be maintained and both periodic solutions and their stability to be analyzed by the same equations, is illustrated in the next Section.

4.2 The mixed Polar-Cartesian representation of complex amplitudes

The following change of variable is introduced:

$$A_n(t) = B_n(t) e^{i\alpha_n(t)} \quad n = 0, 1, \dots, N \quad (21)$$

where $\alpha_n(t)$ are unknown functions of time (except for the dummy $\alpha_0 \equiv 0$). A substitution of Equations (21) in the (non autonomous) Equation (11) leads to the (autonomous) system:

$$\dot{B}_m + i\dot{\alpha}_m B_m = \mathcal{L}_m \left(B_m (B_n \bar{B}_n), \dots, \prod_{n \in \mathcal{N}^\pm} B_n^{l_{smn}}, \dots \right) \quad m = 1, 2, \dots, N \quad (22)$$

if the functions $\alpha_n(t)$ are chosen in such a way that:

$$\sum_{n \in \mathcal{N}^+} k_{rn} \alpha_n + \sigma_r t = 0 \quad r = 1, 2, \dots, R \quad (23)$$

If $R < N$, Equations (23) are not sufficient to determine all the functions α_n , and $L = N - R$ of them (e.g. the first L) remain undetermined. In order to avoid indeterminacies, they are taken as equal to the phases θ_n of the associated amplitudes. By solving Equations (23) for the remaining α_n 's, it follows that:

$$\begin{aligned} \alpha_p &= \theta_p & p &= 1, 2, \dots, L \\ \alpha_q &= \sum_{p=1}^L c_{qp} \theta_p + d_{qr} \sigma_r t & q &= L+1, L+2, \dots, N \end{aligned} \quad (24)$$

where c_{qp} and d_{qr} are constant coefficients. According to this choice, the amplitudes B_p are assumed to be real while the amplitudes B_q are assumed to be complex, i.e.:

$$\begin{aligned} B_p &= \frac{1}{2} a_p & p &= 1, 2, \dots, L \\ B_q &= \frac{1}{2} (u_q + iv_q) & q &= L+1, L+2, \dots, N \end{aligned} \quad (25)$$

The phases θ_p and the associated real amplitudes a_p will be referred to as *principal phases* and *principal amplitudes*, respectively. From Equations (21) (24), and (25) it follows that:

$$\begin{aligned} A_p &= \frac{1}{2} a_p e^{i\theta_p} & p &= 1, 2, \dots, L \\ A_q &= \frac{1}{2} (u_q + iv_q) e^{i\alpha_q} & q &= L+1, L+2, \dots, N \end{aligned} \quad (26)$$

In conclusion, the unknowns of the problem are still $2N$, as in the original problem: the L principal phases θ_p , the L principal amplitudes a_p , and the $2(N-L) = 2R$ components u_q and v_q of the complex amplitudes B_q , measured in bases each rotating with a time law derived from Equation (24₂). With Equations (25), Equations (22) read:

$$\begin{aligned} \dot{a}_p &= \Re_p \left(a_p a_m^2, a_p (u_n + iv_n)^2, \dots, \prod_{m=0}^L a_m^{l_{spm}} \prod_{n=L+1}^N (u_n + iv_n)^{l_{spn}}, \dots \right) \\ a_p \dot{\theta}_p &= \Im_p \left(a_p a_m^2, a_p (u_n + iv_n)^2, \dots, \prod_{m=0}^L a_m^{l_{spm}} \prod_{n=L+1}^N (u_n + iv_n)^{l_{spn}}, \dots \right) \\ \dot{u}_q - \dot{\alpha}_q v_q &= \Re_q \left((u_q + iv_q) a_m^2, (u_q + iv_q) (u_n + iv_n)^2, \dots, \prod_{m=0}^L a_m^{l_{sqm}} \prod_{n=L+1}^N (u_n + iv_n)^{l_{sqn}}, \dots \right) \\ \dot{v}_q + \dot{\alpha}_q v_q &= \Im_q \left((u_q + iv_q) a_m^2, (u_q + iv_q) (u_n + iv_n)^2, \dots, \prod_{m=0}^L a_m^{l_{sqm}} \prod_{n=L+1}^N (u_n + iv_n)^{l_{sqn}}, \dots \right) \end{aligned} \quad (27)$$

in which, due to Equations (24₂):

$$\dot{\alpha}_q = \sum_{p=1}^L c_{qp} \dot{\theta}_p + d_{qr} \sigma_r \quad (28)$$

Equations (27) are referred to as the *mixed form of Equations (11)*. To obtain periodic motions, $\dot{\alpha}_p = \dot{u}_q = \dot{v}_q = 0$ must be enforced, together with $\dot{\theta}_p = \nu_p = \text{const}$. Hence, by solving $2N$ algebraic equations, the $2N$ unknowns (a_p, ν_p, u_q, v_q) are evaluated. By substituting $\theta_p = \nu_p t + \varphi_p$ in Equations (24₂), where the φ_p 's are L arbitrary initial phases, $\alpha_q = \nu_q t + \varphi_q$ is drawn, with the frequency corrections ν_q univocally determined and the initial phases φ_q depending on φ_p .

