Time scaling control for an underactuated biped robot

Christine Chevallereau

Time scaling control for an underactuated biped robot

C. Chevallereau

***Abstract*— This paper presents a control law for the tracking of a cyclic reference trajectory by an under-actuated biped robot. The robot studied is a five-link planar robot. The degree of under-actuation is one during the single support phase. The control law is defined in such a way that only the geometric evolution of the robot is controlled, not the temporal evolution. To achieve this objective, we consider a time scaling control. But, for a given reference path, the temporal evolution during the geometric tracking is completely defined and can be analyzed through the study of the dynamic model. A simple analytical condition is deduced that guarantees convergence towards the cyclic reference trajectory. This condition implies temporal convergence after the geometric convergence. This condition is defined on the cyclic reference path. The control law is stable if the angular momentum around the contact point is greater at the end of the single support phase than at the beginning of the single support phase.**

***Index Terms*— Biped robot, under-actuated system, control, stability, walking, limit cycle.**

I. Introduction

The reduction of the number of actuators of a walking robot is a step towards simpler and cheaper robots. The suppression of the feet and of the actuated ankle seems to be a reasonable way to simplify the mechanical design of the robot. But a consequence is that only purely dynamic walking can be achieved, i.e motion without mechanical equilibrium (neither static nor dynamic). We choose to study a planar biped with only four actuators, two on the haunch, two on the knees. During the single support phase, the configuration of the robot is defined by five independent variables but there are only four actuators. Hence, the robot is under-actuated. This simplification in terms of mechanics makes the design of the control law difficult.

One classical way to control a robotic system consists in two steps. During the first step, an open loop joint reference trajectory is designed. In the second step, a control law is defined to track this reference trajectory. In such a context, a reference trajectory was obtained by an optimization technique for the biped [1] and now a new control law is proposed in this paper for the second step.

There exist various studies about the control of an under-actuated biped. A first method is based on the definition of the reference trajectory for l outputs (where l is the number of actuators), not as a function of time but as a function of a configuration variable independent of the l outputs. With such a control, the configuration of the robot at impact time is the desired configuration but its velocities can differ from the desired velocities. The convergence of the motion towards a cyclic motion is studied numerically using the Poincaré criterion [2], [3]. Another approach involves parameterized reference trajectories. In that case, one derivative of the parameter is used as a supplementary input, as it was shown in [4]-[7]. In [4], the parameter is used to satisfy some constraints on the reaction between the feet and the ground. In [6], a parameter involved in the zero dynamics is used as a supplementary input.

In the present paper like in [2], [3], only the geometric evolution of the robot is controlled, not the temporal evolution. To achieve this objective, a set of reference trajectories parameterized by a virtual time is considered. A time scaling control is defined as in [9] and [10]. The second derivative of the virtual time is considered as a supplementary control input. Thus, we deal with a model for which the number of inputs and the number of independent configuration variables are equal. A non linear control law is defined to ensure the geometrical tracking of the reference path. After the convergence of this control law, the study of the dynamic model provides a relation between the virtual time, its derivatives and the reference path; the evolution of the virtual time can be analyzed. A condition to ensure a convergence towards the cyclic reference trajectory is deduced from this analysis. This condition is defined on the reference path. For the optimal walking of a biped presented in [1] this condition is naturally satisfied. The domain of attraction is also studied; its size also depends on the reference path.

In section 2, the modelling of the robot and an optimal reference trajectory are presented. In section 3, the control law is defined. The evolution of the virtual time is analyzed in section 4 and a condition of convergence is deduced.

Manuscript received ; revised . This work was supported by the CNRS Robea Program

The author is with the Institut de Recherche en Communications et Cybernétique de Nantes, France (e-mail:Christine.Chevallereau@irccyn.ec-nantes.fr)

II. The robot modelling

A. The robot

The biped we consider walks in a vertical *xz* plane. It is composed of a torso and two identical legs. Each leg is composed of two links articulated by a knee. The knees and the hips are one-degree-of-freedom rotational joints. The walk is composed of single support phases separated by impact phases. During the single support phase, the vector $q = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)^T$ (Fig. 1) describes the configuration of the biped. Variables θ_1 and θ_2 describe the stance leg and variables θ_4 and θ_5 describe the swing leg.

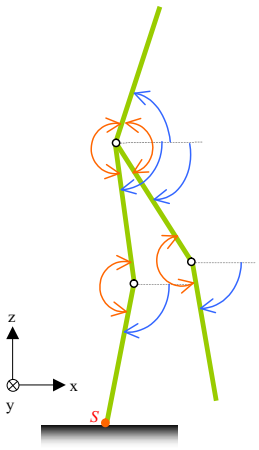

Fig. 1. The studied biped

All links are assumed massive and rigid. The lengths of the thighs and of the shins are 0.4 m. However, their masses are different: 6 Kg for each thigh and 4 kg for each shin. The length of the torso is 0.625 m and its mass is 20 kg. These values correspond to the Robot Rabbit characteristics [11]. The inertia of the links is also taken into account. Γ is the vector of the torques applied at the hip and at the knee joints.

