Dynamics of structures coupled with elastic media - a review of numerical models and methods

D. Clouteau[a,], R. Cottereau[a], G. Lombaert[b]

[a]Laboratoire MSSMat UMR 8579, École Centrale Paris, CNRS, grande voie des vignes, F-92295 Châtenay-Malabry, France
[b]Department of Civil Engineering, K.U.Leuven, Kasteelpark Arenberg 40, B-3001 Leuven, Belgium

Abstract

This paper reviews issues and developments in the field of structure-environment interaction problems, in which the environment is an elastic body, possibly unbounded. It covers in particular the fields of soil-structure interaction, ground-borne noise and vibration emitted by transportation systems and wave diffraction by obstacles in an elastic medium. The general setting for a linear bounded structure coupled to an unbounded linear environment through a bounded interface is first recalled, and the domain decomposition technique classically used for its description is put up. Extensions for the cases of non-linear structure and uncertain environment are then discussed. Finally, the ongoing research in the fields of unbounded interface, moving interface, and multiple interfaces are reviewed and summarized.

Keywords: dynamics, soil-structure interaction, ground-borne vibrations, moving loads, city-site effect

1. Introduction

This paper aims at reviewing some well established theoretical and numerical methods developed in the last few decades to account for the dynamic interaction between structures and their environment. Although some of the reviewed methods can be applied to a wide spectrum of interacting media (*e.g.* fluid-structure interaction [1], or electromechanical coupling), the structure's environment is restricted here to a large and possibly unbounded visco-elastic medium. Under these hypotheses a Lagrangian formulation for both parts can be assumed, together with the assumption of infinitely small transformations. Still, many practical applications fall into this category: i) Soil-Structure Interaction [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]; ii) Ground-borne noise and vibration emitted by transportation systems [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32]; and iii) Surface geophysics [33, 34, 35].

However, the focus here is on methods rather than applications. Given this context, making a clear distinction between what is called the structure and its elastic environment is not straightforward and deserves more precise definitions. From an engineering point of view, the structure can be defined as the mechanical object to be designed and built (*e.g.* a car, a building, a piece of equipment), and the environment as the rest of the world (*e.g.* the road, the soil, the host mechanical system), under the aforementioned restrictions. Though still imprecise, this definition indicates that the structure is an object of limited extent D on which: i) a fair amount of detailed information is available; ii) hight levels of precision and robustness are required; and iii) modifications and improvements may be proposed. On the contrary, the environment is characterized by: iv) its large dimension L compared to that of the structure; v) a moderate to high level of uncertainty on the model and the data; and vi) little possibility of modification. Basically, the structure is the object of engineering design and concern, while the environment is imposed and considered only through its impact on the structure.

Considering the above problem setting, the first obvious dimensionless parameter of the problem is D/L, which is assumed to be small. In addition, the dynamic behavior of the structure is assumed to remain in the low-to-medium frequency range, which implies, under a linear elastic hypothesis and given a typical wavelength in the structure λ,

Email address: `didier.clouteau@ecp.fr` (D. Clouteau)

that the ratio D/λ is not too large. Indeed, hypotheses ii) and iii) are difficult to meet out of that range. Combining these remarks, one obtains that λ/L remains small:

$$D \approx \lambda \ll L \tag{1}$$

These hypotheses on the length scales of the problem provide the basic framework for the general and now traditional modeling approach for this type of structure-environment problem. It considers: i) a bounded linear elastic structure coupled to; ii) an unbounded linear elastic environment along; iii) a bounded interface. For completeness, this model is recalled in section 2. In the following sections, this elementary approach is then generalized in several directions.

Non-linear behavior of the structure and its environment is the first important issue to be addressed. Indeed, most design methods are based on linear or equivalent linear models, which have proven in particular to be efficient in quantifying amplifications due to resonance. Yet, advanced design methods, safety margin studies on existing structures or specific cases call for the consideration of non-linear models for the structure and its nearby environment. These non-linear extensions of the structure-environment coupling will be considered in section 3.

We then consider in section 4 the influence of uncertainties on the coupled structure-environment problem. These uncertainties play a key role in the dynamical behavior of the structure itself (see the review in [36]), but we will concentrate here on the interaction problem. Hence, we will mainly consider stochastic models related to the environment, both in terms of loadings and mechanical properties.

The next three sections consider interaction problems for which the interface between the structure and its environment falls outside of the cases considered before: i) unbounded in section 5; ii) moving in section 6; and iii) multiple in section 7. Typical examples for these three problems include, respectively: i) the interaction between a tunnel and the soil in which it is embedded; ii) the train-track interaction problem; and iii) the city-site coupling effect in which multiple interaction between the buildings of a city modifies a seismic incident field. The first two types of problems are approached through appropriate changes of variables or transformations of the equations, while the last one is tackled through statistical methods.

Finally the conclusion will emphasis the shortcomings of the proposed approach and will draw some perspectives for further developments in this field.

2. Linear dynamics of structure-environment interaction

The aim of this section is to present a general formulation for a structure-environment interaction problem, along with the associated parameters, unknowns and equations, under the assumptions of: linear elasticity, bounded structure and bounded interface. The case of an unbounded interface is considered in section 5 and that of a non-linear structure in section 3.

The classical approach under these simple hypotheses is based on sub-structuring techniques [37, 38] commonly used in structural dynamics [39], fluid-structure [40] and soil-structure interaction [41, 42, 43, 44, 45]. The physical domain is decomposed into two subdomains: the (possibly unbounded) elastic environment, denoted by Ω_e, and the bounded structure, denoted by Ω, as shown in Figure 1. The interface between these domains is denoted by Σ. On the other parts of their boundaries denoted by Γ and Γ_e, free surface boundary conditions are assumed. The extension of this formalism to the case where part of the boundary of either Ω_e or Ω beholds Dirichlet boundary conditions is straightforward.

This decomposition has two important features: i) the behavior of the structure is usually of more interest than that of the environment; and ii) the environment is in practice often infinite or semi-infinite, which calls for a particular numerical approach. Therefore, the two parts of that coupled problem have often been solved with different numerical techniques, which will be briefly described in section 2.5.

2.1. The local problems: structure and environment

The permanent displacement fields and stress fields in Ω_e and Ω due to static loads (the weight for example) are denoted by $\mathbf{u}_{eo}(\mathbf{x})$, $\sigma_{eo}(\mathbf{x})$, $\mathbf{u}_o(\mathbf{x})$ and $\sigma_o(\mathbf{x})$, respectively. These fields are assumed to be known in the following and will play the role of parameters. The dynamic perturbations of the displacement fields due to dynamic loads are denoted by $\mathbf{u}_e(\mathbf{x}, t)$ and $\mathbf{u}(\mathbf{x}, t)$. We assume that they are small enough to allow for a linear approximation of

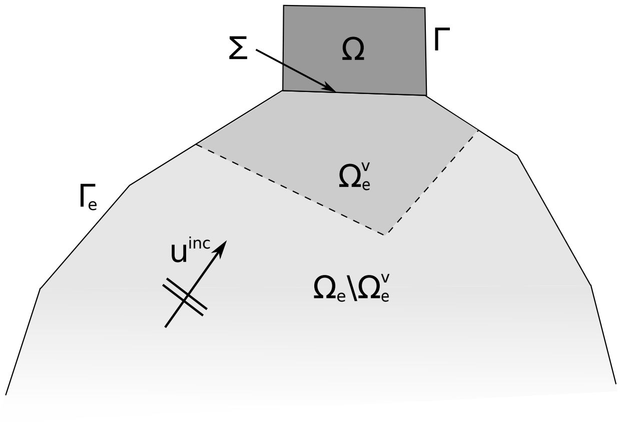

Figure 1: Model layout and notation

the constitutive and equilibrium equations in the vicinity of the static state $(\mathbf{u}_{eo}, \sigma_{eo}; \mathbf{u}_o, \sigma_o)$, and that the nonlinear geometric terms are negligible. Hence, the dynamic perturbations of the stress tensors, denoted $\sigma_\alpha(\mathbf{u})$ can be expressed as linear functions of the dynamic fluctuation of the strain tensors, denoted $\varepsilon(\mathbf{u}_\alpha) = (\nabla\mathbf{u}_\alpha + \nabla\mathbf{u}_\alpha^T)/2$, with $\alpha \in \{e, \}$:

$$\sigma_\alpha(\mathbf{u}_\alpha) = \int_0^t \mathbf{C}_\alpha(t-\tau)\varepsilon(\mathbf{u}_\alpha(\tau))\mathrm{d}\tau, \qquad \alpha \in \{e, \} \tag{2}$$

where ∇ is the gradient operator, $\mathbf{C}_e$ and $\mathbf{C}$ are fourth-order symmetric visco-elastic tensors being the time derivative of the relaxation tensor. In the particular case where the media follow the classical Hooke's Law, these relations simplify to $\sigma_\alpha(\mathbf{u}_\alpha) = \lambda_\alpha(\mathrm{div}\mathbf{u}_\alpha)\mathbf{I}_d + 2\mu_\alpha\varepsilon(\mathbf{u}_\alpha)$, where (λ_e, μ_e) and (λ, μ) are the classical Lamé parameters for the environment and the structure, respectively. $\mathbf{I}_d$ is the 3×3 identity matrix.

Locally in each subdomain Ω_e and Ω, the unknown fields must verify the equilibrium equations

$$\begin{aligned} -\mathbf{Div}\sigma_\alpha(\mathbf{u}_\alpha) + \rho_\alpha\partial_{tt}\mathbf{u}_\alpha = \mathbf{f}_\alpha \quad & \text{in } \Omega_\alpha \\ \sigma_\alpha(\mathbf{u}_\alpha)\mathbf{n} = \mathbf{t}_\alpha \quad & \text{on } \Gamma_\alpha \end{aligned} \tag{3}$$

with homogeneous initial conditions $\mathbf{u}_\alpha(\mathbf{x}, t=0) = 0$ and $\partial_t\mathbf{u}_\alpha(\mathbf{x}, t=0) = 0$, with $\alpha \in \{e, \}$. In these equations, $\mathbf{f}_e$ and $\mathbf{f}$ are body forces, and $\mathbf{t}_e$ and $\mathbf{t}$ are surface tractions. They are assumed to have bounded support over their respective domain of definition. In particular, we define a bounded subset Ω_e^{v} of Ω_e in the vicinity of the structure, and assume that $\mathbf{f}_e = 0$ in $\Omega_e \setminus \Omega_e^{\mathrm{v}}$ and $\mathbf{t}_e = 0$ on $\Gamma_e \setminus (\partial\Omega_e^{\mathrm{v}} \cap \Gamma_e)$. In the case of non-vanishing initial conditions, one can formally recover the above equations by introducing additional body forces $\mathbf{f}_e^0(\mathbf{x}, t)$ and $\mathbf{f}^0(\mathbf{x}, t)$, using the Dirac distribution $\delta(t)$:

$$\mathbf{f}_\alpha^0(\mathbf{x}, t) = -\rho\left(\delta(t)\partial_t\mathbf{u}_\alpha(\mathbf{x}, t=0) - \mathbf{u}_\alpha(\mathbf{x}, t=0)\partial_t\delta(t)\right), \tag{4}$$

2.2. Incident and diffracted fields

In the type of coupled problems considered here, loads can often be accounted for by defining an incident field $\mathbf{u}^{\mathrm{inc}}$ inside Ω_e. This is the case, for example, in seismic or noise engineering. This incident field can be seen as a parameter of the dynamic interaction problem, that has to satisfy some constraints. Firstly, it is assumed that, before the initial time $t = 0$, $\mathbf{u}^{\mathrm{inc}}(\mathbf{x}, t)$ vanishes in the vicinity of the structure Ω_e^{v}. We then assume that $\mathbf{u}^{\mathrm{inc}}$ satisfy the Navier-Lamé equation, free surface boundary conditions (3) in $\Omega_e \setminus \Omega_e^{\mathrm{b}}$ and with any appropriate extension inside Ω_e^{v}. This generates equivalent body forces $\mathbf{f}_d$ and surface tractions $\mathbf{t}_d$ both with bounded support:

$$\begin{aligned} -\mathbf{Div}\sigma_e(\mathbf{u}^{\mathrm{inc}}) + \rho_e\partial_{tt}\mathbf{u}^{\mathrm{inc}} = \mathbf{f}_e - \mathbf{f}_d \quad & \text{in } \Omega_e^{\mathrm{v}} \\ \sigma_e(\mathbf{u}^{\mathrm{inc}})\mathbf{n} = \mathbf{t}_e - \mathbf{t}_d \quad & \text{in } \partial\Omega_e^{\mathrm{v}} \cap \partial\Omega_e \end{aligned} \tag{5}$$

Classically [42], an auxiliary field, called the *diffracted* – or *scattered* – field, and denoted $\mathbf{u}_d$, becomes the new main unknown in Ω_e. It is defined in the environment, as:

$$\mathbf{u}_d = \mathbf{u}_e - \mathbf{u}^{\text{inc}} \qquad \text{in } \Omega_e. \tag{6}$$

and has to satisfy the Navier-Lamé equations (3), for any time $t \geq 0$ replacing $\mathbf{u}_e$, $\mathbf{f}_e$ and $\mathbf{t}_e$ by $\mathbf{u}_d$, $\mathbf{f}_d$ and $\mathbf{t}_d$ respectively and satisfying homogeneous initial conditions $\mathbf{u}_d(\mathbf{x}, t = 0) = 0$ and $\partial_t \mathbf{u}_d(\mathbf{x}, t = 0) = 0$.

2.3. *Frequency domain and coupling equations*

From now on, we will reformulate the equations in the frequency domain as in [42]. We therefore consider the Fourier transforms of the displacements fields $\mathbf{u}$ and $\mathbf{u}_d$, as well as the Fourier transforms of the loads $(\mathbf{u}^{\text{inc}}, \mathbf{f}_d, \mathbf{t}_d, \mathbf{f}, \mathbf{t})$. However, for notational simplicity, both the fields and their Fourier transform will be denoted in the same way. Note that, in the frequency domain, the constitutive relations (2) become simple complex-valued products. It is worth mentioning that Laplace domain approaches are obtained taking $s = i\omega$ and $\Re(s) \geq 0$ where $i^2 = -1$ and $\Re$ is the real part of a complex number. The fields $\mathbf{u}(\mathbf{x}, \omega)$ and $\mathbf{u}_d(\mathbf{x}, \omega)$ now verify, in the frequency domain, the balance of momentum in Ω and Ω_e respectively:

$$\begin{aligned} \mathbf{Div}\sigma(\mathbf{u}) + \mathbf{f} &= -\rho\omega^2\mathbf{u} \quad &&\text{in } \Omega, \\ \sigma(\mathbf{u})\mathbf{n} &= \mathbf{t} \quad &&\text{on } \Gamma, \end{aligned} \tag{7}$$

$$\begin{aligned} \mathbf{Div}\sigma_e(\mathbf{u}_d) + \mathbf{f}_d &= -\rho_e\omega^2\mathbf{u}_d \quad &&\text{in } \Omega_e, \\ \sigma_e(\mathbf{u}_d)\mathbf{n} &= \mathbf{t}_d \quad &&\text{on } \Gamma_e. \end{aligned} \tag{8}$$

In addition the equilibrium equation and the kinematic conditions on the interface Σ read:

$$\sigma(\mathbf{u})\mathbf{n} - \sigma_e(\mathbf{u}_d)\mathbf{n} = \sigma_e(\mathbf{u}^{\text{inc}})\mathbf{n} \qquad \text{on } \Sigma \tag{9}$$

$$\mathbf{u} - \mathbf{u}_d = \mathbf{u}^{\text{inc}} \qquad \text{on } \Sigma. \tag{10}$$

The set of equations (7-10) forms a well posed boundary value problem as long as damping is accounted for (*e.g.* $\Im(\mathbf{C}(\omega))$ is a positive tensor and $\Im(\omega) \leq 0$, $\Im$ denoting the imaginary part of a complex number). In the purely elastic case, radiation conditions have to be applied at infinity when $\Im(\omega) = 0$. They can be formally defined taking the limit $\Im(\omega) \to 0$ [46]. The displacement fields $\mathbf{u}_d$ and $\mathbf{u}$ verifying this system depend linearly on the parameters $(\mathbf{u}^{\text{inc}}, \mathbf{f}_e, \mathbf{t}_e, \mathbf{f}, \mathbf{t})$.

2.4. *Domain decomposition and weak formulation*

Coupled systems given by equations (7-10) are usually solved through dynamic condensation techniques. The basic idea of this approach is the following:

- to define new unknown fields on the interfaces, either displacements or tractions,
- to solve the local boundary value problem (8) and possibly (7) using these fields as boundary conditions on the interface,
- to enforce the remaining equations in a weak sense, e.g. for any trial admissible field on the interface.