Similarly to the original system (11), and an account of the presence of the principal phases, periodic motions are not fixed points for the mixed form of the Equation (11). To remove this drawback it is necessary to eliminate these phases from the equations. However, this operation is not always possible, as will be explained. Let us introduce the following fundamental hypothesis: *the L principal amplitudes a_p do not vanish in any motion*. In this case Equations (27₂) can be divided by $a_p \neq 0$ and stated in standard form $\dot{\theta}_p = \dot{\theta}_p(a_m, u_n, v_n)$. By using these equations in Equations (28), $\dot{\alpha}_q = \dot{\alpha}_q(a_m, u_n, v_n)$ follows. Therefore Equations (27₁), (27₃) and (27₄) become a standard form system $\dot{\mathbf{a}} = \mathbf{f}(\mathbf{a}, t)$ of $M = N + R$ equations in the M unknowns $\mathbf{a} = (a_p, u_q, v_q)^T$. This will be referred to as *the standard form of the Equation (11)*. Once the unknowns have been determined, from Equations (27₂) the evolution of the principal phases θ_p is first derived and, from Equation (24₂), that of the phases α_q is finally obtained.

The standard form of the Equation (11) has the following peculiarities: (a) it has the smallest dimension M (equal to the codimension of the problem) and, (b) it admits periodic motions as fixed points $\dot{\alpha}_p = \dot{u}_q = \dot{v}_q = 0$. In addition, it suffers no problems when the stability of incomplete classes of motion (i.e. $u_h = v_h = 0$ for some h 's) is analyzed. However, to obtain it, it is necessary to select L principal amplitudes that do not vanish in any class of motion. If, in contrast, some of the principal amplitudes vanish in particular classes, the standard form is unable to give correct information about those motions. In these circumstances it would be possible to build more than one standard form, each valid for some classes of motion. However, such a procedure could be inconvenient from a computational point of view.

The geometrical method developed in the previous Section permits us to detect the existence of standard forms *before* writing them. To this end, it must be checked that L amplitudes participate in all steady motions. For example, for the system in Fig. 2, since $N = 5$ and $R = 3$ it is necessary to find $L = 2$ non-vanishing principal amplitudes. However, no such amplitudes exist, since only A_1 participates in all the motions; hence, no unique standard form can be built up for that system. However, if A_1 and A_3 (or A_4) are taken as principal amplitudes, the relevant standard form equations enable us to analyze the stability of five of the seven classes of motion. If the systems in Fig. 3 are instead considered, since $N = 5$ and $R = 4$, it is necessary to find only $L = 1$ non-

vanishing principal amplitudes. One such amplitude (namely A_1) does exist in the resonance $\omega_0 = \omega_1$ but does not exist in the $\omega_0 = 2\omega_1 + \omega_5$ case.

The standard form, when it exists, also suffers the following drawback. Although $a_p \neq 0$ by hypothesis, it can become small in some motions. In these cases, small denominators entailing numerical problems appear in the standard form, since in Equation (27₂) θ'_p is affected by a small coefficient. The problem can be overcome by adopting a *master*- and *slave*-amplitude representation, in which the non-principal (slave) amplitudes admit the principal (master) amplitude as factors. In Section 4.3 some examples are studied to illustrate this procedure.

4.3 Examples

An example of the procedure is given here. The equations governing the free vibrations of a system in the 1:2 resonance condition $\omega_2 = 2\omega_1 + \varepsilon\sigma$ are [2]:

$$\begin{aligned} -2i \left(\dot{A}_1 + \mu_1 A_1 \right) + c_1 A_2 \bar{A}_1 e^{i\sigma t} &= 0 \\ -2i \left(\dot{A}_2 + \mu_2 A_2 \right) + c_2 A_1^2 e^{-i\sigma t} &= 0 \end{aligned} \quad (29)$$

Since $N = 2$, $R = 1$, it is necessary to choose $L = N - R = 1$ principal amplitudes. From Equation (29), it follows that $A_2 \neq 0$ in any motion, while the same property does not hold for A_1 . Therefore A_2 must be taken as the principal amplitude.

Equation (23) reads $\alpha_2 - 2\alpha_1 + \sigma = 0$. By taking $\alpha_2 = \theta_2$, $\alpha_1 = (\theta_2 + \sigma)/2$ follows. Therefore, the mixed representation to be used is:

$$A_1 = \frac{1}{2} (u_1 + iv_1) e^{\frac{i}{2}(\theta_2 + \sigma t)}, \quad A_2 = \frac{1}{2} a_2 e^{i\theta_2} \quad (30)$$

By substituting it in Equations (29), four real equations in the unknowns $(a_2, \theta_2, u_1, v_1)$ are derived. The equation governing the θ_2 -evolution reads:

$$\dot{\theta}_2 = -\frac{c_2}{4a_2} (u_1^2 - v_1^2) \quad (31)$$

which, after substitution into the other three equations, leads to the following standard form:

$$\begin{aligned} \dot{a}_2 &= -\frac{1}{2} (2\mu_2 a_2 - c_2 u_1 v_1) \\ \dot{u}_1 &= -\frac{1}{8a_2} (8\mu_1 a_2 u_1 - 4\sigma a_2 v_1 + 2c_1 a_2^2 v_1 + c_2 u_1^2 v_1 - c_2 v_1^3) \\ \dot{v}_1 &= -\frac{1}{8a_2} (4\sigma a_2 u_1 + 2c_1 a_2^2 u_1 - 2c_2 u_1^3 + 8\mu_1 a_2 v_1 + c_2 u_1 v_1^2) \end{aligned} \quad (32)$$