B. Dynamic modelling

The walking studied is composed of single support phases separated by instantaneous double support phases. In single support, the dynamic model can be written as:

$$A(q)\ddot{q} + H(q,\dot{q}) = D\Gamma \tag{1}$$

where A(5 x 5) is the inertia matrix, H(5 x 1) is the vector of Coriolis, centrifugal and gravity effects and D is a (5 x 4) matrix. The

number of torques is four but there are five independent configuration variables. Thus, a relation independent of the torques can be written. A simple way to find such a relation is to write the equilibrium of the robot in rotation around the stance leg tip S. Because the motion of the biped is planar, the angular momentum is perpendicular to the motion plane. It is denoted σ. The torque around S is produced by the gravity effects only:

$$\dot{\sigma} = mg(x_g - x_s) \tag{2}$$

where m is the mass of the robot, g is the gravity acceleration, x_s is the abscissa of the contact point, x_g is the abscissa of the robot mass center. This equation is characteristic of the under-actuated motion of the robot. It can also be obtained from the dynamic model (1). The matrix D is a (5×4) full rank matrix. Thus there exist (1×5) matrices denoted $D^{\perp}$ such that $D^{\perp}D = 0$. For a special choice of $D^{\perp}$, in fact, σ can be calculated by:

$$\sigma = D^{\perp}A(q)\dot{q} \tag{3}$$

When the swing leg j touches the ground at the end of single support, an impact takes place. This impact is assumed instantaneous and inelastic. This means that the velocity of foot j in contact with the ground is zero after the impact. But during the impact the stance leg i (with $i \neq j$) takes off. The velocities just before and just after impact, denoted $\dot{q}^-$ and $\dot{q}^+$ respectively, are related by a linear equation [12]:

$$\dot{q}^+ = I(q)\dot{q}^- \tag{4}$$

C. A cyclic reference trajectory

The proposed control will be illustrated for the tracking of a cyclic trajectory. This cyclic trajectory has been obtained by an optimization technique described in [1]. The motion velocity of the robot is $1m/s$ and the integral of the norm of the torque is minimized according to the actuators limits. The duration of a step is denoted T. The legs swap roles from one step to the next, thus the optimal trajectory q_r is such that:

$$q_r(T^+) = Eq_r(T^-) \tag{5}$$

where E allows to take into account the exchange of legs, $+$ and $-$ designed respectively just after and just before the impact. The optimal trajectory is known by numerical values for a given sampling time. For each time instant between 0 and T, the joint configuration $q_r(t)$, the joint velocity $\frac{dq_r(t)}{dt}$ and the joint acceleration $\frac{d^2q_r(t)}{dt^2}$ are recorded. The motion is cyclic, thus:

$$q_r(t + kT) = q_r(t)$$

where $0 < t < T$ and k is a positive integer. This relation permits to define the optimal reference trajectory for any time $t > 0$. Condition (4) is satisfied for the impacts.

$$\frac{dq_r(T^+)}{dt} = I(q_r(T^-))\frac{dq_r(T^-)}{dt} \tag{6}$$

The stick diagram corresponding to the studied reference trajectory is presented in Fig. 2.

For this optimal trajectory, the evolution of the angular momentum σ can be calculated as a function of $q_r(t)$ and $\frac{dq_r(t)}{dt}$ and is denoted σ_r.

$$\sigma_r(t) = D^{\perp}A(q_r(t))\frac{dq_r(t)}{dt} \tag{7}$$

σ_r is periodic, it is presented in Fig. 3 for a step.

For the studied reference trajectories, the abscissa of the mass center is an increasing function. The single support begins with $x_g < x_s$ and finishes with $x_g > x_s$. During the first part of the motion, the kinetic momentum decreases (see equation (2)) and its initial