Primal formulation. We derive here a primal formulation, for which we choose a compatible displacement field $\mathbf{u}_\Sigma$ on the interface Σ (*i.e.* that verifies the kinematic condition (10)). The local boundary value problem consists in finding $\mathbf{u}_d$ satisfying equations (8) together with boundary conditions:

$$\mathbf{u}_d = \mathbf{u}_\Sigma - \mathbf{u}^{\text{inc}} \qquad \text{on } \Sigma \tag{11}$$

The last step would then consist in enforcing weakly the equilibrium equations (9) and (7) to obtain the actual values of $\mathbf{u}$, knowing that $\mathbf{u} = \mathbf{u}_\Sigma$ on Σ. Hence, considering the weak enforcement of the equilibrium equation (9) over Σ yields the following variational formulation of the interaction problem for all $\mathbf{v}$ in $\mathbf{V}_\Omega$:

$$\left\{\mathcal{K} + \mathcal{K}_e(\omega) - \omega^2\mathcal{M}\right\}(\mathbf{u}, \mathbf{v}) = \mathcal{L}_e(\omega; \mathbf{v}) + \mathcal{L}(\mathbf{v}) \tag{12}$$

where $\mathbf{V}_\Omega$ is the functional space of fields with finite energy in Ω. $\mathcal{K}$ and $\mathcal{M}$ are the classical stiffness and mass bilinear forms of the structure arising in the Finite Element Method (FEM) [47, 48], and $\mathcal{L}$ is the linear form induced by the forces and tractions in the structure:

$$\mathcal{K}(\mathbf{u},\mathbf{v}) = \int_\Omega \sigma(\mathbf{u}) : \varepsilon(\mathbf{v})\mathrm{d}V, \qquad \mathcal{M}(\mathbf{u},\mathbf{v}) = \int_\Omega \rho\mathbf{u}\cdot\mathbf{v}\mathrm{d}V \tag{13}$$

$$\mathcal{L}(\mathbf{v}) = \int_\Omega \mathbf{f}\cdot\mathbf{v}\mathrm{d}V + \int_\Gamma \mathbf{t}\cdot\mathbf{v}\mathrm{d}S \tag{14}$$

The bilinear form $\mathcal{K}_e$ stands for the dynamic stiffness of the environment. Finally, the linear form $\mathcal{L}_e(\omega;\mathbf{v})$ accounts for the equivalent forces induced by the incident field. After some manipulations, including the application of Betti-Maxwell reciprocity theorem, these two terms can be rewritten as:

$$\mathcal{K}_e(\omega;\mathbf{u},\mathbf{v}) = \int_\Sigma \sigma_e(\mathbf{u}_{\Sigma e})\mathbf{n}\cdot\mathbf{v}\mathrm{d}S, \tag{15}$$

$$\mathcal{L}_e(\omega;\mathbf{v}) = \int_\Sigma \left(\sigma_e(\mathbf{v}_{\Sigma e})\mathbf{u}^{\mathrm{inc}} - \sigma_e(\mathbf{u}^{\mathrm{inc}})\mathbf{v}\right)\cdot\mathbf{n}\mathrm{d}S - \int_{\Omega_e} \mathbf{f}_d\cdot\mathbf{v}_{\Sigma e}\mathrm{d}V - \int_{\Gamma_e} \mathbf{t}_d\cdot\mathbf{v}_{\Sigma e}\mathrm{d}S, \tag{16}$$

where $\mathbf{n}$ is the outward normal vector to the structure Ω and $\mathbf{u}_{\Sigma e}$ is the solution of the local problem over Ω_e with no load except an imposed Dirichlet boundary condition $\mathbf{u}_\Sigma$ over Σ:

$$\begin{aligned} \mathbf{Div}\sigma_e(\mathbf{u}_{\Sigma e}) = -\rho_e\omega^2\mathbf{u}_{\Sigma e} &\qquad \text{in } \Omega_e, \\ \mathbf{u}_{\Sigma e} = \mathbf{u}_\Sigma &\qquad \text{on } \Sigma, \\ \sigma_e(\mathbf{u}_{\Sigma e})\mathbf{n} = \mathbf{0} &\qquad \text{on } \Gamma_e, \end{aligned} \tag{17}$$

and $\mathbf{v}_{\Sigma e}$ is the same for an imposed Dirichlet boundary condition on Σ equal to $\mathbf{v}$. Note that both $\mathbf{u}_{\Sigma e}$ and $\mathbf{v}_{\Sigma e}$ are frequency-dependent. As a consequence of the Betti-Maxwell reciprocity theorem, these bilinear forms are symmetric but not hermitian since damping or radiation conditions are accounted for.

2.5. Numerical considerations

Summing up the previous sections, two problems are left to be approximated:

- the construction of the dynamic stiffness operator and the equivalent forces, by solution of equations (17) for a set of interface displacements $\mathbf{u}_\Sigma$;
- the solution of the variational problem (12), where the influence of the environment is considered through the dynamic stiffness operator and the equivalent body forces.

The approximation and solution of these two problems is usually obtained by means of discretization techniques like the FEM and the boundary element method (BEM) [49, 50, 51]. These methods are based either on domain or boundary discretization, and the governing equations are written as a system of algebraic equations with respect to space and frequency (or time). The choice between the FEM and the BEM depends upon the type of the problem. For bounded domains (*e.g.* the structure) the FEM is a well-established and versatile procedure. In problems involving unbounded domains (*e.g.* the environment), the FEM requires a special handling, for instance through the use of absorbing boundary conditions, perfectly matched layers or infinite elements [52, 53, 54, 55], in order to avoid spurious wave reflections where the mesh is truncated. However the computational cost of the FEM in the context of 3D unbounded domains had been limiting its use for many years and other numerical techniques have been proposed and will be briefly reviewed here. We will come back to the FEM when drawing perspectives in the conclusion.

Apart from several analytical solutions related to ideal problems, the so-called Scaled Finite Element method has also been proposed in the context of Soil-Structure Interaction [56, 57] but has hardly been used in other contexts. The BEM is well known as a powerful procedure for modeling unbounded domains since the radiation condition is satisfied automatically by the Green's functions. In addition, the dimension of the model is reduced by one in the BEM analysis, resulting in a much simpler mesh generation and smaller meshes, specially for three-dimensional problems. Note, however, that although the resulting matrices are much smaller, they are generally fully populated, which hinders their inversion. This drawback has been overcome this last decade by the use of iterative solvers and multipole expansions [58, 59, 60], numerical computation of more complex Green's kernels accounting for heterogeneous domains such a layered soils [61, 62, 63, 64, 65, 11] and accounting for spurious resonance [66] based on pioneering works originally proposed in homogenous acoustics [67, 68] and bulk FEM-BEM coupling [51, 69].

Hence, because of the specificities of the FE et BE methods, many authors have chosen to model the environment using the BEM, and the structure with the FEM, possibly encompassing the part of the environment close to the structure, where non-linear phenomena take place.

When considering the FEM for the discretization, a finite set $\{\phi_k(\mathbf{x})\}_{1\leq k\leq n}$ of piecewise polynomial functions is defined over a mesh defined over the domain Ω. The $n\times n$ stiffness and mass matrices are constructed, for $1 \leq i, j \leq n$, by $\mathrm{K}_{ij} = \mathcal{K}(\phi_j, \phi_i)$ and $\mathrm{M}_{ij} = \mathcal{M}(\phi_j, \phi_i)$ and the loading vector on the structure is constructed, for $1 \leq i \leq n$, by $\mathrm{f}_i = \mathcal{L}(\phi_i)$. When considering the BEM, one uses a finite set $\{\psi_k(\mathbf{x})\}_{1\leq k\leq m}$ of piecewise polynomial functions defined over a mesh defined only over the interface Σ, or over $\Sigma \cup \Gamma_e$. The $m \times m$ dynamic stiffness matrix is then defined as $\mathrm{K}_{e,ij}(\omega) = \mathcal{K}_e(\omega; \psi_j, \psi_i)$, and the equivalent loading vector is defined as $\mathrm{f}_{e,i}(\omega) = \mathcal{L}_e(\omega; \psi_i)$. Including a linear viscous $n \times n$ damping matrix $\mathbf{D}$ leads to the final linear system:

$$\left\{\mathbf{K} + \mathbf{K}_e(\omega) + i\omega\mathbf{D} - \omega^2\mathbf{M}\right\}\mathbf{q}(\omega) = \mathbf{f}_e(\omega) + \mathbf{f}(\omega) \tag{18}$$

to be solved for the $n \times 1$ vector of generalized Degrees of Freedom $\mathbf{q}$ at all frequencies ω or Laplace parameter $s = i\omega$. In this equation, the matrix $\mathbf{K}_e(\omega)$ is only considered over degrees of freedom that are common to the sets $\{\phi_k(\mathbf{x})\}_{1\leq k\leq n}$ and $\{\psi_k(\mathbf{x})\}_{1\leq k\leq m}$. It is then filled by zeros to fit the required $n \times n$ size wherever the degrees of freedom are not common to both sets.

Finally note that reduced bases for both the inner displacement fields ϕ_k (for instance the eigenmodes on a rigid interface) and the interfaces fields ψ_k (rigid body modes and a set of deformation modes) can be sought in order to drastically reduced the size of the coupled system (18) (see [70, 71] for reduction techniques and related component mode analyses).

2.6. Limitations and extensions

Up to now, only standard material has been reported to set up important features and report on limitations to be lifted. These features are summarized in the frequency domain system equation (18). Provided that the structure and its interface with the environment are bounded and assuming a known linear visco-elastic behavior the main result is that an approximate solution for the interaction problem can be sought in a proper functional space of finite energy fields in a bounded frequency range. The main limitation of the proposed frequency domain approach is that it cannot account for non-linear phenomena even if localized within the structure. A time domain version of equation (18) has to be sought with a proper definition and numerical treatment of the related dynamic impedance $\mathbf{K}_e$. The account for uncertainties both on the loads and on the physical properties of the environment is the second issue addressed in section 4. The extensions to unbounded and multiple interfaces addressed in sections 5 and 6 are based on dual formulations.

Dual and mixt formulations. The primal formulation described up to now is often favored as straightforwardly compatible with standard FEM. It also leads to reduced systems when the displacement field on the interface can be expanded on a reduced basis such as rigid body modes. However from the environment point of view, the boundary value problem (17) is a bit artificial and some would find more physically sound to apply tractions τ on the interface:

$$\begin{aligned} \mathbf{Div}\sigma_e(\mathbf{u}_{\Sigma e}) &= -\rho_e\omega^2\mathbf{u}_{\Sigma e} && \text{in } \Omega_e, \\ \sigma_e(\mathbf{u}_{\Sigma e})\mathbf{n} &= \tau && \text{on } \Sigma, \\ \sigma_e(\mathbf{u}_{\Sigma e})\mathbf{n} &= \mathbf{0} && \text{on } \Gamma_e. \end{aligned} \tag{19}$$

Since $\mathbf{u}_{\Sigma e}$ depends linearly on τ, equation (19) defines the compliance operator:

$$\mathbf{u}_{\Sigma e}(\mathbf{x}) = \mathcal{U}_e(\tau)(\mathbf{x}) = \int_\Sigma \boldsymbol{U}_e^G(\mathbf{x}, \mathbf{x}', \omega)\tau(\mathbf{x}')\mathrm{d}S, \qquad \mathbf{x} \in \Sigma, \tag{20}$$

with $\boldsymbol{U}_e^G(\mathbf{x}, \mathbf{x}', \omega)$ the formal Green's kernel of Ω_e for which no closed-form solution is usually available. Solution $\mathbf{u}_e$ then reads:

$$\mathbf{u}_e = \mathbf{u}^{\mathrm{inc}} + \mathcal{U}_e(\tau - \tau^{\mathrm{inc}}) \tag{21}$$

where $\tau^{\mathrm{inc}} = \sigma_e(\mathbf{u}^{\mathrm{inc}})\mathbf{n}$ is the traction field induced on Σ by the incident field. This term only vanish when this incident field satisfies free surface boundary conditions on the interface.

The same approach applied on the structure using a variational framework yields:

$$\left\{\mathcal{K} - \omega^2 \mathcal{M}\right\}(\mathbf{u}, \mathbf{v}) = \mathcal{L}(\mathbf{v}) - \int_{\Sigma} \tau \cdot \mathbf{v} \mathrm{d}S. \tag{22}$$

Defining $\mathbf{u}_o$ as the solution of this equation for $\tau = 0$, one shows that $\mathbf{u} - \mathbf{u}_o$ depends linearly on τ through a compliance operator $\mathcal{U}$:

$$\mathbf{u} = \mathbf{u}_o - \mathcal{U}(\tau). \tag{23}$$

Finally the kinematic coupling along the interface Σ yields:

$$(\mathcal{U} + \mathcal{U}_e)(\tau) = \mathbf{u}_o - \mathbf{u}^{\mathrm{inc}} + \mathcal{U}_e(\tau^{\mathrm{inc}}) \tag{24}$$

which could also be written in a weak sense for all traction fields τ' on Σ. These dual formulations will be used to deal with moving and multiple interfaces.

3. Time-domain formulations for non-linear structures

The aim of this section is to review recent developments to account for the non-linear behavior of the structure, within the framework proposed in section 2. Although the behavior of the structure is now considered non-linear, that of the environment will still be assumed linear (possibly using equivalent linear approaches [72, 73]). Since non-linear dynamic solvers are now widely available [47], the key issue addressed here is the influence of a linear dynamic model of the environment within a time marching scheme. This can be done either through a full FEM, a coupled BEM-FEM in the time domain, or a mixed time-frequency approach.

The first popular set of methods consists in incorporating a large part of the environment in the FE model [74, 75] and using absorbing boundary conditions [76] for the remaining part of the environment. This approach is really only a FEM approach, not considering the coupling in any particular manner. Many investigations have dealt with the stability of FEM schemes, and some unconditionally stable methods have been proposed [77].

The coupling with a time-domain BEM is another classical option [50, 51]. So far, the BEM/FEM coupling procedure for time-domain analysis has been studied by a restricted number of researchers, aiming at developing robust algorithms where the BEM is used to model the unbounded medium. Beskos and coworkers [78, 79] were among the first to investigate two- and three-dimensional flexible foundations, which were modeled by finite elements and coupled to the elastic half-space represented by BEM. Von Estorff [80] developed a general FEM/BEM coupling method for the soil-structure interaction in the time domain. Finally, a work by the same author [81] appears to be the most general and also considers the non-linear behavior of the FEM domain. The stability of BEM schemes has been investigated recently, and some methods have been used to improve it [82, 83]. However, the performance of these methods in coupling BEM/FEM procedures needs to be improved further, as none is unconditionally stable. Note that, even when considering two stable algorithms, if one of them is not unconditionally stable, then the coupled algorithm is generally unstable [84].

Finally, hybrid approaches coupling a frequency (or a Laplace) BEM with a time domain FEM scheme can also be found in the literature. In this way, the main advantages of both methods can be retained and their respective disadvantages eliminated. Moreover, as said before, if the coupling procedure is entirely formulated in the time domain and the effects of the unbounded soil are represented by boundary conditions, an extension of the algorithm to non-linear problems is possible. Using the framework proposed in section 2, these methods reduce to giving a time-domain version of the boundary impedance matrix $\mathbf{K}_e$ and the interaction force vector $\mathbf{f}_e$ originally computed in the frequency or Laplace domain. Two major requirements should be satisfied: causality and stability. Pioneer works in this field have been proposed in [85, 86, 87] in the field of Soil-Structure Interaction. Basically, two types of methods are proposed. The first one consists in finding an approximation of the impedance on a basis of known functions satisfying the two requirements. It can directly lead to an explicit discrete differential model in the time domain. The second class of methods aims at giving a time domain approximation of the related often ill-conditioned convolution operator. Recent developments along these two lines are reviewed in the following subsections.

3.1. Reduced time-domain models for the environment

Whatever the numerical method used to compute the impedance matrix of the linear environment (*e.g.* FEM, BEM, analytical solutions, expansion on cylindrical or spherical harmonics), this matrix is frequency-dependent and is assumed to satisfy the Kramers-Kronig causality condition [88, 89]. From the theoretical point of view it can be expanded on a basis of matrix-valued Hardy functions [90] and thus can be approximated as a matrix-valued polynomial of ω as:

$$\mathbf{K}_e(\omega) = \frac{\mathbf{N}(\omega)}{q(\omega)} \tag{25}$$

The conversion to the time domain can be performed using model identification techniques or the hidden state variables method [91, 92, 93]. In both cases, the rational approximate can be obtained by a wide range of techniques [94, 95, 96, 97]. In the hidden variable method, the rational approximation is reduced to a polynomial of second order by an identification process that is performed without approximation. Then, the impedance matrix $\mathbf{K}_e$ is written as the dynamic condensation of larger mass, stiffness and damping matrices $(\mathbf{M}_e^*, \mathbf{K}_e^*, \mathbf{C}_e^*)$ considering the physical unknowns $\mathbf{q}$ on the interface Σ as well as additional so-called hidden variables $\mathbf{q}_h$:

$$\mathbf{M}_e^* = \begin{bmatrix} \mathbf{M}_e^\Sigma & \mathbf{M}_e^c \\ \mathbf{M}_e^{cT} & \mathbf{M}_e^h \end{bmatrix}, \quad \mathbf{K}_e^* = \begin{bmatrix} \mathbf{K}_e^\Sigma & \mathbf{K}_e^c \\ \mathbf{K}_e^{cT} & \mathbf{K}_e^h \end{bmatrix}, \quad \mathbf{C}_e^* = \begin{bmatrix} \mathbf{C}_e^\Sigma & \mathbf{C}_e^c \\ \mathbf{C}_e^{cT} & \mathbf{C}_e^h \end{bmatrix}. \tag{26}$$

The impedance matrix is then written as the following Schur complement:

$$\mathbf{K}_e(\omega) = (-\omega^2\mathbf{M}_e^\Sigma + i\omega\mathbf{C}_e^\Sigma + \mathbf{K}_e^\Sigma) - (-\omega^2\mathbf{M}_e^c + i\omega\mathbf{C}_e^c + \mathbf{K}_e^c)^T(-\omega^2\mathbf{M}_e^h + i\omega\mathbf{C}_e^h + \mathbf{K}_e^h)^{-1}(-\omega^2\mathbf{M}_e^c + i\omega\mathbf{C}_e^c + \mathbf{K}_e^c), \tag{27}$$

These additional hidden variables $\mathbf{q}_h$ are then added to the original non-linear time domain system, while the dynamic behavior of the linear environment is written in the form of a standard second order differential system. This method shows some stability problems originating from the non-positive-definite property of some matrices. Yet it has been successfully used to develop stochastic models of impedance matrices, replacing matrices $(\mathbf{M}_e^*, \mathbf{K}_e^*, \mathbf{C}_e^*)$ by random matrices [98, 99] or to derive reduced models for complex systems [13]. The stability issue is treated in [100] for the case of a single input-single output (SISO) relationship between force and displacement in the physical or transformed space. The stability is guaranteed *a priori* by enforcing the zeros and poles to lie on the lower-half complex ω plane. The parameter identification of the rational approximation is further developed by directly solving the nonlinear least-squares fitting problem using the hybrid genetic-simplex optimization algorithm, in which the proposed stability condition is enforced by the penalty function method.