Although $a_2 \neq 0$, if it is small in some motions, small denominators appear in Equations (32), as a consequence of Equation (31). To eliminate these terms, the following alternative representation is adopted:

$$A_1 = \frac{1}{2}a_2(u_1 + iv_1)e^{\frac{i}{2}(\theta_2 + \sigma t)}, \quad A_2 = \frac{1}{2}a_2e^{i\theta_2} \quad (33)$$

in which A_1 plays the role of *slave* amplitude and A_2 that of *master* amplitude, since A_1 cannot exist without A_2 . From Equations (29) and (33)

$$\dot{\theta}_2 = -\frac{c_2a_2}{4}(u_1^2 - v_1^2) \quad (34)$$

and

$$\begin{aligned} \dot{a}_2 &= -\frac{1}{2}(2\mu_2a_2 - c_2a_2^2u_1v_1) \\ \dot{u}_1 &= -\frac{1}{8}(8\mu_1u_1 - 8\mu_2u_1 - 4\sigma v_1 + 2c_1a_2v_1 + 5c_2a_2u_1^2v_1 - c_2a_2v_1^3) \\ \dot{v}_1 &= -\frac{1}{8}(4\sigma u_1 + 2c_1a_2u_1 - c_2a_2u_1^3 + 8\mu_1v_1 - 8\mu_2v_1 + 5c_2a_2u_1v_1^2) \end{aligned} \quad (35)$$

follow, as counterparts of Equations (31) and (32). Equations (35) do not suffer the numerical problems of Equations (32); however, it is necessary to use variables (u_1, v_1) and a_2 , which are not of the same order of magnitude. The procedure illustrated here should be generalized for more complex problems. It has already been adopted in Ref. [34], where, however, its use was suggested by the nature of the problem, rather than for reasons of mathematical convenience.

The previous system is considered again, with the mode-2 now excited by a sinusoidal force of frequency $\omega_0 \simeq \omega_2$. The resonance conditions are $\omega_2 = 2\omega_1 + \varepsilon\sigma_1$ and $\omega_0 = \omega_2 + \varepsilon\sigma_2$. The Equations (29) modify as follows [2]:

$$\begin{aligned} -2i\left(\dot{A}_1 + \mu_1A_1\right) + c_1A_2\bar{A}_1e^{i\sigma_1t} &= 0 \\ -2i\left(\dot{A}_2 + \mu_2A_2\right) + c_2A_1^2e^{-i\sigma t} + \frac{1}{2}A_0e^{i\sigma_2t} &= 0 \end{aligned} \quad (36)$$

Since $R = N = 2$, it is $L = N - R = 0$, so that no principal amplitudes need be sought, i.e. all the complex amplitudes must be expressed in Cartesian form. Equations (23) read: $\alpha_2 - 2\alpha_1 + \sigma_1 = 0$, $-\alpha_2 + \sigma_2 = 0$. By solving them $\alpha_1 = (\sigma_1 + \sigma_2)/2$, $\alpha_2 = \sigma_2$ are found and, therefore, the change of variable to be used is:

$$A_1 = \frac{1}{2}(u_1 + iv_1)e^{\frac{i}{2}(\sigma_1 - \sigma_2)t}, \quad A_2 = \frac{1}{2}(u_2 + iv_2)e^{i\sigma_2t} \quad (37)$$

From Equations (36) and (37) the standard form equations follow:

$$\begin{aligned} \dot{u}_1 &= -\frac{1}{4}(4\mu_1u_1 - 2\sigma_1v_1 - 2\sigma_2v_1 + c_1u_2v_1 - c_1u_1v_2) \\ \dot{v}_1 &= -\frac{1}{4}(2\sigma_1u_1 + 2\sigma_2u_1 + c_1u_1u_2 + 4\mu_1v_1 + c_1v_1v_2) \\ \dot{u}_2 &= -\frac{1}{2}(2\mu_2u_2 - c_2u_1v_1 - 2\sigma_2v_2) \\ \dot{v}_2 &= -\frac{1}{4}(a_0 + c_2u_1^2 + 4\sigma_2u_2 - c_2v_1^2 + 4\mu_2v_2) \end{aligned} \quad (38)$$

5. The reconstitution method

In Sect. 2.1 the reconstitution procedure was introduced to merge the complex amplitude Equations (8), depending on different time-scales, into a single Equation (9), in which only the true time t appears. The perturbation parameter ε was then reabsorbed, and the resulting equation analyzed. This is the most commonly used procedure to analyze the behaviour of a dynamical system when one resorts to higher-order approximations. However, other procedures have recently been used in the literature to combine Equations (8) when dealing with specific problems. It is therefore worthwhile discussing them critically, referring to general systems, rather than particular examples.

5.1 Methods of analysis

The starting point of the analysis is Equation (9), in which, however, the complex amplitudes \mathbf{A} are expressed in real form, using either the polar or the polar-Cartesian representation. A reduced set of M equations is thus obtained in the M unknowns $\mathbf{a} = (a_n, \gamma_r)$ or $\mathbf{a} = (a_p, u_q, v_q)$, respectively, having the same form as Equation (9), namely:

$$\begin{aligned} \dot{\mathbf{a}} = & \varepsilon \mathbf{f}_1(\mathbf{a}, a_0; \hat{\boldsymbol{\mu}}) + \varepsilon^2 [\mathbf{f}_2(\mathbf{a}, a_0; \hat{\boldsymbol{\mu}}) + \boldsymbol{\alpha}_1(\mathbf{a}, a_0; \hat{\boldsymbol{\mu}}) \mathbf{f}_1(\mathbf{a}, a_0; \hat{\boldsymbol{\mu}}) \\ & + \boldsymbol{\alpha}_2 \mathbf{f}_{1,\mathbf{a}}(\mathbf{a}, a_0; \hat{\boldsymbol{\mu}}) \mathbf{f}_1(\mathbf{a}, a_0; \hat{\boldsymbol{\mu}}) + \boldsymbol{\alpha}_3 \mathbf{f}_1^2(\mathbf{a}, a_0; \hat{\boldsymbol{\mu}})] + O(\varepsilon^3) \end{aligned} \quad (39)$$

The questions to be answered are two: (1) can the perturbation parameter ε be reabsorbed, thus ignoring the asymptotic origin of Equations (39)? (2) do all the terms in Equations (39) have to be retained in the analysis, or can some of them be omitted? Different answers to these questions lead to different methods of analysis. Accordingly, two main classes, *consistent methods* and *inconsistent methods*, and two sub-classes, *complete methods* and *incomplete methods* are distinguished. In the consistent approach, the asymptotic nature of the reduced dynamical system is taken into account, consistently with the basic assumptions of the perturbation method, whereas in the inconsistent approach this feature is ignored. In the complete methods, all terms deriving from the solvability conditions are retained in the analysis, while in the incomplete methods some of them are neglected. By combining the alternatives, four approaches are identified; these are discussed below.