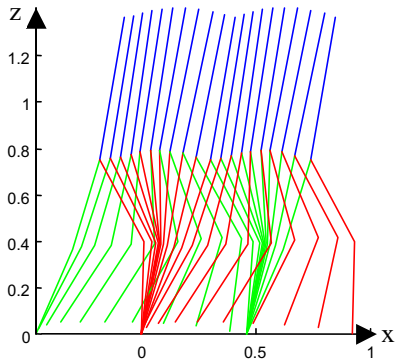

Fig. 2. Two steps of an optimal motion

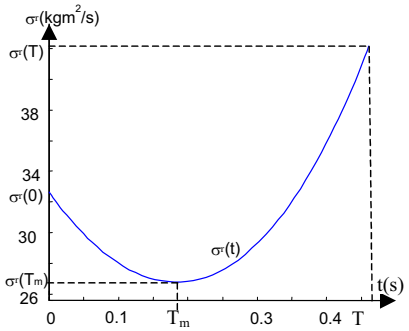

Fig. 3. Angular momentum for one step of the optimal trajectory

angular momentum must be high enough to reach a configuration such that $x_g > x_s$. Otherwise, the robot falls backwards [13]. The gravity makes the robot slow down. The instant such that $x_g = x_s$ and hence $\frac{d\sigma_r(T_m)}{dt} = 0$, is denoted T_m. After T_m, $x_g > x_s$ and the angular momentum increases. The gravity is a help for the motion. Thus, for any cyclic walking of a biped, the evolution of the kinetic momentum is close to the evolution presented in Fig. 3 and never crosses zero.

III. The Proposed Control Law

The optimal trajectory is essentially composed of single support phases. During these phases, the robot is under-actuated. The objective of the control law presented in this section is not to precisely track the reference trajectory but only the associated path: only a geometrical tracking is desired and a time scaling control [9] is used.

A. A set of reference trajectories

We consider a set of trajectories defined by:

$$\begin{cases} q_d(t) = q_r(\tau(t)) \\ \dot{q}_d(t) = \frac{dq_r(\tau(t))}{d\tau}\dot{\tau} \\ \ddot{q}_d(t) = \frac{dq_r(\tau(t))}{d\tau}\ddot{\tau} + \frac{d^2q_r(\tau(t))}{d\tau^2}\dot{\tau}^2 \end{cases} \tag{8}$$

where τ is a function of time called "virtual time". Given $\tau(t)$, a unique trajectory is defined. Any trajectory defined by (8) corresponds to the same path in the joint space as the optimal trajectory, but the evolution of the robot with respect to time may differ. The optimal trajectory belongs to the considered set of reference trajectories with: $\tau = t$, $\dot{\tau} = 1$, $\ddot{\tau} = 0$. The desired configuration of the robot at time t is the optimal configuration at time $\tau(t)$. Thus, for all trajectories defined by (8), the free leg tip does not touch the ground for any time t such that $\tau(t) < T$. But at time t_1 such that $\tau(t_1) = T$, the height of the free leg tip is zero and an impact with the ground takes place. As a

consequence, an impact occurs for step k at time t_k such that $\tau(t_k) = kT$. The configuration at impact is the same as the configuration of the optimal trajectory $q_d(t_k) = q_r(kT)$ but the velocities can differ from the optimal velocities. All the reference trajectories (8) satisfy impact equation (4) and $\dot{\tau}$ is continuous at the impact time. This property is obtained because (i) the optimal trajectory is cyclic and takes into account the impact and (ii) the velocity after the impact is linear with respect to the velocity before the impact (see equation (4)). The second derivative $\ddot{\tau}$ will be treated as a supplementary control input. Thus, the control law will be designed for a system for which the number of outputs and inputs are equal. The chosen outputs are simply the five joint variables q. The control inputs are the four torques Γ plus $\ddot{\tau}$.

B. The control law

The control law is a non-linear control law classically used in robotics. But in order to obtain finite time stabilization around one of the desired trajectories, the feedback function proposed in [3], [14] is used. The tracking errors are defined with respect to the trajectories belonging to (8):

$$\begin{array}{c} e_q(t) = q_r(\tau(t)) - q(t) \\ \dot{e}_q(t) = \frac{dq_r(\tau(t))}{d\tau}\dot{\tau} - \dot{q}(t) \end{array} \tag{9}$$

The desired behavior in closed loop is:

$$\ddot{q} = \ddot{q}_d + \frac{1}{\epsilon^2}\psi \tag{10}$$

where ψ is a vector of 5 components ψ_i(the indices i denotes the i^{th} component of a vector) with:

$$\psi_i = -sign(\epsilon\dot{e}_{q_i})|\epsilon\dot{e}_{q_i}|^{\nu} - sign(\phi_i)|\phi_i|^{\nu} \tag{11}$$

and $0 < \nu < 1$, $\epsilon > 0$, $\phi_i = e_{q_i} + \frac{1}{2-\nu}sign(\epsilon\dot{e}_{q_i})|\epsilon\dot{e}_{q_i}|^{2-\nu}$,

ν and ϵ are parameters to adjust the settling time of the controller. With this choice of close loop behavior, the error e_q converges to zero in finite time [3], [14].