3.2. Convolution techniques

Based on section 2, the time domain discretized non-linear system for the time-dependent DOF vector $\mathbf{q}(t)$ reads:

$$\mathbf{Kq} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{M}\ddot{\mathbf{q}} = \mathbf{f}_\Sigma(\mathbf{q}, t) + \mathbf{f}_{NL}(\mathbf{q}, t) \tag{28}$$

where $\mathbf{f}_{NL}$ stands for the non-linear generalized force vector in the structure and $\mathbf{f}_\Sigma$ is the transient generalized interaction force vector applied on the structure by the environment. It is defined formally as the convolution product between $\mathbf{q}$ and the so-called time impulse displacement matrix $\mathbf{K}_e(t)$, the formal inverse Fourier or Laplace transform of $\mathbf{K}_e(\omega)$:

$$\mathbf{f}_\Sigma(t) = \mathbf{f}_e(t) - \int_0^t \mathbf{K}_e(t-\tau)\mathbf{q}(\tau)\mathrm{d}\tau \tag{29}$$

Hence, convolution methods are hybrid approaches that are mainly based on the recursive discrete-filter or the Inverse Fast Fourier Transform (IFFT). In order to compute $\mathbf{f}_\Sigma(t)$ in the time domain, the dynamic time impulse displacement matrix or time domaine stiffness matrix of the environment $\mathbf{K}_e(t)$ must be computed. Unfortunately, this often leads to unstable results [101], especially when combined with IFFT. Indeed $\mathbf{K}_e(\omega)$ does not converge towards any finite limit at high frequencies, which leads to uncontrolled truncation errors. On the contrary, its inverse $\mathbf{F}_e(\omega) = \mathbf{K}_e^{-1}(\omega)$, known as the dynamic flexibility matrix, does converge to a finite value for large frequencies. Hence, flexibility formulations have been preferred for many years [87, 102]. When coupled with a time-marching scheme, it leads to:

$$\mathbf{f}_\Sigma^n = \mathbf{f}_e^n - \mathbf{K}_e^0\mathbf{q}_n + \sum_{k=0}^{n-1} \mathbf{K}_e^0\mathbf{F}_e^{n-k}(\mathbf{f}_e^k - \mathbf{f}_\Sigma^k) \tag{30}$$

where $\mathbf{F}_e^k$ can be accurately computed using FFT algorithms and where the instantaneous stiffness matrix $\mathbf{K}_e^0$ is the inverse of $\mathbf{F}_e^0$. It is worth to mention that this algorithm requires the storage of both $\mathbf{q}$ and $\mathbf{f}_\Sigma$ at all time steps. It is also pointed out in [103] that the direct integration scheme is often unstable. The θ-method and the introduction of numerical damping in the Newmark scheme have been used to stabilize the system, with no guarantee however neither on the stability nor on the quality of the solution [104].

Finally, we ought to mention some preliminary work on FEM-BEM coupling [105, 106] in the time domain using the Convolution Quadrature Method [107]. It has been shown in [108, 109] that a direct stiffness formulation can be used, along with a sampling of the stiffness matrix in the upper complex ω plane.

4. Uncertainty on the incident field and environment properties

The solution of the interaction equations (12) or (18) depends:

- linearly on the incident displacement and traction fields $\mathbf{u}^{\text{inc}}$ $\sigma(\mathbf{u}^{\text{inc}})\mathbf{n}$ on the interface Σ and equivalent force vectors $\mathbf{f}_e$ and $\mathbf{t}_e$, which all are space-time fields with bounded support in space;
- non linearly on the mechanical properties of the environment $\mathbf{m}_e = (\lambda_e, \mu_e, \rho_e)$, that are generally time independent fields with either bounded or unbounded supports.

In practice, the effective values of these fields of parameters are subject to large uncertainties. In particular, the incident field $\mathbf{u}^{\text{inc}}$ and the mechanical parameters of the environment are usually poorly known and controlled. These uncertainties can be accounted for by modeling these fields as second order stochastic fields. There might be uncertainties associated also to the loads and/or mechanical parameters of the structure and this case (described in another paper of this special issue [36]) will not be addressed here. Hence we will concentrate on the impact of an uncertain environment on the behavior of the structure, and on the methods that address this issue. It should be noted that probabilistic analyses are usually much more involved than their deterministic counterparts, both in terms of data required to parameterize the models [110, 75], and in terms of computational cost. Hence, they are mainly restricted to the most sensitive projects [111, 112, 113].

However, for all structures for which the interface with the environment spans large distances (with respect to some characteristic length of the fluctuations of the properties and incident field), the consideration of the spatial variability of the properties of the environment (and of the incident field) is essential for a rational design. In civil engineering again, this is the case of structures supported on extended footings, as well as for multiply supported structures such as bridges.

4.1. Stochastic Model for the incident field

Accounting for the effect of the uncertainties in the incident field $\mathbf{u}^{\text{inc}}$ on the structural response is easier than accounting for the variability of the properties of the environment, at least when linear models are considered. Indeed, the linear filtering theory then applies, leading to effective engineering techniques, while for non-linear behavior, heavy simulations are often required [114]. In both cases, one of the main difficulties consists in modeling and simulating these uncertainties, with a particular emphasis on the issue of time-dependency. Indeed, these incident fields are often non-stationary (*e.g.* earthquake ground motions), hence calling for a particular treatment.

The time dependency is classically accounted for by assuming that the field is either stationary with respect to time, a deterministic modulation of a stationary process, or an evolutionary stochastic process [115]. These processes $\mathbf{u}^{\text{inc}}(\mathbf{x}, t)$ read:

$$\mathbf{u}^{\text{inc}}(\mathbf{x}, t) = \int_{-\infty}^{+\infty} \mathbf{H}(t, \omega) e^{i\omega t} \hat{\mathbf{u}}_o^{\text{inc}}(\mathbf{x}, \omega) \mathrm{d}\omega \tag{31}$$

with $\hat{\mathbf{u}}_o^{\text{inc}}(\mathbf{x}, \omega)$ a correlation-stationary process (the Fourier Transform of a stationary process with power spectral density $\mathbf{S}_o(\mathbf{x}, \omega)$), and $\mathbf{H}(t, \omega)$ a complex valued time-frequency modulation tensor. This decomposition is not unique even when $\hat{\mathbf{u}}_o^{\text{inc}}(\mathbf{x}, \omega)$ is assumed to be a gaussian white noise (see [116] for an optimal choice in terms of underspread evolutionary spectrum using the Karhunen-Loeve expansion of the correlation operator), but equation (31) conveniently accounts for the case of a deterministic modulation function ($\partial_\omega \mathbf{H} = 0$) and also provides a simple way of sampling the stationary process $\hat{\mathbf{u}}_o^{\text{inc}}(\mathbf{x}, \omega)$ using independent random amplitudes oand phases in the frequency domain. It is also worth noticing that, in addition, since these random amplitudes in the spectral representation of $\hat{\mathbf{u}}_o^{\mathrm{inc}}(\mathbf{x},\omega)$ model the aleatoric uncertainty of the incident field, $\mathbf{S}_o$ and $\mathbf{H}$ can depend on additional random parameters characterizing its epistemic uncertainty. Using this technique, a set a synthetic seismograms fitting given constraints or experimental data basis (see for instance [117] for seismic signals) and time-dependent power spectral density of the structural response can be obtained (see [118] for the case of a deterministic modulation of a stationary signal).

Yet, equation (31) cannot account for non Gaussian non-stationary processes with targeted marginal distributions since it only consists in an extension of the spectral representation. Other modeling and simulation techniques for non-Gaussian Fields with targeted marginal distribution, moments and correlation functions have been proposed based on Polynomial Chaos decomposition [119, 120] or more recently on the information theory [121].

Up to now only time dependency has been discussed. Considering space-dependency, the deterministic wave passage effect and spatial variability have to be taken into account. It is customary to model the resulting stationary space-frequency random field $\hat{\mathbf{u}}_o^{\mathrm{inc}}(\mathbf{x},\omega)$ using a frequency dependent coherency hermitian matrix $\mathbf{\Gamma}(\mathbf{x},\mathbf{x}',\omega)$. The cross correlation between two points $\mathbf{x}$ and $\mathbf{x}'$ then reads:

$$E\left(\hat{\mathbf{u}}_o^{\mathrm{inc}}(\mathbf{x},\omega)\hat{\mathbf{u}}_o^{\mathrm{inc}}(\mathbf{x}',\omega)^H\right) = \mathbf{S}_o^{1/2}(\mathbf{x},\omega)\mathbf{\Gamma}(\mathbf{x},\mathbf{x}',\omega)(\mathbf{S}_o^{1/2}(\mathbf{x}',\omega))^H, \tag{32}$$

where E denotes the ensemble average and ^{H} the conjugate transposed. When considered along a horizontal free surface, the coherency matrix is often decomposed into a lagged coherency depending on the dimensionless parameter $\frac{\omega}{c}\|\mathbf{x}-\mathbf{x}'\|$, with c a characteristic velocity in the environment, and a wave passage effect:

$$\mathbf{\Gamma}(\mathbf{x},\mathbf{x}',\omega) = \left|\mathbf{\Gamma}\left(\frac{\omega\|\mathbf{x}-\mathbf{x}'\|}{c}\right)\right| e^{i\frac{\omega}{c_a}\mathbf{d}_a\cdot(\mathbf{x}-\mathbf{x}')},$$

with $\mathbf{k}_a = \frac{\omega}{c_a}\mathbf{d}_a$ an apparent wavenumber along the free surface corresponding to waves propagating in direction $\mathbf{d}_a$ with a velocity equal to c_a. Hence, in that case, $\hat{\mathbf{u}}_o^{\mathrm{inc}}(\mathbf{x},\omega)$ can be further decomposed as:

$$\hat{\mathbf{u}}_o^{\mathrm{inc}}(\mathbf{x},\omega) = \int_{\mathbb{R}^2} \mathbf{S}_o^{1/2}(\mathbf{x},\omega)\widetilde{\mathbf{\Gamma}}^{1/2}(\mathbf{k},\omega)e^{i\mathbf{k}\cdot\mathbf{x}}\widetilde{\mathbf{W}}(\mathbf{k},\omega)d\mathbf{k} \tag{33}$$

where $\widetilde{\mathbf{\Gamma}}$ is the Fourier Transform of $\mathbf{\Gamma}(\mathbf{x}-\mathbf{x}',0,\omega)$ with respect to its first argument and $\widetilde{\mathbf{W}}$ is the Fourier transform of a space-time white noise satisfying:

$$E\left(\widetilde{\mathbf{W}}(\mathbf{k},\omega)\widetilde{\mathbf{W}}(\mathbf{k}',\omega')\right) = \delta(\mathbf{k},\mathbf{k}')\delta(\omega-\omega')$$

The major drawback of this approach is the underlying assumption of stationary statistically homogeneous process for $\hat{\mathbf{u}}_o^{\mathrm{inc}}(\mathbf{x},\omega)$. The reader is referred to [122] for an extension of coherency models to non-stationary signals. It is worth noticing that due to the dependency of $\mathbf{S}_o$ with respect to $\mathbf{x}$, the random field in (33) is not statistically homogeneous but is a space-modulated homogeneous field. However this term can be incorporated in $\mathbf{H}$ in equation (31) leading to a statistically homogeneous case. Yet a dependency of $\mathbf{S}_o$ on the wavenumber $\mathbf{k}$ would lead to a more general non statistically homogeneous coherency model.

In the earthquake engineering community, many different stationary coherency models have been proposed [123], although no general agreement seems to exist as to which is the most appropriate. Each site - and each event - seems to require a specific model. Even the numerical treatment performed to evaluate the coherency from the raw data seems to have an impact. The first models that were developed [124, 125] were purely empirical and only intended to fit data recorded at seismographs arrays. Other semi-empirical models postulate a structure of the coherency function from the study of analytical models (*e.g.* a very widely used model is based on theoretical results of propagation of shear waves in a random medium [126]) and fit the parameters from recorded data. Finally, the finite size of the source can be modeled, summing up over different arrival angles for waves impinging on a structure [127].

As anticipated, the only information available on the incident field is recorded at the free surface Γ_e [128], whereas it is needed in (16) along the interface with the structure Σ together with the related traction field. Since the incident field in the environment needs to verify the system of equations (3), a stochastic deconvolution [129] has to be performed.

This deconvolution can be applied on the stationary field $\hat{\mathbf{u}}_o^{\text{inc}}(\mathbf{x},\omega)$ under the two following assumptions: i) horizontal statistical homogeneity ($\mathbf{S}_o^{1/2}(\mathbf{x},\omega) = \mathbf{S}_o^{1/2}(\omega)$), ii) layered media and horizontal free surface. Indeed, let $\widetilde{\mathbf{H}}_u(\mathbf{k},\omega,z)$ be the transfer matrix between the displacement at depth z and the displacement at the free surface for a given horizontal wavenumber $\mathbf{k}$ and $\widetilde{\mathbf{H}}_t(\mathbf{k},\omega,z) = \sigma_e(\widetilde{\mathbf{H}}_u e^{i\mathbf{k}\cdot\mathbf{x}})\mathbf{n}e^{-i\mathbf{k}\cdot\mathbf{x}}$ be the corresponding transfer matrix between the traction vector at depth z and the displacement at the free surface. Both can be easily computed using either the reflection-transmission techniques [130] or the stiffness matrix method [131]. Hence, starting from (33) on the free surface, the random displacement and traction fields at any point $\mathbf{x} = (\mathbf{x}_s, z)$ on the interface Σ reads:

$$\hat{\mathbf{b}}_o^{\text{inc}}(\mathbf{x},\omega) = \begin{pmatrix} \hat{\mathbf{u}}_o^{\text{inc}}(\mathbf{x},\omega) \\ \sigma(\hat{\mathbf{u}}_o^{\text{inc}}(\mathbf{x},\omega))\mathbf{n} \end{pmatrix} = \int_{\mathbb{R}^2} \begin{pmatrix} \widetilde{\mathbf{H}}_u(\mathbf{k},\omega,z) \\ \widetilde{\mathbf{H}}_t(\mathbf{k},\omega,z) \end{pmatrix} \mathbf{S}_o^{1/2}(\omega)\widetilde{\mathbf{\Gamma}}^{1/2}(\mathbf{k},\omega)e^{i\mathbf{k}\cdot\mathbf{x}_s}\widetilde{\mathbf{W}}(\mathbf{k},\omega)\mathrm{d}\mathbf{k} \tag{34}$$

This equation yields a well-defined expression for the equivalent force vector in (18), using equation (16) in the stationary case, when one considers samples of the random field $\mathbf{W}$. A non stationary and non linear expression can be obtained taking the inverse Fourier transform of (16) combined with equation (31). However this simulation approach could be highly computationally expensive and reduction techniques are appealing. Hence, one can compute the truncated Kahrunen-Loeve expansion of the covariance of $\hat{\mathbf{b}}_o^{\text{inc}}(\mathbf{x},\omega)$:

$$\mathbf{b}_o^{\text{inc}}(\mathbf{x},\omega) \approx \sum_{k=1}^{N_k} \lambda_k^{\text{inc}}(\omega)\theta_k(\omega)\mathbf{b}_k^{\text{inc}}(\mathbf{x},\omega), \tag{35}$$

where $(\mathbf{b}_k^{\text{inc}}(\mathbf{x},\omega), \lambda_k^{\text{inc}}(\omega))$ are the eigenmodes and eigenvalues of the covariance operator of $\mathbf{b}_o^{\text{inc}}$ on Σ with kernel:

$$\boldsymbol{C}_b^{\text{inc}}(\mathbf{x},\mathbf{x}',\omega) = \int_{\mathbb{R}^2} \begin{pmatrix} \widetilde{\mathbf{H}}_u(\mathbf{k},\omega,z) \\ \widetilde{\mathbf{H}}_t(\mathbf{k},\omega,z) \end{pmatrix} \mathbf{S}_o^{1/2}(\omega)\widetilde{\mathbf{\Gamma}}^{1/2}(\mathbf{k},\omega) \begin{pmatrix} \widetilde{\mathbf{H}}_u(\mathbf{k},\omega,z') & \widetilde{\mathbf{H}}_t(\mathbf{k},\omega,z') \end{pmatrix} e^{i\mathbf{k}\cdot(\mathbf{x}_s-\mathbf{x}_s')}\mathrm{d}\mathbf{k},$$

and $\theta_k(\omega)$ are uncorrelated random variables given by:

$$\theta_k(\omega) = \frac{1}{\lambda_k^{\text{inc}}(\omega)} \int_\Sigma \mathbf{b}_k^{\text{inc}}(\mathbf{x},\omega)\cdot\mathbf{b}_o^{\text{inc}}(\mathbf{x},\omega)\mathrm{d}S.$$

This leads in the stationary case to solving N_k deterministic systems (18) for $\mathbf{q}_k$ with deterministic equivalent forces:

$$[\mathbf{f}_k]_j = \int_\Sigma \left((\sigma_e(\mathbf{v}_{\Sigma e})\mathbf{n})\cdot\mathbf{u}_k^{\text{inc}} - \mathbf{t}_k^{\text{inc}}\cdot\mathbf{v}_{\Sigma e}\right)\mathrm{d}S,$$

with $\mathbf{v}_{\Sigma e} = \phi_j$ on Σ and $\mathbf{b}_k^{\text{inc}} = (\mathbf{u}_k^{\text{inc}}\ \mathbf{t}_k^{\text{inc}})^T$. Spectral analysis and simulation can then be conducted based on the knowledge of the $\mathbf{q}_k$ since, due to linearity, $\mathbf{q}(\omega) \approx \sum_{k=1}^{N_k}\lambda_k^{\text{inc}}(\omega)\theta_k(\omega)\mathbf{q}_k(\omega)$. Based on (31), expression for the non-stationary case can easily be obtained, as shown in [118] for surface foundations and modulated signals.