(a) Complete Inconsistent Method (CIM)

This method was applied in [13], [24]. According to the philosophy of the inconsistent approaches, the reconstituted amplitude equations are dealt with as if they were a *closed-form* representation of the reduced dynamical system, rather than an *asymptotic approximation* of the unknown system. Therefore, the steady-state solutions are found by requiring the right hand member of Equations (39) to vanish in a whole:

$$\varepsilon \mathbf{f}_1 + \varepsilon^2 [\mathbf{f}_2 + \boldsymbol{\alpha}_1 \mathbf{f}_1 + \boldsymbol{\alpha}_2 \mathbf{f}_{1,\mathbf{a}} \mathbf{f}_1 + \boldsymbol{\alpha}_3 \mathbf{f}_1^2] = \mathbf{0} \quad (40)$$

For a fixed ε , Equations (40) are a set of m parameter-dependent nonlinear equations in the m unknown amplitudes. By solving them, if necessary through numerical algorithms,

several paths $\mathbf{a}_s = \mathbf{a}_s(\hat{\boldsymbol{\mu}}; \varepsilon)$ are found. Their stability is analyzed through the variational equation

$$\delta \dot{\mathbf{a}} = \left\{ \varepsilon \mathbf{f}_{1,\mathbf{a}} + \varepsilon^2 [\mathbf{f}_{2,\mathbf{a}} + \alpha_{1,\mathbf{a}} \mathbf{f}_1 + \alpha_1 \mathbf{f}_{1,\mathbf{a}} + \alpha_2 (\mathbf{f}_{1,\mathbf{a}\mathbf{a}} \mathbf{f}_1 + \mathbf{f}_{1,\mathbf{a}}^2) + 2\alpha_3 \mathbf{f}_{1,\mathbf{a}} \mathbf{f}_1] \right\}_{\mathbf{a}=\mathbf{a}_s} \delta \mathbf{a} \quad (41)$$

in which all quantities are evaluated at $\mathbf{a} = \mathbf{a}_s$.

(b) Incomplete Inconsistent Method (IIM)

This differs from the CIM in that $d_1 \mathbf{A}$ and $d_1^2 \mathbf{A}$ terms are omitted from the ε^3 -order solvability conditions (see Equations (8₂)). This procedure was introduced by Lee and Perkins [17] and justified by Lee and Lee [20] as follows: “time derivative terms are non-zero only on their corresponding time scale e.g., d_1 terms are non-zero on the t_1 scale but vanish on the t_2 scale”. The procedure has subsequently been followed by several authors (see [17]-[21]).

The reconstituted amplitude equations (39) simplify as follows:

$$\dot{\mathbf{a}} = \varepsilon \mathbf{f}_1(\mathbf{a}, a_0; \hat{\boldsymbol{\mu}}) + \varepsilon^2 \mathbf{f}_2(\mathbf{a}, a_0; \hat{\boldsymbol{\mu}}) \quad (42)$$

Steady-state solutions are then derived from

$$\varepsilon \mathbf{f}_1 + \varepsilon^2 \mathbf{f}_2 = 0 \quad (43)$$

and their stability analyzed through

$$\delta \dot{\mathbf{a}} = [\varepsilon \mathbf{f}_{1,\mathbf{a}} + \varepsilon^2 \mathbf{f}_{2,\mathbf{a}}]_{\mathbf{a}=\mathbf{a}_s} \delta \mathbf{a} \quad (44)$$

In Refs. [17]–[21] a different procedure is used. Namely, parameters $\boldsymbol{\mu}$ are expanded in series as $\boldsymbol{\mu} = \varepsilon \boldsymbol{\mu}_1 + \varepsilon^2 \boldsymbol{\mu}_2 + \dots$, instead of being ordered as in Equation (4). However, the inverse transformation $\varepsilon \boldsymbol{\mu}_1 + \varepsilon^2 \boldsymbol{\mu}_2 \rightarrow \boldsymbol{\mu}$ is later introduced in the reconstituted equations so that the expansion of the parameters has no role. The procedure illustrated here therefore leads in a more straightforward way to the same results (see [22]).

A slightly modified version (IIM-M) of the method can also be obtained by exchanging the two operations of reconstitution and transformation in real form of the solvability conditions (Equations (8)). If the t_1 -derivatives of the real quantities \mathbf{a} are neglected rather than those of the complex quantities \mathbf{A} , different equations are obtained. For example, if the polar representation is adopted, $d_1 \theta_n = 0$ is assumed in the IIM, entailing $d_1 \gamma_s = \sigma_s \neq 0$ because of Equation (19); this is in contrast to the assumptions $d_1 \gamma_s = 0$ of the IIM-M. If the problem is non-resonant the two versions coincide.