Taking into account expression (8) of the reference trajectory, equation (10) can be rewritten as:

$$\ddot{q} = \frac{dq_r(\tau(t))}{d\tau}\ddot{\tau} + v(\tau, \dot{\tau}, q, \dot{q}) \tag{12}$$

with

$$v(\tau, \dot{\tau}, q, \dot{q}) = \frac{d^2q_r(\tau(t))}{d\tau^2}\dot{\tau}^2 + \frac{1}{\epsilon^2}\psi$$

The dynamic model of the robot is described by equation (1), thus the control law must be such that:

$$A(q)(\frac{dq_r(\tau(t))}{d\tau}\ddot{\tau} + v(\tau, \dot{\tau}, q, \dot{q})) + H(q, \dot{q}) = D\Gamma \tag{13}$$

Because the matrix $A(q)$ is invertible, the desired closed loop behavior is obtained if equation (13) is satisfied.

But the matrix D is not invertible. D is a (5 x 4) full rank matrix, thus its pseudo inverse D^+ is such that $D^+D = I_4$. By definition of D^+ and $D^\perp$, the matrix $\begin{bmatrix} D^\perp \\ D^+ \end{bmatrix}$ is a (5 x 5) invertible matrix. Thus, equation (13) is equivalent to the following set of equations:

$$\begin{array}{l} D^\perp(A(q)(\frac{dq_r(\tau(t))}{d\tau}\ddot{\tau} + v(\tau, \dot{\tau}, q, \dot{q})) + H(q, \dot{q})) = 0 \\ D^+(A(q)(\frac{dq_r(\tau(t))}{d\tau}\ddot{\tau} + v(\tau, \dot{\tau}, q, \dot{q})) + H(q, \dot{q})) = \Gamma \end{array} \tag{14}$$

We can deduce that, in order to obtain the desired closed loop behavior, it is necessary and sufficient to choose:

$$\begin{array}{l} \ddot{\tau} = \frac{-D^\perp(A(q)v(\tau,\dot{\tau},q,\dot{q}) + H(q,\dot{q}))}{D^\perp A(q)\frac{dq_r(\tau(t))}{d\tau}} \\ \Gamma = D^+(A(q)(\frac{dq_r(\tau(t))}{d\tau}\ddot{\tau} + v(\tau, \dot{\tau}, q, \dot{q})) + H(q, \dot{q})) \end{array} \tag{15}$$

The matrix D is constant, thus D^+ and $D^\perp$ are constant and can be calculated off-line. With no singularity nor modelling error, the control law ensures that $q(t)$ converges to $q_r(\tau(t))$. With no initial error, a perfect tracking of $q_r(\tau(t))$ is obtained.

The proposed control law defines $\ddot{\tau}$. The evolution of $\tau(t)$ can be calculated (but not chosen) if $\tau(0)$ and $\dot{\tau}(0)$ are known. We choose $\tau(0) = 0$. $\dot{\tau}(0)$ is defined to minimize the error on the joint velocity $C = |\dot{q}(0) - \dot{q}_r(0)|^2 = |\dot{q}(0) - \frac{dq_r(\tau(0))}{d\tau}\dot{\tau}(0)|^2$. Thus, $\dot{\tau}(0)$ is such that $\frac{dC}{d\tau(0)} = 0$. We obtain:

$$\dot{\tau}(0) = \frac{\dot{q}(0)^t\frac{dq_r(0)}{d\tau}}{\frac{dq_r(0)}{d\tau}^t\frac{dq_r(0)}{d\tau}} \tag{16}$$

1) The singularities for this control law: A singularity occurs if $D^\perp A(q)\frac{dq_r(\tau(t))}{d\tau} = 0$. This expression is not exactly the angular momentum of the robot, which is defined by (3) but, for a trajectory belonging to the set described by equation (8), the condition becomes:

$$D^\perp A(q_r(\tau(t)))\frac{dq_r(\tau(t))}{d\tau} = 0$$

which is exactly

$$\sigma_r(\tau(t)) = 0$$

Thus, for a perfect tracking of the reference path $q(t) = q_r(\tau(t))$, no singularity occurs if the angular momentum for the optimal motion σ_r is always far from zero (see Fig. 3). Consequently, no singularity appears if the tracking errors are small enough.

IV. Convergence towards the optimal trajectory

The control law ensures that the motion of the robot will converge in a finite time towards a reference trajectory described by (8). During the first step, $\ddot{\tau}$ is calculated to zero error e_q and the temporal evolution of the robot depends on e_q. As soon as the control has converged, we have: $e_q(t) = 0$, $q(t) = q_r(\tau(t))$, $\dot{q}(t) = \dot{q}_r(\tau(t), \dot{\tau})$, $\ddot{q}(t) = \dot{q}_r(\tau(t), \dot{\tau}, \ddot{\tau})$ and these properties will be maintained for all subsequent steps. Due to the definition of the reference trajectory (see section III-A), the impact does not introduce any tracking error e_q. The geometric evolution of the robot is known, the temporal evolution of the robot can be studied. In the following section, the behavior of the robot is studied **after** the convergence of the control law that is **when the robot follows a trajectory satisfying (8)**. The robot velocity is $\dot{q}(t) = \frac{dq_r(\tau(t))}{d\tau}\dot{\tau}$ and the optimal velocity is $\frac{dq_r(\tau(t))}{d\tau}$. The difference between the two velocities is proportional to $e = \dot{\tau} - 1$. e is referred to as "velocity difference". The robot converges towards the optimal trajectory if and only if $\dot{\tau}$ converges to 1 or equivalently if and only if e converges to 0.