Approximate methods have often been proposed to simplify this exact but rather complex form. In particular, the kinematics of the interface are often simplified, and the incident field strongly and inappropriately filtered [125, 132, 133, 127]. A more complex integral representation, based on Bycroft [134], is considered in Luco and Wong [126], Luco and Mita [135], Sarkani et al. [136], but a simplification in the form of the interface tractions is added to yield a simpler form.

4.2. *Influence of stochastic environment properties*

The case where the properties of the environment are random is more intricate. If the fluctuations of the properties are small compared to the mean, then a perturbation approach can be applied. When this is not the case, three alternative techniques can be applied, directly extending the approaches followed for deterministic modeling: i) stochastic FEM (for variability in a bounded subdomain); ii) stochastic BEM; and iii) stochastic model of the impedance.

In the first approach, it is assumed that the properties fluctuate only in a bounded region near the structure. Although questionable, this assumption is justified since, even if large, the uncertainties far from the structure may modify the incident field but have a limited impact on the structural response itself. This influence can then be accounted for in an average sense using *effective* elastic properties far from structure instead of *mean* properties. Stochastic FEM representations can then be considered. The main interest of that method is the possibility to vary several properties of the soil at the same time (*e.g.* in a soil-structure interaction problem, in [111], the shear modulus at low strains, the shear modulus-shear strain curve, the structural damping, the structural stiffness, and the incident field are considered simultaneously random), or even to model uncertain layers within the environment [137].

Unfortunately, the Stochastic FEM inherits the deficiencies of the deterministic FEM in modeling unbounded domains. Hence, attempts have been made at coupling a deterministic unbounded environment, that can be modeled for example using a deterministic BEM, and a bounded volume, where the uncertainty and variability in the parameters are considered more important than in the rest of the environment [69, 138, 139, 140]. Alternatively, it is possible to model the fluctuating properties within a bounded but very large domain, but the question of finding adequate absorbing boundary conditions for these heterogeneous media is still open [75].

Another approach consists in considering variability of the random properties of the environment within the BEM. Although attractive, and besides initial work using perturbation techniques, this approach has had little influence due to the difficulty to derive Green's functions for random media [141, 142, 143]. The variability of the interface geometry has also been considered [144] in the context of the BEM.

Finally, some authors have attempted to model directly the influence of the uncertain properties of the environment through a randomization of the boundary impedance. The first attempts have aimed at extending the use of lumped-parameter models and Winkler spring models to the stochastic domain. In particular, the coefficients of these simplified models are randomized, thus leading to random boundary impedances [145, 146, 147, 148, 149], modeled very cost-effectively. The main inconvenient of this approach, is that it is impossible to relate the fluctuations of the physical properties of the environment to the fluctuations of the coefficients, because the latter are usually identified by regression. The link between physical properties of the environment and the coefficients of the model is explicit in some cases, but this is restricted to very simple simple models of the environment (*e.g.* a 1D column model of a soil in [150], with use of the perturbation technique).

A more general approach has been developed in [98, 99]. It consists in extending the description of the boundary impedance matrix seen in section 3.1 and using the nonparametric approach of uncertainties [36]. The use of the nonparametric method renders the interest of the link between model of the properties and model of the boundary impedance irrelevant (although a comparison with a stochastic FEM is presented in [151]), and the random model of the boundary impedance can be checked to be appropriate mathematically and is adequately parameterized. The application of this method to the design of a nuclear plant is presented in [152].

5. Interaction through an unbounded interface

A basic assumption in the above mentioned deterministic and stochastic techniques is that the structure and the fluctuating heterogeneous soil region remain bounded. Thus for very long structures such as tunnels, boreholes, tracks, roads, sheet piles, dykes or hills, some additional developments have to be proposed.

In current practice, these very long structures are assumed to be translationally invariant, and thus only two-dimensional models are considered. However, the incident fields are not translationally invariant. The importance of inclined incident fields with respect to the invariant axis has in particular been recognized by geophysicists [153, 154]. In earthquake engineering, these inclined incident fields have a strong impact on the behavior of connected structures and networks (*e.g.* bridges, pipelines), because these waves generate differential displacements in the longitudinal direction [155]. The finite correlation length of recorded seismic motion indicates that these inclined incident waves always exist in practice. For traffic-induced vibrations or borehole geophysics, the load can be modeled as a fixed or moving source, and the problem is then far from being translationally invariant. The extension of the preceding numerical techniques to translationally invariant geometries with non translationally invariant loads has been described in particular in [156, 157, 158, 159, 160].

This technique is usually referred to as a two-and-a-half-dimensional approach and has been used in the past to study problems such as the seismic response of tunnels [161, 162, 163] and the amplification of ground motion in arbitrarily shaped cylindrical alluvial valleys [164]. More recently, this technique has been used by many researchers to study train-track interaction [165, 28] and ground-borne vibrations due to road and railway traffic [26, 166, 24, 167, 168]. An analytical solution of the two-and-a-half dimensional Green's function of a full space has been presented by Tadeu and Kausel [169]. This solution can be used in a two-and-a-half-dimensional BEM formulation. Coupled two-and-a-half dimensional FEM and BEM have been presented for railway traffic at grade and in tunnels [170, 171], as well as to study vibration mitigation by open and in-filled trenches [172]. Recently, a two-and-a-half dimensional coupled FEM-BEM based on the Green's functions of a horizontally layered halfspace has been presented [173] to avoid meshing of the free surface as required when using full space Green's functions.

Translational invariance of the geometrical and mechanical properties of the model is often a coarse approximation and for many applications a periodic assumption is better suited. The proposed numerical techniques have been extended to periodic models in [159] with stochastic loadings [158] or stochastic constitutive parameters [174].

The main purpose of this section is to show that, when the entire domain, or some parts of it, satisfy invariance properties, the numerical techniques proposed in section 2 can still be used on a reference cell, as long as the proper integral transform is applied on the unknown fields and on the loads. In particular, this means that restrictions on the invariance of the domain do not imply restrictions on the loads. Note that the procedure which is described below corresponds to the Fourier transform when time-dependent problems have been transformed to the frequency domain, using an argument of invariance with respect to time. Of course, these techniques can only be applied for linear problems.

5.1. General space-wavenumber transform

In order to give a general overview of these techniques, the general framework of infinite groups of isometries will be used. Let $\mathcal{T}$ be a group of isometries through which the domain and the parameters are left unchanged ($\mathcal{T} = \{\tau : \Omega_t \to \Omega_t / \tau(\Omega_t) = \Omega_t ; \tau(\mathbf{m}) = \mathbf{m}\}$). Hence, domain Ω_t can formally be decomposed as a product $\Omega_t = \widetilde{\Omega}_t \times \Omega_{\mathcal{T}}$, where $\widetilde{\Omega}_t$ is the generator (the cross section for 2.5D models or the generic cell for periodic ones) and $\Omega_{\mathcal{T}}$ is the set of indices of $\mathcal{T}$. For 2.5D models, $\Omega_{\mathcal{T}}$ is the set of coordinates along the invariant axis, namely the real line $\mathbb{R}$. For periodic models, it is the set of cell indices, namely $\mathbb{Z}$.

Using this framework, the expression of $\widetilde{\mathbf{u}}$, the integral transform of any field $\mathbf{u}$ with respect to the space variables belonging to $\Omega_{\mathcal{T}}$, is then given by:

$$\widetilde{\mathbf{u}}(\widetilde{\mathbf{x}}, \mathbf{k}) = \int_{\Omega_{\mathcal{T}}} \mathbf{u}(\widetilde{\mathbf{x}} + \mathbf{x}_{\mathcal{T}}) e^{i\langle \mathbf{k}, \mathbf{x}_{\mathcal{T}} \rangle} \mathrm{d}\mathbf{x}_{\mathcal{T}}, \qquad \widetilde{\mathbf{x}} \in \widetilde{\Omega}_t, \tag{36}$$

where $\mathbf{k}$ is the wavenumber belonging to $\Omega'_{\mathcal{T}}$, the dual of $\Omega_{\mathcal{T}}$ for the duality product $\langle ., . \rangle$. In particular, the Fourier transform of $\mathbf{u}$ with respect to p directions $\{\mathbf{e}_j\}_{1 \leq i \leq p < n}$:

$$\widetilde{\mathbf{u}}(\widetilde{\mathbf{x}}, \mathbf{k}) = \int_{\mathbb{R}^p} \mathbf{u}(\widetilde{\mathbf{x}} + x_1 \mathbf{e}_1 + ... x_p \mathbf{e}_p) e^{i(k_1 x_1 + ... k_p x_p)} \mathrm{d}x_1 ... \mathrm{d}x_p, \tag{37}$$

is obtained taking $\Omega_{\mathcal{T}} = \Omega'_{\mathcal{T}} = \mathbb{R}^p$,

$$\widetilde{\mathbf{x}} = \sum_{j=p+1}^{n} x_j \mathbf{e}_j, \qquad \mathbf{x}_{\mathcal{T}} = \sum_{j=1,p} x_j \mathbf{e}_j, \qquad \langle \mathbf{k}, \mathbf{x}_{\mathcal{T}} \rangle = \sum_{j=1,p} k_j x_j.$$

The Floquet transform [175] of $\mathbf{u}$ for a $\mathbf{l} = l\mathbf{e}_1$ periodic domain defined as:

$$\widetilde{\mathbf{u}}(\widetilde{\mathbf{x}}, \mathbf{k}) = \sum_{n \in \mathbb{Z}} \mathbf{u}(\widetilde{\mathbf{x}} + nl\mathbf{e}_1) e^{inkl} l, \tag{38}$$

is obtained taking $\widetilde{\Omega}_t = \{\widetilde{\mathbf{x}} \in \Omega / 0 \leq \tilde{x}_1 < l\}$, $\Omega_{\mathcal{T}} = \{nl\mathbf{e}_1, n \in \mathbb{Z}\}$, $\Omega'_{\mathcal{T}} = [0, \frac{2\pi}{l}]$, and $\langle \mathbf{k}, \mathbf{x}_{\mathcal{T}} \rangle = knl$ (note that the Floquet transform falls back to the Fourier transform with $nl = x$ when the period l tends to 0).

Most of these transformations are widely used in order to solve boundary value problems on symmetric domains, especially in the FEM [176], and their use for boundary integral equations is well known in physics [177]. In earthquake engineering and structural dynamics, the Fourier Transform has been associated with the BEM [154, 178], using the analytical Fourier transform of Green's function of an homogeneous elastic medium or that of a layered half-space [156, 157, 173]. Fourier series for axisymmetrical or non-axisymmetrical loads on axisymmetrical domains have been used for example in [179, 180], and in [181] within a substructuring context. The main difficulty here is the accurate computation of the axisymmetrical Green's function, which does not have a closed-form solution in elastodynamic problems. Axisymmetric Green's functions for a layered half-space are presented in [182, 183]. The Floquet Transform has been recently used to model periodic domains [184] and results given in [11, 30, 31, 32, 158, 159, 174] are summarized here as an example of integral transform used in connection with Domain Decomposition and boundary integral equations.

5.2. *Invariant operators*

The above mentioned integral transforms expand any field $\mathbf{u}$ defined on a periodic domain into a finite or infinite set of fields $\widetilde{\mathbf{u}}$ defined on the reference cell $\widetilde{\Omega}_t$, that can themselves be extended on the whole domain Ω using the invariance properties:

$$\widetilde{\mathbf{u}}(\widetilde{\mathbf{x}} + \mathbf{x}_{\mathcal{T}}) = e^{-i\langle \mathbf{k}, \mathbf{x}_{\mathcal{T}} \rangle} \widetilde{\mathbf{u}}(\widetilde{\mathbf{x}}). \tag{39}$$

Moreover, once the fields $\widetilde{\mathbf{u}}$ are known, an inverse transform can be defined in order to recover the original field $\mathbf{u}$:

$$\mathbf{u}(\widetilde{\mathbf{x}} + \mathbf{x}_{\mathcal{T}}) = \frac{1}{(2\pi)^p} \int_{\Omega'_{\mathcal{T}}} \widetilde{\mathbf{u}}(\widetilde{\mathbf{x}}, \mathbf{k}) e^{-i\langle \mathbf{k}, \mathbf{x}_{\mathcal{T}} \rangle} \mathrm{d}\mathbf{k}. \tag{40}$$

Then, using the linearity of the original equations, a boundary value problem for $\mathbf{u}$, on a periodic domain Ω_t and with applied loads $\mathbf{f}$, can be equivalently solved as boundary value problems for the $\widetilde{\mathbf{u}}$, on domain $\widetilde{\Omega}_t$ and with applied loads $\widetilde{\mathbf{f}}$. With the invariance property (39), the problem can then be restricted to $\widetilde{\Omega}_t$ provided that $\widetilde{\mathbf{u}}$ satisfies these invariance properties on the interfaces between cells.

5.3. *Domain decomposition on invariant domains*

With the previously mentioned properties, equations (8-10) still apply on the reference cell, provided that the integral transform is applied to all fields and that generalized boundary conditions (39) are satisfied by all fields on the interfaces between cells. These generalized boundary conditions can be transformed to standard periodicity conditions writing all fields $\widetilde{\mathbf{u}}$ as follows:

$$\widetilde{\mathbf{u}}(\widetilde{\mathbf{x}}) = e^{-i\langle \mathbf{k}, \widetilde{\mathbf{x}} \rangle}\, \widehat{\mathbf{u}}(\widetilde{\mathbf{x}}), \tag{41}$$

where $\widehat{\mathbf{u}}$ satisfies standard periodicity conditions. However, this transformation will change equations (8-10) when expressed as functions of $\widehat{\cdot}$ fields since the derivations along the invariant axis have to be applied both on $\widehat{\mathbf{u}}$ and on the exponential term. This is actually what is done when the Fourier Transform is applied. However it is believed that keeping $\widetilde{\cdot}$ fields as unknowns is simpler and safer. Moreover, it gives the correct limit when L tends to 0 as long as constraint (39) is accounted for.

Thus the standard domain decomposition technique can be applied on the reference cell $\widetilde{\Omega}_t$ and local boundary value problems (8) and (11) on $\widetilde{\Omega}_e$ can be defined as long as generalized boundary conditions are satisfied by the local fields. Thanks to the conditions satisfied by both the coupling field $\widetilde{\mathbf{u}_\Sigma}$ and the virtual field $\widetilde{\mathbf{v}_\Sigma}$, integrals on the interfaces between cells disappear because they are of opposite signs due to the orientation of the normal vector. Finally, equation (12) still holds adding $\widetilde{\cdot}$ on all fields and performing the integral (13-15) on $\widetilde{\Omega}$ and $\widetilde{\Sigma}$. The nice feature is that the fields that have to be approximated are now supported only on bounded domains or interfaces.

The approximation of equation (12) is certainly the most difficult step since the basis functions $\widetilde{\phi_k}(\widetilde{\mathbf{x}})$ or $\widetilde{\psi_k}(\widetilde{\mathbf{x}})$ have to satisfy the periodicity conditions (39) on interfaces between cells. A way to build such a basis has been proposed in the context of component mode synthesis [158]. The second difficult step consists in solving the local boundary value problems (17) for $\widetilde{\mathbf{u}_\Sigma}$ using boundary integral equation and Boundary Elements, keeping in mind that these fields have to satisfy the periodicity conditions (39). This step is achieved using Green's functions satisfying the same periodicity conditions (39), either deriving the corresponding close form solution or applying the general transform (36) to the non-periodic Green's function. However, one should notice that the numerical approximate can show rather poor convergence due to the singularity of the Green's functions.

5.4. *Non-invariant unbounded interfaces*

Invariant domains or subdomains are not so easy to find in nature and the scope of applicability of techniques described in this section may not appear to be as wide as expected even when combined with domain decomposition techniques and BEM for complex media. Nevertheless it can be argued that such techniques are able to account for the most important physical phenomena and in particular guided waves [185, 186, 187, 188]. However these waves are very sensitive to perturbations of the medium and the consequences of the loss of invariance have to be investigated and improvements sought. A few guidelines are given below.

Statistically Homogeneous Random Medium. Surprisingly, for strongly perturbed domains the invariant approach is still effective. Indeed for a domain showing random perturbations that are homogenous in space and the covariance depending only on the separation distance, the invariant property is satisfied in a statistical sense. Then periodic models with a period much larger than the correlation length are shown to be very good approximations of the original model as far as ensemble averages are concerned [189, 190].

Weakly Perturbed Invariant Domains. As long as the perturbations around the perfectly invariant case remain small, iterative solutions have been considered. In particular when the problem under consideration can be described as an invariant domain with perturbations at some distance from the invariant interface the interaction problem on this interface can be decoupled from the propagation problem in the medium [188, 191]. Similar techniques can be applied when the surface is weakly perturbed or curved. This technique has been widely used in geophysics to model seismic experiments in boreholes [178, 192, 193, 194].

Truncated Invariant Domains. The perturbation approach fails when the invariant domain is truncated since the perturbation is not weak -compact in the mathematical sense- and the three dimensional problem has to be studied. Some arguments have then to be found to justify the approximation by a numerical model. Some mathematical developments have been proposed in [195, 196] whereas some numerical experiments suggest rules for such an approximation (see [197] for the case of borehole geophysics).