(c) Complete Consistent Method (CCM)

This was introduced in a systematic way by Rahman and Burton [14], after Luongo, Rega and Vestroni [36] had used it in a particular case. In this method the reconstituted amplitude equations are dealt with as an asymptotic approximation of the reduced

dynamical system, corrected up to a certain power of ε . The steady-state solutions and the eigenvalues of the variational equation are therefore consistently sought as series expansions corrected up to the same ε -order. As a first step, parameters $\boldsymbol{\mu}$ in Equations (39) are expanded in series of ε (i. e. $\hat{\boldsymbol{\mu}}(\varepsilon) = \boldsymbol{\mu}_1 + \varepsilon\boldsymbol{\mu}_2 + \dots$), so that the reconstituted amplitude equations read:

$$\begin{aligned} \dot{\mathbf{a}} = & \varepsilon \mathbf{f}_1(\mathbf{a}, a_0; \boldsymbol{\mu}_1) + \varepsilon^2 [\mathbf{f}_2(\mathbf{a}, a_0; \boldsymbol{\mu}_1) + \mathbf{f}_{1,\boldsymbol{\mu}}(\mathbf{a}, a_0; \boldsymbol{\mu}_1)\boldsymbol{\mu}_2 \\ & + \alpha_1(\mathbf{a}, a_0; \boldsymbol{\mu}_1)\mathbf{f}_1(\mathbf{a}, a_0; \boldsymbol{\mu}_1) + \alpha_2 \mathbf{f}_{1,\mathbf{a}}(\mathbf{a}, a_0; \boldsymbol{\mu}_1)\mathbf{f}_1(\mathbf{a}, a_0; \boldsymbol{\mu}_1) + \alpha_3 \mathbf{f}_1^2(\mathbf{a}, a_0; \boldsymbol{\mu}_1)] + O(\varepsilon^3) \end{aligned} \quad (45)$$

By requiring that *amplitudes \mathbf{a} be stationary for any ε* , i. e. $\dot{\mathbf{a}} = \mathbf{0} \forall \varepsilon$, the ε - and ε^2 -terms in Equations (45) must vanish separately, i.e.,

$$\begin{aligned} \mathbf{f}_1(\mathbf{a}, a_0; \boldsymbol{\mu}_1) &= \mathbf{0} \\ \mathbf{f}_{1,\boldsymbol{\mu}}(\mathbf{a}, a_0; \boldsymbol{\mu}_1)\boldsymbol{\mu}_2 + \mathbf{f}_2(\mathbf{a}, a_0; \boldsymbol{\mu}_1) &= \mathbf{0} \end{aligned} \quad (46)$$

where Equation (46₁) has been accounted for in deriving Equation (46₂). Conditions (46) express the vanishing of the amplitude time-derivatives on the different slow scales, namely $d_1\mathbf{a} = 0$, $d_2\mathbf{a} = 0$. Equation (46₁) is a set of M *nonlinear* equations in the amplitudes \mathbf{a} and in the first-order part $\boldsymbol{\mu}_1$ of the parameters $\boldsymbol{\mu}$. They can be solved with respect to $\boldsymbol{\mu}_1$ for fixed \mathbf{a} , to furnish $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_1(\mathbf{a})$. *The procedure therefore entails expanding exactly M control parameters, i.e. the same number as the codimension of the problem.* By substituting the solution in Equations (46₂), a set of *linear* equations in the $\boldsymbol{\mu}_2$ unknowns is found, from which $\boldsymbol{\mu}_2 = \boldsymbol{\mu}_2(\mathbf{a})$ is derived. By moving to higher orders, *linear* equations in $\boldsymbol{\mu}_3, \boldsymbol{\mu}_4, \dots$ are still derived. Finally

$$\boldsymbol{\mu} = \varepsilon \boldsymbol{\mu}_1(\mathbf{a}) + \varepsilon^2 \boldsymbol{\mu}_2(\mathbf{a}) + \dots \quad (47)$$

is obtained.

In the method illustrated, steady-state solutions are described asymptotically in the form $\boldsymbol{\mu} = \boldsymbol{\mu}(\mathbf{a})$. However, steady-state solutions are frequently sought in the more convenient form $\mathbf{a} = \mathbf{a}(\boldsymbol{\mu})$, so that Equations (47) have to be inverted (see [22]). As an alternative, steady-state solutions can be found directly in the form $\mathbf{a} = \mathbf{a}(\boldsymbol{\mu})$ simply by *expanding the amplitudes \mathbf{a} , instead of the parameters $\hat{\boldsymbol{\mu}}$, in the steady version of Equations (39).*

Concerning stability analysis, the variational equation built up on Equations (45), is

$$\delta \dot{\mathbf{a}} = \left\{ \varepsilon \mathbf{f}_{1,\mathbf{a}} + \varepsilon^2 \left[\mathbf{f}_{2,\mathbf{a}} - \mathbf{f}_{1,\boldsymbol{\mu}\mathbf{a}} \mathbf{f}_{1,\boldsymbol{\mu}}^{-1} \mathbf{f}_2 + \alpha_1 \mathbf{f}_{1,\mathbf{a}} + \alpha_2 \mathbf{f}_{1,\mathbf{a}}^2 \right] \right\}_{\mathbf{a}=\mathbf{a}_s} \delta \mathbf{a} \quad (48)$$

where Equations (46) have been used. Equation (48) is studied in detail in [29]. It should be noted that α_3 -terms play no role.