A. Evolution of the virtual time

During the single support phase, the robot is under-actuated. Thus it cannot follow any trajectory described by (8). The control law imposes the path of the robot in the joint space. The gravity defines the motion of the robot along this path. The motion of the robot, in turn, can be deduced from the evolution of τ. To study the evolution of τ, it is sufficient to analyze the angular momentum σ. The angular momentum σ is linear with respect to the component of $\dot{q}$ and the real velocity vector $\dot{q}(t)$ is proportional to the optimal velocity vector $\dot{q}_r(\tau(t))$. Thus, the angular momentum can be expressed by:

$$\sigma(t) = \sigma_r(\tau(t))\dot{\tau} \tag{17}$$

Using this equation, the derivative of the angular momentum can be written as:

$$\dot{\sigma}(t) = \frac{d\sigma_r(\tau(t))}{d\tau}\dot{\tau}^2 + \sigma_r(\tau(t))\ddot{\tau} \tag{18}$$

But the derivative of the angular momentum depends only on the configuration of the robot (see equation (2)). Since the configuration of the robot is the optimal one $q(t) = q_r(\tau(t))$, we have:

$$\dot{\sigma}(t) = mg(x_g(q(t)) - x_s) = mg(x_g(q_r(\tau(t))) - x_s) \quad (19)$$

We can also write equation (2) for the optimal trajectory. Thus, we deduce that: $\dot{\sigma}(t) = \frac{d\sigma_r(\tau(t))}{d\tau}$ and equation (18) becomes:

$$\frac{d\sigma_r(\tau(t))}{d\tau}(\dot{\tau}^2 - 1) + \sigma_r(\tau(t))\ddot{\tau} = 0 \quad (20)$$

The "velocity difference" was defined as: $e = \dot{\tau} - 1$, so its derivative is $\dot{e} = \ddot{\tau}$. From equation (20), the behavior of the velocity difference is:

$$\dot{e} = -\frac{\frac{d\sigma_r(\tau(t))}{d\tau}}{\sigma_r(\tau(t))}(\dot{\tau} + 1)e \quad (21)$$

Since $(\dot{\tau} + 1) = \dot{\tau}(\frac{2+e}{1+e})$, equation (21) is rewritten as:

$$\frac{\dot{e}}{e}(\frac{1+e}{2+e}) = -\frac{\dot{\sigma}_r(\tau(t))}{\sigma_r(\tau(t))} \quad (22)$$

σ_r is a periodic function, but σ_r is discontinuous at the impact time. So equation (22) can only be integrated over one step. For $t_k < t < t_{k+1}$, the integration during step k gives:

$$\int_{t_k}^{t} \frac{\dot{e}(s)}{e(s)}(\frac{1+e(s)}{2+e(s)})ds = -\int_{t_k}^{t} \frac{\dot{\sigma}_r(\tau(s))}{\sigma_r(\tau(s))}ds \quad (23)$$

$$\frac{1}{2}[Log((e+1)^2 - 1)]_{t_k}^{t} = -[Log(\sigma_r(\tau))]_{t_k}^{t} \quad (24)$$

To simplify the notation, the velocity difference at the beginning of step k, $e(t_k)$, is denoted e_k. Using the initial condition, we have, for $t_k < t < t_{k+1}$:

$$e(t) = \sqrt{1 + e_k(e_k+2)\left(\frac{\sigma_r(0)}{\sigma_r(\tau(t))}\right)^2} - 1 \quad (25)$$

this function includes a square root and is defined only if $e_k > e_{min}$ with:

$$e_{min} = -1 + \sqrt{1 - \left(\frac{\sigma_r(T_m)}{\sigma_r(0)}\right)^2} \quad (26)$$

A typical evolution of $e(t)$ is presented in Fig. 4.

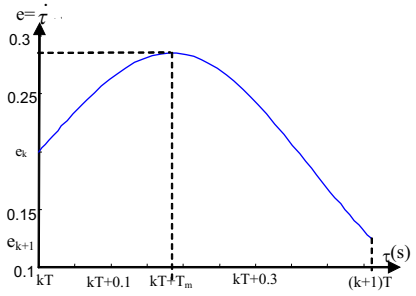

Fig. 4. Typical evolution of the velocity difference $e = \dot{\tau} - 1$ for one step

- For $kT < \tau < kT + T_m$, the mass center of the robot is behind the leg tip. Thus, the gravity makes the robot slow down, that is $\sigma_r(\tau)$ decreases and thus $|e|$ increases.
- For $\tau = kT + T_m$, $|e|$ has a maximum denoted e_k^m.

$$e(t_m^k) = \sqrt{1 + e_k(e_k+2)\left(\frac{\sigma_r(0)}{\sigma_r(T_m)}\right)^2} - 1 \quad (27)$$

- For $T_m + kT < \tau < (k+1)T$, the mass center of the robot is in front of the leg tip. Thus the gravity accelerates the motion, that is $\sigma_r(\tau)$ increases and thus $|e|$ decreases.