6. Interaction through a moving interface

A particular problem in the interaction of a bounded structure and its unbounded environment is found when the interface is moving. Such problems occur when road or railway vehicles interact with their supporting infrastructure. In these analyses, a lot of emphasis has gone to physical phenomena occuring when the train speed approaches or exceeds the Rayleigh wave velocity in the soil [25, 198, 199]. This may occur for high speed trains running on tracks supported by soft soils and give rise to high vibrations and track displacements, affecting track stability and safety.

Early research on the subject of moving loads has lead to the formulation of closed-form solutions for loads with constant intensity moving on an unbounded homogeneous elastic full space [200] or halfspace [201, 202, 203, 204, 205, 206]. Numerical techniques are usually required for studying problems with more complicated loading or environment. For the particular case of a horizontally layered halfspace, the Fourier transform techniques outlined in the previous section have been used to study the response to moving loads, thereby exploiting the translationally-invariant character of the domain [207, 208, 209, 210]. Such solutions are particularly useful for verifying numerical solutions of more complicated problems.

The interaction between a moving vehicle and its supporting environment may be studied with the aim of quantifying the vehicle response or the effect the vehicle has on its environment. Determining the vehicle response is important *e.g.* for the study of passenger comfort, stability of railway vehicles [199], risk of derailment, and loading of vehicle components, whereas the response of the supporting infrastructure is of interest for track and road design [160] and environmental problems of ground-borne vibration [18, 19, 20, 22, 23, 24, 26, 27, 29, 30, 211] and noise [16, 21, 212, 213, 214]. The focus of the study will determine the detail of modeling in both structures, as well as the kind of excitation mechanisms considered. The total load applied by the vehicle on its environment can be decomposed into a static load component and a dynamic load component. As the vehicle is moving, the static load component also becomes a dynamic load. The static moving load generally dominates the response of the track or the road as well as the soil in its immediate vicinity. In the usual case where the vehicle speed is below the wave velocities in the environment, the environment displays a quasi-static response traveling with the vehicle. For the dynamic response of the vehicle and problems of ground-borne vibration and noise in the environment, however, the dynamic load component is to be considered. In the case of railway traffic, dynamic train-track interaction results from several excitation mechanisms, such as the spatial variation of the support stiffness and the wheel and track unevenness [212, 215, 216]. Due to its irregular nature, track unevenness is usually modeled as a stationary random process.

Since line infrastructure such as roads and railway tracks is generally characterized by a regular geometry, the techniques outlined in the previous section are well suited to analyze the dynamic interaction between the infrastructure and the underlying unbounded layered soil [156]. Often, the supporting infrastructure is assumed to be translationally-invariant. This assumption also holds in the case where the infrastructure has a periodic geometry (*e.g.* classical ballasted railway tracks) and the wavelength in the system is sufficiently large compared to the period. Because of their high computational efficiency and relatively modest modeling effort, techniques for translationally-invariant domains have been applied by a large number of researchers to study dynamic train-track interaction [199], ground-borne vibration due to railway traffic at grade [24, 26, 166, 170, 217] and railway traffic in tunnels [167, 168, 171, 218]. As stated earlier, alternative methods based on the FEM require appropriate procedures [219] to account for the unbounded domain and to avoid spurious reflections at the boundaries. Although translationally-invariant models may be used for studying low-frequency ground-borne vibration due to railway traffic, they are not sufficient for studying the dynamic track response at higher frequencies for track structures exhibiting periodicity. In this case, the previously discussed Floquet transform may be applied [30, 32, 220]. Note that the full non-linear train-track-tunnel interaction has been proposed in [221], using the aforementioned compliance formulation in the time domain. In order to prevent instabilities to develop a regularization method is also proposed.

A brief recapitulation of numerical techniques for analyzing the interaction of bounded structures with an unbounded environment is made next. Further to the assumptions previously outlined, it is assumed that i) the vehicle is moving at a constant speed v in a fixed direction $\mathbf{e}_2$ and ii) the environment is translationally-invariant in the direction $\mathbf{e}_2$. This allows for the use of a Fourier transform of all field variables in the direction $\mathbf{e}_2$. The discussion is therefore limited to translationally-invariant domains and problems such as vehicles crossing transition zones are not considered. The assumption of a constant velocity is not very restrictive since the speed of the vehicle remains almost constant over the typical period of the dynamic phenomena at stake. Anyhow, this assumption is much less restrictive for practical application compared to the second one on translation invariance. This considerably simplifies the analysis as the response in the moving frame of reference is stationary when the load is stationary in the deterministic or stochastic sense. The response in a fixed frame of reference remains, however, nonstationary due to the load motion. A (partial) generalization of the results for periodic domains can be found in [32]. In the following, it is first outlined how the response of the environment to static and dynamic moving loads is analyzed. Next, it is shown how the dynamic load component can be computed from the solution of the vehicle-track interaction problem.

6.1. Response of the environment to moving loads

It is first assumed that the load applied by the moving bounded structure (the vehicle) on its environment (the supporting infrastructure) is known. For simplicity, a point load is considered here, so that the load position can be written as $\mathbf{x}_s(t) = v\mathbf{e}_2 t$. The load intensity is denoted by the vector $\mathbf{g}(t)$. The load distribution $\mathbf{f}_e(\mathbf{x}, t)$ acting on the environment is written using the Dirac delta function δ as $\delta(\mathbf{x} - \mathbf{x}_s(t))\mathbf{g}(t)$. The aim is now to solve the equilibrium equation in the subdomain Ω_e with the prescribed load distribution $\mathbf{f}_e(\mathbf{x}, t)$. A solution to the moving load problem is easily obtained in terms of the Green's displacement tensor $\mathbf{U}_e^{\mathrm{G}}(\mathbf{x}, \mathbf{x}', t - t')$ of the environment. Each element $[\mathbf{U}_e^{\mathrm{G}}]_{ij}(\mathbf{x}, \mathbf{x}', t - t')$ of the tensor represents the displacement at the position $\mathbf{x}'$ in the direction $\mathbf{e}_j$ due to a unit impulse load in the direction $\mathbf{e}_i$ at time t' and position $\mathbf{x}$. Note that a closed-form solution for the Green's displacement tensor is generally not available for most environments of interest but it is formally defined in the frequency domain by equations (19) and (20). The solution for the displacement field $\mathbf{u}_e(\mathbf{x}, t)$ is found as follows:

$$\mathbf{u}_e(\mathbf{x}, t) = \int_{-\infty}^{t} \int_{\Omega_e} \mathbf{U}_e^{\mathrm{G}}(\mathbf{x}, \mathbf{x}', t - t')\mathbf{f}_e(\mathbf{x}', t')\, \mathrm{d}\mathbf{x}'\, \mathrm{d}t'. \tag{42}$$

Introducing the load distribution $\mathbf{f}_e(\mathbf{x}', t') = \delta(\mathbf{x}' - \mathbf{x}_s(t'))\mathbf{g}(t')$ eliminates the integration with respect to space:

$$\mathbf{u}_e(\mathbf{x}, t) = \int_{-\infty}^{t} \mathbf{U}_e^{\mathrm{G}}(\mathbf{x}, \mathbf{x}_s(t'), t - t')\mathbf{g}(t')\, \mathrm{d}t'. \tag{43}$$

Due to the translationally-invariant character of the environment Ω_e in the direction $\mathbf{e}_2$, the Green's displacement tensor $\mathbf{U}_e^{\mathrm{G}}(\mathbf{x}, \mathbf{x}', t - t')$ is equal to $\mathbf{U}_e^{\mathrm{G}}(\tilde{\mathbf{x}}, \tilde{\mathbf{x}}' + x_2'\mathbf{e}_2 - x_2\mathbf{e}_2, t - t')$ with $\tilde{\mathbf{x}} = x_1\mathbf{e}_1 + x_3\mathbf{e}_3$:

$$\mathbf{u}_e(\mathbf{x}, t) = \int_{-\infty}^{t} \mathbf{U}_e^{\mathrm{G}}(\tilde{\mathbf{x}}, \tilde{\mathbf{x}}_s + vt'\mathbf{e}_2 - x_2\mathbf{e}_2, t - t')\mathbf{g}(t')\,\mathrm{d}t'. \tag{44}$$

By applying first a Fourier transform with respect to x_2 and next a Fourier transform with respect to the time t, the following expression is derived from equation (44):

$$\tilde{\mathbf{u}}_e(\tilde{\mathbf{x}}, k_2, \omega) = \tilde{\mathbf{U}}_e^{\mathrm{G}}(\tilde{\mathbf{x}}, \tilde{\mathbf{x}}_s, -k_2, \omega)\mathbf{g}(\omega - k_2 v), \tag{45}$$

where $\tilde{\mathbf{U}}_e^{\mathrm{G}}(\tilde{\mathbf{x}}, \tilde{\mathbf{x}}', k_2, t - t')$ denotes the Fourier transform of the Green's tensor $\mathbf{U}_e^{\mathrm{G}}(\tilde{\mathbf{x}}, \tilde{\mathbf{x}}' + x_2'\mathbf{e}_2 - x_2\mathbf{e}_2, t - t')$ with respect to $(x_2' - x_2)$. In equation (45), the shift between the frequency $\omega - k_2 v$ at the source and the frequency ω at the receiver gives rise to the Doppler effect. An inverse Fourier transform allows obtaining the displacement field in the frequency domain:

$$\mathbf{u}_e(\mathbf{x}, \omega) = \frac{1}{2\pi} \int_{\mathbb{R}} \tilde{\mathbf{U}}_e^{\mathrm{G}}(\tilde{\mathbf{x}}, \tilde{\mathbf{x}}_s, -k_2, \omega)\mathbf{g}(\omega - k_2 v) e^{-ik_2 x_2}\,\mathrm{d}k_2. \tag{46}$$

Equation (46) is usually preferred over equation (44) because the techniques of the previous section for translationally-invariant domains offer a solution for the Green's tensor in terms of the wavenumber k_2 and the radial frequency ω.

A similar expression can be derived for the Fourier transform of the displacement field $\mathbf{u}_e'(\hat{\mathbf{x}}, t) \equiv \mathbf{u}_e(\hat{\mathbf{x}} + vt, t)$ in the moving frame of reference $\hat{\mathbf{x}} = \mathbf{x} - vt\mathbf{e}_2$ starting from the evaluation of equation (44) at $\mathbf{x} = \hat{\mathbf{x}} + vt\mathbf{e}_2$:

$$\mathbf{u}_e'(\hat{\mathbf{x}}, \omega) = \mathbf{C}(\omega)\mathbf{g}(\omega), \qquad \text{with} \qquad \mathbf{C}_e(\omega) = \frac{1}{2\pi} \int_{\mathbb{R}} \tilde{\mathbf{U}}_e^{\mathrm{G}}(\tilde{\mathbf{x}}, \tilde{\mathbf{x}}_s, -k_2, \omega + k_2 v) e^{-ik_2 \hat{x}_2}\,\mathrm{d}k_2, \tag{47}$$

where $\mathbf{C}_e(\omega)$ represents the compliance of the environment. This shows that the Doppler effect will not be observed in the moving frame of reference but that the compliance depends on velocity and the frequency.

6.2. *Interaction between the moving bounded structure and the unbounded environment*

In problems of ground-borne vibration due to railway traffic, the dynamic interaction between the vehicle and the track needs to be considered in order to compute the intensity of the moving loads. In this case, the bounded structure Ω is the vehicle and the interface Σ between the bounded structure and the environment is moving in a fixed frame of reference. The interface Σ usually consists of a finite set of contact surfaces through which the vehicle interacts with the supporting infrastructure. In the following, it is assumed that the contact surfaces do not depend on time and are small, so that they can be approximated by contact points. This implies that a perfect, time independent contact is considered, excluding *e.g.* occurrence of loss of contact between the vehicle and the track. In the expressions, a single contact point is considered, as would be the case when a single axle model is used for the vehicle. The results are easily generalized for the case of multiple contact points [26, 187, 222, 223]. A more general formulation considering finite sized contact surfaces is found in [187].

The position of the single contact point in the fixed frame of reference is time dependent and equal to $\mathbf{x}_{\mathrm{c}}(t) = v\mathbf{e}_2 t$, whereas in the Galilean frame of reference $\hat{\mathbf{x}} = \mathbf{x} - vt\mathbf{e}_2$ that follows the vehicle, the position $\hat{\mathbf{x}}_{\mathrm{c}} = \mathbf{0}$ is independent of the time t. Since the contact between the vehicle and the supporting infrastructure is assumed to be perfect, the following compatibility equation can be written in the moving frame of reference:

$$\mathbf{u}_e(\hat{\mathbf{x}}_{\mathrm{c}}, t) + \mathbf{u}_{\mathrm{w/r}}(\hat{\mathbf{x}}_{\mathrm{c}} + vt\mathbf{e}_2) = \mathbf{u}(\hat{\mathbf{x}}_{\mathrm{c}}, t), \tag{48}$$

where $\mathbf{u}(\hat{\mathbf{x}}_c, t)$ is the displacement of the vehicle at the contact point $\mathbf{x}_{\mathrm{c}}(t)$ and $\mathbf{u}_{\mathrm{w/r}}(\hat{\mathbf{x}}_{\mathrm{c}} + vt\mathbf{e}_2)$ represents the combined wheel and rail (or road) unevenness at the position of the contact point. This equation, formulated in the frequency domain, is used to calculate the unknown load intensity $\mathbf{g}(t)$. For a linear vehicle model, a linear relation between the displacement at the contact point $\mathbf{u}_{\mathrm{v}}(\hat{\mathbf{x}}_{\mathrm{c}}, \omega)$ and the load intensity $\mathbf{g}(\omega)$ can be derived as in equation (23):

$$\mathbf{u}(\hat{\mathbf{x}}_{\mathrm{c}}, \omega) = -\mathbf{C}(\omega)\mathbf{g}(\omega), \tag{49}$$

where $\mathbf{C}_\mathrm{v}(\omega)$ is termed the vehicle compliance and the minus sign on the right hand side is due to the convention where the load intensity $\mathbf{g}(t)$ is chosen positive when acting in the positive direction on the supporting infrastructure. In the particular case of translationally-invariant structures, a similar relation is derived from equation (47). By introducing equations (49) and (47) in the compatibility equation (48), the following expression is derived for the Fourier transform of the load intensity $\mathbf{g}(\omega)$:

$$[\mathbf{C}_\mathrm{v}(\omega) + \mathbf{C}_e(\omega)]\,\mathbf{g}(\omega) \quad = \quad -\frac{1}{v}\tilde{\mathbf{u}}_{\mathrm{w/r}}\left(\tilde{\mathbf{x}}_\mathrm{c}, -\frac{\omega}{v}\right), \tag{50}$$

where $\tilde{\mathbf{u}}_{\mathrm{w/r}}(\tilde{\mathbf{x}}, k_2)$ is the Fourier transform of $\mathbf{u}_{\mathrm{w/r}}(\mathbf{x})$ with respect to x_2. The combined unevenness $\tilde{\mathbf{u}}_{\mathrm{w/r}}(\tilde{\mathbf{x}}, k_2)$ is evaluated at a wavenumber $k_2 = -\omega/v$, showing that unevenness with a wavelength $\lambda_2 = 2\pi/k_2$ leads to excitation at a radial frequency $\omega = 2\pi v/\lambda_2$. Furthermore, equation (50) shows how the load intensity $\mathbf{g}(\omega)$ depends on the vehicle and track compliance, respectively. In the case of road traffic, the vehicle compliance is usually much larger than the road compliance, so that the latter can be disregarded in the calculation of the dynamic vehicle loads. In the case of railway traffic, however, both terms have a similar order of magnitude and equation (50) allows accounting for interaction phenomena such as resonance of the unsprung mass on the track. Equation (50) has been used recently by many researchers studying ground-borne vibration due to railway traffic with track models either derived following the techniques outlined in the previous section [26, 187, 222, 223] or using FEM models involving a finite part of track [23].

7. Interaction through multiple interfaces

The last issue addressed in this paper is the cross-interaction between many structures through a common linear elastic environment. Several of the aforementioned methods can be used, simply considering the generic structure as a set of disconnected sub-structures. The only limitation is then the computational cost, related to the boundedness in space of the set of substructures. This approach has been used in particular to study the interaction between the geological site and the buildings in a city by means of pure FEM [224, 225], FEM-BEM coupling [14, 35, 226, 227, 228], or the interaction of a large structure with attached small resonant objects [229]. Periodic models such as those presented in section 5 have been used to extend these analyses to an infinite number of structures [7]. Yet, these approaches often yield unrealistic results since wave propagation in periodic wave guides is strongly modified by the presence of small perturbations [230, 231]. This major drawback of periodic models can be reduced accounting for larger cells incorporating a natural variability. This leads to more reliable results, both in terms of mean response and fluctuation levels [174].

When the number of substructures becomes very large and both their location and dynamic properties are highly uncertain, deterministic numerical simulations are extremely expensive and bring little insight into the physical phenomena at stake. Theoretical models have recently been developed and have proven to be very effective in those cases. A deterministic homogenization technique has been proposed in [232, 233], but statistical methods are often preferred. As far as the interaction between a large structure and a large set of local resonators is concerned, a structural fuzzy model has been proposed in [234]. Coming back to the formalism presented in section 2, the set of random oscillators plays the role of a random linear environment. This leads to a stochastic model for the dynamic stiffness matrix $\mathbf{K}_e(\omega)$, which however shows only short range coupling. More recently, methods based on the Ward Identity have proven to be quite effective as long as no localization occurs [235].