(d) Incomplete Consistent Method (ICM)

This has only been applied in Ref. [37]. In the spirit of the incomplete methods, $d_1 \mathbf{a}$ and $d_1^2 \mathbf{a}$ terms are omitted in the reconstituted amplitude Equations (45), which therefore read:

$$\dot{\mathbf{a}} = \varepsilon \mathbf{f}_1(\mathbf{a}, a_0; \boldsymbol{\mu}_1) + \varepsilon^2 [\mathbf{f}_2(\mathbf{a}, a_0; \boldsymbol{\mu}_1) + \mathbf{f}_{1,\boldsymbol{\mu}}(\mathbf{a}, a_0; \boldsymbol{\mu}_1) \boldsymbol{\mu}_2] \quad (49)$$

In contrast to the IIM (see Equations (42)), steady-state solutions are required to satisfy Equations (49) for any ε . Therefore Equations (46) and the same solution as the CCM are recovered. Stability is instead analyzed on the variational equation based on Equations (49):

$$\delta \dot{\mathbf{a}} = \left\{ \varepsilon \mathbf{f}_{1,\mathbf{a}} + \varepsilon^2 \left[\mathbf{f}_{2,\mathbf{a}} - \mathbf{f}_{1,\boldsymbol{\mu}\mathbf{a}} \mathbf{f}_{1,\boldsymbol{\mu}}^{-1} \mathbf{f}_2 \right] \right\}_{\mathbf{a}=\mathbf{a}_s} \delta \mathbf{a} \quad (50)$$

which differs from Equations (48) in the absence of α_1 - and α_2 -terms.

5.2 Discussion

A number of considerations regarding the four methods of analysis illustrated are now discussed.

- (1) If steady-state solutions are sought through an inconsistent method, an ordering violation occurs. In fact, if the non-polynomial inconsistent solutions are expanded in series, powers of ε greater than the highest power present in the equations are found. These higher-order terms are incorrect, since they do not represent the series expansion of the (unknown) exact solution. On the contrary, consistent methods furnish the correct coefficients of the series expansion, up-to the equation order. Indeed, inconsistent terms can sometimes improve the truncated solution; however, when this happens it cannot be predicted in advance.
- (2) Inconsistent methods are usually applied in the literature by first expanding the parameters and then recombining them in the reconstituted equation. The expansion therefore plays no role and can be avoided by using the procedure illustrated.
- (3) The t_1 -derivatives accounted for in the CIM affect steady-state solutions. This result appears to be *qualitatively incorrect*, since the steady amplitudes, which are constant in time, cannot depend on their time-derivatives. Paradoxically, by using the CIM, static equilibrium positions of a damped mechanical system have been found to depend on the damping coefficient. This drawback is avoided in the IIM simply by neglecting $d_1 \mathbf{a}$ and $d_1^2 \mathbf{a}$, however, this omission is arbitrary and affects stability.
- (4) For a particular class of systems, which is encountered fairly frequently in applications [18], [17], [20], the IIM and CCM furnish the same steady-state solutions. This occurs whenever the parameters appear linearly in the first-order part of the reconstituted amplitude equations and are absent in the higher-order parts.
- (5) The slow-time amplitude derivatives omitted in incomplete methods influence the stability of steady-state solutions, since they describe the contribution to the acceleration of higher-order harmonics as well as of the passive coordinates possibly

triggered by the active ones. In [22] a consistent approach is used to show that the omission of these derivatives entails an error of ε^2 -order on the critical amplitude if it is associated with a dynamic bifurcation, while no error exists at the ε^2 -order if the bifurcation is of a static type. However the error persists in all non-critical states.

- (6) Consistent methods sometimes fail in stability analysis. This happens whenever the second approximation is responsible for the occurrence of a critical condition which is absent in the first approximation, i.e. it entails *qualitative* changes. In these cases an inconsistent approach can instead give correct qualitative information. The failure of the consistent approach depends on the occurrence of an ordering violation caused by higher-order terms prevailing over lower-order terms. This type of problem could be analyzed more easily using a different approach, ordering competitive terms at the same level in the perturbation scheme (see e.g., [38]).

5.3 Example

A Duffing/Van der Pol oscillator in primary resonance with an external excitation is considered, having equation:

$$\ddot{q} + q - \nu \dot{q} + c_1 q^3 + c_2 \dot{q}^3 = p \cos(1 + \mu)t \quad (51)$$

where μ and ν are control parameters. When $\nu = 0$, the homogeneous linear part of Equation (51) admits a couple of purely imaginary eigenvalues; when $\mu = 0$, the oscillator is in resonance with the external excitation. The codimension of the problem is $M = 2$. By ordering the control parameters as $\mu = \varepsilon \hat{\mu}$, $\nu = \varepsilon \hat{\nu}$, the excitation amplitude (imperfection parameter) as $p = \varepsilon^2 \hat{p}$ and expanding the displacement as $q = \varepsilon^{1/2} q_1 + \varepsilon^{3/2} q_2 + \varepsilon^{5/2} q_3 + \dots$, the perturbation equations are:

$$\begin{aligned} d_0^2 q_1 + q_1 &= 0 \\ d_0^2 q_2 + q_2 &= \hat{\nu} d_0 q_1 - c_1 q_1^3 - c_2 (d_0 q_1)^3 - 2d_0 d_1 q_1 + \hat{p} \cos(t_0 + \hat{\mu} t_1) \\ d_0^2 q_3 + q_3 &= \hat{\nu} (d_0 q_2 + d_1 q_1) - 3c_1 q_1^2 q_2 - 3c_2 (d_0 q_1)^2 (d_1 q_1 + d_0 q_2) \\ &\quad - 2d_0 d_1 q_2 - (d_1^2 + 2d_0 d_2) q_1 \end{aligned} \quad (52)$$