Just before impact $k+1$, the robot velocity satisfies (8). Thus we have:

$$\dot{q}(t_{k+1}^-) = \frac{dq_r((k+1)T^-)}{d\tau}\dot{\tau}(t_{k+1}^-)$$

After impact, the robot velocity is calculated by impact model (4):

$$\dot{q}(t_{k+1}^+) = I(q((k+1)T^-))\dot{q}(t_{k+1}^-)$$

Since the discontinuity is properly taken into account in the cyclic reference trajectory (6), we have $\dot{\tau}(t_{k+1}^-) = \dot{\tau}(t_{k+1}^+)$. The value of $\dot{\tau}$ is continuous during the impact and the velocity difference e is also continuous.

The different steps can be taken into account using the following iterative equation:

$$e_{k+1} = \sqrt{1 + e_k(e_k+2)\left(\frac{\sigma_r(0)}{\sigma_r(T)}\right)^2} - 1 \quad (28)$$

Remark: Minimum initial velocity for one step. The definition of parameterized reference trajectories implicitly assumes that the parameter is monotonic. In the case studied, the parameter is a virtual time so it must increase. In fact, if τ decreases, this means that the robot moves backwards. The evolution of τ is monotonic if $\dot{\tau} > 0$ or $e > -1$. Since $e(t)$ is expressed by equation (25), $e(t)$ is always greater than -1 when it is defined. Thus, the condition of monotonicity is $e_k > e_{min}$. For the studied optimal trajectory, the minimum velocity difference is $e_{min} = -0.42$.

B. Temporal convergence

With the proposed control for the robot and the gait studied, the motion of the robot converges to the set of reference trajectories (8) in a finite time that can be chosen to be less than the duration of one step.

Theorem 1: The robot motion tends towards the optimal reference trajectory if and only if $e_1 > e_{min}$ and $\left(\frac{\sigma_r(0)}{\sigma_r(T)}\right) < 1$.

Proof: Since $\frac{\sigma_r^2(0)}{\sigma_r^2(\tau(t))}$ is a bounded periodic function, equation (25) shows that $e(t)$ for $t_k < t < t_{k+1}$ converges to 0 if and only if e_k converges to 0. Thus, it is necessary and sufficient to prove that e_k converges to 0 asymptotically.

The error e_{k+1} is related to the error e_k by (28). Thus, by definition of the absolute value of an expression, the following inequality is satisfied:

$$\sqrt{1 + (e_k^2 - 2|e_k|)\frac{\sigma_r(0)^2}{\sigma_r(T)^2}} \leq e_{k+1}+1 \leq \sqrt{1 + (e_k^2 + 2|e_k|)\frac{\sigma_r(0)^2}{\sigma_r(T)^2}}$$

If $\frac{\sigma_r(0)}{\sigma_r(T)} < 1$, we have:

$$\sqrt{1 + e_k^2\frac{\sigma_r(0)^2}{\sigma_r(T)^2} - 2|e_k|\frac{\sigma_r(0)}{\sigma_r(T)}} \leq e_{k+1} + 1 \leq \sqrt{1 + e_k^2\frac{\sigma_r(0)^2}{\sigma_r(T)^2} + 2|e_k|\frac{\sigma_r(0)}{\sigma_r(T)}}$$

or

$$1 - |e_k|\frac{\sigma_r(0)}{\sigma_r(T)} \leq e_{k+1} + 1 \leq 1 + |e_k|\frac{\sigma_r(0)}{\sigma_r(T)}$$

it can be deduced that:

$$|e_{k+1}| \leq |e_k|\frac{\sigma_r(0)}{\sigma_r(T)}$$

or:

$$|e_{k+1}| \leq \left(\frac{\sigma_r(0)}{\sigma_r(T)}\right)^{k-1}|e_1| \quad (29)$$

thus $e_k \to 0$ when $k \to \infty$.

If $\frac{\sigma_r(0)}{\sigma_r(T)} > 1$, similar calculations give $|e_{k+1}| \geq \frac{\sigma_r(0)}{\sigma_r(T)}|e_k|$. If $|e_1| \neq 0$, $|e_k| \to \infty$ when $k \to \infty$ and the robot motion diverges.