When applied to the interaction of a random collection of structures with an elastic environment, general theoretical methods for wave propagation in random media can be used [236, 237, 238, 239, 240, 241]. The first step consists in analyzing the scattering properties of each individual structure. The approach used in section 2 can be used, and it has been shown in [227] that the total cross-section S and the scattering cross-section S_s are proportional to the imaginary parts of the equivalent forces and dynamic stiffness given in equations (15) and (16):

$$p_i S = \omega \Im \left\{\mathcal{L}_e(\omega; \mathbf{v}^*_{\Sigma e})\right\}, \qquad p_i S_s = \omega \Im \left\{\mathcal{K}_e(\omega; \mathbf{v}_{\Sigma e}, \mathbf{v}^*_{\Sigma e})\right\} \tag{51}$$

where * indicates the complex conjugate, $\Im\{\cdot\}$ the imaginary part, and $p_i = \rho c v^2$ the power surface density of the incident wave of speed c and velocity amplitude v along the wave front. In addition, the local interaction problem has been used [189] to compute the impurity operator $\mathbf{\Lambda}$ in the Lippmann-Schwinger equation:

$$\mathbf{u}_e = \mathbf{u}^{\text{inc}} + \mathcal{U}_e^G \mathbf{\Lambda} \mathbf{u}_e. \tag{52}$$

Here $\mathcal{U}_e^G$ can be the integral operator of the elastic environment defined in equation (20), with prescribed traction on the interface, or any other integral operator satisfying the field equations in Ω_e (see [69] for a dual variational form of this equation). When $\mathcal{U}_e^G$ is defined by equation (20), with free surface conditions on Σ for the incident field, one has, for each structure, an equation of the form (23) with $\mathbf{\Lambda}^{-1} = \mathcal{U}^G$. Under the assumption that the impurity operator is random, the combination of the average of (52) and the fluctuation around this average leads to the deterministic Dyson equation, that yields the value of the mean field $\underline{u}_e$:

$$\underline{u}_e = \mathbf{u}^{\text{inc}} + \mathcal{U}_e^G \mathcal{M} \underline{\mathbf{u}}_e, \qquad \mathcal{M} = \mathcal{P}\mathbf{\Lambda}(\mathcal{I} - \mathcal{U}_e^G(\mathcal{I} - \mathcal{P})\mathbf{\Lambda})^{-1}), \tag{53}$$

with $\mathcal{P}$ and $\mathcal{I}$ are the expectation and identity operators, respectively. Similar developments for the correlation lead to the Bethe-Salpeter equation. However, these equations are formally defined as infinite sums, with possibly divergent series of operators, so that the truncation of the series may be unstable. Yet, in the context of elastic waves, the approximation of these equations using the Foldy [242] and the Ladder approximations have proven to compare effectively with full numerical models [189, 190, 227]. A numerical approximation using the polynomial Chaos expansion has also been proposed in [243], as well as methods based on Radiative Transfer approaches [244].

8. Conclusion

This review paper aimed at showing that the important developments made during these past two decades in the field of interaction between structures and their elastic environment can be cast in the general framework proposed in section 2. The main interest of this framework is to be independent of the numerical technique at stake in each subdomain: FEM, BEM, series expansions or analytical developments. In addition, it has been efficiently extended to account for randomness (section 4), non-linear structure (section 3) and unbounded (section 5), moving (section 6) and multiple (section 7) interfaces. Several applications of these techniques in different fields have also been reported, showing its high versatility in engineering. Finally, it is worth noticing that it can be extended to fluid-structure interaction, especially when a Lagrangian approach is chosen (as proposed in [1]).

This approach also shows some downsides. The first criticism to be addressed concerns the decomposition between the structure and the environment, which is somehow artificial. Although well justified in many cases, especially in fluid-structure interaction, this decomposition can be considered as useless from a numerical point of view. Indeed, a global FEM model for the linear or non-linear dynamic analysis is a quite valuable option using modern commercial FEM codes and computers.

The second criticism, somehow related, is that the proposed substructure method has been fostered by the development of BEM methods to account for unbounded environments. Yet, even though many researchers have tried in the nineties to rise the BEM as an alternative to the FEM, it turned out to be competitive (in terms of efficiency as well as software availability) only in specific fields, among which linear dynamic problems in unbounded domains. The recent development of multi-pole solvers for harmonic problems is a recent attempt to reach the efficiency of domain decomposition solvers in the FEM community [245]. Finally, the development of the Perfectly Matched Layers has also weakened the superiority of the BEM to account for radiation conditions at infinity [55], especially for transient analysis and when coupled with Spectral Elements [246, 247]. To the authors' knowledge, the BEM cannot compete for this type of application especially when the medium is strongly heterogeneous. But the FEM also has its own limitations, for example in terms of efficiency and accuracy when coupling Spectral Elements to standard non-linear FEM codes or accounting for randomly heterogeneous media [248].

Hence, the need for coupling techniques between specific solvers for the structures (possibly standard FE) and fast linear solvers for the environment (the SEM is a good candidate) still remains. To this end, and considering a broader perspective, these fast solvers for the environment can be seen as generalized BEM techniques, as proposed in the Fictitious Domain approach [249]. Volume coupling techniques [250] are also to be considered. But, in any case, time-domain approaches will probably be favored in the future, both for non-linear analyses or for efficiency reasons when targeting broadband transient loads.

Finally, it is believed that the proposed formalism remains highly valuable for the physical analysis of the phenomena at hand and the account of uncertainties [36], especially when adapted levels of description of the random dimension are sought (see [140] as a first attempt). But the main challenge in the context of dynamic phenomena will certainly be the coupling of different models at different scales with different accuracy requirements.

References

[1] J.-S. Schotté, R. Ohayon, Linearized formulation for fluid-structure interaction. application to the vibrations of an elastic structure containing a fluid with a free surface, Journal of Sound and Vibration (2012). Accepted for publication.

[2] R. W. Clough, J. Penzien, Dynamics of structures, MacGraw-Hill, 1975.

[3] J. P. Wolf, Dynamic soil-structure interaction, Prentice-Hall, 1985.

[4] R. Paolucci, Soil-structure interaction effects on an instrumented building in Mexico city, European Earthquake Engineering 3 (1993) 33–44.

[5] F. C. de Barros, J. E. Luco, Identification of foundation impedance functions and soil properties from vibration tests of the Hualien containment model, Soil Dynamics and Earthquake Engineering. 14 (1995) 229–248.

[6] H. Antes, C. C. Spyrakos, Soil-structure interaction, in: D. Beskos, S. Anagnostopoulos (Eds.), Computer analysis and design of earthquake resistant structures. a handbook, Computational Mechanics Publications, 1996, pp. 271–332.

[7] A. Wirgin, P.-Y. Bard, Effects of buildings on the duration and amplitude of ground motion in Mexico City, Bulletin of the Seismological Society of America 86 (1996) 914–920.

[8] T. K. Datta, Seismic response of buried pipelines: a state-of-the-art review, Nuclear Engineering Design 192 (1999) 271–284.

[9] H. Hao, S. R. Zhang, Spatial ground motion effect on relative displacement of adjacent building structures, Earthquake Engineering and Structural Dynamics 28 (1999) 333–349.

[10] S. C. Dutta, R. Roy, A critical review on idealization and modeling for interaction among soil-foundation-structure system, Computer & Structures 80 (2002) 1579–1594.

[11] D. Clouteau, D. Aubry, Computational soil-structure interaction, in: W. S. Hall, G. Oliveto (Eds.), Boundary element methods for soil-structure interaction, Kluwer Academic Publishers, 2004, pp. 61–125.

[12] J. E. Luco, F. C. de Barros, Assessment of predictions of the response of the Hualien containment model during forced vibration tests, Soil Dynamics and Earthquake Engineering. 24 (2004) 1013–1035.

[13] R. Taherzadeh, D. Clouteau, R. Cottereau, Simple formulas for the dynamic stiffness of pile groups, Earthquake Engineering and Structural Dynamics 38 (2009) 1665–1685.

[14] D. Clouteau, D. Broc, G. Devésa, V. Guyonvarh, P. Massin, Calculation methods to Structure-Soil-Structure interaction (3SI) for embedded buildings: application to NUPEC tests, Soil Dynamics and Earthquake Engineering. 32 (2012) 129–142.

[15] U. Sandberg, Low noise road surfaces. a state-of-the-art review, Journal of the Acoustical Society of Japan 20 (1999) 1–17.

[16] D. J. Thompson, C. J. Jones, A review on the modelling of wheel/rail noise generation, Journal of Sound and Vibration 231 (2000) 519–536.

[17] C. Steele, A critical review of some traffic noise prediction models, Applied Acoustics 62 (2001) 271–287.

[18] C. H. Chua, K. W. Lo, T. Balendra, Building response due to subway train traffic, Journal of Geotechnical Engineering 121 (1995) 747–754.

[19] H. E. Hunt, Stochastic modelling of traffic-induced ground vibration, Journal of Sound and Vibration 144 (1991) 53–70.

[20] H. Grundmann, M. Lieb, E. Trommer, The response of a layered half-space to traffic loads moving along its surface, Archives of Applied Mechanics 69 (1999) 55–67.

[21] G. P. Wilson, H. J. Saurenman, J. T. Nelson, Control of groundborne noise and vibration, Journal of Sound and Vibration 87 (1983) 339–350.

[22] G. Lombaert, G. Degrande, D. Clouteau, Numerical modelling of free field traffic-induced vibrations, Soil Dynamics and Earthquake Engineering. 19 (2000) 473–488.

[23] L. Auersch, The excitation of ground vibration by rail traffic: theory of vehicle-track-soil interaction and measurements on high-speed lines, Journal of Sound and Vibration 284 (2005) 103 – 132.

[24] X. Sheng, C. J. Jones, M. Petyt, Ground vibration generated by a load moving along a railway track, Journal of Sound and Vibration 228 (1999) 129–156.

[25] C. Madshus, A. M. Kaynia, High-speed railway lines on soft ground: dynamic behaviour at critical train speed, Journal of Sound and Vibration 231 (2000) 689–701.

[26] G. Lombaert, G. Degrande, J. Kogut, S. François, The experimental validation of a numerical model for the prediction of railway induced vibrations, Journal of Sound and Vibration 297 (2006) 512–535.

[27] V. Krylov, Calculation of the low-frequency ground vibrations from railway trains, Applied Acoustics 42 (1994) 199–213.

[28] H. Dieterman, A. Metrikine, The equivalent stiffness of a halfspace interacting with a beam. Critical velocities of a moving load along the beam, European Journal of Mechanics A/Solids 15 (1996) 67–90.

[29] L. Pyl, G. Degrande, D. Clouteau, Validation of a source-receiver model for road traffic induced vibrations in buildings. Part II: the receiver model, Journal of Engineering Mechanics 130 (2004) 1394–1406.

[30] G. Degrande, D. Clouteau, R. Othman, M. Arnst, H. Chebli, R. Klein, P. Chatterjee, B. Janssens, A numerical model for ground-borne vibrations from underground railway traffic based on a periodic finite element/boundary element formulation, Journal of Sound and Vibration 293 (2006) 645–666.

[31] H. Chebli, R. Othman, D. Clouteau, 3D periodic BE-FE model for various transportation structures interacting with soil, Computer and Geotechnics 35 (2008) 22–32.

[32] H. Chebli, R. Othman, D. Clouteau, Response of periodic structures due to moving loads, Comptes Rendus Mécanique 334 (2006) 347–352.

[33] M. Dravinski, T. K. Mossessian, Scattering of plane harmonic P, SV and Rayleigh waves by dipping layers of arbitrary shape, Bulletin of the Seismological Society of America 77 (1987) 212–235.

[34] L. Geli, P. Y. Bard, B. Jullien, The effect of topography on earthquake ground motion : a review and new results, Bulletin of the Seismological Society of America 78 (1988) 42–63.
[35] M. Kham, J.-F. Semblat, P.-Y. Bard, P. Dangla, Seismic site-city interaction: Main governing phenomena through simplified numerical models, Bulletin of the Seismological Society of America 96 (2006) 1934–1951.
[36] C. Soize, Stochastic modeling of uncertainties in computational structural dynamics - recent theoretical advances, Journal of Sound and Vibration (2012). Accepted for publication.
[37] A. Quarteroni, A. Valli, Domain decomposition methods for partial differential equations, Oxford University Press, 1999.
[38] A. Toselli, O. B. Widlund, Domain decomposition methods - algorithms and theory, Springer, 2005.
[39] M. C. Bampton, R. R. Craig, Coupling of substructures for dynamic analyses, AIAA Journal 6 (1968) 1313–1319.
[40] R. Ohayon, C. Soize, Structural acoustics and vibration: mechanical models, variational formulations and discretization, Elsevier, 1998.
[41] H. Wong, J. E. Luco, Dynamic response of rigid foundations of arbitrary shape, Earthquake Engineering and Structural Dynamics 4 (1976) 579–587.
[42] I. Herrera, J. Bielak, Soil-structure interaction as a diffraction problem, in: Proceedings of the Sixth World Conference on Earthquake Engineering, pp. 10–14.
[43] J. A. Gutierrez, A. Chopra, A substructure method for earthquake analysis of structures including structure-soil interaction, Earthquake Engineering and Structural Dynamics 6 (1978) 51–69.
[44] J. Bielak, P. Christiano, On the effective seismic input for non-linear soil-structure interaction systems, Earthquake Engineering and Structural Dynamics 12 (1984) 107–119.
[45] M. G. Cremonini, P. Christiano, J. Bielak, Implementation of effective seismic input for soil-structure interaction systems, Earthquake Engineering and Structural Dynamics 16 (1988) 615–625.
[46] J. Harris, Linear elastic waves, Cambridge University Press, 2001.
[47] J. C. Simo, T. J. Hughes, Computational inelasticity, Interdisciplinary Applied Mathematics, Springer, 1998.
[48] O. C. Zienkiewicz, R. L. Taylor, The Finite Element Method: the basis, Butterworth-Heinemann, 5 edition, 2000.
[49] G. D. Manolis, D. E. Beskos, Boundary element methods in elastodynamics, Unwin Hyman Ltd, 1988.
[50] J. Dominguez, Boundary Elements in dynamics, Computer Mechanics Publications, 1993.
[51] M. Bonnet, Boundary integral equations methods for Solids and Fluids, John Wiley and sons, 1999.
[52] D. Givoli, Numerical Methods for Problems in Infinite Domains, Studies in Applied Mechanics, Elsevier, 1992.
[53] I. Harari, A unified variational approach to domain-based computation of exterior problems of time-harmonic acoustics, Applied Numerical Mathematics 27 (1998) 417–441.
[54] S. V. Tsynkov, Numerical solution of problems on unbounded domains. a review, Applied Numerical Mathematics 27 (1998) 465–532.
[55] F. Bérenger, A perfectly matched layer for the absorption of electromagnetics waves, Journal of Computational Physics 114 (1994) 185–200.
[56] J. P. Wolf, Unit-impulse response matrix of unbounded medium by infinitesimal finite-element cell method, Computer Methods in Applied Mechanics and Engineering 122 (1995) 251–272.
[57] J. P. Wolf, S. Song, Some cornerstones of dynamic soil-structure interaction, Engineering Structures 24 (2002) 13–28.
[58] V. Rokhlin, Rapid solution of integral equations of classical potential theory, Journal of Computational Physics 60 (1985) 187–207.
[59] C. Lage, C. Schwab, Wavelet Galerkin algorithms for boundary integral equation, SIAM Journal on Scientific Computing 20 (1999) 2195–2222.
[60] S. Chaillat, M. Bonnet, J.-F. Semblat, A multi-level fast multipole BEM for 3-D elastodynamics in the frequency domain, Computer Methods in Applied Mechanics and Engineering 197 (2008) 4233–4249.
[61] E. Kausel, R. Peek, Dynamic loads in the interior of a layered stratum - an explicit solution, Bulletin of the Seismological Society of America 72 (1982) 1459–1481.
[62] J. E. Luco, R. J. Apsel, On the Green's functions for a layered half-space, part I and II, Bulletin of the Seismological Society of America 73 (1983) 909–950.
[63] F. Chapel, Boundary element method applied to linear soil structure interaction on a heterogeneous soil, Earthquake Engineering and Structural Dynamics 15 (1987) 815–829.
[64] D. Aubry, D. Clouteau, A regularized boundary element method for stratified media, in: G. Cohen, al. (Eds.), First international conference on Mathematical and Numerical Aspects of Wave Propagation, SIAM, pp. 660–668.
[65] R. Y. Pak, B. B. Guzina, Three-dimensional Green's functions for a multilayered half-space in displacement potentials, ASCE Journal of the Engineering Mechanics Division 128 (2002) 449–461.
[66] L. Pyl, D. Clouteau, G. Degrande, A weakly singular boundary integral equation in elastodynamics for heterogeneous domains mitigating fictitious eigenfrequencies, Engineering Analyses with Boundary Elements 28 (2004) 1493–1513.
[67] H. Brakhage, P. Werner, Über das Dirichletsche außenraumproblem für die Helmholtzsche schwingungsgleichung (About the exterior Dirichlet problem for the Helmholtz equation), Archiv der Mathematik 16 (1965) 325–329. In German.
[68] A. Burton, G. Miller, Application of integral equation methods to numerical solution of some exterior boundary-value problems, Proceedings of the Royal Society of London 323 (1971) 201.
[69] E. Savin, D. Clouteau, Elastic wave propagation in a 3-D unbounded random heterogeneous medium coupled with a bounded medium. Application to seismic soil-structure interaction (SSSI), International Journal for Numerical Methods in Engineering 54 (2002) 607–630.
[70] E. Balmès, Optimal Ritz vectors for component mode synthesis using the singular value decomposition, AIAA Journal 34 (1996) 1256–1260.
[71] G. Kergourlay, E. Balmes, D. Clouteau, Model reduction for efficient FEM/BEM coupling, in: P. Sas, D. Moens (Eds.), Noise and Vibration Engineering, volume 1-3, KU Leuven, pp. 1167–1174.
[72] C. K. Lee, H. Shiojiri, K. Hanada, T. T. Nakagawa, A method for nonlinear dynamic response analysis of semi-infinite foundation soil, Soil Dynamics and Earthquake Engineering. 25 (2005) 649–659.
[73] D. Pitilakis, D. Clouteau, Equivalent linear substructure approximation of soil-foundation-structure interaction: model presentation and validation, Bulletin of Earthquake Engineering 8 (2010) 257–282.