By solving Equations (52), $q_1 = a \cos(t + \theta)$, $q_2 = \frac{a^3}{32} [c_1 \cos(3t + 3\theta) + c_2 \sin(3t + 3\theta)]$ and the solvability conditions are obtained [22]. The reconstituted amplitude equations, in real polar form, read:

$$\begin{aligned} \dot{a} &= \frac{\varepsilon}{2} \left(a \hat{\nu} - \frac{3}{4} c_2 a^3 + \hat{p} \sin \gamma \right) \\ &\quad + \varepsilon^2 \left[\frac{3}{64} c_1 c_2 a^5 + \frac{1}{2} (\hat{\mu} - d_1 \gamma) \left(a \hat{\nu} - \frac{9}{4} c_2 a^3 - 2d_1 a \right) + \frac{1}{2} a d_1^2 \gamma \right] \\ \dot{\gamma} &= \varepsilon \left(\hat{\mu} - \frac{3}{8} c_1 a^2 + \frac{\hat{p}}{2a} \cos \gamma \right) \\ &\quad + \varepsilon^2 \left[\frac{3}{256} (3c_2^2 - c_1^2) a^4 - \frac{1}{2} d_1 a \left(\frac{3}{4} c_2 a - \frac{\nu}{a} \right) - \frac{1}{2a} d_1^2 a + \frac{1}{2} (\hat{\mu} - d_1 \gamma)^2 \right] \end{aligned} \quad (53)$$

where $\gamma = \widehat{\mu}t_1 - \theta$ is the phase difference between the excitation and the response, $d_1^2 a$ and $d_1^2 \gamma$ can be expressed in terms of a and γ by differentiating the first-order part of Equations (53).

If a complete method (CCM, CIM) is adopted, all terms in Equations (53) have to be retained. If an incomplete method (ICM, IIM) is instead followed, $d_1 a = d_1^2 a = 0$ and $d_1 \theta = d_1^2 \theta = 0$ must be taken. Hence, $d_1 \gamma = \widehat{\mu}$ and $d_1^2 \gamma = 0$ must be posed in Equations (53). In contrast, if the modified approach is followed (IIM-M), then $d_1 a = d_1^2 a = d_1 \gamma = d_1^2 \gamma = 0$ are directly substituted in Equations (53). In all methods, consistency requires that Equations (53) be solved in series. By letting $\widehat{\mu} = \mu_1 + \varepsilon\mu_2$, $\gamma = \gamma_1 + \varepsilon\gamma_2$ and solving the resulting set of perturbation equations, steady solutions in the form $\mu = \mu(a, \nu)$ and $\gamma = \gamma(a, \nu)$ are derived. In contrast, if an inconsistent procedure is followed, the parameter ε is reabsorbed and the nonlinear equations are solved directly.

In order to make a quantitative comparison between different solutions furnished by the four previous methods, numerical values of the auxiliary parameters c_1 and c_2 and of the excitation amplitude p are chosen, namely $c_1 = 1 / 30$, $c_2 = 1 / 60$, $p = 1/5$; moreover, the control parameter ν is kept fixed at $\nu = 1/40$. The steady-state amplitude a and the phase γ are then plotted vs μ . Figure 5 shows the first approximation, common to all methods. It is found that the amplitude reaches a limit point at A , saddle-node bifurcations occur at points B and C and a Hopf bifurcation manifests itself at H . For increasing μ 's, the steady solution loses stability at B , regains it at C , and again becomes unstable at H .

The second approximation is next considered according to the methods illustrated above. The results are shown in Figure 5. It is seen that the second approximation entails only quantitative modifications of the curves, i.e. the first approximation captures all the qualitative aspects of the phenomenon. It is remarkable that the IIM produces only small

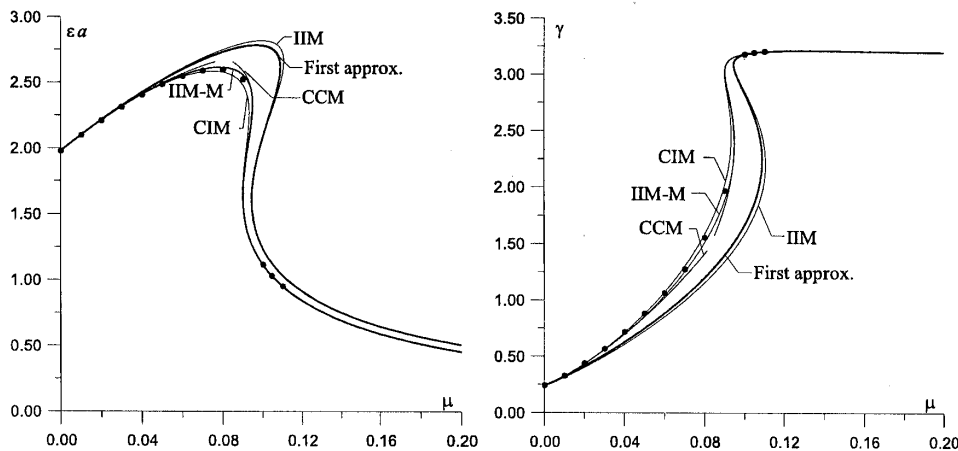


Figure 5. Amplitude and phase response vs detuning in the second approximation; • numerical results.

modifications with respect to the first approximation. The remaining three methods, in contrast, give curves that are very close to each other and fairly distant from the first approximation curve, above all at higher amplitudes. The CCM expansion is not valid near the limit point A , where the coefficients matrix of the unknowns γ_2 and μ_2 appearing in Equations (46) becomes singular. However, this drawback does not entail errors at points B and C , where instability occurs, although B is fairly close to A . Vice versa, if a and γ , instead of μ and γ , were expanded in series, the method would fail precisely at points B and C . The problem would be avoided completely if ν were expanded in series, instead of being kept constant.