Condition $e_1 > e_{min}$ ensures that $\tau(t)$ increases during the second step. If $\frac{\sigma_r(0)}{\sigma_r(T)} < 1$, since $e_{min} < 0$, the inequality $e_k > e_{min}$ is satisfied for all k, thus the function $\tau(t)$ increases for all $t > 0$. ■

1) Convergence ratio: The convergence of the control law can also be shown using a Poincaré section as in [2] and [3]. Equation (28) allows to draw easily the evolution of e_{k+1} as a function of e_k. The linearization of equation (28) around $e_k = 0$ defines the convergence ratio for one step around the cyclic motion. We have: $e_{k+1} = \left(\frac{\sigma_r(0)}{\sigma_r(T)}\right)^2 e_k$. The lower the ratio $\left(\frac{\sigma_r(0)}{\sigma_r(T)}\right)^2$, the faster the convergence of the control law.

2) Maximum initial velocity: The study of the control law defines some minimum initial velocity given by e_{min}. No maximum velocity appears in this study. But if the initial velocity is too large, the centrifugal forces are higher than the gravitational forces and a take-off of the robot is observed. In the case studied, the maximum velocity difference is $e_{max} = 0.505$. Consequently, the domain of attraction is defined by $-0.42 < e_1 < 0.505$. The size of the attraction domain is important for the practical application. The larger this domain, the more robust the control law. This domain is also interesting to study possible changes of velocity for the robot walking. In the definition of the cyclic reference trajectory [1], a constraint on the minimal value of the reaction force has been taken into account. A minimal value of $100N$ has been chosen. This choice has a direct effect on the maximal initial velocity.

C. Simulation results

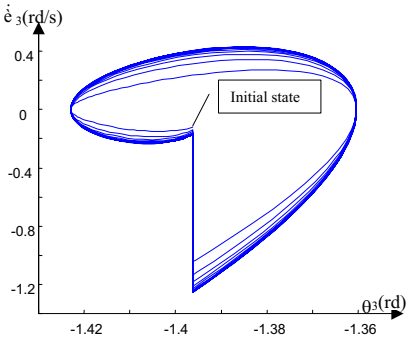

Fig. 5. Phase plane evolution of the torso orientation during 15 steps

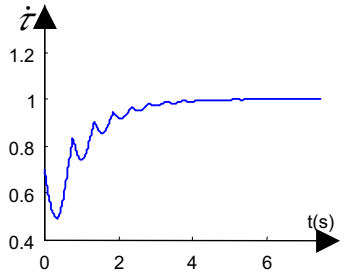

Fig. 6. Evolution of $\dot{\tau}$ during 15 steps

The behavior obtained in simulation for a large initial velocity error is presented. The initial state of the robot does not belong to the set of reference trajectories. During the first step, the tracking error on the parameterized reference trajectory converges to zero. At the end of the first step there is no configuration error but the cyclic velocity is not yet reached. After this first step, the robot follows the parameterized reference trajectory without error and converges towards the optimal trajectory. In figure 5, the torso evolution is shown for 15 steps in its phase plane. In figure 6 the evolution of $\dot{\tau}$ is presented. $\dot{\tau}$ converges towards the function $\dot{\tau} = 1$. The robot motion converges to the cyclic reference trajectory. The increasing and decreasing phases of the velocity difference $e = \dot{\tau} - 1$ for each step appear clearly. During the first step, the evolution of $\dot{\tau}$ is slightly perturbed.

D. Scope of our method

Unlike the Honda Robot [15], the robot studied has no feet. The proposed approach is developed especially for robots with point contact with the ground. Methods based on the ZMP cannot be used for such robots. There are many such devices (anchored so that they cannot fall aside) and all belong to the category of under-actuated systems in single support. They have only one passive joint. Our control law will be tested on our prototype [11] during next year. The extension to the spatial motion of a biped is straightforward if the degree of under-actuation is still one, by considering the angular momentum around the passive axis. The problem is more difficult for a higher degree of under-actuation. To the author's knowledge, this problem is still open.

For a robot that has point contact with the ground, the introduction of non-instantaneous double support phases in the walk is not very interesting from an energetic point of view. Since the contact points are fixed (the heel cannot take-off), these phases cannot prepare the subsequent single support phases. However, these phases can help to stabilize the motion because the robot is over-actuated during double supports. The proposed control law can be extended without any difficulty to the tracking of cyclic walking trajectories including finite duration double support phases (not instantaneous). The proposed strategy will be applied during single support phases only. An important characteristics of the control is that no configuration error exists at the end of single support phases. The distance between the leg tips is the expected distance. During double support phases, the robot is not under-actuated. Thus a classical control can be used. The conditions of the theorem $e_1 > e_{min}$ and $\left(\frac{\sigma_r(0)}{\sigma_r(T)}\right) < 1$ become sufficient conditions only (not necessary) because the double support phase also permits to decrease the velocity difference.