[74] J. O. Robertsson, R. Laws, C. Chapman, J.-P. Vilotte, E. Delavaud, Modelling of scattering of seismic waves from a corrugated rough sea surface: a comparison of three methods, Geophysical Journal International 167 (2006) 70–76.
[75] Q.-A. Ta, D. Clouteau, R. Cottereau, Modeling of random anisotropic elastic media and impact on wave propagation, European Journal of Computational Mechanics 19 (2010) 241–253.
[76] F. Magoulès, I. Harari, Absorbing boundary conditions, Computer Methods in Applied Mechanics and Engineering 195 (2006) 3551–3902.
[77] N. M. Newmark, A method of computation for structural dynamics, ASCE Journal of the Engineering Mechanics Division 85 (1959) 67–94.
[78] D. L. Karabalis, D. E. Beskos, Dynamic response of three-dimensional flexible foundations by time domain BEM and FEM, Earthquake Engineering and Structural Dynamics 24 (1985) 91–101.
[79] C. C. Spyrakos, D. E. Beskos, Dynamic response of flexible strip foundation by boundary and finite elements, Earthquake Engineering and Structural Dynamics 5 (1986) 84–96.
[80] O. von Estorff, M. J. Prabucki, Dynamic response in the time domain by coupled boundary and finite elements, Computational Mechanics 6 (1990) 35–46.
[81] O. von Estorff, M. Firuziaan, Coupled BEM/FEM approach for nonlinear soil/structure interaction, Engineering Analyses with Boundary Elements 24 (2000) 715–725.
[82] A. Peirce, E. Siebrits, Stability analysis and design of time-stepping schemes for general elastodynamic boundary element models, International Journal for Numerical Methods in Engineering 40 (1997) 319–342.
[83] G. Y. Yu, W. J. Mansur, J. A. Carrer, S. T. Lie, A more stable scheme for BEM/FEM coupling applied to two-dimensional elastodynamics, Computer & Structures 79 (2001) 811–823.
[84] S. T. Lie, G. Yu, C. Fan, Further improvement to the stability of the coupling BEM/FEM scheme for 2-D elastodynamic problems, Computational Mechanics 25 (2000) 468–476.
[85] J. P. Wolf, D. R. Somaini, Approximate dynamic model of embedded foundation in time domain, Earthquake Engineering and Structural Dynamics 14 (1986) 683–703.
[86] G. R. Darbre, J. P. Wolf, Criterion of stability and implementation issues of hybrid frequency time domain procedure for non-linear dynamic analysis, Earthquake Engineering and Structural Dynamics 16 (1988) 569–581.
[87] J. P. Wolf, Soil-structure interaction analysis in the time domain, Prentice-Hall, 1988.
[88] H. A. Kramers, La diffusion de la lumière par les atomes (The diffusion of light by atoms), Resoconto del Congresso Internazionale dei Fisici 2 (1927) 545–557. In French.
[89] R. d. Kronig, On the theory of dispersion of x-rays, Journal of the Optical Society of America 12 (1926) 547–557.
[90] L. B. Pierce, Hardy functions, Master's thesis, Princeton University, 2001.
[91] F. Chabas, C. Soize, Modeling mechanical subsystems by boundary impedance in the finite element method, La Recherche Aérospatiale (english edition) 5 (1987) 59–75.
[92] R. Cottereau, D. Clouteau, C. Soize, Construction of a probabilistic model for the soil impedance matrix using a non-parametric method, in: C. Soize, G. I. Schüeller (Eds.), EURODYN 2005: Proceedings of the 6th International Conference on Structural Dynamics, Millpress Science Publishers, 2005, pp. 841–846.
[93] R. Cottereau, Probabilistic models of impedance matrices. Application to dynamic soil-structure interaction, Ph.D. thesis, École Centrale Paris, France, 2007.
[94] R. J. Allemang, D. Brown, A unified matrix polynomial approach to modal identification, Journal of Sound and Vibration 211 (1998) 301–322.
[95] R. Pintelon, J. Schoukens, System Identification: A Frequency Domain Approach, IEEE Press, Piscataway, 2001.
[96] M. van Barel, A. Bultheel, A parallel algorithm for discrete least squares rational approximation, Numerische Mathematik 63 (1992) 99–121.
[97] A. Bultheel, A. Cuyt, et al., Generalizations of orthogonal polynomials, Journal of Computational and Applied Mathematics 179 (2005) 57–95.
[98] R. Cottereau, D. Clouteau, C. Soize, S. Cambier, Probabilistic nonparametric model of impedance matrices. Application to the seismic design of a structure, European Journal of Computational Mechanics 15 (2006) 131–142.
[99] R. Cottereau, D. Clouteau, C. Soize, Construction of a probabilistic model for impedance matrices, Computer Methods in Applied Mechanics and Engineering 196 (2007) 2252–2268.
[100] X. Du, M. Zhao, Stability and identification for rational approximation of frequency response function of unbounded soil, Earthquake Engineering and Structural Dynamics 39 (2010) 165–186.
[101] A. Nieto-Ferro, D. Clouteau, N. Greffet, G. Devésa, Approche hybride Laplace-temps pour les calculs d'interaction dynamique (Hybrid Laplace-time approach for dynamic interaction computations), in: CSMA2011: 10ème Colloque National en Calcul des Structures, p. on CDROM. In French.
[102] H. R. Masoumi, S. François, G. Degrande, A non-linear coupled finite element-boundary element model for the prediction of vibrations due to vibratory and impact pile driving, International Journal for Numerical and Analytical Methods in Geomechanics 33 (2009) 245–274.
[103] S. François, Nonlinear modelling of the response of structures due to ground vibrations, Ph.D. thesis, Department of Civil Engineering, Katholieke Universiteit Leuven, 2008.
[104] H. R. Masoumi, Numerical modeling of free field vibrations due to pile driving, Ph.D. thesis, Katholieke Universiteit Leuven, 2008.
[105] L. Gaul, M. Schanz, A comparative study of three boundary element approaches to calculate the transient response of viscoelastic solids with unbounded domains, Computer Methods in Applied Mechanics and Engineering 179 (1999) 111–123.
[106] M. Schanz, H. Antes, Aplication of 'operational Quadrature Methods' in Time Domain Boundary Element Methods, Meccanica 32 (2006) 179–186.
[107] C. Lubich, Convolution quadrature and discretized operational calculus II, Numerische Mathematik 52 (1988) 413–425.
[108] W. Moser, H. Antes, G. Beer, A Duhamel integral based approach to one-dimensional wave propagation analysis in layered media, Computational Mechanics 35 (2005) 115–126.
[109] W. Moser, H. Antes, G. Beer, Soil-structure interaction and wave propagation problems in 2D by a Duhamel integral based approach and

the convolution quadrature method, Computational Mechanics 36 (2005) 431–443.
[110] C. Soize, Non-gaussian positive-definite matrix-valued random fields for elliptic stochastic partial differential operators, Computer Methods in Applied Mechanics and Engineering 195 (2006) 26–64.
[111] D. M. Ghiocel, R. G. Ghanem, Stochastic finite element analysis of seismic soil-structure interaction, Journal of Engineering Mechanics 128 (2002) 66–77.
[112] P. Labbé, H. Noé, Stochastic approach for the seismic design of nuclear power plant equipment, Nuclear Engineering Design 129 (1991) 367–379.
[113] C. Desceliers, C. Soize, S. Cambier, Non-parametric-parametric model for random uncertainties in non-linear structural dynamics: application to earthquake engineering, Earthquake Engineering and Structural Dynamics 33 (2004) 315–327.
[114] Y. Lin, G. Q. Cai, Probabilistic Structural Dynamics, McGraw-Hill, 1995.
[115] M. Priestley, Power spectral analysis of non-stationary random processes, Journal of Sound and Vibration 6 (1967) 86–97.
[116] G. Matz, F. Hlawatsch, W. Kozek, Generalized evolutionary spectral analysis and the Weyl spectrum of nonstationary random processes, IEEE Transactions on Signal Processing 45 (1997) 1520–1534.
[117] G. Pousse, L. F. Bonilla, F. Cotton, L. Margerin, Nonstationary stochastic simulation of strong ground motion time histories including natural variability: Application to the K-net Japanese database, Bulletin of the Seismological Society of America 96 (2006) 2103–2117.
[118] E. Savin, Influence of free field variability on linear soil-structure interaction (SSI) by indirect integral representation, Earthquake Engineering and Structural Dynamics 32 (2003) 49–69.
[119] B. Puig, F. Poirion, C. Soize, Non-gaussian simulation using Hermite polynomial expansion: convergences and algorithms, Probabilistic Engineering Mechanics 17 (2002) 253–264.
[120] F. Poirion, B. Puig, Simulation of non-gaussian multivariate stationary processes, International Journal for Numerical Methods in Engineering 45 (2010) 587–597.
[121] C. Soize, Information theory for generation of accelerograms associated with shock response spectra, Computer Methods in Applied Mechanics and Engineering 25 (2010) 334–347.
[122] G. Matz, F. Hlawatsch, Time-frequency coherence analysis of non stationary random processes, in: 10th IEEE Workshop on Statistical Signal and Array Processing, pp. 554–558.
[123] A. Zerva, V. Zervas, Spatial variation of seismic ground motions: an overview, Applied Mechanics Review 55 (2002) 271–297.
[124] R. S. Harichandran, E. H. Vanmarke, Stochastic variation of earthquake ground motion in space and time, Journal of Engineering Mechanics 112 (1986) 154–174.
[125] M. Hoshiya, K. Ishii, Evaluation of kinematic interaction of soil-foundation systems by a stochastic model, Soil Dynamics and Earthquake Engineering. 2 (1983) 128–134.
[126] J. E. Luco, H. L. Wong, Response of a rigid foundation to spatially random ground motion, Earthquake Engineering and Structural Dynamics 14 (1986) 891–908.
[127] A. L. Pais, E. Kausel, Stochastic response of rigid foundations, Earthquake Engineering and Structural Dynamics 19 (1990) 611–622.
[128] A. Der Kiureghian, A coherency model for spatially varying ground motion, Earthquake Engineering and Structural Dynamics 25 (1996) 99–111.
[129] E. Kausel, A. Pais, Stochastic deconvolution of earthquake motions, Journal of Engineering Mechanics 113 (1987) 266–277.
[130] B. L. N. Kennett, Seismic wave propagation in stratified media, Cambridge University Press, 1983.
[131] M. Schevenels, S. Francois, G. Degrande, An elastodynamics toolbox for matlab, Computers and Geosciences 35 (2009) 1752–1754.
[132] R. S. Harichandran, Stochastic analysis of a rigid foundation filtering, Earthquake Engineering and Structural Dynamics 15 (1987) 889–899.
[133] A. S. Veletsos, A. M. Prasad, Seismic interaction of structures and soils: stochastic approach, Journal of Structural Engineering 115 (1989) 935–956.
[134] G. N. Bycroft, Soil-foundation interaction and differential ground motion, Earthquake Engineering and Structural Dynamics 8 (1980) 397–404.
[135] J. E. Luco, A. Mita, Response of circular foundation to spatially random ground motion, Journal of Engineering Mechanics 113 (1987) 1–15.
[136] S. Sarkani, L. D. Lutes, S. Jin, C. Chan, Stochastic analysis of seismic structural response with soil-structure interaction, Structural Engineering Mechanics 8 (1999) 53–72.
[137] R. G. Ghanem, W. Brzakala, Stochastic finite-element analysis of soil layers with random interface, Journal of Engineering Mechanics 122 (1996) 361–369.
[138] E. Savin, Effet de la variabilité du sol et du champ incident en interaction sismique sol-structure (Effect of soil and incident field variability on seismic soil structure interaction), Ph.D. thesis, Ecole Centrale Paris, France, 1999. In French.
[139] R. Cottereau, H. Ben Dhia, D. Clouteau, Localized modeling of uncertainty in the Arlequin framework, in: R. Langley, A. Belyaev (Eds.), Vibration Analysis of Structures with Uncertainties, IUTAM Bookseries, Springer, 2010, pp. 477–488.
[140] R. Cottereau, D. Clouteau, H. Ben Dhia, C. Zaccardi, A stochastic-deterministic coupling method for continuum mechanics, Computer Methods in Applied Mechanics and Engineering 200 (2011) 3280–3288.
[141] G. D. Manolis, C. Z. Karakostas, A Green's function method to SH-wave motion in a random continuum, Engineering Analyses with Boundary Elements 27 (2003) 93–100.
[142] M. Schevenels, G. Lombaert, G. Degrande, S. Francois, A probabilistic assessment of resolution in the SASW test and its impact on the prediction of ground vibrations, Geophysical Journal International 172 (2008) 262–275.
[143] M. Schevenels, G. Lombaert, G. Degrande, D. Degrauwe, B. Schoors, The Green's functions of a vertically inhomogeneous soil with a random dynamic shear modulus, Probabilistic Engineering Mechanics 22 (2007) 100–111.
[144] R. Honda, Stochastic BEM with spectral approach in elastostatic and elastodynamic problems with geometrical uncertainty, Engineering Analyses with Boundary Elements 29 (2005) 415–427.
[145] E. Melerski, Thin shell foundation resting on a stochastic soil, Journal of Structural Engineering 114 (1988) 2692–2709.
[146] F. Toubalem, B. Zeldin, F. Thouverez, P. Labbé, Vertical excitation of stochastic soil-structure interaction systems, Journal of Geotechnical

and Geoenvironmental Engineering 125 (1999) 349–356.
[147] S. Jin, L. D. Lutes, S. Sarkani, Response variability for a structure with soil-structure interactions and uncertain soil properties, Probabilistic Engineering Mechanics 15 (2000) 175–183.
[148] L. D. Lutes, S. Sarkani, S. Jin, Response variability of an SSI system with uncertain structural and soil properties, Engineering Structures 22 (2000) 605–620.
[149] D. Breysse, H. Niandou, S. Elachachi, L. Houy, A generic approach to soil-structure interaction considering the effects of soil heterogeneity, Géotechnique 55 (2005) 143–150.
[150] F. Toubalem, F. Thouverez, P. Labbé, Contribution in the study of a transfer function in a medium with random characteristics, in: M. Lemaire, J.-L. Favre, A. Mébarki (Eds.), Applications of Statistics and Probability - Civil Engineering and Reliability. Proceedings of the ICASP7 Conference, A.A. Balkema, 1995, pp. 1197–1201.
[151] R. Cottereau, D. Clouteau, C. Soize, Parametric and nonparametric models of the impedance matrix of a random medium, European Journal of Computational Mechanics 17 (2008) 881–892.
[152] R. Cottereau, D. Clouteau, C. Soize, Probabilistic impedance of foundation: impact on the seismic design on uncertain soils, Earthquake Engineering and Structural Dynamics 37 (2008) 899–918.
[153] H. A. Pedersen, B. L. Brun, D. Hatzfeld, M. Campillo, P.-Y. Bard, Ground motion amplitude across ridges, Bulletin of the Seismological Society of America 84 (1994) 1786–1800.
[154] H. A. Pedersen, F. J. Sanchez-Sesma, M. Campillo, Three-dimensional scattering by two-dimensional topographies, Bulletin of the Seismological Society of America 84 (1994) 1169–1183.
[155] A. Zerva, On the spatial variation of seismic ground motions and its effects on lifelines, Engineering Structures 16 (1994) 534–546.
[156] D. Aubry, D. Clouteau, G. Bonnet, Modelling of wave propagation due to fixed or mobile dynamic sources, in: N. Chouw, G. Schmid (Eds.), Workshop Wave '94, Wave propagation and Reduction of Vibrations, Berg-verlag, 1994, pp. 109–121.
[157] G. Lombaert, G. Degrande, D. Clouteau, Numerical modelling of free field traffic induced vibrations, Soil Dynamics and Earthquake Engineering. 19 (2000) 473–488.
[158] D. Clouteau, D. Aubry, M.-L. Elhabre, E. Savin, Periodic and stochastic BEM for large structures embedded in an elastic half-space, Chapman & Hall, CRC Press, London, pp. 91–102.
[159] D. Clouteau, D. Aubry, M.-L. Elhabre, Periodic BEM and FEM-BEM coupling : application to seismic behaviour of very long structures, Computational Mechanics 25 (2000) 567–577.
[160] H. Chebli, R. Othman, L. Schmitt, Dynamic response of high-speed ballasted railway tracks: 3D periodic model and in situ measurements, Soil Dynamics and Earthquake Engineering. 28 (2008) 118–131.
[161] A. Stamos, D. Beskos, 3-D seismic response analysis of long lined tunnels in half-space, Soil Dynamics and Earthquake Engineering. 15 (1996) 111–118.
[162] R. N. Hwang, J. Lysmer, Response of buried structures to traveling waves, Journal of Geotechnical and Geoenvironmental Engineering 107 (1981) 183–200.
[163] F. C. de Barros, J. E. Luco, Diffraction of obliquely incident waves by a cylindrical cavity embedded in a layered viscoelastic half-space, Soil Dynamics and Earthquake Engineering. 12 (1993) 159–171.
[164] K. Khair, S. Datta, A. Shah, Amplification of obliquely incident seismic waves by cylindrical alluvial valleys of arbitrary cross-sectional shape. Part I. Incident P and SV waves, Bulletin of the Seismological Society of America 79 (1989) 610–630.
[165] A. Metrikine, K. Popp, Instability of vibrations of an oscillator moving along a beam on an elastic half-space, European Journal of Mechanics A/Solids 18 (1999) 331–349.
[166] X. Sheng, C. J. Jones., M. Petyt, Ground vibration generated by a harmonic load acting on a railway track, Journal of Sound and Vibration 225 (1999) 3–28.
[167] J. A. Forrest, H. E. Hunt, A three-dimensional tunnel model for calculation of train-induced ground vibration, Journal of Sound and Vibration 294 (2006) 678–705.
[168] M. F. Hussein, H. E. Hunt, A numerical model for calculating vibration from a railway tunnel embedded in a full-space, Journal of Sound and Vibration 305 (2007) 401–431.
[169] A. J. Tadeu, E. Kausel, Green's functions for two-and-a-half-dimensional elastodynamic problems, Journal of Engineering Mechanics 126 (2000) 1093–1096.
[170] X. Sheng, D. J. Jones, D. J. Thompson, Prediction of ground vibration from trains using the wavenumber finite and boundary element methods, Journal of Sound and Vibration 293 (2006) 575–586.
[171] L. Andersen, C. J. Jones, Coupled boundary and finite element analysis of vibration from railway tunnels – a comparison of two- and three-dimensional models, Journal of Sound and Vibration 293 (2006) 611–625.
[172] L. Andersen, S. Nielsen, Reduction of ground vibration by means of barriers or soil improvement along a railway track, Soil Dynamics and Earthquake Engineering. 25 (2005) 701–716.
[173] S. François, M. Schevenels, G. Lombaert, P. Galvín, G. Degrande, A 2.5D coupled FE-BE methodology for the dynamic interaction between longitudinally invariant structures and a layered halfspace, Computer Methods in Applied Mechanics and Engineering 199 (2010) 1536–1548.
[174] D. Clouteau, D. Aubry, Modifications of the ground motion in dense urban areas, Journal of Computational Acoustics 9 (2001) 1659–1675.
[175] G. Floquet, Sur les équations différentielles linéaires à coefficients périodiques (On linear differential equations with periodic coefficients), Annales Scientifiques de l'Ecole Normale, Série 2 12 (1883) 47–88. In French.
[176] A. Bossavit, Symmetry, groups and boundary value problems : a progressive introduction to non commutative harmonic analysis of partial differential equations in domains with geometrical symmetry, Computer Methods in Applied Mechanics and Engineering 56 (1986) 167–215.
[177] J. Guy, B. Mangeot, Use of group theory in various integral equations, SIAM Journal on Applied Mathematics 40 (1981) 390–399.
[178] D. Aubry, D. Clouteau, J. Svay, The seismic horizontal borehole problem, a mixed BEM-FEM modal approach in the frequency domain, in: 2nd International Conference on Mathematical and Numerical Aspects of Wave Propagation, SIAM, Philadelphia, 1993, pp. 10–19.