The second-order perturbative solutions are compared with exact numerical solutions obtained by directly integrating the equation of motion (51) and performing an FFT of the regime response. The amplitudes of the fundamental harmonic for different μ 's in the stable zones are marked in Figure 5. It is found that the CIM and IIM-M give an excellent approximation of the exact solutions over the whole range, while the small errors in the CCM are due to the presence of the amplitude limit point. The IIM, in contrast, is affected by errors at large amplitudes.

An eigenvalue analysis was performed in Ref. [22]. It was found that the CCM and IIM solutions are different, but virtually coincident at points B and C and slightly different at H .

6. Conclusions

Some computational problems in multiple scale analysis have been discussed, namely: (a) classes of motion evaluation, (b) the proper form of the amplitude equations, and, (c) reconstitution techniques. The three problems have been analyzed in the light of the existing literature, and the following conclusions are drawn.

- (1) The multiple scale method is an efficient tool to analyze the behaviour of nonlinear dynamical systems with weak nonlinearities. It works as a *reduction method*, by lowering the dimensions of the original system to its *codimension*.
- (2) The qualitative form of the complex amplitude equations can be derived *before writing and solving the perturbation equation*, simply by looking at the resonance conditions. All terms are classified as *improper resonant* and *primary* or *secondary proper resonant terms*, according to their origin in the resonance conditions.
- (3) Classes of motion are defined as invariant subspaces of the amplitude space. It is proved that their existence depends only on the primary resonant terms, the remaining terms having only a quantitative effect on the motion. Moreover, higher-order asymptotic solutions do not modify the classes of motion of a system once the primary terms have appeared in the perturbation procedure, unless other resonances are encountered.
- (5) A simple and efficient algorithm (*the Resonance Coefficients Method, RCM*) is implemented, that is able to find all the existing classes of motion simply by analyzing the resonance conditions. The method works well for general systems, for which all the nonlinear terms are assumed to be different from zero. Particular systems, for which some terms may be absent, usually possess more classes of motion than general systems.

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- (6) The resonance conditions are interpreted as constraint relations for the classes of motion, since they prevent the occurrence of some classes. If a system is nonresonant, all the classes are admitted; the larger the number of resonances, the smaller the number of classes. A constraint degree can be attributed to each resonance, equal to the number of classes it inhibits. The index allows the resonances to be ordered hierarchically: resonances with a lower degree of constraint are usually unessential to the evaluation of the classes, but general rules are not available.
- (7) A geometrical approach to the problem of evaluating the classes of motion is also illustrated. The method is based on a set representation of the classes of motion, consisting in the following two steps: (a) the set of the classes of motion admitted by each individual resonance (family) is first built up; (b) the interactions among families are studied recursively in pairs, by applying simple rules generated by a unique theorem (*the Class Diagram Method, CDM*). The method furnishes class diagrams which efficiently synthesize the couplings existing among the amplitudes as well as their importance in describing the motion.
- (8) The structure of the variational equations governing the stability of steady (periodic or bi-periodic) solutions is then analyzed. Three types of perturbation are identified: (a) *in-class perturbations*, i.e. perturbations of the amplitudes participating in the motion; (b) *non-resonant* and (c) *resonant out-of-class perturbations*, i.e. perturbations of the amplitudes not participating in the motion directly or not involved in resonance with the active amplitudes, respectively. It is proved that the Jacobian matrix is block diagonal and does not contain coupling terms among the three classes of perturbation. Finally, based on the class diagrams, it is also possible to gain information on possible further uncouplings of the Jacobian block associated with the resonant out-of-class perturbations. It is found that improper resonant terms that are unessential to the class evaluation do nonetheless contribute to the stability of steady motion.
- (9) To perform *quantitative* analysis, the complex amplitude equations must first be transformed in real quantities and then reduced to their codimension. In this form they are autonomous and admit periodic solutions as fixed points. If the variable transformation is made by using polar components, the resulting real equations will be in non-standard form, so that problems arise when the stability of an (incomplete) class of motion is analyzed. It is shown here that a mixed polar-Cartesian representation permits *standard form of equations* to be obtained. However, the method works well only if a sufficient number of amplitudes (called principal) *do not vanish in any class of motion*. The class diagram technique illustrated above makes it possible to ascertain in advance whether or not the amplitude equations admit such a standard form.
- (10) *The reconstitution procedure*, i.e. the procedure to combine in a unique equation (the reduced dynamical system) the solvability conditions derived at each step of the perturbation procedure, is finally discussed. The approaches followed in the literature are classified as Complete Inconsistent Method (CIM), Incomplete Inconsistent Method (IIM), Complete Consistent Method (CCM) and Incomplete Consistent Method (ICM). In consistent methods the asymptotic nature of the reduced dynamical system is consistently taken into account, whereas in inconsistent methods the reduced dynamical system is dealt with as if it were

expressed in closed form. In the complete methods, all the terms deriving from the analysis are retained, while in incomplete methods the amplitude derivatives $d_1 \mathbf{a}$ and $d_1^2 \mathbf{a}$, appearing in the higher-order solvability conditions, are neglected.

- (11) A critical comparison among the methods is performed, showing how the omission of the slow derivatives in the incomplete methods entails errors both in the steady solution and in their stability, if these are dynamic. Complete methods, however, if consistent, suffer a violation of order that sometimes leads to qualitatively incorrect results; if they are consistent, they suffer failures around the bifurcation points of the lower approximations. Therefore complete methods should be used, but care should be taken to the drawbacks highlighted above.
- (12) Examples are worked out for each problem addressed to illustrate the procedures and to compare numerical results.

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