This work is presently theoretical but the relation between the reference trajectories and the performance of the control law presented in this paper is really important for practical applications. It has been shown that the ratio $\frac{\sigma_r(T)}{\sigma_r(0)}$ characterizes the convergence towards the cyclic reference trajectory. This value is essential for the proposed control law. The ratio $\frac{\sigma_r(T_m)}{\sigma_r(0)}$ is also very important because it limits the admissible velocities and constitutes a rating of robustness.

The condition for convergence, the rate of convergence and the minimal initial velocity difference are defined by very simple analytical expressions. Consequently, some bounds on these expressions can be fixed during the design of optimal trajectories to obtain robust optimal reference trajectories.

V. Conclusion

The robot studied is a planar biped mechanically simple, low cost and under-actuated during the single support phases. A cyclic reference trajectory satisfying the dynamic equations is assumed to be known. The reference trajectory is composed of single support phases separated by passive impacts. It is defined by position, velocity and acceleration as functions of time.

An original control law for the tracking of a desired joint reference trajectory has been proposed and analyzed. The degree of under-actuation is compensated by one degree of freedom in the tracking

of the reference trajectory because only a geometrical tracking is defined, not a temporal tracking. When the control has converged, the time evolution of the robot can be deduced from the dynamic model and the properties of the reference path. It has been proved that a stable tracking of the reference trajectory is obtained if and only if the reference path is such that $\frac{\sigma_r(0)}{\sigma_r(T)} < 1$ where σ_r is the angular momentum around the contact point with the ground for the cyclic motion and T is the expected duration of the single support.

The proposed control law ensures only a geometrical tracking of the reference trajectory. However, if the aforementioned property is satisfied, the temporal tracking of the reference trajectory is naturally obtained.

This result is very interesting because the condition can be easily tested or used to modify a given reference trajectory. It has been observed that, for the robot presented in this paper, the optimal motion for an energetic criterion naturally satisfies the convergence condition.

REFERENCES

[1] C. Chevallereau and Y. Aoustin. Optimal reference trajectories for walking and running of a biped robot. *Robotica*, vol.19, N5, pages 557-569, 2001.

[2] Y. Aoustin and A.M Formal'sky. Design of reference trajectory to stabilize desired nominal cyclic gait of a biped. *ROMOCO'99*, pages 159–164, Kiekrz, Poland, 1999.

[3] J. W. Grizzle, G. Abba, and F. Plestan. Asymptotically stable walking for biped robots: analysis via systems with impulse effects. *IEEE Trans. on Automat. Contr.*, vol.46, N1, pages 51–64, 2001.

[4] C. Chevallereau, L.Adouane, On-line reference trajectory adaptation for the control of a planar biped. *Proc Conf. on Climbing and Walking Robots CLAWAR 2002*, pages 427-436, Paris, France, 2002.

[5] F. Gubina and H. Hemami and R. McGhee On the dynamic stability of biped locomotion. *IEEE Trans. on Biomedical engineering*, vol.21, N2, pages 102–108, 1974.

[6] C. Canudas de Witt. and B. Espiau and C. Urrea Orbital stabilization of underactuated mechanical systems. *Proc IFAC World Congress 2002*, Barcelona, July 2002.

[7] P.B Wieber Modelling and control of an anthropomorphic walking robot. *Ph D Thesis*, Ecole des Mines de Paris, Dec 2000.

[8] C. Azevedo Trajectory tracking for an under-actuated double inverted pendulum. *Master-thesis*, in French, INRIA Rhone Alpes, June 1999.

[9] O. Dahl and L. Nielsen Torque-limited path following by online trajectory time scaling. *IEEE Trans. on Robotics and Automation*, vol.6, N5, pages 554–561, 1990.

[10] H. Arai and K.Tanie and N. Shiroma Time-scaling Control of an Underactuated Manipulator. *J. of Robotic Systems*, vol.15, N9, pages 525–536, 1998.

[11] www-lag.ensieg.inpg.fr/PRC-Bipedes

[12] J. Furusho and M. Sano and M. Sakaguchi and E. Koizumi. Realization of Bounce Gait in a Quadruped Robot with Articular-Joint-Type Legs. *Proc. of IEEE Conf on Robotics and automation*,Capri, Italy, pages 697-702, 1995.

[13] S. Kajita and K. Tani. Experimental Study of Biped Dynamic Walking in the Linear Inverted Pendulum Mode. *Proc. IEEE Int. Conf. on Robotics & Automation*, pages 2885–2891, 1995.

[14] S.P. Bhat and D.S. Bernstein Continious finite time stabilization of the translational and rotational double integrators. *IEEE Trans. on Automat. Contr*, vol.43, pages 678–682, 1998.

[15] www.honda-p3.com