[179] J. P. Rizzo, D. J. Shippy, A boundary integral approach to potential and elasticity problems for axisymmetric bodies with arbitrary boundary conditions, Mechanics Research Communications 6 (1979) 99–103.

[180] J. M. Emperador, J. Dominguez, Dynamic response of axisymmetric embbedded foundations, Earthquake Engineering and Structural Dynamics 18 (1989) 1105–1117.

[181] H. Wang, P. K. Banerjee, Multi-domain general axisymmetric stress analysis by BEM, International Journal for Numerical Methods in Engineering 30 (1990) 115–131.

[182] R. J. Apsel, J. E. Luco, Torsional response of a rigid embedded foundation, ASCE Journal of the Engineering Mechanics Division 102 (1976) 957–970.

[183] M. Bouchon, A numerical simulation of the acoustic and elastic wavefields radiated by a source on a fluid-filled borehole embedded in a layered medium, Geophysics 58 (1993) 475–481.

[184] T. Abboud, V. Mathis, J.-C. Nedelec, Diffraction of an electromagnetic travelling wave by a periodic structure, in: G. Cohen (Ed.), Third international conference on Mathematical and Numerical Aspects of Wave Propagation, SIAM, 1995, pp. 412–421.

[185] J. Svay, Equations intégrales en espace à invariance unidirectionnelle. Application à la modélisation de la sismique en puits horizontal (2.5D integral equations: application to horizontal borehole geophysics), Ph.D. thesis, Ecole Centrale Paris, France, 1995. In French.

[186] D. Clouteau, G. Degrande, G. Lombaert, Some theoretical and numerical tools to model traffic induced vibrations, in: N. Chouw, G. Schmid (Eds.), WAVE2000 - Wave Propagation, Moving Load, Vibration Reduction, Balkema, Rotterdam, 2000, pp. 13–27.

[187] D. Clouteau, G. Degrande, G. Lombaert, Numerical modelling of traffic induced vibrations, Meccanica 36 (2001) 401–420.

[188] S. Gupta, M. F. M. Hussein, G. Degrande, H. E. M. Hunt, D. Clouteau, A comparison of two numerical models for the prediction of vibrations from underground railway traffic, Soil Dynamics and Earthquake Engineering. 27 (2007) 608–624.

[189] G. Lombaert, D. Clouteau, The resonant multiple wave scattering in the seismic response of a city, Waves in Random and Complex Media 16 (2006) 205–230.

[190] G. Lombaert, D. Clouteau, Elastodynamic wave scattering by finite-sized resonant scatterers at the surface of a horizontally layered halfspace, Journal of the Acoustical Society of America 125 (2009) 2041–2052.

[191] M. F. M. Hussein, H. E. M. Hunt, L. Rikse, S. Gupta, G. Degrande, J. P. Talbot, S. Francois, M. Schevenels, Using the PiP model for fast calculation of vibration from a railway tunnel in a multi-layered half-space, in: B. Schulte-Werning, D. Thompson, P. E. Gautier, C. Hanson, B. Hemsworth, J. Nelson, T. Maeda, P. DeVos (Eds.), Noise and Vibration Mitigation for Rail transportation systems, volume 99, pp. 136–142. 9th International Workshop on Railway Noise, Munich, Germany, 2007.

[192] D. Aubry, D. Clouteau, J. Svay, Tube waves in horizontal boreholes, in: Proc. SEG Los Angeles, volume BG1.8, pp. 28–31.

[193] A. Baroni, Modélisation du couplage sol-fluide pour la sismique entre puits (Modelling of soil-fluid coupling for cross-hole geophysics), Ph.D. thesis, Ecole Centrale Paris, France, 1996. In French.

[194] D. Clouteau, A. Baroni, D. Aubry, Boundary integrals and ray method coupling for seismic borehole modeling, in: J. A. DeSanto (Ed.), Fourth international conference on Mathematical and Numerical Aspects of Wave Propagation, p. 768.

[195] A.-S. Bonnet-Ben Dhia, A. Tillequin, A generalized mode matching method for the junction of open waveguides, in: A. Bermudez (Ed.), Fifth international conference on Mathematical and Numerical Aspects of Wave Propagation, SIAM, 2000, pp. 399–402.

[196] S. Fliss, P. Joly, Exact boundary conditions for time-harmonic wave propagation in locally perturbed periodic media, Applied Numerical Mathematics 59 (2009) 2155–2178.

[197] J. Dompierre, Equations intégrales en axisymétrie généralisée. Application à la sismique entre puits (Axisymmetric Boundary integral equations: application to cross-hole geophysics), Ph.D. thesis, Ecole Centrale Paris, France, 1993. In French.

[198] K. Adolfsson, B. Andréasson, P.-E. Bengtson, A. Bodare, C. Madshus, R. Massarch, G. Wallmark, P. Zackrisson, High speed lines on soft ground. Evaluation and analyses of measurements from the West Coast Line, Technical Report, Banverket, Sweden, 1999.

[199] A. Metrikine, S. Verichev, J. Blauwendraad, Stability of a two-mass oscillator moving on a beam supported by a visco-elastic half-space, International Journal of Solids and Structures 42 (2005) 1187–1207.

[200] G. Eason, J. Fulton, I. Sneddon, The generation of waves in an infinite elastic solid by variable body forces, Philosophical Transactions of the Royal Society of London A248 (1956) 575–607.

[201] D. Ang, Transient motion of a line load on the surface of an elastic half-space, Quaterly Journal of Applied Mathematics 18 (1960) 251–256.

[202] J. Mandel, A. Avramesco, Déplacements produits par une charge mobile à la surface d'un semi-espace élastique (Displacements produced by a moving load above an elastic half-space), Comptes rendus hebdomadaires des séances de l'académie des sciences Paris. Serie D: Sciences Naturelles 252 (1961) 3730–3732. In French.

[203] M. Papadopoulos, The use of singular integrals in wave propagation problems; with application to the point source in a semi-infinite elastic medium, Philosophical Transactions of the Royal Society of London A276 (1963) 204–237.

[204] R. Payton, An application of the dynamic Betti-Rayleigh reciprocal theorem to moving point loads in elastic media, Quaterly Journal of Applied Mathematics 21 (1964) 299–313.

[205] G. Eason, The stresses produced in a semi-infinite solid by a moving surface force, International Journal of Engineering Sciences 2 (1965) 581–609.

[206] D. Lansing, The displacements in an elastic half-space due to a moving concentrated normal load, NASA TR R-238, Langley Research Center, National Aeronautics and Space Administration, 1966.

[207] G. Müller, Bewegter Streifenlast auf dem elastisch-isotropen Halbraum (Moving load over an isotropic elastic half-space), Acustica 72 (1990) 47–53. In German.

[208] G. Müller, H. Huber, Dynamische Bodenbeanspruchungen infolge bewegter Lasten (Dynamic reaction of soils under moving loads), Bauingenieur 66 (1991) 375–380. In German.

[209] F. C. de Barros, J. E. Luco, Moving Green's functions for a layered visco-elastic half-space, Technical Report, Department of Applied Mechanics and Engineering Sciences of the University of California, La Jolla, 1992.

[210] F. C. de Barros, J. E. Luco, Response of a layered visco-elastic halfspace to a moving point load, Wave Motion 19 (1994) 189–210.

[211] C. H. Chua, T. Balendra, K. W. Lo, Groundborne vibrations due to trains in tunnels, Earthquake Engineering and Structural Dynamics 21 (1992) 445–460.

[212] M. Heckl, G. Hauck, R. Wettschureck, Structure-borne sound and vibration from rail traffic, Journal of Sound and Vibration 193 (1996) 175–184.
[213] D. J. Thompson, B. Hemsworth, N. Vincent, Experimental validation of the twins prediction program for rolling noise, part 1: Description of the model and method, Journal of Sound and Vibration 193 (1996) 123–135.
[214] D. J. Thompson, P. Fodiman, H. Mahé, Experimental validation of the twins prediction program for rolling noise, part 2: results, Journal of Sound and Vibration 193 (1996) 137–147.
[215] K. Knothe, S. L. Grassie, Modelling of railway track and vehicle/track interaction at high frequencies, Vehicle Structural Dynamics 22 (1993) 209–262.
[216] J. C. Nielsen, A. Igeland, Vertical dynamic interaction between train and track-influence of wheel and rail imperfections, Journal of Sound and Vibration 187 (1995) 825–839.
[217] G. Lombaert, G. Degrande, B. Vanhauwere, B. Vandeborght, S. François, The control of ground borne vibrations from railway traffic by means of continuous floating slabs, Journal of Sound and Vibration 297 (2006) 946–961.
[218] J. A. Forrest, H. E. Hunt, Ground vibration generated by trains in underground tunnels, Journal of Sound and Vibration 294 (2006) 706–736.
[219] T. Ekevid, N.-E. Wiberg, Wave propagation related to high-speed train. A scaled boundary FE-approach for unbounded domains, Computer Methods in Applied Mechanics and Engineering 191 (2002) 3947–3964.
[220] D. Clouteau, M. Arnst, T. Al-Hussaini, G. Degrande, Freefield vibrations due to dynamic loading on a tunnel embedded in a stratified medium, Journal of Sound and Vibration 283 (2005) 173–199.
[221] K. Müller, H. Grundmann, S. Lenz, Nonlinear interaction between a moving vehicle and a plate elastically mounted on a tunnel, Journal of Sound and Vibration 310 (2008) 558 – 586.
[222] X. Sheng, C. J. Jones, D. J. Thompson, A comparison of a theoretical model for quasi-statically and dynamically induced environmental vibration from trains with measurements, Journal of Sound and Vibration 267 (2003) 621–635.
[223] X. Sheng, C. J. Jones, D. J. Thompson, A theoretical model for ground vibration from trains generated by vertical track irregularities, Journal of Sound and Vibration 272 (2004) 937–965.
[224] J.-P. Groby, C. Tsogka, A. Wirgin, Simulation of seismic response in a city-like environment, Soil Dynamics and Earthquake Engineering. 25 (2005) 487–504.
[225] J. Bielak, K. Loukakis, Y. Hisada, C. Yoshimura, Domain reduction method for three-dimensional earthquake modeling in localized regions, part I: theory, Bulletin of the Seismological Society of America 93 (2003) 817–824.
[226] N. Mezher, Modélisation numérique et quantification de l'effet sismique Site-Ville (Numerical modeling and quantification of city-site effect), Ph.D. thesis, Ecole Centrale Paris, France, 2004. In French.
[227] O. Ishizawa, Diffraction élasto-dynamique en milieu aléatoire : application à l'étude de l'effet site-ville (Elasto-dynamic diffraction in random media: application to the study of city-site effect), Ph.D. thesis, Ecole Centrale Paris, France, 2006. In French.
[228] G. Lombaert, D. Clouteau, A comparison of the city-site effect for the case of Nice and Mexico City, in: C. Soize, G. I. Schüeller (Eds.), EURODYN 2005: Proceedings of the 6th International Conference on Structural Dynamics, Millpress Science Publishers, 2005, pp. 1329–1334.
[229] M. Strasberg, D. Feit, Vibration damping of large structures induced by attached small resonant structures, Journal of the Acoustical Society of America 99 (1996) 335–344.
[230] P. Sheng, Z. Q. Zhang, B. White, G. Papanicolaou, Multiple scattering noise in one dimension: universality through localization-length scaling, Physical Review Letters 57 (1986) 1000–1003.
[231] M. Castanier, C. Pierre, Individual and interactive mechanisms for localization and dissipation in mono-coupled nearly-periodic structure, Journal of Sound and Vibration 168 (1993) 479–505.
[232] E. Sanchez-Palencia, Non-homogeneous media and vibration theory, Springer-Verlag, 1980.
[233] C. Boutin, P. Roussillon, Assessment of the urbanization effect on seismic response, Bulletin of the Seismological Society of America 94 (2004) 251–268.
[234] C. Soize, A model and numerical-method in the medium frequency-range for vibroacoustics predictions using the theory of structural fuzzy, Journal of the Acoustical Society of America 94 (1993) 849–865.
[235] R. L. Weaver, Equipartition and mean-square responses in large undamped structures, Journal of the Acoustical Society of America 110 (2001) 894–903.
[236] A. Askar, A. S. Cakmak, Seismic waves in random media, Probabilistic Engineering Mechanics 3 (1988) 124–129.
[237] M. Lax, Multiple scattering of waves, Physical Review 23 (1951) 287–310.
[238] M. Lax, Multiple scattering of waves. II. the effective field in dense systems, Physical Review 85 (1952) 621–629.
[239] U. Frisch, La propagation des ondes en milieu aléatoire et les équations stochastiques (Wave propagation in random media and stochastic equations), Annales d'Astrophysique 29 (1966) 645–682. In French.
[240] L. Ryzhik, G. Papanicolaou, J. B. Keller, Transport equations for elastic and other waves in random media, Wave Motion 24 (1996) 327–370.
[241] P. Sheng, Introduction to wave scattering, localization and mesoscopic phenomena, Academic Press, 1998.
[242] L. L. Foldy, The multiple scattering of waves. I. general theory of isotropic scattering by randomly distributed scatterers, Physical Review 67 (1945) 107–119.
[243] D. Clouteau, E. Savin, D. Aubry, Dynamic soil-structure interaction : stochastic simulations, Meccanica 36 (2001) 379–399.
[244] S. Chandrasekhar, Radiative Transfer, Oxford University Press, 1960.
[245] C. Farhat, A. Macedo, M. Lesoinne, F.-X. Roux, F. Magoules, A. de la Bourdonnaie, Two-level domain decomposition methods with Lagrange multipliers for the fast iterative solution of acoustic scattering problems, Computer Methods in Applied Mechanics and Engineering 184 (2000) 213–239.
[246] Y. Maday, A. T. Patera, Spectral element methods for the incompressible Navier-Stokes equations, in: A. K. Noor, J. T. Oden (Eds.), State-of-the-art surveys on computational mechanics, ASME, 1989, pp. 71–143.
[247] G. Festa, J.-P. Vilotte, The Newmark scheme as a velocity-stress time staggering: An efficient PML for spectral element simulations of

elastodynamics, Geophysical Journal International 161 (2005) 789–812.
[248] Q.-A. Ta, Modélisation des propriétés mécaniques anisotropes aléatoires et impacts sur la propagation des ondes élastiques (Modeling of random anisotropic mechanical properties and impacts on elastic wave propagation), Ph.D. thesis, Ecole Centrale Paris, Châtenay-Malabry, France, 2010. In French.
[249] R. Glowinski, T. W. Pan, J. Periaux, A fictitious domain method for Dirichlet problem and applications, Computer Methods in Applied Mechanics and Engineering (1994) 283–303.
[250] H. Ben Dhia, Multiscale mechanical problems: the Arlequin method, Comptes Rendus de l'Académie des Sciences de Paris, Série II b 326 (1998) 899–904.